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Sharp regularity for degenerate fully nonlinear equations

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Abstract

The purpose of this thesis is to obtain new regularity results for degenerate fully nonlinear equations.

In order to keep the text as self-contained as possible, we begin by studying uniformly elliptic equations. Next we study degenerate fully nonlinear free transmission problems, where the degeneracy rate varies in the domain. We prove optimal pointwise regularity depending on the degeneracy rate. Our arguments consist of perturbation methods, relating our problem to a homogeneous, fully nonlinear, uniformly elliptic equation, as studied in the first part of the thesis.

Finally, we examine Hamilton-Jacobi equations driven by fully nonlinear degenerate elliptic operators in the presence of superlinear Hamiltonians. By exploring the Ishii-Jensen inequality, we prove that viscosity solutions are locally Lipschitz-continuous, with estimates depending on the structural conditions of the problem. We conclude this chapter with an application of our findings to a two-phase free boundary problem.

Resumo

O objetivo desta tese é obter novos resultados de regularidade para equações degeneradas completamente não lineares. Por forma a manter o texto o mais auto-contido possível, começamos por apresentar um estudo completo das equações uniformemente elípticas.

De seguida, estudamos problemas degenerados não lineares de transmissão livre, onde a degenerecência varia no domínio. Provamos regularidade ótima pontual, dependendo da degenerecência. Os nossos argumentos consistem em métodos de perturbação, relacionando o nosso problema com um homogéneo, não linear, uniformemente elíptico, como o estudado na primeira parte da tese.

Finalmente, examinamos equações de Hamilton-Jacobi controladas por um operador degenerado não linear, na presença de um Hamiltoniano super-linear. Explorando a desigualdade de Ishii-Jensen, provamos que soluções viscosas são Lipschitz-contínuas, com estimativas dependendo das condições estruturais do problema. Concluímos este capítulo com uma aplicação dos nossos resultados a um problema com fronteira livre de duas fases.

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Chapter 1

Introduction

This thesis is aimed at obtaining regularity results for some nonlinear equations. These results are useful in the sense that they yield higher regularity than the one necessary to define the solutions to the equation. The simplest example to illustrate this fact is the Laplace equation: if $u \in C^2(B_1)$ is a solution of the equation

$$\Delta u = 0, \text{ in } B_1,$$

then we actually know that $u \in C^\infty(B_1)$. The remarkable fact about this assertion is that all the partial derivatives of u have to be continuous, even the ones that do not appear in the equation. This is the idea behind a regularity result: we start by assuming that $u \in C^2$ so that we can define it to be a solution, and we obtain the higher regularity $u \in C^\infty$. Note that u may not be differentiable, or even continuous, up to the boundary ∂B_1.

We will consider a weak notion of solution, called *viscosity solution*, introduced by Crandall and Lions. This very weak notion allows for solutions to be nowhere differentiable, while providing very general existence and uniqueness theorems. For a very comprehensive exposition of the basic theory of viscosity solutions to fully nonlinear equations, we recommend the article [19].

Regularity results for equations in nondivergence form have been the subject of study for many years, starting with the linear equation

$$\mathrm{Tr} A(x) D^2 u = f(x), \text{ in } B_1. \tag{1.0.1}$$

If u is a bounded solution of (1.0.1), then we have the following results:

1. (Cordes-Nirenberg). Let $0 < \alpha < 1$ and assume that $\|A - I\|_{L^\infty(B_1)} \leq \delta = \delta(\alpha)$, for a small δ. Then $u \in C^{1,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\right);$$

2. (Schauder). If A and f belong to $C^\alpha(\overline{B})$ then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\right);$$

3. (Calderón-Zygmund). If A is continuous in B_1 and $f \in L^p(B_1)$, for some $1 < p < \infty$, then $u \in W^{2,p}(\overline{B}_{1/2})$ and
$$\|u\|_{W^{2,p}(\overline{B}_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right).$$

For a complete study of these results, we recommend the book [31].

In chapter 3 we consider extensions of these results to the fully nonlinear case, following the ideas in [15]. We present a comprehensive study of uniformly elliptic equations of the form

$$F(D^2u,x) = f(x), \quad \text{for } x \in B_1. \tag{1.0.2}$$

We prove the Aleksandrov-Bakelman-Pucci estimate, and use it to obtain a Harnack inequality and Hölder continuity of solutions; we obtain uniqueness of solutions by a variation of Jensen's uniqueness theorem in [38]. We conclude this chapter by proving interior $C^{1,\alpha}$ regularity of solutions of (1.0.2). This is done by a perturbation argument relating our equation to the simpler one $F(D^2u,x_0) = 0$, with x_0 fixed, for which we have $C^{1,\alpha}$ regularity available.

The two problems we consider involve degenerate elliptic equations which do not fall under the scope of the previous chapter. The equations we consider can be written in the more general form

$$F(D^2u,Du,u,x) = f, \quad \text{for } x \in B_1.$$

We present a maximum principle, Proposition 4.4.1, which can be found in [19]. This result can be applied to a large family of degenerate problems to obtain Hölder continuity.

In Chapter 4 we consider the following degenerate equation

$$|Du|^{\beta(x,u,Du)}F\left(D^2u\right) = f(x) \qquad \text{in } B_1, \tag{1.0.3}$$

where $\beta \geq 0$, F is uniformly elliptic and f is bounded and continuous.

Problems of the form

$$|Du|^{\beta}F\left(D^2u\right) = f$$

belong to a larger class of equations studied in a series of papers by Birindelli and Demengel, starting with the singular case in [7]. The degenerate case was also considered in [8, 9]. An important development concerning higher regularity for degenerate fully nonlinear equations was put forward in [34]. In that paper, the authors obtain local $C^{1,\alpha}$ regularity, for

$$\alpha \in (0,\alpha_0) \quad \text{and} \quad \alpha \leq \frac{1}{1+\beta},$$

with α_0 corresponding to the C^{1,α_0} regularity of the homogeneous equation $F\left(D^2u\right) = 0$. In [34, Lemma 6], the authors provide a connection between the homogeneous degenerate equation and the corresponding homogeneous uniformly elliptic equation. This step unlocks a higher regularity class which they access via a tangential path.

The methods introduced in [34] resonated, launching new perspectives in the theory of degenerate fully nonlinear equations. In [12], the authors consider the equation

$$|Du|^{\beta(x)} F\left(D^2u\right) = f(x),$$

where β is allowed to change sign, and obtain local $C^{1,\alpha}$ regularity, where

$$\alpha \in (0,\alpha_0) \quad \text{and} \quad \alpha \leq \frac{1}{1+\|\beta_+\|_\infty + \|\beta_-\|_\infty},$$

with β_+ and β_- corresponding to the positive and negative parts of β, respectively. The estimates obtained in [12] are independent of the continuity modulus of β.

In [32], the authors consider a degeneracy law depending on the sign of the solution. They study the equation

$$|Du|^{\beta_+\chi\{u>0\}+\beta_-\chi\{u<0\}} F\left(D^2u\right) = f(x),$$

which has constant degeneracy rates at each of the phases $\{\pm u > 0\}$, but has a discontinuity across the free boundary $\partial\{u=0\}$. They obtain local $C^{1,\alpha}$ regularity, for

$$\alpha \in (0,\alpha_0) \quad \text{and} \quad \alpha \leq \frac{1}{1+\max\{\beta_-,\beta_+\}}.$$

The authors also establish existence of solutions via Perron's method combined with a fixed point argument.

Finally, we mention the recent paper [29], where the author considered the following equation

$$\begin{cases} \left[|Du|^{\beta_u(x)} + a(x)\chi_{\{u>0\}}|Du|^{\beta_1} + b(x)\chi_{\{u<0\}}|Du|^{\beta_2}\right] F(D^2u) = f, & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\beta_u(x) = \beta^+\chi_{\{u>0\}} + \beta^-\chi_{\{u<0\}}$. In this setting, the author proves existence and uniqueness of solutions and obtain local $C^{1,\alpha}$ regularity.

In Chapter 4, we will present two main results which improve the regularity results in the aforementioned literature. First, we present a very simple proof of local regularity under very general assumptions on $\beta(x,u,Du)$. Indeed, we only require β to be well-defined, nonnegative and bounded from above. The second and main result consists of a pointwise regularity result under more restrictive assumptions on β.

Obtaining a pointwise regularity result, that is, a regularity result which varies over the domain, is useful when the optimal regularity of the solutions depends intrinsically on the structure of the problem, which varies over the domain. As can be seen from the previously mentioned literature, there is a clear dependence in the exponent α of the optimal regularity of solutions and the degeneracy rate β.

In Chapter 5, we consider a problem which belongs to a different class of degenerate equations, where the degeneracy is intrinsic to the operator F itself. This introduces a significant difficulty, as one does not have a solid theory to fall back to, in contrast with equation (1.0.3). We study a fully nonlinear Hamilton-Jacobi equation of the form

$$F(D^2u)+H(Du,x)=f(x) \quad \text{in} \quad \Omega\subset\mathbb{R}^d, \tag{1.0.4}$$

where $F:S(d)\to\mathbb{R}$ is degenerate elliptic and the Hamiltonian $H=H(p,x)$ satisfies standard growth and continuity conditions. These conditions include the following Hamiltonian

$$H(p,x)=a(x)\left(1+|p|^2\right)^{\frac{m}{2}}+V(x).$$

We prove that viscosity solutions to (1.0.4) are locally of class $C^{0,1}$. In addition, we examine a two-phase free boundary problem driven by the operator in (1.0.4). In this context, our findings include the existence of solutions and regularity estimates across the free boundary. The conditions we impose on the structure of the problem are fairly general and cover important examples, such as Bellman and Isaacs equations.

Hamilton-Jacobi (HJ) equations of second-order often relate to stochastic optimal control problems [30]. In this context, the value function of the optimization problem is a viscosity solution of the associated HJ equation. For developments at the intersection of viscosity solutions, Hamilton-Jacobi equations and optimal control, we refer the reader to [4, 5, 17, 20, 36, 37]. Concretely, (1.0.4) arises in the following context.

Suppose the state of a system is constrained to the open and bounded subset $\Omega\subset\mathbb{R}^d$ and driven by the stochastic differential problem

$$\begin{cases} \mathrm{d}X_t=\alpha(X_t)\mathrm{d}t+\sqrt{2}\sigma(X_t)W_t & \text{for} \quad t>t_0 \\ X_{t_0}=x\in\Omega, \end{cases} \tag{1.0.5}$$

where $\alpha:\Omega\to\mathbb{R}^d$ is a feedback control in some admissible class $\mathscr{A}$, $\sigma:\Omega\to\mathbb{R}^{d^2}$ is a matrix-valued map accounting for the volatility, W_t is a d-dimensional Brownian motion adapted to a given stochastic basis, and $t_0>0$ and $x\in\Omega$ are fixed. The cost functional of the problem is given by

$$J(x,\alpha):=\mathbb{E}^x\left[\int_{t_0}^{\tau_x}L(x_t,\alpha_t)\mathrm{d}t+\Psi(x_{\tau_x})\right], \tag{1.0.6}$$

where $\mathbb{E}^x$ denotes the conditional expectation for $X_{t_0}=x$, τ_x is the exit-time of the trajectory starting in x at t_0, the function $L:\Omega\times\mathbb{R}^d\to\mathbb{R}$ is a given Lagrangian, and $\Psi:\partial\Omega\to\mathbb{R}$ is an exit cost. The value function of the optimal control problem (1.0.5)-(1.0.6), defined as

$$u(x):=\inf_{\alpha\in\mathscr{A}}J(x,\alpha),$$

is a viscosity solution to

$$\begin{cases} \mathrm{Tr}\left[\left(\sigma(x)^T\sigma(x)\right)D^2u\right]+H(Du,x)=0 & \text{in} \quad \Omega, \\ u=\Psi & \text{on} \quad \partial\Omega, \end{cases}$$

where H is the Legendre transform of the Lagrangian L, and the boundary condition is met in some appropriate sense. If we allow the volatility $\sigma(\cdot)$ to vanish somewhere in Ω, the associated HJ equation becomes degenerate.

The study of (1.0.4) in the case $F \equiv \mathrm{Tr}$ appears, for instance, in [47]. In that paper, the author proves the existence of classical solutions for the problem under Neumann boundary conditions and natural growth regimes on the Hamiltonian H. The role of Neumann (or oblique) boundary conditions relates to state-constrained optimal control problems, as they encode a reflection at the boundary; we refer the reader to [48].

An in-depth account of state-constrained optimal control problems is the subject of [45], where the authors examine (1.0.4) in the uniformly elliptic setting. In addition to establishing local Lipschitz continuity of the solutions, the authors connect the characterisation of boundary conditions with the growth regime of the Hamiltonian. Indeed, for sub-quadratic Hamiltonians, solutions blow up as they approach $\partial\Omega$, requiring a relaxed notion of boundary condition. However, in the (strictly) super-quadratic case, solutions are globally Hölder continuous. See [6] for related developments.

The regularity theory available for (1.0.4) in the degenerate elliptic case advanced substantially with the contributions in [16]. Among the findings in that paper, we highlight the Hölder continuity of *subsolutions* in the presence of super-quadratic Hamiltonians. The remarkable aspect of this result is in the one-sided requirement entailed by the subsolution condition. To properly appreciate the minimality of such an assumption, we briefly recall the Krylov-Safonov theory. Concerning uniformly elliptic fully nonlinear operators, the latter implies Hölder continuity provided a *two-sided control* is available. Indeed, let $\mathscr{P}^{\pm}$ be the Pucci extremal operators and $C > 0$ be a constant; if $u \in C(\Omega)$ is a viscosity solution to

$$\mathscr{P}^{-}(D^2u) \leq C \quad \text{in} \quad \Omega$$

and

$$\mathscr{P}^{+}(D^2u) \geq -C \quad \text{in} \quad \Omega,$$

the Krylov-Safonov theory ensures the Hölder continuity of u. However, if one of the former inequalities fails to hold, the theory is no longer available; see, for instance, [15, 49, 55]. We notice the results in [16] also include global Hölder continuity for subsolutions and Lipschitz regularity for the solutions of the homogeneous problem.

In [2], the authors study (1.0.4), considering $F(M,x) = \mathrm{Tr}(A(x)M)$, for a degenerate elliptic matrix-valued map $A : \Omega \to \mathbb{R}^{d^2}$. The findings in [2] advance the general theory of Hamilton-Jacobi equations, as they cover a general maximum principle, Lipschitz continuity for the solutions of the homogeneous equation with explicit estimates (in terms of the matrix A and the structure of H), and state-constrained boundary conditions. We notice the Lipschitz continuity result in [2] relies on the maximum principle [19, Theorem 3.2] and explores the connection of the trace operator and eigenvalues.

Our contribution is two-fold. By developing an intrinsically nonlinear argument, we prove that viscosity solutions to (1.0.4) are Lipschitz continuous with estimates. Then we examine a consequence of our regularity result to a two-phase free boundary problem and prove the existence of solutions, with estimates in Hölder spaces.

Chapter 2

Preliminary results

2.1 Basic notation and terminology

We will denote by $\mathbb{R}^d$ the d-dimensional Euclidean space with norms

$$|x| = \sqrt{|x_1|^2 + ... + |x_d|^2},$$
$$|x|_\infty = \max\{|x_1|, ..., |x_d|\}.$$

By $B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$ we denote the open ball centered at x_0 with radius r. For simplicity, we denote $B_r := B_r(0)$.

By $Q_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0|_\infty < r/2\}$ we denote the open cube centered at x_0 with side length r. We also denote $Q_r := Q_r(0)$.

$\mathrm{diam}(\Omega)$ and $|\Omega|$ will denote the diameter and the d-dimensional Lebesgue measure of Ω, respectively.

A function L is called *affine* if $L(x) - L(0)$ is a linear function.

If u is a real-valued function, we denote by u_+ and u_- its positive and negative parts, respectively, so that $u = u_+ - u_-$.

We will make use of the multi-index terminology: if $\alpha_1, ..., \alpha_d \in \mathbb{N}_0$ and $\alpha = (\alpha_1, ..., \alpha_d)$, we call $|\alpha| = \alpha_1 + ... + \alpha_d$ and define the derivative

$$D^\alpha u(x) := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_d^{\alpha_d}} u(x).$$

2.2 Results in real analysis

In this section we include some useful results for real analysis.

Let $f : \Omega \to \mathbb{R}$ be a measurable function and denote by $\mu_f(t)$ its distribution function, that is,

$$\mu(t) = \mu_f(t) := |\{x \in \Omega : |f(x)| \geq t\}|. \tag{2.2.1}$$

This function satisfies the following properties (see also [31, Lemma 9.7]).

Lemma 2.2.1. *For any $p > 0$ and $|f|^p \in L^1(\Omega)$, it holds*

$$\mu(t) \leq t^{-p} \int_{\Omega} |f|^p \tag{2.2.2}$$

and

$$\int_{\Omega} |f|^p = p \int_0^{\infty} t^{p-1} \mu(t)\, dt. \tag{2.2.3}$$

Proof. Clearly, for every $t > 0$, it holds

$$\begin{aligned} \mu(t) t^p &\leq \int_{\{|f| \geq t\}} |f|^p \\ &\leq \int_{\Omega} |f|^p. \end{aligned}$$

For (2.2.3), we start by assuming that $p = 1$. Then

$$\int_{\Omega} |f|\, dx = \int_{\Omega} \int_0^{|f|} dt\, dx.$$

We want to apply Fubini's theorem to interchange the integration order. Note that we can rewrite the domain of integration

$$\begin{aligned} &\{(x,t) \in \Omega \times \mathbb{R} : 0 \leq t \leq |f(x)|, x \in \Omega\} \\ =&\{(x,t) \in \Omega \times \mathbb{R} : t \in \mathbb{R}, |f(x)| \geq t\}, \end{aligned}$$

and so

$$\begin{aligned} \int_{\Omega} |f|\, dx &= \int_0^{\infty} \int_{\{|f| \geq t\}} dy\, dt \\ &= \int_0^{\infty} |\{|f| \geq t\}|\, dt \end{aligned}$$

which proves the case $p = 1$. For an arbitrary $p > 0$, call $g = |f|^p \in L^1(\Omega)$ and $s = t^p$. Then by the previous argument and a simple change of variables we conclude

$$\begin{aligned} \int_{\Omega} g\, dx &= \int_0^{\infty} \mu_g(s) ds \\ &= \int_0^{\infty} |\{g > s\}|\, ds \\ &= \int_0^{\infty} |\{|f|^p > t^p\}|\, p t^{p-1} dt. \end{aligned}$$

□

The following lemma will be instrumental when obtaining Hölder continuity (see [31, Lemma 8.23] for the proof).

Lemma 2.2.2. *Let ω be a nondecreasing function on the interval $(0,R_0]$ satisfying, for all $R \le R_0$, the inequality*

$$\omega(\tau R) \le \gamma\omega(R) + \sigma(R),$$

where σ is also nondecreasing and $0 < \gamma, \tau < 1$. Then, for every $\mu \in (0,1)$ and $R < R_0$, we have

$$\omega(R) \le C\left(\left(\frac{R}{R_0}\right)^{\alpha}\omega(R_0) + \sigma(R^{\mu}R_0^{1-\mu})\right) \tag{2.2.4}$$

where $C = C(\gamma,\tau)$ and $\alpha = \alpha(\gamma,\tau,\mu)$ are positive constants.

2.3 Characterization of Hölder spaces

Hölder spaces constitute an important subspace of the space of continuously differentiable functions, and they correspond to the desired regularity in all the results we will discuss in this thesis.

We say $u \in C^{k,\alpha}(\Omega)$ if $u \in C^k(\Omega)$ and the following holds

$$|u|_{C^{k,\alpha}(\Omega)} := \max_{|\beta|=k}\sup_{x\neq y}\frac{|D^{\beta}u(x) - D^{\beta}u(y)|}{|x-y|^{\alpha}} < \infty. \tag{2.3.1}$$

The space $C^{k,\alpha}(\Omega)$ becomes a Banach space when equipped with the norm

$$||u||_{C^{k,\alpha}(\Omega)} = ||u||_{C^k(\Omega)} + |u|_{C^{k,\alpha}(\Omega)}. \tag{2.3.2}$$

We are particularly interested in the spaces $C^{0,\alpha}$, which we call C^{α}, and $C^{1,\alpha}$. We give an equivalent characterization of these spaces.

Proposition 2.3.1. *A function u is in $C^{\alpha}(\Omega)$ if and only if, for every $x \in \Omega$, there exists C_x and a uniform constant $K > 0$ such that*

$$||u - C_x||_{L^{\infty}(B_r(x))} \le Kr^{\alpha}. \tag{2.3.3}$$

Furthermore, if $|C_x| + 2K \le M$ for every $x \in \Omega$, then

$$||u||_{C^{\alpha}(\Omega)} \le M.$$

A similar characterization holds for the $C^{1,\alpha}$ spaces. We present the proof only for this case.

Proposition 2.3.2. *A function u is in $C^{1,\alpha}(\Omega)$ if and only if, for every $x \in \Omega$, there exists an affine function $\ell_x(y) = a_x + b_x \cdot (y - x)$ and a uniform constant $K > 0$ such that*

$$||u - \ell_x||_{L^{\infty}(B_r(x))} \le Kr^{1+\alpha}. \tag{2.3.4}$$

Furthermore, if $|a_x| + |b_x| + 4K \le M$ for every $x \in \Omega$, then

$$||u||_{C^{1,\alpha}(\Omega)} \le M.$$

Proof. For the direct implication, assume that $u \in C^{1,\alpha}(\Omega)$ and let $K = |u|_{C^{1,\alpha}(\Omega)}$. Define $\ell_x(y) = u(x) + Du(x) \cdot (y-x)$. Then, by Taylor's formula with Lagrange remainder,

$$u(y) = u(x) + Du(\xi) \cdot (y-x),$$

for some $\xi \in [x,y]$ (the line segment). Then

$$u(y) = u(x) + Du(x) \cdot (y-x) + (Du(\xi) - Du(x)) \cdot (y-x).$$

Hence,

$$\begin{aligned} |u(y) - \ell_x(y)| &= |u(y) - u(x) - Du(x) \cdot (y-x)| = \\ &= |(Du(\xi) - Du(x)) \cdot (y-x)| \leq \\ &\leq K|\xi - x|^{\alpha}|y-x| \leq K|y-x|^{1+\alpha}. \end{aligned}$$

Now, for any fixed $x \in \Omega$ and $r > 0$, taking $y \in B_r(x)$, clearly $|y-x|^{1+\alpha} \leq r^{1+\alpha}$ and (2.3.4) follows readily.

For the reverse implication, we start by defining the useful rescaling

$$f_r(x) = \frac{1}{r} f(rx).$$

Let x, y be two points a distance r apart, and by translation, assume that $x = -y$. Then, noting that $B_{r/2} \subset B_r(x), B_r(y)$,

$$\begin{aligned} &||\ell_{x,r/2} - \ell_{y,r/2}||_{L^\infty(B_1)} \leq ||u_{r/2} - \ell_{x,r/2}||_{L^\infty(B_1)} + ||u_{r/2} - \ell_{y,r/2}||_{L^\infty(B_1)} = \\ =&\frac{2}{r} \sup_{z \in B_1} |u(rz/2) - \ell_x(rz/2)| + \frac{2}{r} \sup_{z \in B_1} |u(rz/2) - \ell_y(rz/2)| = \\ =&\frac{2}{r} \sup_{z \in B_{r/2}} |u(z) - \ell_x(z)| + \frac{2}{r} \sup_{z \in B_{r/2}} |u(z) - \ell_y(z)| \leq \\ \leq&\frac{2}{r} \sup_{z \in B_r(x)} |u(z) - \ell_x(z)| + \frac{2}{r} \sup_{z \in B_r(y)} |u(z) - \ell_y(z)| \leq 4Kr^{\alpha}, \end{aligned}$$

assumption (2.3.4) being used in the last inequality.

We now see how the $L^\infty(B_1)$ norm controls the coefficients of an affine function. Let $l(x) = a + b \cdot x$ satisfy $||l||_{L^\infty(B_1)} \leq C$. Then, choosing $x = 0$ we get $|a| \leq C$. Choosing now $x = \frac{b}{|b|(1+\varepsilon)} \in B_1$, we get

$$\begin{aligned} \left| a + \frac{|b|}{1+\varepsilon} \right| \leq C &\implies \frac{|b|}{1+\varepsilon} \leq |a| + C \\ &\implies |b| \leq 2C(1+\varepsilon). \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we get $|b| \leq 2C$.

Now we apply this reasoning to our estimate. We have that

$$(l_{x,r/2} - l_{y,r/2})(z) = \frac{2}{r}(a + (b_x - b_y) \cdot rz) = 2a/r + 2(b_x - b_y) \cdot z,$$

where $a = \ell_x(0) - \ell_y(0)$. Our previous calculation tells us that

$$|b_x - b_y| \leq 4Kr^{\alpha}. \tag{2.3.5}$$

Finally, from (2.3.4) and the Taylor formula for $u \in C^1$, we can check that $a_x = u(x)$ and $b_x = Du(x)$. Indeed, writing $L_x(y) = u(x) + Du(x) \cdot (y - x)$, we get

$$u(y) = L_x(y) + o(r),$$

for $y \in B_r(x)$. Hence, $||u - L_x||_{L^\infty(B_r(x))} = o(r)$. So

$$\begin{aligned} ||L_x - \ell_x||_{L^\infty(B_r(x))} &\leq ||u - \ell_x||_{L^\infty(B_r(x))} + ||u - L_x||_{L^\infty(B_r(x))} \leq \\ &\leq Kr^{1+\alpha} + o(r) = o(r). \end{aligned}$$

Then, in particular,

$$|u(x) - a_x| = |L_x(x) - \ell_x(x)| = o(r)$$

hence, letting $r \to 0$, we get that $a_x = u(x)$ and so

$$L_x(y) - \ell_x(y) = (Du(x) - b_x)(y - x).$$

By contradiction, assume $b_x \neq Du(x)$. Then, we can fix

$$y = x + \frac{Du(x) - b_x}{2|Du(x) - b_x|} r \ \in B_r(x)$$

to get

$$|Du(x) - b_x| \leq \frac{o(r)}{r} \to 0$$

as $r \to 0$, hence $b_x = Du(x)$, which, together with (2.3.5), gives $|u|_{C^{1,\alpha}(\Omega)} \leq 4K$. The final expression follows from the definition of the norm in $C^{1,\alpha}(\Omega)$.

□

From Proposition 2.3.2 we see that a function belongs to the space $C^{1,\alpha}$ if it has an affine expansion at every point which approximates it very well, in every small scale. This is a much more useful characterization than the one given by the definition, because it is something we can quantify by finding this approximation ℓ_x and proving it satisfies the required estimate. However, it is impractical to prove that estimate (2.3.4) holds for every $0 < r < 1$. Instead, we prefer the following discrete characterization.

Proposition 2.3.3. *Suppose we can find $r < 1$ and sequences of affine functions $\ell_k(x) = a_k + b_k \cdot x$, such that*

$$\|u - \ell_k\|_{L^\infty(B_{r^k}(x_0))} \leq K r^{k(1+\alpha)}. \tag{2.3.6}$$

Then $u \in C^{1,\alpha}(x_0)$ with constant $C(r)K$.

The usefulness of this characterization lies in the fact that we can prove (2.3.6) using an induction argument. This is the core of many articles in regularity theory by perturbation methods, which we study in the next chapter. The main idea can be described as follows: assume, as our induction hypothesis, that estimate (2.3.6) holds up to k, and define the following rescalling

$$v_k(x) = \frac{u - \ell_k}{K r^{k(1+\alpha)}}(r^k x + x_0), \qquad \text{for } x \in B_1.$$

This rescaling v_k allows us to measure how the PDE behaves in a small scale around the point x_0. If this rescaling parameter α is properly chosen, v_k will solve an adequate PDE which will yield the next iteration of the induction step. In some cases, this microscopic behaviour of the PDE improves over each zooming iteration and in these cases, we are able to obtain a better estimate than (2.3.6), the more we zoom in. Therefore, to be able to capture the optimal regularity for these cases, we developed a new characterization of Hölder spaces, which we present in the following proposition.

Proposition 2.3.4. *Suppose we can find $r < 1$, sequences of affine functions $\ell_k(x) = a_k + b_k \cdot x$ and exponents $\alpha_k \uparrow \alpha$, such that $(\alpha_k - \alpha) = o(k)$ and*

$$\|u - \ell_k\|_{L^\infty(B_{r^k}(x_0))} \leq K r^{k(1+\alpha_k)}. \tag{2.3.7}$$

Then $u \in C^{1,\alpha}(x_0)$ with constant $C(r)K$ and $0 < \alpha < 1$.

Proof. Assume without loss of generality that $x_0 = 0$. The idea is that $\ell_k \to \ell$ uniformly, where ℓ satisfies the desired characterization.

Consider the first order scaling $f_r(x) := \frac{1}{r} f(rx)$. We have, by assumption,

$$\begin{aligned}
\|\ell_{k+1} - \ell_k\|_{L^\infty(B_{r^{k+1}})} &\leq \|u - \ell_{k+1}\|_{L^\infty(B_{r^{k+1}})} + \|u - \ell_k\|_{L^\infty(B_{r^{k+1}})} \\
&\leq K r^{(k+1)(1+\alpha_{k+1})} + K r^{k(1+\alpha_k)} \\
&\leq K r^{k(1+\alpha_k)} \left(r^{k(\alpha_{k+1} - \alpha_k)} + 1 \right) \\
&\leq 2K r^{k(1+\alpha_k)},
\end{aligned}$$

since $\alpha_{k+1} - \alpha_k \geq 0$ and $r < 1$. First order scaling gives

$$\|(\ell_{k+1})_{r^k} - (\ell_k)_{r^k}\|_{L^\infty(B_1)} \leq 2K r^{k\alpha_k}.$$

Clearly, this estimate in B_1 implies the following estimates on the coefficients

$$\begin{aligned}
|a_{k+1} - a_k| &\leq K r^{k(1+\alpha_k)}, \\
|b_{k+1} - b_k| &\leq K r^{k\alpha_k}.
\end{aligned} \tag{2.3.8}$$

Since these are Cauchy sequences, we have that

$$a_k \to a, \quad b_k \to b,$$

respectively in $\mathbb{R}$ and $\mathbb{R}^d$. It now follows that $\ell_k \to \ell$ in $L^\infty(B_1)$, where $\ell = a + b \cdot x$. Using these estimates, we get

$$\begin{aligned} \|u-\ell\|_{L^\infty(B_{r^k})} &\leq \|u-\ell_k\|_{L^\infty(B_{r^k})} + |a_k - a| + r^k|b_k - b| \\ &\leq CKr^{k(1+\alpha_k)}, \end{aligned}$$

where C depends only on r. Now, we apply the usual discretization strategy: for an arbitrary $0 < R < 1$, there exists $k \in \mathbb{N}$ such that $r^{k+1} \leq R < r^k$. Then,

$$\begin{aligned} \|u-\ell\|_{L^\infty(B_R)} \leq \|u-\ell\|_{L^\infty(B_{r^k})} &\leq CKr^{k(1+\alpha_k)} \\ &\leq r^{k(\alpha_k-\alpha)}CKr^{k(1+\alpha)}. \end{aligned}$$

Now we use the fast convergence $\alpha_k \to \alpha$, so that

$$\lim_{k\to\infty} |k(\alpha_k - \alpha)| = 0.$$

Then $r^{k(\alpha_k-\alpha)} < C_1$ and therefore we get the desired inequality

$$\|u-\ell\|_{L^\infty(B_R)} \leq C(r)KR^{1+\alpha}.$$

□

The previous proposition will be especially useful to obtain $C^{1,\alpha(\cdot)}$ regularity, when α is a function. To define this space, note that the characterization in Proposition 2.3.2 is pointwise, and therefore is easily adapted to the case where α is a function.

Definition 2.3.5. *A function u is in $C^{1,\alpha(\cdot)}(\Omega)$ if for every $x \in \Omega$, there exists an affine function $\ell_x(y) = a_x + b_x \cdot (y - x)$ and a uniform constant $K > 0$ such that*

$$||u - \ell_x||_{L^\infty(B_r(x))} \leq Kr^{1+\alpha(x)}.$$

Furthermore, we define

$$||u||_{C^{1,\alpha(\cdot)}(\Omega)} := \sup_{x\in\Omega} |a_x| + \sup_{x\in\Omega} |b_x| + 4K.$$

2.4 Viscosity solutions

The concept of viscosity solutions was introduced by Crandall and Lions in [20]. The virtues of this definition are that it allows for merely continuous functions to be solutions of PDEs and provides very general existence and uniqueness results.

Definition 2.4.1 (Viscosity solution)**.** *We say that $u \in C(B_1)$ is a viscosity subsolution (resp. supersolution) of*

$$F(D^2u, Du, u, x) = f(x), \text{ in } B_1, \tag{2.4.1}$$

or we write $F(x,u,Du,D^2u) \leq f(x)$ (resp. $\geq f(x)$), if the following condition holds:

If $x_0 \in B_1$, $\varphi \in C^2(B_1)$ and $u - \varphi$ has a local maximum (resp. minimum) at x_0, then

$$F\left(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0\right) \leq f(x_0), (\text{ resp. } \geq f(x_0)).$$

We sometimes refer to the pair of vector and matrix $(D\varphi(x_0), D^2\varphi(x_0))$ satisfying the prior property as a subjet (resp. superjet), adopting the notation in [20].

We say u is a viscosity solution, if it is both a subsolution and a supersolution.

This weaker notion of solution is consistent with the classical one, as illustrated in the following example.

Example 2.4.2. Assume that $F = -\operatorname{Tr}$, $f \equiv 0$ and $u \in C^2$ is a classical subsolution of

$$-\Delta u(x) \leq 0, \quad \text{for } x \in B_1.$$

We want to check that u is also a viscosity subsolution of this equation. Indeed, let $x_0 \in B_1$ be arbitrary, and take $\varphi \in C^2(B_1)$ such that $u - \varphi$ has a local maximum at x_0. Then

$$D^2(u - \varphi)(x_0) \leq 0.$$

Therefore

$$-\Delta(u - \varphi)(x_0) = -\operatorname{Tr}(D^2u(x_0) - D^2\varphi(x_0)) \geq 0.$$

Since $-\Delta u(x_0) \leq 0$, we obtain $-\Delta\varphi(x_0) \leq 0$, as intended. Since x_0 was arbitrary, we see that u is a viscosity subsolution. Similarly, we obtain the same result for supersolutions and we can conclude that classical solutions are also viscosity solutions.

The reciprocal implication is even simpler: If we assume that $u \in C^2(B_1)$ is a viscosity solution of the Laplace equation, then it is a classical solution. Indeed, we can just take $\varphi = u$ and noting that $u - u \equiv 0$ has local maxima and minima at every point, we get immediately from the definition that u solves the equation at every point. Hence these notions of solution are equivalent, provided we have enough regularity to define classical solutions.

Example 2.4.3. To understand why we use the term *subsolution*, let u solve the equation

$$\begin{cases} -\Delta u(x) = f, & \text{for } x \in B_1, \\ u = g, & \text{for } x \in B_1, \end{cases}$$

and let v be a strict subsolution, that is

$$\begin{cases} -\Delta v(x) < f, & \text{for } x \in B_1, \\ v = g, & \text{for } x \in B_1. \end{cases}$$

Then the function $\omega := v - u$ is a strict subsolution of the homogeneous problem

$$\begin{cases} -\Delta \omega(x) < 0, & \text{for } x \in B_1, \\ \omega = 0, & \text{for } x \in B_1. \end{cases}$$

Hence, by the maximum principle, ω is strictly negative inside B_1, implying that the subsolution v is strictly below the solution u in B_1.

The following monotocity assumption is crucial in our study

$$F(M,p,r,x) \leq F(N,p,s,x), \quad \text{whenever } N \leq M, r \leq s. \tag{2.4.2}$$

This assumption is composed of the two following assumptions

$$F(M,p,r,x) \leq F(N,p,r,x), \quad \text{whenever } N \leq M, \tag{2.4.3}$$

and

$$F(M,p,r,x) \leq F(M,p,s,x), \quad \text{whenever } r \leq s. \tag{2.4.4}$$

Condition (2.4.3) is called *degenerate ellipticity*. When combined with (2.4.4) we obtain (2.4.2) and we say that F is *Proper*. We will also always assume that F is continuous w.r.t. all its variables.

The following results are useful, although simple to prove.

Proposition 2.4.4. *Let $u \in C(B_1)$. The following are equivalent.*

1. *u is a viscosity subsolution of* (2.4.1)*;*
2. *Let $x_0 \in B_1$ and A be a small neighborhood of x_0 contained in B_1. Take $\varphi \in C^2(A)$ such that*

$$\varphi \geq u \text{ in } A \quad \text{and} \quad u(x_0) = \varphi(x_0), \tag{2.4.5}$$

   then $F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq f(x_0)$;
3. *Same as 2, with φ being a quadratic polynomial.*

Proof. Clearly *1* $\Longrightarrow$ *2* and *2* $\Longrightarrow$ *3* hold. We only prove that *3* $\Longrightarrow$ *1*. Let $x_0 \in B_1$ be an arbitrary point, A be a small neighborhood of x_0 and take $\varphi \in C^2(A)$ such that $u - \varphi$ has a maximum at x_0 in the set A. For a small $\varepsilon > 0$, define the polynomial

$$\begin{aligned} P_\varepsilon(x) := & \, u(x_0) + D\varphi(x_0) \cdot (x - x_0) \\ & + \frac{1}{2}(x - x_0) \cdot D^2\varphi(x_0)(x - x_0) - \frac{\varepsilon}{2}|x - x_0|^2. \end{aligned}$$

Since $\varphi \in C^2(A)$, we can write

$$\begin{aligned}\varphi(x) =& u(x_0) + D\varphi(x_0) \cdot (x - x_0) \\ &+ \frac{1}{2}(x - x_0) \cdot D^2\varphi(x_0)(x - x_0) + o(|x - x_0|^2).\end{aligned}$$

Therefore, for $r_\varepsilon > 0$ sufficiently small, we have $P_\varepsilon(x) \leq \varphi(x) \leq u(x)$ for every $x \in B_{r_\varepsilon}(x_0)$ with equality if $x = x_0$. Hence P_ε is a quadratic polynomial that touches u from below at the point x_0. By *3*, we have

$$F(D^2\varphi(x_0) - \varepsilon I, D\varphi(x_0), u(x_0), x_0) \leq f(x_0).$$

Recalling that F is continuous, we can take the limit $\varepsilon \to 0^+$ to obtain

$$F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq f(x_0),$$

hence we conclude that u is a viscosity subsolution of (2.4.1). □

Remark 2.4.5. Whenever property (2.4.5) holds, we say φ touches u from below at the point x_0.

Similar statements also hold for viscosity supersolutions. Indeed, if u is a viscosity subsolution of $F(D^2u, x) = f(x)$, then the function $v = -u$ is a viscosity supersolution of $G(D^2v, x) = -f(x)$, where

$$G(M, x) := -F(-M, x)$$

is still (λ, Λ)-elliptic, that is, it is uniformly elliptic with ellipticity constants λ and Λ.

2.5 Punctually second order differentiability

As we saw in the previous section, we are interested in touching our solution from above and below by paraboloids. In this section we will prove that we can get important information about the regularity of a function by studying the properties of the paraboloids that touch it.

We say that P is a convex paraboloid of opening M whenever

$$P(x) = \ell(x) + \frac{M}{2}|x|^2, \tag{2.5.1}$$

where ℓ is an affine function. We say P is a concave paraboloid of opening M if $-P$ satisfies (2.5.1).

If u is a continuous function defined on Ω and A is an open subset of Ω, for x_0 we can define

$$\overline{\Theta}(u, A)(x_0) \tag{2.5.2}$$

to be the infimum of all positive constants M for which there exists a convex paraboloid of opening M that touches u from above at x_0 in A. We define (2.5.2) to be $+\infty$ if no such constant M exists. One can check that $\overline{\Theta}(u, A)$ is a Borel measurable function on A.

Using concave paraboloids that touch u from below instead, we similarly define

$$\underline{\Theta}(u,A)(x_0) \in [0,\infty].$$

We finally consider

$$\Theta(u,A)(x_0) = \max\{\overline{\Theta}(u,A)(x_0), \underline{\Theta}(u,A)(x_0)\} \leq \infty.$$

In the next result we import boundedness from $\Theta(u,A)$ into the second derivatives of u. Note that the proof differs substantially from the reference [15, Proposition 1.2]. Indeed, we get a better constant in the inequality at the cost of the proof being much longer.

Proposition 2.5.1. *Let $u \in C(\Omega)$ and B be a convex domain such that $\overline{B} \subset \Omega$. Let $r > 0$ and define*

$$\Theta(u,r) := \Theta(u, \Omega \cap B_r(x))(x), \quad \text{for } x \in \overline{B}.$$

Assume that, for some constant K, $\Theta(u,r)(x) \leq K$ for any $x \in \overline{B}$. Then $u \in C^{1,1}(\overline{B})$ and

$$|Du(x) - Du(y)| \leq \|\Theta(u,r)\|_{L^\infty(B)} |x-y|, \quad \text{for } x,y \in \overline{B}. \tag{2.5.3}$$

Proof. Since $\Theta(u,r) \leq K$, then for every fixed x,

$$\begin{aligned} u(x) + b_1 \cdot (y-x) - \frac{K}{2}|y-x|^2 &\leq u(y) \\ \leq u(x) + b_2 \cdot (y-x) + \frac{K}{2}|y-x|^2, &\quad \text{for } y \in B_r(x). \end{aligned}$$

We start by proving that $b_1 = b_2$. In fact, since

$$(b_1 - b_2) \cdot (y-x) \leq K|y-x|^2,$$

if we assume that $b_1 \neq b_2$, we can take $0 < \delta < r$ and

$$y = x + \delta \frac{b_1 - b_2}{|b_1 - b_2|} \in B_r(x),$$

implying $|b_1 - b_2| \leq K\delta$, which is a contradiction since we can take δ to be arbitrarily small. Hence we have

$$-\frac{K}{2}|y-x|^2 \leq u(y) - u(x) - b_1 \cdot (y-x) \leq \frac{K}{2}|y-x|^2$$

and therefore u is differentiable at every point $x \in \overline{B}$ and $b_1 = Du(x)$. Additionally, we have that, for almost every $x \in \overline{B}$,

$$|Du(x) - Du(y)| \leq K_1|x-y|, \quad \text{for } y \in B_r(x), \tag{2.5.4}$$

where $K_1 := \|\Theta(u,r)\|_{L^\infty(B)} \leq K$. We will prove that (2.5.4) actually holds for every $x \in \overline{B}$. By contradiction, assume there exists $x_0 \in \overline{B}$ such that

$$|Du(x_0) - Du(y)| > K_1|x_0 - y|, \quad \text{for } y \in B_r(x_0). \tag{2.5.5}$$

Let $\mathscr{A} := \{x \in \overline{B} : \Theta(u,r)(x) \leq K_1\}$. Since $\mathscr{A}$ is dense in $\overline{B}$, there exists a sequence $(x_n)_n$ converging to x_0 with $x_n \in \mathscr{A}$ and therefore for every $0 < \delta \ll r$ there exists n_0 such that for every $n > n_0$, $B_{r-\delta}(x_0) \subset B_r(x_n)$. Thus

$$\begin{aligned} \sup_{y \in B_{r-\delta}(x_0)} |Du(x_0) - Du(y)| \leq & \sup_{y \in B_r(x_n)} |Du(x_n) - Du(y)| \\ \leq & K_1 r. \end{aligned}$$

So for every $y \in B_{r-\delta}(x_0)$,

$$|Du(x_0) - Du(y)| \leq \frac{r}{r-\delta} K_1 |x_0 - y|.$$

Since we can take δ to be arbitrarily small, we obtain a contradiction with (2.5.5).

We now extend this local estimate to the whole set $\overline{B}$. For this purpose, let $x, y \in \overline{B}$ be arbitrary. Take N to be the smallest integer which is larger than or equal to $|x-y|/r$, and define

$$z_k := \frac{k}{N} x + \left(1 - \frac{k}{N}\right) y.$$

Then $|z_k - z_{k-1}| \leq r$ and

$$\begin{aligned} |Du(x) - Du(y)| \leq & \sum_{k=1}^{N} |Du(z_k) - Du(z_{k-1})| \\ \leq & K \sum_{k=1}^{N} |z_k - z_{k-1}| = K|x-y|, \end{aligned}$$

where the equality holds because z_k are collinear. □

When u has improved regularity a stronger statement than *3* in Proposition 2.4.2 holds. For this purpose, we introduce the following definition.

Definition 2.5.2. *We say that a continuous function u in Ω is punctually second order differentiable at $x_0 \in \Omega$ if there exists a paraboloid P_{x_0} such that*

$$u(x) = P_{x_0}(x) + o(|x - x_0|^2), \quad \text{as } x \to x_0,$$

which means that $|u(x) - P_{x_0}(x)|\,|x - x_0|^{-2} \to 0$ as $x \to x_0$. We define $D^2u(x_0) := D^2 P_{x_0}$.

In this case, *3.* in Proposition 2.4.2 holds for this particular polynomial P_{x_0}. The proof is identical to Proposition 2.4.2 and shall be omitted.

Lemma 2.5.3. *Assume that u is a viscosity subsolution of* (3.1.1) *in B_1 and that u is punctually second order differentiable at $x_0 \in B_1$. Then*

$$F(D^2u(x_0), x_0) \leq f(x_0).$$

The following result consists of two assertions. The first one is due to Rademacher, see [27, Section 5.8.3, Theorem 6]. The second assertion is called the area formula for Lipschitz maps, see [28, Section 3.3.2, Theorem 1].

Theorem 2.5.4. *Let $H : \overline{B}_R \subset \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz map. Then H is differentiable almost everywhere in B_R.*

Let $A \subset B_R$ be such that $|B_R \setminus A| = 0$ and H is twice differentiable at every point in A. Then

$$|H(B_R)| \leq \int_A |\det DH|,$$

where DH denotes the Jacobian matrix of H.

Chapter 3

Uniformly elliptic equations

3.1 Introduction

This chapter concerns the study of fully nonlinear uniformly elliptic equations of the form

$$F(D^2u,x) = f, \quad \text{for } x \in B_1, \tag{3.1.1}$$

where $f : B_1 \to \mathbb{R}$ is a continuous function.

In order to make this thesis as self-contained as possible and to provide a more detailed explanation of the results presented in the book [15], which the author believes can be challenging for a Ph.D. student, we closely follow its contents in this chapter. By doing so, we hope to provide a comprehensive understanding of the concepts and results presented in the book and to facilitate the reader's comprehension of the research presented in this thesis.

We start with the following essential assumption.

Definition 3.1.1. *We say F is uniformly elliptic if there are two positive constants $0 < \lambda \leq \Lambda$, called ellipticity constants, such that*

$$\lambda \|N\| \leq F(M,x) - F(M+N,x) \leq \Lambda \|N\|,$$

for every $M,N \in S(d)$ where $N \geq 0$.

We write $N \geq 0$ whenever N is a non-negative definite matrix. We consider the matrix norm induced by the euclidean norm on $\mathbb{R}^d$, i.e.,

$$\|N\| := \sup_{|x|=1} Nx.$$

Recall that every $N \in S(d)$ can be uniquely decomposed as $N = N^+ - N^-$ where $N^+, N^- \geq 0$ and $N^+N^- = 0$. We then have the following equivalent characterization.

Lemma 3.1.2. *F is uniformly elliptic if and only if*

$$F(M,x) \leq F(M+N,x) + \Lambda \|N^+\| - \lambda \|N^-\|.$$

Proof. We start with the direct implication. Let $M,N \in S(d)$ be arbitrary, write $N = N^+ - N^-$ as above and call $M' := M - N^-$. Then

$$\begin{aligned} F(M,x) - F(M+N,x) =& F(M,x) - F(M+N^+ - N^-,x) \\ =& F(M'+N^-,x) - F(M'+N^+,x) \\ =& F(M'+N^-,x) - F(M',x) + F(M',x) - F(M'+N^+,x) \\ \leq& -\lambda \|N^-\| + \Lambda \|N^+\|, \end{aligned}$$

as intended.

For the indirect implication, let $N \geq 0$. Then

$$F(M,x) - F(M+N,x) \leq \Lambda \|N\|.$$

Also,

$$F(M+N,x) - F((M+N)-N,x) \leq -\lambda \|N\|,$$

which implies

$$F(N,x) - F(M+N,x) \geq \lambda \|N\|.$$

Combining both, we get that F is uniformly elliptic. □

Proposition 3.1.3. *Let u and v be viscosity subsolutions of* (3.1.1). *Then $h := \max\{u,v\}$ is also a viscosity subsolution of* (3.1.1).

Proof. Let $\varphi \in C^2(B_1)$ touch h from above at the point x_0 (we can assume this without loss of generality by Proposition 2.4.2 item 2). Supposing $h(x_0) = u(x_0)$, then φ touches u from above at x_0 and since u is a viscosity subsolution, $F(D^2\varphi(x_0),x_0) \leq f(x_0)$, as intended. □

The following result relates to extension of supersolutions.

Proposition 3.1.4. *Let Ω and Ω_1 be bounded domains such that $\overline{\Omega} \subset \Omega_1$. Suppose that $u \in C(\Omega_1)$ is a viscosity supersolution of $F(D^2u,x) = f(x)$ in Ω_1 and $v \in C(\overline{\Omega})$ is a viscosity supersolution of $F(D^2v,x) = g(x)$ in Ω.*

Assume that $v \geq u$ on $\partial\Omega \cap \Omega_1$ and let

$$\omega = \begin{cases} u & \text{in } \Omega_1 \setminus \Omega, \\ \min(u,v) & \text{in } \overline{\Omega} \end{cases}$$

and

$$h = \begin{cases} f, & \text{in } \Omega_1 \setminus \Omega, \\ \inf(f,g), & \text{in } \Omega. \end{cases}$$

Then ω is a viscosity supersolution in Ω_1 of $F(D^2\omega,x) = h(x)$.

Proof. Let $\varphi \in C^2$ touch ω from below at $x_0 \in \Omega_1$. If $\omega(x_0) = u(x_0)$ then φ touches u from below at x_0, and therefore $F(D^2\varphi(x_0), x_0) \geq f(x_0) \geq h(x_0)$. If $u(x_0) < \omega(x_0) = v(x_0)$ then $x_0 \in \Omega$ (by the assumption that $v \geq u$ on $\partial\Omega \cap \Omega_1$) and φ touches v from below at x_0. It follows that $F(D^2\varphi(x_0), x_0) \geq g(x_0) \geq h(x_0)$. □

An important result concerns the closedness of the family of viscosity solutions.

Proposition 3.1.5. *Let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence of (λ,Λ)-elliptic operators and let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of viscosity subsolutions of $F_k(D^2u_k, x) \leq f(x)$ in B_1. Assume that F_k converges uniformly in compact sets of $S(d)\times B_1$ to F and that u_k converges locally uniformly to $u \in C(B_1)$. Then F is (λ,Λ)-elliptic and*

$$F(D^2u, x) \leq f(x).$$

Proof. We start by proving that F is (λ,Λ)-elliptic. Let $M, N \in S(d)$ and assume $N \geq 0$. For every $\varepsilon > 0$, there exists k_0 such that for every $k \geq k_0$, we have $|F_k(M,x) - F(M,x)| \leq \varepsilon\|M\|$. Hence

$$\begin{aligned} F(M,x) - F(M+N,x) \leq& F_k(M,x) - F_k(M+N,x) + \varepsilon(\|M+N\| + \|M\|) \\ \leq& (\Lambda+\varepsilon)\|N\| + 2\varepsilon\|M\|. \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we obtain

$$F(M,x) - F(M+N,x) \leq \Lambda\|N\|.$$

Similarly we can prove the lower bound. We now prove that u is a viscosity subsolution of $F(D^2u, x) \leq f(x)$. Let $x_0 \in B_1$ be arbitrary and consider the paraboloid P that touches u from above at x_0. Take $\varepsilon > 0$ arbitrarily small and define

$$P_\varepsilon(x) = P(x) + \frac{\varepsilon}{2}|x - x_0|^2.$$

Since $u_k \to u$ locally uniformly, then for every $\delta > 0$ there exists k_0 such that for every $k \geq k_0$,

$$\sup_{x\in B_{9/10}} |u(x) - u_k(x)| \leq \delta.$$

Then, by continuity, $u_k - P_\varepsilon$ attains a local maximum at some point x_k, so we can write

$$\begin{aligned} (u_k - P_\varepsilon)(x_k) \geq& (u_k - P_\varepsilon)(x_0) \\ \geq& u(x_0) - P_\varepsilon(x_0) - \delta = -\delta. \end{aligned}$$

Furthermore,

$$P_\varepsilon(x_k) \geq u(x_k) + \frac{\varepsilon}{2}|x_k - x_0|^2,$$

and so

$$\begin{aligned}(u_k - P_\varepsilon)(x_k) &\leq u_k(x_k) - u(x_k) - \frac{\varepsilon}{2}|x_k - x_0|^2 \\ &\leq \delta - \frac{\varepsilon}{2}|x_k - x_0|^2.\end{aligned}$$

Combining both estimates, we get

$$|x_k - x_0|^2 \leq \frac{4\delta}{\varepsilon}$$

Fixing $\delta = \frac{\varepsilon^3}{4}$, thus also fixing k_0, we get

$$|x_k - x_0| \leq \varepsilon.$$

Since $u_k - P_\varepsilon$ has a local maximum at x_k and u_k is a viscosity subsolution of $F_k(D^2 u_k, x) \leq f$, we get

$$F_k(D^2 P + \varepsilon I, x_k) \leq f.$$

As $\varepsilon \downarrow 0$, we get $\delta \downarrow 0$ and $k \to \infty$ and thus

$$F(D^2 P, x) \leq f$$

since F is continuous and $x_k \to x$.

□

Remark 3.1.6. The previous Proposition is very useful when combined with Arzelà-Ascoli Theorem. Indeed, if $\{F_k\}_{k\in\mathbb{N}}$ is a sequence of (λ, Λ)-elliptic operators, then they are equicontinuous and equibounded on compact sets of $S(d)$. Therefore, there exists a subsequence which converges to some F. If we can prove similar properties for the solutions u_k, then we can again use Arzelà-Ascoli Theorem to conclude that $u_k \to u$ locally uniformly, again up to a subsequence. Then Proposition 3.1.5 implies that the limit of the solutions solves a uniformly elliptic equation, which is extremely useful.

3.2 Pucci operators and the class S of solutions

In this section, we are interested in replacing any equation of the form (1.0.2) with a couple of inequalities which will depend only on the ellipticity constants λ and Λ. To motivate this idea, we start with the linear equation

$$-\operatorname{Tr}(A(x) D^2 u) = f(x) \tag{3.2.1}$$

where $A(\cdot)$ is a matrix function with eigenvalues in the interval $[\lambda,\Lambda]$. Then, we can write

$$\lambda \operatorname{Tr} D^2u^- - \Lambda \operatorname{Tr} D^2u^+ \le -\operatorname{Tr}(A(x)D^2u),$$
$$\Lambda \operatorname{Tr} D^2u^- - \lambda \operatorname{Tr} D^2u^+ \ge -\operatorname{Tr}(A(x)D^2u),$$

and so, by defining the Pucci extremal operators

$$\mathscr{P}^-_{\lambda,\Lambda}(M) := \lambda \operatorname{Tr} M^- - \Lambda \operatorname{Tr} M^+,$$
$$\mathscr{P}^+_{\lambda,\Lambda}(M) := \Lambda \operatorname{Tr} M^- - \lambda \operatorname{Tr} M^+,$$

we see that u also satisfies the following inequalities

$$\mathscr{P}^-_{\lambda,\Lambda}(D^2u) \le f,$$
$$\mathscr{P}^+_{\lambda,\Lambda}(D^2u) \ge f.$$

This argument can be generalized to any (λ,Λ)-elliptic operator F, provided

$$F(0,x) \equiv 0, \quad \text{for } x \in B_1. \tag{3.2.2}$$

We shall always assume (3.2.2) holds true. This is not restrictive since we can rewrite (3.1.1) as

$$G(D^2u,x) := F(D^2u,x) - F(0,x) = f(x) - F(0,x).$$

Before proving this fact, we start with some important definitions and basic properties.

Definition 3.2.1. *We denote by $\underline{S}(\lambda,\Lambda,f)$ the space of viscosity subsolutions of the extremal equation*

$$\mathscr{P}^-_{\lambda,\Lambda}(D^2u) \le f.$$

Similarly, we denote by $\overline{S}(\lambda,\Lambda,f)$ the space of viscosity supersolutions of the extremal equation

$$\mathscr{P}^+_{\lambda,\Lambda}(D^2u) \ge f.$$

We also define

$$S(\lambda,\Lambda,f) := \underline{S}(\lambda,\Lambda,f) \cap \overline{S}(\lambda,\Lambda,f)$$

and

$$S^*(\lambda,\Lambda,f) := \underline{S}(\lambda,\Lambda,|f|) \cap \overline{S}(\lambda,\Lambda,-|f|).$$

Remark 3.2.2. For simplicity, we will often omit the ellipticity constants in all the objects we have just defined. We also use the notation $S(0) = S$ and similarly for the remaining classes of solutions. We have the simple inclusion $S(f) \subset S^*(f)$ and $S = S^*$.

We present some basic properties of the Pucci operators.

Lemma 3.2.3.

(1) $\mathscr{P}^-(M) \le \mathscr{P}^+(M)$*;*

(2) If $\lambda' \le \lambda \le \Lambda \le \Lambda'$*, then*

$$\mathscr{P}^+_{\lambda,\Lambda}(M) \le \mathscr{P}^+_{\lambda',\Lambda'}(M)$$

and

$$\mathscr{P}^-_{\lambda,\Lambda}(M) \ge \mathscr{P}^-_{\lambda',\Lambda'}(M);$$

(3) $\mathscr{P}^-(M) = -\mathscr{P}^+(-M)$*;*

(4) $\mathscr{P}^\pm(\alpha M) = \alpha \mathscr{P}^\pm(M)$ *whenever* $\alpha \ge 0$*;*

(5) $\mathscr{P}^-(M) + \mathscr{P}^+(N) \le \mathscr{P}^+(M+N) \le \mathscr{P}^+(M) + \mathscr{P}^+(N)$*;*

(6) $\mathscr{P}^-(M) + \mathscr{P}^-(N) \le \mathscr{P}^-(M+N) \le \mathscr{P}^-(M) + \mathscr{P}^+(N)$*;*

(7) If $N \ge 0$ *then* $-\Lambda\|N\| \le \mathscr{P}^-(N) \le \mathscr{P}^+(N) \le -d\lambda\|N\|$*;*

(8) $\mathscr{P}^\pm$ *are uniformly elliptic operators with ellipticity constants* λ *and* $d\Lambda$*.*

Similar properties also hold for the classes of solutions.

Lemma 3.2.4.

(1) If $\lambda' \le \lambda \le \Lambda \le \Lambda'$ *then* $S(\lambda,\Lambda,f) \subset S(\lambda',\Lambda',f)$*. The same holds for* $\underline{S}$*,* $\overline{S}$ *and* S^**;*

(2) If $u \in \underline{S}(f)$ *then* $-u \in \overline{S}(-f)$*;*

(3) Let $\alpha > 0$*,* $r > 0$*,* $u \in \underline{S}(f)$ *and define* $v(x) = \alpha u(x/r)$*, for* $x \in B_r$*. Then* $v \in \underline{S}(\alpha f(x/r)/r^2)$*;*

(4) Let $u \in \underline{S}(f)$*,* $\varphi \in C^2(B_1)$ *and* $\mathscr{P}^+(D^2\varphi(x)) \le g(x)$*, for* $x \in B_1$*. Then* $u - \varphi \in \underline{S}(f-g)$*.*

The following proposition generalizes the argument in the beginning of this section to the fully nonlinear case.

Proposition 3.2.5. *Let u be a viscosity subsolution of* $F(D^2u,x) \le f$ *(resp. supersolution of* $F(D^2u,x) \ge f$*). Then*

$$u \in \underline{S}\left(\frac{\lambda}{d},\Lambda,f\right) \quad (\text{resp. } u \in \overline{S}\left(\frac{\lambda}{d},\Lambda,f\right)).$$

More generally, for any $\varphi \in C^2(B_1)$*, we have*

$$u - \varphi \in \underline{S}\left(\frac{\lambda}{d},\Lambda,f(x) - F(D^2\varphi(x),x)\right)$$

$$(\text{resp. } u - \varphi \in \overline{S}\left(\frac{\lambda}{d},\Lambda,f(x) - F(D^2\varphi(x),x)\right)).$$

Proof. We only prove the second statement since it implies the first one by choosing $\varphi = 0$ and recalling (3.2.2).

Let $\varphi \in C^2(B_1)$ be an arbitrary function. We want to prove that

$$u - \varphi \in \underline{S}\left(\frac{\lambda}{d}, \Lambda, f(x) - F(D^2\varphi(x), x)\right),$$

that is, if $\psi \in C^2(B_1)$ is a function which touches $u - \varphi$ from above at x_0, then

$$\mathscr{P}^-_{\lambda/d,\Lambda}(D^2\psi(x_0)) \leq f(x_0) - F(D^2\varphi(x_0), x_0).$$

Note that in this case the function $\varphi + \psi$ touches u from above at x_0. Hence

$$\begin{aligned} f(x_0) \geq & F(D^2\varphi(x_0) + D^2\psi(x_0), x_0) \\ \geq & F(D^2\varphi(x_0), x_0) - \Lambda \|D^2\psi(x_0)^+\| + \lambda \|D^2\psi(x_0)^-\| \\ \geq & F(D^2\varphi(x_0), x_0) - \Lambda \operatorname{Tr} D^2\psi(x_0)^+ + \frac{\lambda}{d} \operatorname{Tr} D^2\psi(x_0)^- \\ = & F(D^2\varphi(x_0), x_0) + \mathscr{P}^-_{\lambda/d,\Lambda}(D^2\psi(x_0)), \end{aligned}$$

which is the desired inequality. In the first inequality, we used that u is a viscosity subsolution of $F(D^2u, x) \leq f$ and the second inequality follows from Lemma 3.1.2. □

3.3 Examples of fully nonlinear equations

1. ***Isaacs Equations*** Let $\mathscr{A}$ and $\mathscr{B}$ be any index sets and $A_{\alpha,\beta}(\cdot)$ be matrix functions with eigenvalues in $[\lambda, \Lambda]$. Then Isaac's equation is of the form

$$F(D^2u, x) := \sup_{\beta \in \mathscr{B}} \inf_{\alpha \in \mathscr{A}} (-\operatorname{Tr} A_{\alpha,\beta}(x) D^2u - f_{\alpha,\beta}(x)) = 0.$$

   This important class of equations arises in stochastic differential games, see [54]. We stress the fact that these operators are fully nonlinear, lack differentiability and are non-convex/concave.

2. ***Bellman Equations*** In the particular case when $|\mathscr{B}| = 1$, we get the simpler equation

$$F(D^2u, x) := \inf_{\alpha \in \mathscr{A}} (-\operatorname{Tr} A_\alpha(x) D^2u - f_\alpha(x)) = 0.$$

   Since we are taking the infimum of linear functions, $F(\cdot, x)$ is a concave function. This operator describes the optimal cost in a stochastic control problem, see [42].

3.4 Alexandroff-Bakelman-Pucci estimate

The Alexandroff-Bakelman-Pucci estimate, often abbreviated ABP estimate, is a form of maximum principle for weak solutions of nonlinear elliptic equations. It provides a pointwise estimate on solutions in terms of a measure theoretic quantity.

We start with some definitions and properties regarding convex functions.

Definition 3.4.1. *Let ω be a function defined in a set $A \subset \mathbb{R}^d$ and let $x_0 \in A$. If ℓ is an affine function that touches ω from below at x_0 in A, then we say that ℓ is a supporting hyperplane of ω at x_0.*

Remark 3.4.2. We recall that if A is an open convex set and ω is a convex function defined on A, then it admits a supporting hyperplane at every point $x_0 \in A$, however this hyperplane may not be unique (in case ω is not C^1). The existence of this supporting hyperplane is guaranteed by Hahn-Banach theorem applied to the open convex set $\{(x,y) \in A \times \mathbb{R} : y > \omega(x)\}$.

Definition 3.4.3. *Let v be a continuous function in an open convex set A. The convex envelope of v in A is defined as*

$$\begin{aligned}\Gamma_v(x) &:= \sup_{\omega} \{\omega(x) : \omega \le v \text{ in } A, \, \omega \text{ convex in } A\} \\ &= \sup_{\ell} \{\ell(x) : \ell \le v \text{ in } A, \, \ell \text{ is affine}\}\end{aligned}$$

Clearly Γ_v is a convex function in A. The set $\{v = \Gamma_v\}$ is called the (lower) contact set of v. The points in the contact set are called contact points.

The classical ABP estimate states that any strong supersolution u of the linear elliptic problem

$$\begin{cases} -\operatorname{Tr}(A(x)D^2u) \ge f, & \text{for } x \in B_1, \\ u(x) \ge 0, & \text{for } x \in \partial B_1, \end{cases}$$

satisfies the estimate

$$\sup_{B_1} u^- \le C \left(\int_{\{\Gamma_u = u\}} \left(\frac{f^-}{\mathscr{D}^*} \right)^d \right)^{\frac{1}{d}},$$

where $\mathscr{D}^*$ corresponds to the geometrical average of the eigenvalues of A. For a proof of this result see [31, Section 9.1]. In this chapter we will present an adaptation of the classical ABP estimate to viscosity solutions of fully nonlinear equations, developed by Caffarelli in [13, 14], stated as follows.

Theorem 3.4.4. *Let $u \in \overline{S}(f)$ in B_R where f is a continuous and bounded function. Assume that u is continuous in $\overline{B}_R$ and $u \ge 0$ on ∂B_R. Then*

$$\sup_{B_R} u^- \le CR \left(\int_{B_R \cap \{u = \Gamma_u\}} (f^-)^d \right)^{\frac{1}{d}}. \tag{3.4.1}$$

Here we have extended u by zero outside of B_R, so that $-u^-$ is continuous in B_{2R}; Γ_u corresponds to the convex envelope of $-u^-$ in B_{2R} and C is a universal constant.

The fact that the source term f only contributes over the contact set should not be overlooked and will play an important role in the future. The main difficulty in proving Theorem 3.4.3 for a viscosity supersolution is the fact that u might be very irregular. However, we will prove that Γ_u is $C^{1,1}$ and so the proof of the classical ABP estimate for strong solutions applies. The following result proves this regularity at contact points.

Lemma 3.4.5. *Let $u \in \overline{S}(f)$ in B_δ. Assume that f is bounded and continuous, and that φ is a convex function in B_δ such that $0 \le \varphi \le u$ in B_δ and $0 = \varphi(0) = u(0)$. Then*

$$\varphi(x) \le C\left(\sup_{B_\delta} f^-\right)|x|^2, \quad \text{for } x \in B_{\nu\delta}, \tag{3.4.2}$$

where $\nu < 1$ and C are positive universal constants.

Proof. Fix $0 < r \le \delta/4$ and define

$$\overline{C} = \frac{1}{r^2}\sup_{B_r} \varphi.$$

Since φ is convex, it must attain its supremum at the boundary, therefore there must exist $x_0 \in \partial B_r$ such that

$$\overline{C}r^2 = \varphi(x_0).$$

Since the set $B := \{x \in B_\delta \,:\, \varphi(x) \le \overline{C}r^2\}$ is convex and includes B_r, the same reasoning implies that $x_0 \in \partial B$. It follows from Remark 4 that there exists a supporting hyperplane H to B at x_0. Since $x_0 \in \partial B \cap \partial B_r$, this H is unique and corresponds to the tangent hyperplane in $\mathbb{R}^d$ to B_r that passes through x_0. Indeed, if there exist two distinct hyperplanes supporting B at x_0, then B has to “curve in a non C^1 way" at x_0, and since $x_0 \in B_r \subset B$, it has to curve “outwards", which violates the convexity property.

Thus

$$\varphi \ge \overline{C}r^2, \quad \text{for } x \in H \cap B_\delta. \tag{3.4.3}$$

For convenience and without loss of generality, we can assume that $x_0 = (0,\dots,0,r)$ and $H = \{x \in \mathbb{R}^d \,:\, x_d = r\}$. We write $x = (x', x_d)$ where $x' \in \mathbb{R}^{d-1}$.

Consider the open set $A := B_{\delta/2} \cap \{-r < x_d < r\}$ obtained by cutting $B_{\delta/2}$ by H and $-H$. Then $\partial A = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= \overline{B}_{\delta/2} \cap \{x_d = r\},\\ A_2 &= \overline{B}_{\delta/2} \cap \{x_d = -r\},\\ A_3 &= \partial\overline{B}_{\delta/2} \cap \{-r \le x_d \le r\}. \end{aligned}$$

By (3.4.3) and since $\varphi \ge 0$ in B_δ, we get

$$\varphi \ge \overline{C}r^2 \quad \text{in } A_1 \quad \text{and} \quad \varphi \ge 0 \quad \text{in } A_2 \cup A_3. \tag{3.4.4}$$

To complete the proof, it suffices to obtain an upper bound for $\overline{C}$. To do so, we will construct a barrier function P which curves upwards “too much" in the direction x_d and is therefore a subsolution of some extremal equation. It will additionally curve downwards in the remaining directions x' to guarantee that it touches u from below at some point $y \in A$. Since $u \in \overline{S}(f)$, P will be a supersolution to the extremal equation

$$\mathscr{P}^+(D^2P) \geq f(y).$$

Combining both extremal equations will yield the desired estimate. For this purpose, define the paraboloid

$$P(x) = \frac{\overline{C}}{8}(x_d + r)^2 - 4\overline{C}\frac{r^2}{\delta^2}|x'|^2.$$

In A_3, $\delta^2/4 = |x|^2 \leq |x'|^2 + r^2 \leq |x'|^2 + \delta^2/16$. Hence

$$4\overline{C}\frac{r^2}{\delta^2}|x'|^2 \geq \frac{3}{4}\overline{C}r^2.$$

Therefore $P \leq \overline{C}r^2$ in A_1 and $P \leq 0$ in $A_2 \cup A_3$. Combining this with (3.4.4) we get $P \leq \varphi \leq u$ in ∂A. On the other hand, $P(0) > 0 = \varphi(0) = u(0)$. It follows that $u - P$ attains its minimum at some interior point $y \in A$ and so an appropriate vertical translation of P, call it again P, touches u from below at $y \in A$. Since $u \in \overline{S}(f)$, at the point y it must hold

$$\mathscr{P}^+(D^2P) \geq f(y) \geq -\sup_{B_\delta} f^-. \tag{3.4.5}$$

Since

$$D^2P = \operatorname{diag}\left(-8\overline{C}\frac{r^2}{\delta^2}, \dots, -8\overline{C}\frac{r^2}{\delta^2}, \frac{\overline{C}}{4}\right),$$

we can explicitly compute

$$\mathscr{P}^+(D^2P) = -\frac{\lambda\overline{C}}{4} + \Lambda(d-1)8\overline{C}\frac{r^2}{\delta^2}.$$

Taking

$$r \leq \frac{\delta}{8}\sqrt{\frac{\lambda}{(d-1)\Lambda}},$$

we get

$$\mathscr{P}^+(D^2P) \leq -\frac{\lambda\overline{C}}{8}.$$

Combining this with (3.4.5), we conclude that

$$\overline{C} \leq \frac{8}{\lambda}\sup_{B_\delta} f^-.$$

Therefore $\sup_{B_r} \varphi \leq (8/\lambda)(\sup_{B_\delta} f^-)r^2$. Taking $r = |x|$ we obtain (3.4.2) with

$$\nu = \frac{1}{8}\sqrt{\frac{\lambda}{(d-1)\Lambda}}.$$

□

The following result relates the pointwise behavior of a function u with a measure theoretical behaviour of its convex envelope Γ_u. Note that we do not require $u \in \overline{S}(f)$, therefore this result is independent of the differential equation. See [31, Section 9.1] for a more detailed proof.

Lemma 3.4.6. *Let $u \in C(\overline{B}_R)$ satisfy $u \geq 0$ on ∂B_R and let Γ_u be defined as in Theorem 3.4.3. Assume further that $\Gamma_u \in C^{1,1}(\overline{B}_R)$.*

Then there exists a set $A \subset B_R$ such that $|B_R \setminus A| = 0$, Γ_u is second order differentiable at any $x \in A$ and

$$\sup_{B_R} u^- \leq C(d)R\left(\int_A \det D^2\Gamma_u\right)^{\frac{1}{d}}, \tag{3.4.6}$$

where $C(d)$ is a dimensional constant.

Proof. Since the proof if trivial if $u \geq 0$, we may assume that $u^- \not\equiv 0$. From $u^- = 0$ on ∂B_R we have that

$$M := \sup_{B_R} u^- = u^-(x_0) > 0,$$

for some $x_0 \in B_R$. The idea of the proof is to use the area formula (see Theorem 2.5.4) to the Lipschitz map $D\Gamma_u$. The main step is to obtain

$$B_{M/3R} \subset D\Gamma_u(B_R), \tag{3.4.7}$$

i.e., for every $y \in B_{M/3R}$, there exists $x \in B_R$ such that $y = D\Gamma_u(x)$. Since Γ_u is convex and continuously differentiable, $D\Gamma_u(x)$ corresponds to the gradient of the corresponding hyperplane of Γ_u at x. For each number C and each vector $\xi \in B_{M/3R}$, consider the affine function $L(x) = C + \xi \cdot (x - x_0)$. We will prove that for every $\xi \in B_{M/3R}$, there exists C such that L touches u from below at a point $x^* \in B_R$. Indeed, if this is not true, then there exists C such that L touches u from below at $x^* \notin B_R$. Since $-u^- \equiv 0$ outside B_R, $x^* \in \partial B_{2R}$ and $L(x^*) = u(x^*) = 0$. Note also that

$$-M = u(x_0) > L(x_0) = C.$$

Hence

$$\begin{aligned} L(x^*) &< -M + \xi \cdot (x^* - x_0) \\ &\leq -M + |\xi|\,|x^* - x_0| \\ &\leq -M + \frac{M}{3R}3R = 0 \end{aligned}$$

which is a contradiction. This proves (3.4.7). By considering the measures of these sets, we get

$$C(d)\frac{M^d}{R^d} \le |D\Gamma_u(B_R)|. \tag{3.4.8}$$

By Theorem 2.5.4 we know that $D\Gamma_u$ is differentiable almost everywhere in B_R. That is, $D\Gamma_u$ is differentiable at any $x \in A$ with $|B_R \setminus A| = 0$. Therefore, applying the area formula (see again Theorem 2.5.4) and recalling that since Γ_u is convex, $D^2\Gamma_u(x) \ge 0$ for every $x \in A$,

$$|D\Gamma_u(B_R)| \le \int_A \det D^2\Gamma_u.$$

Combining this with (3.4.8) we get

$$C(d)\frac{M^d}{R^d} \le \int_A \det D^2\Gamma_u.$$

□

In order to conclude Theorem 3.4.3 from Lemma 3.4.5, it suffices to prove that $\Gamma_u \in C^{1,1}(\overline{B}_R)$ and $\det D^2\Gamma_u(x) = 0$ a.e. $x \in B_R \setminus \{u = \Gamma_u\}$. Lemma 3.4.4 already addresses the contact points. The following result studies the remaining points. Note that again we don't assume that u belongs to $\overline{S}(f)$.

Lemma 3.4.7. *Let $u \in C(\overline{B}_R)$ such that $u \ge 0$ on ∂B_R and let Γ_u be defined as in Theorem 3.4.3. Fix $K > 0$ and $0 < r \le R$ and assume that for any $x_0 \in \overline{B}_R \cap \{u = \Gamma_u\}$ there exists a convex paraboloid of opening K that touches Γ_u by above at x_0 in $B_r(x_0)$; that is, with the terminology of Section 2.5,*

$$\overline{\Theta}(\Gamma_u, B_r(x_0))(x_0) \le K, \quad \textit{for } x_0 \in \overline{B}_d \cap \{u = \Gamma_u\}. \tag{3.4.9}$$

Then $\Gamma_u \in C^{1,1}(\overline{B}_R)$ and hence there exists a set $A \subset B_R$ such that $|B_R \setminus A| = 0$ and Γ_u is second order differentiable at any $x \in A$. Moreover, we have that

$$\sup_{B_R} u^- \le C(d)R\left(\int_{A\cap\{u=\Gamma_u\}} \det D^2\Gamma_u\right)^{\frac{1}{d}},$$

where $C(d)$ is a dimensional constant.

Recall that Γ_u is defined as the convex envelope in B_{2R} of $-u^-$, after extending $u = 0$ outside $\overline{B}_R$. Therefore $(\Gamma_u)_{|\partial B_{2R}} \equiv 0$ and by convexity, $\Gamma_u \le 0$ in $\overline{B}_{2R}$. Since Theorem 3.4.3 is trivial when $u \ge 0$, we can (and do) assume that there exists some point $z \in B_R$ such that $u(z) < 0$ and so $\Gamma_u < 0$ in B_{2R}. Note also that Γ_u has a supporting hyperplane at every point in $\overline{B}_{2R}$, i.e., for any $x_0 \in \overline{B}_{2R}$, there exists an affine function ℓ that touches Γ_u from below at $x_0 \in \overline{B}_{2R}$ – see Remark 4 when $x_0 \in B_{2R}$; if $x_0 \in \partial B_{2R}$ the existence of ℓ follows from the fact that Γ_u in polar coordinates is affine with respect to r for $R \le r \le 2R$. Indeed, it is clear that $\{u = \Gamma_u\} \subset B_R$ since $u \equiv 0$ in $B_{2R} \setminus B_R$ and $\Gamma_u < 0$ in B_{2R}. Let now $e \in \partial B_1$ and $R \le r \le 2R$. We can write $r = \lambda R + (1-\lambda)2R$ with $\lambda = 2 - r/R$ and we want to prove that

$$\begin{aligned}\Gamma_u(re) =& \lambda\Gamma_u(Re) + (1-\lambda)\Gamma_u(2Re)\\ =& \left(2-\frac{r}{R}\right)\Gamma_u(Re).\end{aligned}$$

Defining the affine function

$$f_e(r) = \left(2-\frac{r}{R}\right)\Gamma_u(Re),$$

by convexity of Γ_u, it holds $\Gamma_u(re) \le f_e(r)$. Since f_e is also convex and $f(r) \le 0 = u(re)$ for $R \le r \le 2R$, by maximality of Γ_u it must hold $\Gamma_u(re) \ge f_e(r)$, which implies that Γ_u is affine with respect to r.

Now we present the proof of Lemma 3.4.6.

Proof of Lemma 3.4.6. By Lemma 3.4.5 it suffices to prove that $\Gamma_u \in C^{1,1}(\overline{B}_R)$ and

$$\det D^2\Gamma_u(x) = 0, \quad \text{a.e. } x \in B_R \setminus \{u = \Gamma_u\}. \tag{3.4.10}$$

To prove this, we will observe that for a.e. $x \in B_R \setminus \{u = \Gamma_u\}$, Γ_u behaves linearly in at least one direction e at a neighborhood of x hence in this direction $\partial_e^2\Gamma_u = 0$ and therefore (3.4.10) follows.

As we noted before, for any $y_0 \in \overline{B}_R \cap \{u = \Gamma_u\}$, there is a hyperplane L_0 that touches Γ_u from below at y_0; by (3.4.9), there is a paraboloid that touches Γ_u from above at y_0. It follows that $u \in C^{1,1}$ at y_0, that $L_0(y) = \Gamma_u(y_0) + D\Gamma_u(y_0)\cdot(y-y_0)$ and

$$\Theta(\Gamma_u, B_r(y_0))(y_0) \le K, \quad \text{for } y_0 \in \overline{B}_R \cap \{u = \Gamma_u\}. \tag{3.4.11}$$

Step 1 – Let $x_0 \in \overline{B}_R \setminus \{u = \Gamma_u\}$ and let L be the supporting hyperplane for Γ_u at x_0 in $\overline{B}_{2R}$. We claim the following

(a) x_0 belongs to a simplex S with vertices $x_1, \dots, x_{d+1}$, (i.e. S is the convex hull of the set $\{x_1, \dots, x_{d+1}\}$) and $L = \Gamma_u$ in this simplex. The points $x_1, \dots, x_{d+1}$ need not be all distinct. Moreover, all vertices x_i are in $B_R \cap \{u = \Gamma_u\}$, except for possibly one $x_{d+1} \in \partial B_{2R}$.

(b) If we write $x_0 = \sum_1^{d+1} \lambda_i x_i$, where λ_i are non-negative and add up to 1, then $\lambda_i \ge 1/(3d)$ for at least one index i for which $x_i \in B_R \cap \{u = \Gamma_u\}$.

We start by proving the claim. Since Γ_u is convex, it has a supporting hyperplane at x_0, i.e., there exists an affine function L that touches Γ_u from below at x_0. By maximality of Γ_u, this L has to touch $-u^-$ at at least one point in $\overline{B}_{2R}$, since otherwise we could slide up the function L until we touch $-u^-$ thus violating the maximality of Γ_u. Therefore the closed convex hull C of $\{x \in \overline{B}_{2R} : L(x) = -u^-(x)\}$ is nonempty.

Assume that $x_0 \notin C$. Let $a = \operatorname{dist}(x_0, C)$ and $y_0 \in C$ be the only point such that $|x_0 - y_0| = a$. Define

$$\ell(x) = \frac{a}{2} - \frac{y_0 - x_0}{2}\cdot(x - x_0).$$

Then $\ell(x_0) > 0$ and for every $y \in C$,

$$\ell(y) \leq \ell(y_0) < 0.$$

There exists a small $\varepsilon > 0$ such that if we define the open set $C_\varepsilon := \{x \in \overline{B}_{2R} : \operatorname{dist}(x, C) < \varepsilon\}$, ℓ is still negative in C_ε. Furthermore, since $-u^- - L > 0$ outside of C, there exists a small $\varepsilon_1 > 0$ such that $-u^- - L > \varepsilon_1$ outside of C_ε. Let $M := \max_{\overline{B}_{2R}} \ell$ and $\delta = \varepsilon_1/M$. Then $L + \delta\ell \leq -u^-$ in $\overline{B}_{2R}$.

However, by maximality of Γ_u,

$$\Gamma_u(x_0) \geq (L + \delta\ell)(x_0) > L(x_0)$$

which contradicts the fact that $\Gamma_u(x_0) = L(x_0)$. Hence $x_0 \in C$.

We can therefore apply Caratheodory's theorem which implies that x_0 is the convex combination of $d+1$ points $x_1, \ldots, x_{d+1}$ in $\{x \in \overline{B}_{2R} : L(x) = -u^-(x)\}$. We easily check that there is at most one $x_i \in \partial B_{2R}$, since otherwise there is a line segment $[x_i, x_j] \subset \overline{B}_{2R}$ such that for every y in the open line segment (x_i, x_j), $0 = L(y) \leq \Gamma_u(y)$ which contradicts the fact that $\Gamma_u < 0$ in B_{2R}. For the same reason, there is no x_i in $B_{2R} \setminus B_R$. This proves part *(a)* of the claim.

We now prove *(b)*. If all x_i belong to B_R, then $\lambda_i \geq 1/(d+1) \geq 1/(3d)$ for at least one index i. If on the other hand $x_{d+1} \in \partial B_{2R}$ and $\lambda_i < 1/(3d)$ for every $i = 1, \ldots, d$, then $\lambda_{d+1} > 2/3$ and therefore

$$\begin{aligned} |x_0| \geq & \lambda_{d+1}|x_{d+1}| - \sum_{i=1}^{d} \lambda_i |x_i| \\ > & \frac{2}{3} 2R - \frac{1}{d} dR = R \end{aligned}$$

which is a contradiction, since $x_0 \in \overline{B}_R$. This concludes Step 1 which implies that for every $x_0 \in B_R \setminus \{u = \Gamma_u\}$, there exists an open line segment through x_0 where Γ_u is affine. Therefore to complete the proof it suffices to prove that Γ_u is twice differentiable a.e. in $\overline{B}_R$. We do this by proving that $\Gamma_u \in C^{1,1}(\overline{B}_R)$.

Step 2 – Let x_0 and L be as in Step 1 and take $h \in \mathbb{R}^d$ with $|h| < R$. By *(b)*, we can relabel x_i such that $x_1 \in B_R \cap \{u = \Gamma_u\}$ and $\lambda_1 \geq 1/(3d)$ and write

$$x_0 + h = \lambda_1 \left(x_1 + \frac{h}{\lambda_1} \right) + \lambda_2 x_2 + \cdots + \lambda_{d+1} x_{d+1}.$$

Since L touches Γ_u from below and Γ_u is convex, we have

$$L(x_0 + h) \leq \Gamma_u(x_0 + h) \leq \lambda_1 \Gamma_u \left(x_1 + \frac{h}{\lambda_1} \right) + \lambda_2 \Gamma_u(x_2) + \cdots + \lambda_{d+1} \Gamma_u(x_{d+1}).$$

Supposing that $|h| < r/(3d)$, we have $|h|/\lambda_1 < r$. Recall that L is a supporting hyperplane for Γ_u at $x_1 \in B_R \cap \{u = \Gamma_u\}$ and $L(x) = \Gamma_u(x_1) + D\Gamma_u(x_1) \cdot (x - x_1)$; this together with (3.4.11) applied with

y_0 replaced by x_1 give

$$L(x_0+h) \le \Gamma_u(x_0+h) \le \lambda_1 \left(L\left(x_1 + \frac{h}{\lambda_1}\right) + \frac{K}{2}\left|\frac{h}{\lambda_1}\right|^2 \right) + \lambda_2 L(x_2) + \cdots + \lambda_{d+1} L(x_{d+1}).$$

Therefore

$$\Theta(\Gamma_u, B_{r/(3d)}(x_0))(x_0) \le 3dK, \quad \text{for } x_0 \in \overline{B}_R \setminus \{u = \Gamma_u\},$$

which combined with (3.4.11) gives

$$\Theta(\Gamma_u, B_{r/(3d)}) \le 3dK$$

for every $x \in \overline{B}_R$. We can now apply Proposition 2.5.1 which implies that $\Gamma_u \in C^{1,1}$.

To complete the proof, let $x \in A \cap (B_R \setminus \{u = \Gamma_u\})$. Recall that $|B_R \setminus A| = 0$ and Γ_u is twice differentiable in A. Since there exists a line segment $[x-h, x+h]$ for some $|h| \ll 1$ where Γ_u is affine, it follows

$$h \cdot D^2\Gamma_u(x)h = 0.$$

This combined with the fact that $D^2\Gamma_u$ is a symmetric non-negative definite matrix implies that it has at least one eigenvalue which is zero, therefore (3.4.10) follows.

□

We are now ready to prove Theorem 3.4.3.

Proof of Theorem 3.4.3. We claim that

$$\overline{\Theta}(\Gamma_u, B_{\nu R}(x_0))(x_0) \le C \sup_{B_R} f^-, \quad \text{for } x_0 \in \overline{B}_R \cap \{u = \Gamma_u\} \tag{3.4.12}$$

and

$$\det D^2\Gamma_u(x_0) \le C f^-(x_0), \quad \text{a.e. } x_0 \in B_R \cap \{u = \Gamma_u\}, \tag{3.4.13}$$

where C and $\nu < 1$ are positive universal constants. Theorem 3.4.3 follows immediately from this claim using Lemma 3.4.6 with $r = \nu R$ and $K = C \sup_{B_R} f^-$.

We proceed by proving the claim. Take $x_0 \in \overline{B}_R \cap \{u = \Gamma_u\}$ and let L be the supporting hyperplane for Γ_u at x_0. Proposition 3.1.4 applied with $F = \mathscr{P}^+$ implies that $-u^- = \min(u, 0) \in \overline{S}(\lambda, \Lambda, f^- \chi_{B_R})$ in B_{2R}.

Since L is affine, $\Gamma_u - L$ is convex and $-u^- - L \in \overline{S}(\lambda, \Lambda, f^- \chi_{B_R})$ in B_{2R}. We also have (for every $0 < \delta < R$) that $B_\delta(x_0) \subset B_{2R}$, $0 \le \Gamma_u - L \le -u^- - L$ in $B_\delta(x_0)$ and equalities hold at x_0. Applying

Lemma 3.4.4 with u replaced by $-u^- - L$ we get

$$L(x) \le \Gamma_u(x) \le L(x) + C\left(\sup_{B_\delta(x_0)} f^- \chi_{B_R}\right)|x-x_0|^2,$$

for every $x_0 \in B_{\nu\delta}(x_0)$. This implies (3.4.12) and letting $\delta \to 0$ by continuity of f we get (3.4.13) which proves the claim. □

As an immediate corollary we obtain the *maximum principle* for viscosity solutions.

Corollary 3.4.8. *Let $u \in C(\overline{\Omega})$. Then*

(1) if $u \in \underline{S}$ and $u \le 0$ on $\partial\Omega$ then $u \le 0$ in Ω.

(2) if $u \in \overline{S}$ and $u \ge 0$ on $\partial\Omega$ then $u \ge 0$ in Ω.

3.5 Harnack inequality and Hölder continuity

This section is dedicated to obtaining the Harnack inequality for uniformly elliptic equations. It consists of a quantitative version of the strong maximum principle in the sense that it quantifies how far from zero a nonnegative solution of an elliptic equation is allowed to go. This quantity is measured by the L^d norm of the source term f plus the value of u at a particular point, multiplied by a universal constant.

We start with two important tools: a barrier function which controls the behavior of u and a corollary of the Calderón-Zygmund cube decomposition. With these tools, we can use the ABP estimate obtained in the previous section in an appropriate iterative way to obtain an L^ε (or weak Harnack) estimate for u, that is, we obtain an exponential decay for the distribution function of u. Since this estimate is universal, we can apply the same reasoning to the function $C_1 - C_2 u$ and obtain the Harnack inequality for u. As immediate consequences of this proof, we obtain the weak Harnack inequality and a local maximum principle.

We conclude this section with the first regularity result for solutions of uniformly elliptic equations. Indeed, we prove that functions in the class $S(f)$ are Hölder continuous in the interior of their domain. The importance of this result is two-fold: first of all, it provides the first compactness result for solutions, opening the way to use Proposition 3.1.5, recall also Remark 2. Secondly, we emphasize that this regularity result holds for the large class $S(f)$, which we recall, corresponds to the class of subsolutions and supersolutions of the extremal Pucci equations. As we will see later in this chapter, it turns out that if u is a viscosity solution of $F(D^2u, x) = 0$, then the first order quotient differences of u, defined as $D_h u(x) = \frac{u(x+h)-u(x)}{|h|}$ which approximate it's first derivatives, belong to the class S and thus are Hölder continuous with universal estimates (in particular, independent of h). Therefore, we again resort to Proposition 3.1.5 combined with Remark 2 to take the limit $|h| \to 0$ and get $u \in C^1$ and $Du \in C^\alpha$, that is, $u \in C^{1,\alpha}$.

3.5.1 Barrier function and cube decomposition

We start by constructing a barrier function.

Lemma 3.5.1. *Given the constants $0 < \lambda \le \Lambda$, there exists a smooth function φ in $\mathbb{R}^d$ and positive universal constants C and $M > 1$ such that*

$$\begin{aligned} \varphi \geq 0, &\quad \textit{in} \quad \mathbb{R}^d \setminus B_{2\sqrt{d}}, && (3.5.1)\\ \varphi \leq -2, &\quad \textit{in} \quad Q_3, && (3.5.2)\\ \mathscr{P}^-(D^2\varphi) \geq -C\xi &\quad \textit{in} \quad \mathbb{R}^d, && (3.5.3) \end{aligned}$$

where $0 \le \xi \le 1$ is a continuous function in $\mathbb{R}^d$ with $\operatorname{supp}\xi \subset \overline{Q}_1$. Recall that $\overline{Q}_1 \subset \overline{Q}_3 \subset B_{2\sqrt{d}}$.

Moreover, $\varphi \geq -M$ in $\mathbb{R}^d$.

Proof. Let $\alpha := \max\{1, (d-1)\Lambda/\lambda - 1\}$. Note that

$$B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{3\sqrt{d}/2} \subset B_{2\sqrt{d}}.$$

Define

$$\varphi(x) := M_1 - M_2|x|^{-\alpha}, \quad \text{for } x \in \mathbb{R}^d \setminus B_{1/4};$$

we fix

$$M_1 := \frac{2^{1-\alpha}}{\left(\frac{3}{2}\right)^{-\alpha} - 2^{-\alpha}} > 0$$

and

$$M_2 := \frac{2\,d^{\alpha/2}}{\left(\frac{3}{2}\right)^{-\alpha} - 2^{-\alpha}} > 0,$$

such that

$$\varphi_{|\partial B_{2\sqrt{d}}} \equiv 0 \quad \text{and} \quad \varphi_{|\partial B_{3\sqrt{d}/2}} \equiv -2,$$

therefore (3.5.1) holds. We further extend φ smoothly to all $\mathbb{R}^d$ such that (3.5.2) holds. Note that this extension depends only on d, λ and Λ.

We finally check (3.5.3) by computing the hessian in polar coordinates $(r, \theta_1, \dots, \theta_{d-1})$. Since φ is radially symmetric we can assume that $x = (r, 0, \dots, 0)$. For $r \geq 1/4$ we have

$$\begin{aligned} &\partial_{ij}\varphi = 0 && \text{if } i \neq j,\\ &\partial_{ii}\varphi = M_2\alpha r^{-\alpha-2} && \text{if } i > 1,\\ &\partial_{11}\varphi = -M_2\alpha(1+\alpha)r^{-\alpha-2}. \end{aligned}$$

By the definition of $\mathscr{P}^-$ and the choice of α we get, for $|x| \geq 1/4$

$$\begin{aligned} \mathscr{P}^-(D^2\varphi)(x) =& \lambda \operatorname{Tr} D^2\varphi^- - \Lambda \operatorname{Tr} D^2\varphi^+\\ =& \lambda M_2\alpha(1+\alpha)|x|^{-\alpha-2} - (d-1)\Lambda M_2\alpha|x|^{-\alpha-2}\\ =& \lambda\alpha M_2|x|^{-\alpha-2}\big((1+\alpha) - (d-1)\Lambda/\lambda\big) > 0. \end{aligned}$$

Since the extension of φ depends only on universal constants, we further have for $|x| \leq 1/4$,

$$\mathscr{P}^-(D^2\varphi)(x) \geq -C(d,\lambda,\Lambda).$$

We now take $0 \leq \xi \leq 1$ smooth such that $\xi \equiv 1$ in $B_{1/4}$ and $\xi \equiv 0$ outside $B_{1/2}$ so that (3.5.3) holds. $\square$

Next we introduce the second tool we will need in the remaining of this section. It is a corollary of the Calderón-Zygmund cube decomposition. We start by introducing some terminology. We start with the unit cube Q_1. The first iteration consists of splitting this cube into 2^d identical cubes of half length and we then repeat this iteration to each subcube. The cubes obtained this way are called dyadic cubes.

If Q is a dyadic cube different from Q_1, we say that a cube is the predecessor of Q, and denoted by $\tilde{Q}$, if Q is one of the 2^d cubes obtained from dividing $\tilde{Q}$.

Lemma 3.5.2. *Let $A \subset B \subset Q_1$ be measurable sets and $0 < \delta < 1$ satisfying*

(a) $|A| \leq \delta$,

(b) If Q is a dyadic cube such that $|A \cap Q| > \delta|Q|$, then $\tilde{Q} \subset B$.

Then $|A| \leq \delta|B|$.

Proof. We use the Calderón-Zygmund cube decomposition technique of the L^1 function χ_A; see section 9.3 of [31]. The idea is to subsequently divide A into very small cubes until they all satisfy the first part of (b). Since

$$\frac{|Q_1 \cap A|}{|Q_1|} = |A| \leq \delta,$$

Q_1 does not satisfy (b), so we divide Q_1 into 2^d dyadic cubes and we keep all the “good" subcubes, i.e. the ones that satisfy (b). If Q is a “bad" cube, that is if it satisfies

$$\frac{|Q \cap A|}{|Q|} \leq \delta,$$

then we again split Q into 2^d dyadic cubes and repeat the argument. In this way, we pick a family $Q^1, Q^2, \ldots$ of “good" cubes satisfying

$$\frac{|Q^i \cap A|}{|Q^i|} > \delta, \quad \forall i.$$

We now prove that this family of cubes exhausts A, in measure. If $x \notin \bigcup Q^i$, then x belongs to an infinite number of closed dyadic cubes Q with diameter tending to zero, so we can write $Q \to x$.

Applying the Lebesgue differentiation theorem to χ_A at x, we get that

$$\begin{aligned}\chi_A(x) &= \lim_{Q\to x}\frac{1}{|Q|}\int_Q \chi_A\, d\lambda\\ &= \lim_{Q\to x}\frac{|Q\cap A|}{|Q|}\leq \delta < 1.\end{aligned}$$

Since

$$\chi_A(x) = \begin{cases} 1, & \text{for } x\in A,\\ 0, & \text{for } x\notin A,\end{cases}$$

we see that the set of points in A which don't belong to $\bigcup Q_i$ has measure zero.

Consider the family of predecessors of the cubes Q^i. Since we can have repeated cubes, we relabel them so that $\{\tilde{Q}^i\}_{i\in\mathbb{N}}$ are pairwise disjoint. Recalling that we only divided the "bad" cubes, we see that

$$\frac{|\tilde{Q}^i\cap A|}{|\tilde{Q}^i|}\leq \delta.$$

Furthermore, from (b) we have that $\tilde{Q}^i\subset B$, for any $i\geq 1$. Hence

$$A\subset \bigcup_{i\geq 0}\tilde{Q}^i\subset B.$$

We conclude that

$$|A|\leq \sum |\tilde{Q}^i\cap A|\leq \delta\sum|\tilde{Q}^i| = \delta|\cup\tilde{Q}^i|\leq \delta|B|.$$

□

3.5.2 Harnack inequality

We present the following inequality for viscosity solutions of uniformly elliptic equations.

Theorem 3.5.3. *Let $u\in S^*(f)$ in Q_1 satisfy $u\geq 0$ in Q_1, where f is continuous and bounded in Q_1. Then*

$$\sup_{Q_{1/2}} u\leq C\left(\inf_{Q_{1/2}} u + \|f\|_{L^d(Q_1)}\right),$$

where $C>1$ is a universal constant.

The proof of this theorem will have important consequences, which we detail later. Recall that $S^*(f) = S^*(\lambda,\Lambda,f) = \underline{S}(\lambda,\Lambda,|f|)\cap\overline{S}(\lambda,\Lambda,-|f|)$. It is convenient to instead prove this result under a smallness regime.

Lemma 3.5.4. *Let $u \in S^*(f)$ in $Q_{4\sqrt{d}}$, $u \in C(\overline{Q}_{4\sqrt{d}})$ satisfy $u \geq 0$ in $Q_{4\sqrt{d}}$, where f is continuous and bounded in $Q_{4\sqrt{d}}$. Assume that $\inf_{Q_{1/4}} u \leq 1$ and $\|f\|_{L^d(Q_{4\sqrt{d}})} \leq \varepsilon_0$. Then $\sup_{Q_{1/4}} u \leq C$, where ε_0 and C are universal constants.*

Indeed Theorem 3.5.3 follows from Lemma 3.5.4 by the following reasoning. Take $u \in S^*(f) \cap C(\overline{Q}_{4\sqrt{d}})$ with $u \geq 0$ in $Q_{4\sqrt{d}}$. For any $\delta > 0$, consider the function

$$u_\delta := u \left(\inf_{Q_{1/4}} u + \delta + \frac{\|f\|_{L^d}}{\varepsilon_0} \right)^{-1}.$$

By the properties of the Pucci operators, we have

$$u_\delta \in S^* \left(f \left(\inf_{Q_{1/4}} u + \delta + \frac{\|f\|_{L^d}}{\varepsilon_0} \right)^{-1} \right).$$

Moreover, u_δ falls in the smallness regime of Lemma 3.5.4 which gives, after letting $\delta \to 0$,

$$\sup_{Q_{1/4}} u \leq C \left(\inf_{Q_{1/4}} u + \|f\|_{L^d(Q_{4\sqrt{d}})} \right). \tag{3.5.4}$$

To get Theorem 3.5.3, we proceed with the standard covering argument: Take $x_0, x_1 \in \overline{Q}_{2\sqrt{d}}$ such that $u(x_0) = \sup_{Q_{2\sqrt{d}}} u$ and $u(x_1) = \inf_{Q_{2\sqrt{d}}} u$. Consider an overlapping chain of cubes $Q^1, \ldots, Q^N$ connecting x_0 to x_1, where N is bounded by a constant which depends only on d. Note that since they overlap, we have for every i that $\inf_{Q^i} u \leq \sup_{Q^{i+1}} u$. Therefore

$$\begin{aligned}
u(x_0) =& \sup_{Q^1} u \leq C \left(\inf_{Q^1} u + \|f\|_{L^d(Q_{4\sqrt{d}})} \right) \\
\leq& C \left(\sup_{Q^2} u + \|f\|_{L^d(Q_{4\sqrt{d}})} \right) \\
\leq& C \left(C \left(\inf_{Q^2} u + \|f\|_{L^d(Q_{4\sqrt{d}})} \right) + \|f\|_{L^d(Q_{4\sqrt{d}})} \right) \\
=& C^2 \inf_{Q^2} u + (C+1) \|f\|_{L^d(Q_{4\sqrt{d}})} \\
\leq& \cdots \leq C^N \inf_{Q^N} u + \sum_{i=0}^{N-1} C^i \|f\|_{L^d(Q_{4\sqrt{d}})} \\
\leq& \overline{C} \left(u(x_1) + \|f\|_{L^d(Q_{4\sqrt{d}})} \right),
\end{aligned}$$

where $\overline{C}$ is still universal. After a simple rescaling we obtain Theorem 3.5.3.

The idea of the proof of Harnack inequality is the following. Let u be as in Lemma 3.5.4 and let $\lambda_u(t) := |\{u > t\} \cap Q_1|$ be its distribution function. It suffices to prove that

$$\lambda_u(t) \equiv 0, \quad \text{for } t \geq C, \tag{3.5.5}$$

with C universal. This will be done through two steps:

(a) First, we prove some power decay for λ_u, i.e., $\lambda_u(t) < Ct^{-\varepsilon}$, for some universal C and ε. This is accomplished using Lemmas 3.5.1, 3.5.2 and the ABP estimate.

(b) Then, using step (a), we prove the same decay for all $C_1 - C_2 u$ (where C_1 and C_2 are positive constants) at every scale, that is, every small cube. From this, we get $\lambda_u \equiv 0$ for $t \geq C$.

Remark 3.5.5. Step (b) is the difficult part of the proof and corresponds to Lemma 3.5.7. The main idea is the following: we prove (3.5.5) by contradiction, by noting that if there exists a point x_0 such that $u(x_0) > C$ is universally large, then for a universally small cube Q containing x_0, $\sup_Q u \geq \nu u(x_0)$, where $\nu > 1$ is also universal. That is, there exists a point x_1 close to x_0 satisfying $u(x_1) > \nu C$. After rescaling and translating, we can iterate this result, getting a sequence of points $x_n \to x_\infty \in Q_{1/2}$ such that $u(x_n) \to +\infty$, which is a contradiction since u is continuous.

We start with some lemmas. Here f will always be a continuous and bounded function on $Q_{4\sqrt{d}}$.

Lemma 3.5.6. *There exist universal constants $\varepsilon_0 > 0$, $0 < \mu < 1$ and $M > 1$, such that if $u \in \overline{S}(-|f|)$ in $Q_{4\sqrt{f}}$, $u \in C(\overline{Q}_{4\sqrt{d}})$ and f satisfy*

$$u \geq 0, \quad \text{in } Q_{4\sqrt{d}}, \tag{3.5.6}$$

$$\inf_{Q_3} u \leq 1 \quad \text{and} \tag{3.5.7}$$

$$\|f\|_{L^d(Q_{4\sqrt{d}})} \leq \varepsilon_0, \tag{3.5.8}$$

then

$$|\{u \leq M\} \cap Q_1| > \mu. \tag{3.5.9}$$

Proof. Let φ be the barrier function introduced in Lemma 3.5.1 and define $\omega := u + \varphi$. Recall that $B_{2\sqrt{d}} \subset Q_{4\sqrt{d}}$, φ is smooth and $\mathscr{P}^-(D^2\varphi) \geq -C\xi$. By property (4) of Lemma 3.2.3 we have

$$\omega \in \overline{S}(-|f| - C\xi) \quad \text{in } B_{2\sqrt{d}}.$$

We have that $\omega \in C(\overline{B}_{2\sqrt{d}})$ and $\omega \geq 0$ on $\partial B_{2\sqrt{d}}$ by (3.5.6) and (3.5.1). We also have that $\inf_{Q_3} \omega \leq -1$ by (3.5.7) and (3.5.2). Applying the ABP estimate in Theorem 3.4.3 to ω we get

$$\begin{aligned} 1 \leq & C\left(\int_{\{\omega=\Gamma_\omega\}\cap B_{2\sqrt{d}}} (|f| + C\xi)^d\right)^{1/d} \\ \leq & C\|f\|_{L^d(Q_{4\sqrt{d}})} + C|\{\omega = \Gamma_\omega\} \cap Q_1|^{1/d}. \end{aligned} \tag{3.5.10}$$

In the second inequality, we used that $0 \leq \xi \leq 1$ and $\operatorname{supp}\xi \subset Q_1$. Let now ε_0 be so small that $C\|f\|_{L^d(Q_{4\sqrt{d}})} \leq C\varepsilon_0 \leq 1/2$. Then we get

$$\begin{aligned} \frac{1}{2} \leq & C|\{\omega = \Gamma_\omega\} \cap Q_1|^{1/d} \\ \leq & C|\{u \leq M\} \cap Q_1|^{1/d}. \end{aligned}$$

□

We emphasize the importance in the previous proof that $\|f\|_{L^d(\{\omega=\Gamma_\omega\})}$ appears in the right hand side of the ABP estimate and that the contact set $\{\omega=\Gamma_\omega\}$ carries important information about the function ω. In this case, we have used that $\omega \le 0$ implies $u \le M$, in the contact set of ω. Next we iterate Lemma 3.5.5.

Remark 3.5.7. Note the following useful properties of dyadic cubes. If Q is a dyadic cube then $Q = Q_{1/2^i}(x_0)$, for some $i \ge 0$ and $x_0 \in Q_1$; moreover,

$$Q_{4\sqrt{d}/2^i}(x_0) \subset Q_{4\sqrt{d}}, \text{ and}$$
$$i \ge 0 \implies \tilde{Q} \subset Q_{3/2^i}(x_0).$$

Lemma 3.5.8. *Let u be as in Lemma 3.5.5. Then*

$$|\{u \ge M^k\} \cap Q_1\}| \le (1-\mu)^k, \tag{3.5.11}$$

for $k = 1, 2, \ldots$, where M and μ are as in Lemma 3.5.6.

Therefore, we have

$$|\{u \ge t\} \cap Q_1| \le R t^{-\varepsilon}, \quad \text{for } t > 0, \tag{3.5.12}$$

where R and ε are positive universal constants.

Proof. We proceed by induction. The case $k = 1$ is already proved in Lemma 3.5.5. Suppose now (3.5.11) holds up to $k-1$, and define the sets

$$A = \{u > M^k\} \cap Q_1, \quad B = \{u > M^{k-1}\} \cap Q_1.$$

If we manage to prove that

$$|A| \le (1-\mu)|B|, \tag{3.5.13}$$

then (3.5.11) follows by the induction hypothesis. To apply Lemma 3.5.2 with $\delta = 1-\mu$, we first verify that the assumptions are verified. Clearly $A \subset B \subset Q_1$ and $|A| \le |\{u > M\} \cap Q_1| \le 1-\mu$, by Lemma 3.5.5. It remains to prove that condition (b) holds; that is, we need to show that if $Q = Q_{1/2^i}(x_0)$ is a dyadic cube such that

$$|A \cap Q| > (1-\mu)|Q| \tag{3.5.14}$$

then the predecessor satisfies $\tilde{Q} \subset B$. Arguing by contradiction, suppose this is not the case. Then we can take

$$\tilde{x} \in \tilde{Q} \quad \text{such that} \quad u(\tilde{x}) \le M^{k-1}. \tag{3.5.15}$$

Consider the change of coordinates

$$x = x_0 + \frac{1}{2^i} y, \quad y \in Q_1, \quad x \in Q := Q_{1/2^i}(x_0)$$

and the function

$$\tilde{u}(y) := u(x)/M^{k-1}.$$

Applying Lemma 3.5.5 to $\tilde{u}$ will yield the next iteration and complete the induction argument. Indeed, assuming for now that $\tilde{u}$ satisfies the hypothesis of this Lemma, then by (3.5.9) we get

$$\mu \le |\{\tilde{u}(y) \le M\} \cap Q_1| = 2^{id}|\{u(x) \le M^k\} \cap Q|.$$

Hence $|Q \setminus A| > \mu|Q|$, contradicting (3.5.14).

It remains to prove that $\tilde{u}$ satisfies the hypothesis of Lemma 3.5.5. By Remark 6 we have $\tilde{u}(y) \in \overline{S}(f(x)/(2^{2^i}M^{k-1}) =: \tilde{f}(y))$ in $Q_{4\sqrt{d}}$ and

$$x \in \tilde{Q} \implies y \in Q_3.$$

Therefore $\tilde{u} \ge 0$ and $\inf_{Q_3} \tilde{u} \le u(\tilde{x})/M^{k-1} \le 1$ by (3.5.15). Finally,

$$\|\tilde{f}\|_{L^d(Q_{4\sqrt{d}})} = \frac{2^i}{2^{2^i}M^{k-1}}\|f\|_{L^d(Q_{4\sqrt{d}})} \le \|f\|_{L^d(Q_{4\sqrt{d}})} \le \varepsilon_0.$$

To conclude the proof, note that (3.5.12) follows immediately from (3.5.11) by taking $R = (1-\mu)^{-1}$ and ε satisfying $1-\mu = M^{-\varepsilon}$.

□

Next we present the final lemma before the proof of Lemma 3.5.4, which will be used inside a contradiction argument. Recall that if $u \in S^*(f) \subset \underline{S}(|f|)$ then $-u \in \overline{S}(-|f|)$.

Lemma 3.5.9. *Let $u \in \underline{S}(|f|)$ in $Q_{4\sqrt{d}}$. Assume that f satisfies* (3.5.6) *and u satisfies* (3.5.12).

Then there exist universal constants $M_0 > 1$ and $\sigma > 0$ such that, for ε as in (3.5.12) *and $\nu = M_0/(M_0 - 1/2) > 1$, the following holds:*

If $j \ge 1$ is an integer and x_0 satisfies

$$|x_0|_\infty \le 1/4 \tag{3.5.16}$$

and

$$u(x_0) \ge \nu^{j-1}M_0, \tag{3.5.17}$$

then

$$Q^j := Q_{l^j}(x_0) \subset Q_1 \quad \text{and} \quad \sup_{Q^j} u \ge \nu^j M_0,$$

where $l_j = \sigma M_0^{-\varepsilon/d}\nu^{-\varepsilon j/d}$.

Proof. Take $\sigma > 0$ and then $M_0 > 1$ such that

$$\frac{1}{2}\sigma^d > d2^\varepsilon(4\sqrt{d})^d \tag{3.5.18}$$

and

$$\sigma M_0^{-\varepsilon/d} + RM_0^{-\varepsilon} \leq \frac{1}{2}, \tag{3.5.19}$$

with R and ε given by (3.5.12). By (3.5.19) we get $l_j \leq 1/2$ which combined with (3.5.16) implies

$$Q_{l_j/(4\sqrt{d})}(x_0) \subset Q_{l_j}(x_0) = Q^j \subset Q_1. \tag{3.5.20}$$

We argue by contradiction, assuming that $\sup_{Q^j} u < \nu^j M_0$ and show that this leads to a contradiction. By (3.5.20) and (3.5.12) we have that

$$\begin{aligned} \left|\left\{u \geq \nu^j \frac{M_0}{2}\right\} \cap Q_{l_j/(4\sqrt{d})}(x_0)\right| \leq & \left|\left\{u \geq \nu^j \frac{M_0}{2}\right\} \cap Q_1\right| \\ \leq & R\nu^{-j\varepsilon}\left(\frac{M_0}{2}\right)^{-\varepsilon}. \end{aligned} \tag{3.5.21}$$

Consider the change of coordinates

$$x = x_0 + \frac{l_j}{4\sqrt{d}}y, \quad y \in Q_{4\sqrt{d}}, \quad x \in Q^j$$

and define the function

$$v(y) = \frac{\nu M_0 - \nu^{1-j}u(x)}{(\nu-1)M_0}.$$

This change of coordinates defines bijections between the following sets.

$$\begin{aligned} & x \in Q_{l_j}(x_0)\,[\text{resp. } Q_{3l_j/(4\sqrt{d})}, Q_{l_j/(4\sqrt{d})}(x_0)] \\ \Longleftrightarrow\ & y \in Q_{4\sqrt{d}}\,[\text{resp. } Q_3, Q_1]. \end{aligned}$$

We start by claiming that v satisfies the hypothesis of Lemma 3.5.5. Then we can apply Lemma 3.5.6 and get

$$|\{v(y) > M_0\} \cap Q_1\}| \leq RM_0^{-\varepsilon}.$$

By the implication

$$u(x) < \nu^j M_0/2 \implies v(y) > \nu\frac{\nu - 1/2}{\nu - 1} = M_0$$

we get that

$$\left|\left\{u(x) < \nu^j \frac{M_0}{2}\right\} \cap Q_{l_j/(4\sqrt{d})}(x_0)\right| \leq \left(\frac{l_j}{4\sqrt{d}}\right)^d RM_0^{-\varepsilon}.$$

Since this inequality and (3.5.21) produce a dichotomy, we must have

$$\left(\frac{l_j}{4\sqrt{d}}\right)^d \leq \left(\frac{l_j}{4\sqrt{d}}\right)^d RM_0^{-\varepsilon} + R\nu^{-j\varepsilon}\left(\frac{M_0}{2}\right)^{-\varepsilon}.$$

Since $RM_0^{-\varepsilon} \leq 1/2$, we get

$$\frac{1}{2}\left(\frac{l_j}{4\sqrt{d}}\right)^d \leq R\nu^{-j\varepsilon}\left(\frac{M_0}{2}\right)^{-\varepsilon}.$$

By the definition of l_j, we finally get

$$\frac{1}{2}\sigma^d \leq R2^{\varepsilon}(4\sqrt{d})^d,$$

which is a contradiction with (3.5.18).

We now prove the claim that v satisfies the conditions of Lemma 3.5.5. Clearly

$$v(y) \in \overline{S}\left[-\left(\frac{l_j}{4\sqrt{d}}\right)^2 \left(\nu^{j-1}(\nu-1)M_0\right)^{-1} |f(x)| =: \tilde{f}(y)\right] \cap C(\overline{Q}_{4\sqrt{d}}),$$

for $y \in Q_{4\sqrt{d}}$. Also $v > 0$ in $Q_{4\sqrt{d}}$ since by assumption of the contradiction $\sup_{Q^j} u < \nu^j M_0$. Furthermore, (3.5.17) implies that $\inf_{Q_3} v \leq 1$. We finally prove $\tilde{f}$ still satisfies (3.5.9).

$$\begin{aligned}\|\tilde{f}\|_{L^d(Q_{4\sqrt{d}})} =& \frac{\sigma M_0^{-\varepsilon/d}\nu^{-\varepsilon j/d}}{4\sqrt{d}\nu^{j-1}(\nu-1)M_0}\|f(x)\|_{L^d(Q^j)} \\ \leq& \frac{\sigma M_0^{-\varepsilon/d}\nu^{-\varepsilon j/d}}{4\sqrt{d}\nu^{j-1}(\nu-1)M_0}\varepsilon_0\end{aligned}$$

Using (3.5.19), $\nu > 1$ and $\nu = 2(\nu-1)M_0$, we get

$$\|\tilde{f}\|_{L^d(Q_{4\sqrt{d}})} \leq \frac{\nu^{-\varepsilon j/d}}{8\sqrt{d}\nu^{j-1}(\nu-1)M_0}\varepsilon_0 \leq \frac{\nu^{-\varepsilon j/d}}{4\sqrt{d}\nu^j}\varepsilon_0 \leq \varepsilon_0.$$

This verifies the claim which completes the proof. □

We are finally ready to prove Harnack Inequality.

Proof of Lemma 3.5.4. By the assumptions of Lemma 3.5.4, u satisfies the assumptions of Lemma 3.5.5 and therefore it satisfies the L^{ε} estimate (3.5.12).

Recall that $l_j = \sigma M_0^{-\varepsilon/d} \nu^{-\varepsilon j/d}$, for $j \in \mathbb{N}$. Since the associated series is convergent, there exists a large integer j_0, depending only on the universal constants σ, M, ε, d and ν, such that

$$\sum_{j \geq j_0} l_j \leq \frac{1}{4}. \tag{3.5.22}$$

We will prove, by contradiction, that

$$\sup_{Q_{1/4}} u \leq \nu^{j_0-1} M_0. \tag{3.5.23}$$

This will conclude the proof since $C = \nu^{j_0-1} M_0$ is universal. We suppose that (3.5.23) is not true. Then, there exists some point x_{j_0} satisfying

$$|x_{j_0}|_\infty \leq \frac{1}{8} \quad \text{and} \quad u(x_{j_0}) \geq \nu^{j_0-1} M_0.$$

Thus we are in conditions of applying Lemma 3.5.7, yielding a point x_{j_0+1} such that

$$|x_{j_0+1} - x_{j_0}|_\infty \leq \frac{l_{j_0}}{2} \quad \text{and} \quad u(x_{j_0+1}) \geq \nu^{j_0} M_0.$$

Calling $u_1 := \nu^{-1} u$, we easily see that u_1 still satisfies the assumptions of Lemma 3.5.7. We can therefore iterate this argument, thus getting a sequence of points $(x_j)_{j \geq j_0}$ satisfying

$$|x_{j+1} - x_j|_\infty \leq \frac{l_j}{2} \quad \text{and} \quad u(x_{j+1}) \geq \nu^j M_0, \quad \text{for } j \geq j_0.$$

Note also that

$$\begin{aligned} |x_j| \leq & |x_{j_0}| + \sum_{k=j_0}^{j-1} |x_{k+1} - x_k| \\ \leq & \frac{1}{8} + \sum_{k \geq j_0} \frac{l_k}{2} \leq \frac{1}{4} \end{aligned}$$

by (3.5.22) and thus (3.5.16) is always verified.

Since $(x_j)_j$ is a Cauchy sequence, it must converge to some point $x_\infty \in \overline{Q}_{1/2}$. By continuity, we get

$$u(x_\infty) = \lim_{j \to \infty} u(x_j) \geq \lim_{j \to \infty} \nu^j M_0 = +\infty.$$

This contradiction implies that

$$\sup_{Q_{1/4}} u \leq \nu^{j_0-1} M_0$$

which completes the proof. □

The proof of Harnack Inequality implies the following important results. The first one is called *weak Harnack inequality*, and requires u to be merely a nonnegative supersolution. The second result is called local maximum principle and applies to subsolutions.

Theorem 3.5.10.

(1) Let $u \in \overline{S}(\lambda, \Lambda, f)$ in Q_1, where f is continuous and bounded in Q_1. Then

$$\|u\|_{L^{p_0}(Q_{1/4})} \le C \left(\inf_{Q_{1/2}} u + \|f\|_{L^d(Q_1)} \right),$$

where $p_0 > 0$ and C are universal constants.

(2) Let $u \in \underline{S}(\lambda, \Lambda, f)$ in Q_1, where f is continuous and bounded in Q_1. Then, for any $p > 0$,

$$\sup_{Q_{1/2}} u \le C(p) \left(\|u^+\|_{L^p(Q_{3/4})} + \|f\|_{L^d(Q_1)} \right),$$

where $C(p)$ is a constant depending only on d, λ, Λ and p.

Proof. Part (1) follows immediately from Lemma 3.5.6 combined with Lemma 2.2.1; recall also the argument leading to (3.5.4).

To verify (2), we start by assuming that $u \in \underline{S}(f) \subset \underline{S}(|f|)$ in $Q_{4\sqrt{d}}$, $\|f\|_{L^d(Q_{4\sqrt{d}})} \le \varepsilon_0$ and $\|u^+\|_{L^\varepsilon(Q_1)} \le R^{1/\varepsilon}$, where ε_0 is as in Lemma 3.5.5, and R and ε are as in Lemma 3.5.6. Then by (2.2.2) in Lemma 2.2.1, we have

$$|\{u \ge t\} \cap Q_1| \le t^{-\varepsilon} \int_{Q_1} \left(u^+\right)^\varepsilon \le R t^{-\varepsilon}, \quad \text{for } t > 0,$$

that is, u satisfies (3.5.12). Therefore, Lemma 3.5.7 still applies and hence the proof of Lemma 3.5.4 gives

$$\sup_{Q_{1/4}} u \le C.$$

Rescalling u we get

$$\sup_{Q_{1/4}} u \le C \left(\|u^+\|_{L^\varepsilon(Q_1)} + \|f\|_{L^d(Q_{4\sqrt{d}})} \right)$$

for any $u \in \underline{S}(f)$ in $Q_{4\sqrt{d}}$. By rescaling, we get (2) for $p = \varepsilon$. If $p > \varepsilon$ we can apply Hölder inequality and get

$$\|u^+\|_{L^\varepsilon(Q_{3/4})} \le |Q_{3/4}|^{\frac{1}{r}} \|u^+\|_{L^p(Q_{3/4})}, \quad \frac{1}{r} + \frac{1}{p} = \frac{1}{\varepsilon}.$$

Hence

$$\begin{aligned}\sup_{Q_{1/2}} u \leq & C\left(|Q_{3/4}|^{\frac{1}{r}}\|u^+\|_{L^p(Q_{3/4})} + \|f\|_{L^d(Q_1)}\right)\\ \leq & C\left(\|u^+\|_{L^p(Q_{3/4})} + \|f\|_{L^d(Q_1)}\right),\end{aligned}$$

since $|Q_{3/4}|^{\frac{1}{r}} < 1$.

The proof for any $p < \varepsilon$ follows by interpolation.

□

An important consequence of the weak Harnack inequality in Theorem 3.5.8 is the following *strong maximum principle* for supersolutions.

Proposition 3.5.11. *Let $u \in \overline{S}$ in Q_1. Assume that $u \geq 0$ in Ω and $u(x_0) = 0$ for some $x_0 \in Q_1$. Then $u \equiv 0$ in Q_1.*

Remark 3.5.12. By Proposition 3.2.4, if u is a viscosity supersolution of $F(D^2u, x) = 0$ with $F(0, x) \equiv 0$, then $u \in \overline{S}(\lambda/d, \Lambda, 0)$ and therefore u also satisfies the strong maximum principle.

As another important consequence of the Harnack Inequality, we have the following Liouville type theorem.

Corollary 3.5.13. *Let $u \in S(0)$ in $\mathbb{R}^d$ be bounded from below (or above). Then u is constant.*

Proof. We can assume, without loss of generality, that $\inf u = 0$. Then, for every $\varepsilon > 0$, there exists a point x_0 such that $u(x_0) \leq \varepsilon$. In every ball $B_{2R}(x_0)$, we have by Harnack inequality that

$$\sup_{B_{2R}(x_0)} u \leq C\varepsilon.$$

Let $x \in \mathbb{R}^d$ be an arbitrary point and take R so large that $x \in B_{2R}(x_0)$. Since the constant C is universal (hence independent of R), we have $u(x) \leq C\varepsilon$. Letting $\varepsilon \to 0$ we conclude the proof. □

3.5.3 C^α regularity for the class $S^*(f)$

The following result is a very important application of the Harnack inequality. It was first discovered by De Giorgi in 1957 and Nash in 1958 for linear equations of the form $Lu = D_i(a_{ij}(x)D_ju)$ where the coefficients were assumed to be merely bounded in L^∞. It follows easily from the Harnack inequality in Theorem 3.5.3 combined with the instrumental Lemma 2.2.2.

Proposition 3.5.14. *Let $u \in S^*(f)$ in Q_1. Then*

(1) For a universal constant $\gamma < 1$, it holds

$$\operatorname{osc}_{Q_{1/2}} u \leq \gamma \operatorname{osc}_{Q_1} u + 2\|f\|_{L^d(Q_1)}.$$

(2) Furthermore, $u \in C^\alpha(\overline{Q}_{1/2})$ and

$$\|u\|_{C^\alpha(\overline{Q}_{1/2})} \leq C\left(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^\infty(Q_1)}\right),$$

where $0 < \alpha < 1$ *and* $C > 0$ *are universal constants.*

Proof. We start by proving *(1)*. Define $m_r := \inf_{Q_r} u$, $M_r := \sup_{Q_r} u$ and $o_r := \operatorname{osc}_{Q_r} u = M_r - m_r$. We apply the Harnack inequality, Theorem 3.5.3, to the nonnegative functions $u - m_1$ and $M_1 - u$ in Q_1 (which clearly belong to $S^*(f)$), recall property (3) of Lemma 3.2.2). We get

$$M_{1/2} - m_1 \le C\left(m_{1/2} - m_1 + \|f\|_{L^d(Q_1)}\right)$$

and

$$M_1 - m_{1/2} \le C\left(M_1 - M_{1/2} + \|f\|_{L^d(Q_1)}\right)$$

Adding both estimates we obtain

$$o_{1/2} + o_1 \le C\left(o_1 - o_{1/2} + 2\|f\|_{L^d(Q_1)}\right),$$

that is,

$$o_{1/2} \le \frac{C-1}{C+1} o_1 + \frac{2C}{C+1}\|f\|_{L^d(Q_1)}$$

which concludes (1).

We now proceed with (2). Recalling the scaling property (3) in Lemma 3.2.3 we get, for every $0 < R < 1$

$$\operatorname{osc}_{Q_{R/2}} u \le \gamma \operatorname{osc}_{Q_R} u + 2R^2\|f\|_{L^d(Q_R)}.$$

We can easily check that the assumptions of Lemma 2.2.2 are satisfied, with $R_0 = 1$, $\omega(t) = o_t$, $\sigma(t) = 2t^2\|f\|_{L^d(Q_t)}$, $\gamma = (C-1)/(C+1)$ and $\tau = 1/2$. It yields

$$\operatorname{osc}_{Q_R} u \le C\left(R^\alpha \operatorname{osc}_{Q_1} u + 2R^{2\mu}\|f\|_{L^d(Q_{R^\mu})}\right)$$

for every $0 < R < 1$ and $0 < \mu < 1$. Hence,

$$\sup_{x,y\in Q_R} |u(x) - u(y)| \le R^\alpha C\left(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^d(Q_1)}\right),$$

that is, $u \in C^\alpha(Q_{1/2})$ with

$$\|u\|_{C^\alpha(Q_{1/2})} \le C\left(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^d(Q_1)}\right),$$

where $0 < \alpha < 1$ and $C > 0$ are universal constants.

□

Remark 3.5.15. Using a standard covering argument, it is possible to state Proposition 3.5.11 in balls $B_{1/2}$ and B_1 instead of cubes $Q_{1/2}$ and Q_1, respectively.

As we mentioned previously, we can combine Hölder continuity of solutions with the closedness of the family of viscosity solutions (recall Proposition 3.1.5) via Arzelà-Ascoli Theorem to get the following important compactness result.

Proposition 3.5.16. *Let $(F_k)_{k\in\mathbb{N}}$ be a sequence of (λ,Λ)-elliptic operators, $(u_k)_{k\in\mathbb{N}}$ be a sequence of viscosity solutions in Ω of $F_k(D^2u_k,x) = f_k(x)$.*

Assume that (F_k) converges uniformly in compact sets of $S(d)\times\Omega$ to F, that $(u_k)_k$ is universally bounded in compact sets of Ω and that $\|f_k - f\|_{L^\infty(\Omega)} \to 0$.

Then there exists $u \in C(\Omega)$ and a subsequence of $(u_k)_k$ that converges locally uniformly to u. Moreover, $F(D^2u,x) = f(x)$ in the viscosity sense, in Ω.

3.6 Uniqueness of solutions

This section is devoted to proving uniqueness of solution of the fully nonlinear Dirichlet problem

$$\begin{cases} F(D^2u) = 0, & \text{for} \quad x\in B_1, \\ u = \varphi, & \text{for} \quad x \in \partial B_1. \end{cases}$$

In the next section we will see how this result implies interior $C^{1,\alpha}$ regularity for this homogeneous problem.

We begin by defining a class of approximate solutions as introduced by Jensen in [38]. Let $u\in C(B_1)$ and H be an open set with $\overline{H}\subset B_1$. We define, for $\varepsilon > 0$, the *upper ε-envelope* of u (with respect to H) as

$$u^\varepsilon(x_0) = \sup_{x\in\overline{H}} \left\{ u(x) + \varepsilon - \frac{1}{\varepsilon}|x-x_0|^2 \right\}, \qquad \text{for } x_0 \in H.$$

To get a geometric intuition of this definition, consider at each point $x\in H$ the corresponding concave paraboloid of opening $2/\varepsilon$ and vertex at $(x, u(x)+\varepsilon)$. Then the graph of u^ε corresponds to the upper envelope of this family of paraboloids. Therefore, this approximation will smoothen the function u from below, but not from above. To see this, consider the following examples.

Example 3.6.1. Let $u(x) = -|x|$ for $x\in\mathbb{R}$ and consider

$$u^\varepsilon(x_0) = \sup_{x\in\overline{H}} \left\{ -|x| + \varepsilon - \frac{1}{\varepsilon}|x-x_0|^2 \right\}.$$

Then clearly $u^\varepsilon(0) = \varepsilon$. For $x_0 > 0$, let x_0^* correspond to the point where the supremum is attained. Since $x_0^* \geq 0$, we can calculate explicitly that $x_0^* = (x_0 - \varepsilon/2)_+$ and so, for $0 < x_0 < \varepsilon/2$ we get that

$$u^\varepsilon(x_0) = \varepsilon - \frac{1}{\varepsilon}x_0^2.$$

By symmetry, this expression holds for $|x_0| < \varepsilon/2$. Therefore the upper envelope has removed the edge at the origin and we obtain a smooth function.

Example 3.6.2. On the other hand, let $u(x) = |x|$ for $x \in \mathbb{R}$ and consider

$$u^\varepsilon(x_0) = \sup_{x \in \overline{H}} \left\{ |x| + \varepsilon - \frac{1}{\varepsilon}|x - x_0|^2 \right\}.$$

Then for $x_0 > 0$ we get that the supremum is obtained at the point $x_0^* = x_0 + \varepsilon/2$ and so

$$u^\varepsilon(x_0) = \left| x_0 + \frac{\varepsilon}{2} \right| + \varepsilon - \frac{\varepsilon}{4} = x_0 + \frac{5\varepsilon}{4}.$$

On the other hand, if $x_0 < 0$ then $x_0^* = x_0 - \varepsilon/2$ and

$$u^\varepsilon(x_0) = \left| x_0 - \frac{\varepsilon}{2} \right| + \varepsilon - \frac{\varepsilon}{4} = -x_0 + \frac{5\varepsilon}{4}.$$

Therefore, for any x_0, we have

$$u^\varepsilon(x_0) = |x_0| + \frac{5\varepsilon}{4},$$

which is still only $C^{0,1}$ at the origin.

We formulate the previous ideas in the following theorem.

Theorem 3.6.3. *Let u^ε be the upper ε-envelope of u. Then,*

(a) $u^\varepsilon \in C(H)$ and $u^\varepsilon \downarrow u$ uniformly in H as $\varepsilon \to 0$.

(b) For any $x_0 \in H$ there exists a concave paraboloid of opening $\varepsilon/2$ that touches u^ε from below at $x_0 \in H$. Hence u^ε is in $C^{1,1}$ from below in H. In particular, u^ε is punctually second order differentiable at almost every point of H.

(c) Let u be a viscosity subsolution of $F(D^2u) = 0$ in B_1 and H_1 be an open set such that $\overline{H_1} \subset H$. Then, for $\varepsilon \le \varepsilon_0$ (where ε_0 depends only on u, H and H_1), u^ε is a subsolution of $F(D^2u) = 0$ in H_1; in particular, since u^ε is punctually twice differentiable, $F(D^2u^\varepsilon(x)) \le 0$ a.e. $x \in H_1$.

Similarly we can define the lower envelope u_ε which will converge to u from below. Of course the smoothening effect will be reversed, as hinted by Examples 3 and 4. We proceed with some properties of u^ε

Lemma 3.6.4. *Let $x_0, x_1 \in H$. Then*

(1) There exists $x_0^* \in \overline{H}$ such that $u^\varepsilon(x_0) = u(x_0^*) + \varepsilon - |x_0^* - x_0|^2/\varepsilon$;

(2) $u^\varepsilon(x_0) \ge u(x_0) + \varepsilon$;

(3) $|u^\varepsilon - u^\varepsilon(x_1)| \le (3/\varepsilon)$ diam $H\,|x_0 - x_1|$;

(4) If $0 < \varepsilon < \varepsilon'$ then $u^\varepsilon(x_0) \le u^{\varepsilon'}(x_0)$;

(5) $|x_0^* - x_0| \le \varepsilon \operatorname{osc}_H u$;

(6) $0 < u^\varepsilon(x_0) - u(x_0) < u(x_0^*) - u(x_0) + \varepsilon$.

Proof. *(1)*, *(2)*, *(4)* and *(6)* are immediate. To obtain *(3)* let $x \in \overline{H}$ and note that

$$\begin{aligned}
u_\varepsilon(x_0) \geq & u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_0|^2 \\
\geq & u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_1|^2 - \frac{1}{\varepsilon}|x_0 - x_1|^2 - \frac{2}{\varepsilon}|x - x_1||x_0 - x_1| \\
\geq & u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_1|^2 - \frac{1}{\varepsilon} \operatorname{diam} H\, |x_0 - x_1| - \frac{2}{\varepsilon} \operatorname{diam} H\, |x_0 - x_1| \\
= & u(x) + \varepsilon - \frac{1}{\varepsilon}|x - x_1|^2 - \frac{3}{\varepsilon} \operatorname{diam} H\, |x_0 - x_1|.
\end{aligned}$$

Taking the supremum of x over H we get

$$u^\varepsilon(x_0) \geq u^\varepsilon(x_1) - \frac{3}{\varepsilon} \operatorname{diam} H\, |x_0 - x_1|$$

as intended.

Combining *(1)* and *(2)* we obtain *(5)* by noting that

$$\begin{aligned}
\frac{1}{\varepsilon}|x_0^* - x_0|^2 = & u(x_0^*) - u(x_0)| \\
\leq & u(x_0^*) - u(x_0) \\
\leq & \operatorname{osc}_H u.
\end{aligned}$$

□

It will be useful to define the second differential quotient

$$\Delta_h^2 u(x_0) := \frac{u(x_0 + h) + u(x_0 - h) - 2u(x_0)}{|h|^2}$$

for $h \in \mathbb{R}^d$ which is well-defined provided $x_0, x_0 + h, x_0 - h \in \Omega$, where Ω is the convex domain of u. Note that if

$$P(x) := -\frac{K}{2}|x - \overline{x}|^2,$$

then

$$\Delta_h^2 P(x_0) = -\frac{K}{2}$$

If additionally P touches u from below at x_0, then

$$\begin{aligned}
\Delta_h^2 u(x_0) = & \frac{u(x_0 + h) + u(x_0 - h) - 2u(x_0)}{|h|^2} \\
\geq & \frac{P(x_0 + h) + P(x_0 - h) - 2P(x_0)}{|h|^2} \\
= & \Delta_h^2 P(x_0) = -\frac{K}{2}.
\end{aligned}$$

Therefore, since Δ_h^2 is linear,

$$\Delta_h^2(u-P) \geq 0.$$

Now we prove that if $\Delta_h^2 v \geq 0$ for every h, then v is convex. Indeed, let $x, y \in \Omega$, call $x_0 := (x+y)/2 \in \Omega$ and $h := (x-y)/2$. Then

$$0 \leq \Delta_h^2 v = \frac{v(x)+v(y)-2v\left(\frac{x+y}{2}\right)}{\left|\frac{x-y}{2}\right|},$$

thus

$$v\left(\frac{x+y}{2}\right) \leq \frac{v(x)+v(y)}{2}$$

which implies that v is convex. The following result, called the Alexandroff-Buselman-Feller theorem, concerns with the regularity of convex functions (see Theorem 1 in section 6.4 of [28]), recall Definition 2.5.2.

Theorem 3.6.5. *Let u be a convex function in B_1. Then u is punctually second order differentiable at every point $x_0 \in B_1$.*

Combining this result with the previous argument, we obtain the following

Proposition 3.6.6. *Let u be a continuous function in a convex domain Ω and assume that at every point $x \in \Omega$ there exists a concave paraboloid with opening K which touches u from below at x. Then*

$$u(x) + \frac{K}{2}|x|^2$$

is convex in Ω. In particular, u is punctually second order differentiable at almost every point $x \in \Omega$.

We are now ready to prove Theorem 3.6.1.

Proof of Theorem 3.6.1. Property *(3)* in Lemma 3.6.2 implies that u^ε are continuous (actually Lipschitz). By *(4)*, u^ε decreases as $\varepsilon \to 0$. By *(5)* and *(6)*

$$|u^\varepsilon(x_0) - u(x_0)| \leq |u(x_0^*) - u(x_0)| + \varepsilon \leq \varepsilon(\operatorname{osc}_H u + 1)$$

which converges to 0 as $\varepsilon \to 0$, uniformly in x_0. This implies (a).

To prove (b), note that

$$P_0(x) := u(x_0^*) + \varepsilon - \frac{1}{\varepsilon}|x - x_0^*|^2 \leq u^\varepsilon(x), \quad \text{for } x \in H,$$

with equality at $x = x_0$, hence P_0 touches u^ε from below at x_0, which implies the first assertion of (b). Proposition 3.6.4, applied with $\Omega = B$ where B is any ball contained in H, implies that u^ε is punctually second order differentiable a.e. in H.

We finally prove (c). Let $x_0 \in H_1$ and $P(x)$ be a paraboloid that touches u from above at x_0. It suffices to prove that $F(D^2P) \leq 0$. Since $F(D^2u) \leq 0$, we want to construct a perturbation Q of P

which touches u from above. This approximation is given by

$$Q(x) = P(x + x_0 - x_0^*) + \frac{1}{\varepsilon|x_0 - x_0^*|} - \varepsilon.$$

Note that $D^2Q = D^2P$. By property (5) of Lemma 3.6.2, there is $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $x_0 \in H_1$, then $x_0^* \in H$. Take $x \in H$ sufficiently close to x_0^* such that $x + x_0 - x_0^* \in G$. Then

$$u^\varepsilon(x + x_0 - x_0^*) \geq u(x) + \varepsilon - \frac{1}{\varepsilon}|x_0 - x_0^*|^2.$$

Therefore, again for x sufficiently close to x_0^*,

$$u(x) \leq P(x + x_0 - x_0^*) + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 - \varepsilon = Q(x)$$

and

$$\begin{aligned} Q(x_0^*) &= P(x_0) + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 - \varepsilon \\ &= u^\varepsilon(x_0) + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 - \varepsilon \\ &= u(x_0^*), \end{aligned}$$

since x_0^* corresponds to the point where the supremum is attained. Therefore Q touches u from above at x_0^* and so

$$0 \geq F(D^2Q) = F(D^2P),$$

as intended. The last statement in (c) follows from (b) and Lemma 2.5.3.

□

Remark 3.6.7. It was essential in our argument that if u solves the equation $F(D^2u) = 0$, then $v(x) = u(x + x_0) + y_0$ still solves the same equation. Therefore, this reasoning would not work for equations of the type $F(D^2u, x) = 0$ or $F(D^2u, u) = 0$.

We proceed with the most important result of this chapter – the uniqueness of solution of the homogeneous equation with Dirichlet boundary conditions. This result is a variation of the Jensen's method.

Theorem 3.6.8. *Let u be a viscosity subsolution of $F(D^2\omega) = 0$ in Ω and v be a viscosity supersolution of $F(D^2\omega) = 0$ in Ω. Then*

$$u - v \in \underline{S}\left(\frac{\lambda}{d}, \Lambda\right), \text{ in } \Omega.$$

We recall that if either u or v are C^2, then the result follows immediately from Proposition 3.2.4. Therefore, we will instead prove that $u^\varepsilon - v_\varepsilon \in \underline{S}\left(\frac{\lambda}{d}, \Lambda\right)$ in Ω, for sufficiently small ε and use the closedness of this class to conclude the result for $u - v$. Although the envelopes u^ε and v_ε are not C^2, they are differentiable almost everywhere and can be touched by paraboloids from below and above, respectively, at every point.

Proof. We fix H and H_1 such that $\overline{H}_1 \subset H \subset \overline{H} \subset \Omega$; we will start by proving that for ε small enough, $u^\varepsilon - v_\varepsilon \in \underline{S}(\lambda/d, \Lambda)$ in H_1. For this purpose, let P be a paraboloid that touches $u^\varepsilon - v_\varepsilon$ from above at x_0 in $\overline{B}_r(x_0) \subset H_1$. We are done once we prove that $\mathscr{P}^-_{\lambda/d,\Lambda}(D^2P) \leq 0$. We may assume that $\overline{B}_{2r}(x_0) \subset H$. Take $\delta > 0$ and define the auxiliary function

$$\omega(x) := v_\varepsilon(x) - u^\varepsilon(x) + P(x) + \delta|x - x_0|^2 - \delta r^2.$$

We will apply the area formula in Lemma 3.4.6 to the function ω, so we proceed by checking that the assumptions are verified. We have that $\omega \geq 0$ on $\partial B_r(x_0)$ and by (b) in Theorem 3.6.1 we know that for any $x \in \overline{B}_r(x_0)$ there exists a convex paraboloid P^x of opening K, independent of x, that touches ω from above at x in $B_r(x)$. Furthermore $\omega(x_0) < 0$ hence $\sup_{B_r(x_0)} \omega^- > 0$. We are thus ready to use Lemma 3.4.6 to ω in $B_R := B_r(x_0)$. With the notation therein, we have that if $x \in \overline{B}_r \cap \{\omega = \Gamma_\omega\}$, then P^x also touches Γ_ω by above at x in $B_r(x)$. We have that

$$0 < \int_{B_r(x_0)\cap\{\omega=\Gamma_\omega\}} \det D^2\omega. \tag{3.6.1}$$

By (b) in Theorem 3.6.1 we also get that there exists $A \subset B_r(x_0)$ such that $|B_r(x_0) \setminus A| = 0$ where both u^ε and v_ε are punctually twice differentiable. By (c) in Theorem 3.6.1,

$$F(D^2v_\varepsilon(x)) \geq 0 \text{ and } F(D^2u^\varepsilon(x)) \leq 0, \text{ for } x \in A. \tag{3.6.2}$$

Since Γ_ω is convex and $\Gamma_\omega \leq \omega$, then the points $x \in A \cap \{\omega = \Gamma_\omega\}$ correspond to maxima of the function $\omega - \Gamma_\omega$, thus $D^2\omega(x) \geq D^2\Gamma_\omega(x) \geq 0$, that is,

$$D^2\omega \text{ is nonnegative definite at } A \cap \{\omega = \Gamma_\omega\}. \tag{3.6.3}$$

It follows from (3.6.1) that there exists a point $x_1 \in A \cap \{\omega = \Gamma_\omega\}$ and at this point we have, by (3.6.2) and (3.6.3) that

$$\begin{aligned}
0 \geq& F(D^2u^\varepsilon(x_1)) = F(D^2v_\varepsilon(x_1) - D^2\omega(x_1) + D^2P + 2\delta I)\\
\geq& F(D^2v_\varepsilon(x_1) + D^2P + 2\delta I)\\
\geq& F(D^2v_\varepsilon(x_1) + D^2P) - 2\Lambda\delta\\
\geq& F(D^2v_\varepsilon(x_1)) - \Lambda\|(D^2P)^+\| + \lambda\|(D^2P)^-\| - 2\Lambda\delta\\
\geq& -\Lambda\|(D^2P)^+\| + \lambda\|(D^2P)^-\| - 2\Lambda\delta\\
\geq& \Lambda \operatorname{Tr}(D^2P)^+ + \frac{\lambda}{d}\operatorname{Tr}(D^2P)^+ - 2\Lambda\delta\\
=& \mathscr{P}^-_{\lambda/d,\Lambda}(D^2P) - 2\Lambda\delta.
\end{aligned}$$

Letting $\delta \to 0$ concludes that $u^\varepsilon - v_\varepsilon \in \underline{S}\left(\frac{\lambda}{d}, \Lambda\right)$ in Ω, for sufficiently small ε. By the closedness of this class (recall Proposition 3.1.5), we conclude the result for $u - v$.

□

Corollary 3.6.9. *The Dirichlet problem*

$$\begin{cases} F(D^2u) = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases}$$

has at most one viscosity solution $u \in C(\Omega)$.

This corollary follows immediately from Theorem 3.6.5 combined with the maximum principle for viscosity subsolutions, Corollary 3.4.7.

3.7 $C^{1,\alpha}$ regularity for solutions of $F(D^2u) = 0$

In this section we will present the proof of $C^{1,\alpha}$ regularity of solutions to the homogeneous equation $F(D^2u) = 0$. Note that, without additional conditions on the operator F, this regularity is optimal, as proven by the counter-examples in [50–53].

The following result is an immediate consequence of Theorem 3.6.5. Denote by $\Omega_h := \{x \in \Omega : d(x, \partial\Omega) > h\}$.

Proposition 3.7.1. *Let u be a viscosity solution of $F(D^2u) = 0$ in Ω. Let $h > 0$ and $e \in \mathbb{R}^d$ with $|e| = 1$. Then*

$$u(x+he) - u(x) \in S\left(\frac{\lambda}{d}, \Lambda\right) \qquad \text{in } \Omega_h.$$

Next we present an auxiliary result relating an estimate on the difference quotient of a function and its Hölder regularity.

Proposition 3.7.2. *Let $u \in C(\Omega) \cap L^\infty(\Omega)$. Suppose there exists $K > 0$ and $\alpha \in (0,1)$ such that*

$$\sup_{B_r(x_0)} |u(x_0 + x) - 2u(x_0) + u(x_0 - x)| \leq Kr^{1+\alpha},$$

for every $0 < r < 1$ and every $x_0 \in \Omega$ with $B_r(x_0) \subset \Omega$. Then $u \in C^{1,\alpha}_{loc}(\Omega)$. In addition, for every $\Omega' \Subset \Omega$ there exists $C_0 > 0$ for which

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C_0 \left(\|u\|_{L^\infty(\Omega)} + K\right).$$

Proof. We divide the proof in 3 steps:

(i) $u \in C^{0,1}(\Omega)$ and thus u is punctually differentiable at almost every point;

(ii) For almost every $x \in \Omega$, there exist an affine function ℓ_x such that

$$\|u - \ell_x\|_{L^\infty(B_r(x))} \leq Kr^{1+\alpha}$$

and $|\ell_x| \leq C\left(\|u\|_{L^\infty(\Omega)} + K\right)$;

(iii) Property (ii) actually holds for every $x \in \Omega$.

(i) is clear by definition of the space $C^{0,1}(\Omega)$. To prove (ii), note that since u is punctually differentiable at almost every point, we can define

$$\mathscr{A} := \{x \in \Omega : u \text{ is punctually differentiable at } x\}$$

and $|\Omega \setminus \mathscr{A}| = 0$. Therefore, for every $x \in \mathscr{A}$ and $|h|$ sufficiently small,

$$u(x+h) = u(x) + Du(x) \cdot h + o(|h|)$$

and

$$u(x-h) = u(x) - Du(x) \cdot h + o(|h|).$$

Adding both we get

$$u(x+h) - 2u(x) + u(x-h) = o(|h|)$$

which combined with the assumption gives the estimate

$$o(|h|) \leq K|h|^{1+\alpha}.$$

This proves (ii) with $\ell_x(y) = u(x) - Du(x) \cdot (y-x)$. Clearly $|\ell_x| \leq C(\|u\|_{L^\infty(\Omega)} + K)$ holds.

We finally prove (iii). Let $x \in \Omega$ and $r > 0$ be arbitrary. By contradiction, assume that there exists x_0 such that (ii) does not hold, that is, assume that for every affine function ℓ there exists $r > 0$ such that

$$\|u - \ell\|_{B_r(x_0)} > Kr^{1+\alpha}. \tag{3.7.1}$$

Since (ii) holds for every $x \in \mathscr{A}$ which is dense in $B_r(x_0)$, there exists a sequence of points $x_n \to x_0$ and a sequence of affine functions $(\ell_n)_n$ such that $|\ell_n| \leq C\left(\|u\|_{L^\infty(\Omega)} + K\right)$ and

$$\|u - \ell_n\|_{L^\infty(B_r(x_n))} \leq Kr^{1+\alpha},$$

for every $r > 0$. Furthermore $\ell_n \to \ell_0$ locally uniformly for some affine function ℓ_0, up to a subsequence. So, for every $\varepsilon > 0$ there exists n_1 such that if $n > n_1$ then $\|\ell_n - \ell_0\|_{L^\infty(B_r(x_n))} \leq \varepsilon$. Since $x_n \to x_0$, for every $\delta > 0$, there exists $n_2 > 0$ such that if $n > n_2$, then

$$B_{r-\delta}(x_0) \subset B_r(x_n).$$

Let $n_0 = \max\{n_1, n_2\}$ and take $n > n_0$. Then

$$\begin{aligned}\sup_{y \in B_{r-\delta}} |u(y) - \ell_0(y)| \leq & \sup_{B_r(x_n)} |u(y) - \ell_n(y)| + \|\ell_n - \ell_0\|_{L^\infty(B_r(x_n))} \\ \leq & Kr^{1+\alpha} + \varepsilon\end{aligned}$$

So, for every $y \in B_{r-\delta}(x_0)$ we get

$$|u(y) - \ell_0(y)| \leq \left(\frac{r}{r-\delta}\right)^{1+\alpha} K|y - x_0|^{1+\alpha} + \varepsilon.$$

Since we can take δ and ε to be arbitrarily small, we get a contradiction with (3.7.1).

Once we have proven (iii), the result follows from Proposition 2.3.3.

□

The following result will allow us to bootstrap the Hölder regularity obtained in Section 3.5.3 and get $C^{1,\alpha}$ regularity.

Lemma 3.7.3. *Let $0 < \alpha < 1$, $0 < \beta \leq 1$ and $K > 0$ be constants. Let $u \in L^\infty([-1,1])$ satisfy $\|u\|_{L^\infty([-1,1])} \leq K$. Define for $h \in \mathbb{R}$ with $0 < |h| \leq 1$,*

$$v_{\beta,h} = \frac{u(x+h) - u(x)}{|h|^\beta}, \quad x \in I_h$$

where $I_h := [-1, 1-h]$ if $h > 0$ and $I_h := [-1-h, 1]$ if $h < 0$. Assume that $v_{\beta,h} \in C^\alpha(I_h)$ and $\|v_{\beta,h}\|_{C^\alpha(I_h)} \leq K$, for any $0 < |h| \leq 1$. It then holds

(1) If $\alpha + \beta < 1$, then $u \in C^{\alpha+\beta}([-1,1])$ and $\|u\|_{C^{\alpha+\beta}([-1,1])} \leq CK$;

(2) If $\alpha + \beta > 1$, then $u \in C^{0,1}([-1,1])$ and $\|u\|_{C^{0,1}([-1,1])} \leq CK$;

(3) If in particular $\beta = 1$, then $u \in C^{1,\alpha}([-1,1])$ and $\|u\|_{C^{1,\alpha}([-1,1])} \leq CK$;

where C in (1), (2), (3), (as well as inside the proof) depends only on $\alpha + \beta$.

Proof. By the symmetry of the problem with respect to the change of variables $x \to -x$, it suffices to bound $|u(x+\varepsilon) - u(x)|$ for $-1 \leq x \leq 0$, $\varepsilon > 0$ and $x + \varepsilon \leq 1$.

Fix the integer i satisfying $x + 2^i\varepsilon \leq 1 \leq x + 2^{i+1}\varepsilon$ and define $\tau_0 := 2^i\varepsilon$. Then $-1 \leq x \leq x + \tau_0 \leq 1$ and

$$1/2 \leq \tau_0 \leq 2. \tag{3.7.2}$$

Define also

$$\omega(\tau) := u(x+\tau) - u(x), \quad \text{for } 0 < \tau \leq \tau_0.$$

Then it holds

$$\begin{aligned}
|\omega(\tau) - 2\omega(\tau/2)| \leq& |u(x+\tau) - 2u(x+\tau/2) + u(x)| \\
\leq& \left(\frac{\tau}{2}\right)^\beta |v_{\beta,\tau/2}(x+\tau/2) - v_{\beta,\tau/2}(x)| \\
\leq& K\left(\frac{\tau}{2}\right)^{\alpha+\beta},
\end{aligned}$$

since $\|v_{\beta,\tau/2}\|_{C^\alpha([-1,1-\tau/2])} \le K$ by hypothesis. By the same reasoning, we get

$$\begin{aligned}
|\omega(\tau_0) - 2\omega(\tau_0/2)| &\le CK\tau_0^{\alpha+\beta} \\
|2\omega(\tau_0/2) - 2^2\omega(\tau_0/2^2)| &\le CK2^{1-(\alpha+\beta)}\tau_0^{\alpha+\beta} \\
&\dots \\
|2^{i-1}\omega(\tau_0/2^{i-1}) - 2^i\omega(\tau_0/2^i)| &\le CK2^{(i-1)(1-(\alpha+\beta))}\tau_0^{\alpha+\beta}.
\end{aligned}$$

By triangular inequality, we get

$$\begin{aligned}
|\omega(\tau_0) - 2^i\omega(\varepsilon)| =& |\omega(\tau_0) - 2^i\omega(\tau_0/2^i)| \\
\le& CK\tau_0^{\alpha+\beta}\sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))}.
\end{aligned}$$

Since $2^{-i} = \tau_0^{-1}\varepsilon \le 2\varepsilon$ by (3.7.2), and $\|u\|_{L^\infty([-1,1])} \le K$, we have that

$$\begin{aligned}
|\omega(\varepsilon)| \le& 2^{-i}|\omega(\tau_0)| + CK2^{-i}\tau_0^{\alpha+\beta}\sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))} \\
\le& 4K\varepsilon + CK\varepsilon\tau_0^{\alpha+\beta-1}\sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))}.
\end{aligned}$$

Now if $\alpha+\beta < 1$, we get

$$\begin{aligned}
|\omega(\varepsilon)| \le& 4K\varepsilon + CK\varepsilon\tau_0^{\alpha+\beta-1}2^{i(1-(\alpha+\beta))} \\
=& 4K\varepsilon + CK\varepsilon^{\alpha+\beta} \\
\le& CK\varepsilon^{\alpha+\beta}.
\end{aligned}$$

That is,

$$|u(x+\varepsilon) - u(x)| \le CK\varepsilon^{\alpha+\beta},$$

and we get (1).

If $\alpha+\beta > 1$, then we instead have

$$\begin{aligned}
|\omega(\varepsilon)| \le& 4K\varepsilon + CK\varepsilon\tau^{\alpha+\beta-1} \\
\le& CK\varepsilon,
\end{aligned}$$

and we get (2).

If in particular $\beta = 1$, then $v_{1,h}$ corresponds to the difference quotient of u. Since $v_{1,h} \in C^\alpha(I_h)$ we have

$$\left|\frac{u(x+h) - u(x)}{h} - \frac{u(y+h) - u(y)}{h}\right| \le K|x-y|^\alpha,$$

for every $x,y \in I_h$ and $0 < |h| < 1$. Fixing $y = x - h$ we get

$$|u(x+h) - 2u(x) + u(x-h)| \leq K|h|^{1+\alpha},$$

placing u in the setting of Proposition 3.7.2 which gives $u \in C^{1,\alpha}([-1,1])$ with $\|u\|_{C^{1,\alpha}([-1,1])} \leq CK$, thus concluding (3).

□

We now bootstrap Lemma 3.7.3 to get interior $C^{1,\alpha}$ regularity. Recall that because of the ABP in Theorem 3.4.3, u is bounded in B_1.

Corollary 3.7.4. *Let u be a viscosity solution of $F(D^2u) = 0$ in B_1. Then $u \in C^{1,\alpha}(\overline{B}_{1/2})$ with the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + |F(0)|\right),$$

where $0 < \alpha < 1$ and C are universal constants.

Proof. Fix $e \in \mathbb{R}^d$ with $|e| = 1$ and $0 < h < 1/8$. By Proposition 3.7.1, for every $0 < \beta \leq 1$ it holds

$$v_\beta(x) = \frac{1}{h^\beta}(u(x+he) - u(x)) \in S(\lambda/d, \Lambda), \quad \text{in } B_{7/8}.$$

Therefore by Proposition 3.5.11 properly scaled we have the following interior Hölder estimate for v_β

$$\|v_\beta\|_{C^\alpha(\overline{B}_r)} \leq C(r,s)\|v_\beta\|_{L^\infty(B_{(r+s)/2})} \leq C(r,s)\|u\|_{C^{0,\beta}(\overline{B}_s)}, \tag{3.7.3}$$

where $C(r,s)$ depends on universal constants, r and s, $0 < r < s \leq 7/8$, $0 < h < (s-r)/2$ and α is universal.

By making α smaller if necessary, we can assume it is not of the form $\alpha = 1/k$ for any $k \in \mathbb{N}$, and therefore there exists a universal integer i such that $i\alpha < 1 < (i+1)\alpha$. Since $u \in S(\lambda/d, \Lambda, F(0))$ (by Proposition 3.2.4) we can again apply Proposition 3.5.11 and get the interior estimate for u

$$\|u\|_{C^\alpha(\overline{B}_{7/8})} \leq C\left(\|u\|_{L^\infty(B_1)} + |F(0)|\right) =: CK;$$

where $K = \|u\|_{L^\infty(B_1)} + |F(0)|$. Applying once again (3.7.3) with $\beta = \alpha$ and $r = r_1 < s = 7/8$ we get

$$\|v_\alpha\|_{C^\alpha(\overline{B}_{r_1})} \leq C(r_1)\|u\|_{C^\alpha(\overline{B}_{7/8})} \leq C(r_1)K,$$

where $0 < h < (7/8 - r_1)/2$ and $C(r_1)$ depends only on universal constants and r_1. This estimate places v_α in the assumptions of Lemma 3.7.3 which can now be applied in line segments parallel to e. We get

$$\|u\|_{C^{2\alpha}(\overline{B}_{r_2})} \leq C(r_1, r_2)K, \quad \text{for} \quad r_2 < r_1.$$

We now repeat the argument, first applying (3.7.3) and then Lemma 3.7.3 with $\beta = 2\alpha$, thus getting $u \in C^{3\alpha}(\overline{B}_{r_4})$. We can iterate this argument i times and since $i\alpha < 1 < (i+1)\alpha$, by (2) of Lemma 3.7.3 we will arrive at

$$\|u\|_{C^{0,1}(\overline{B}_{3/4})} \leq CK.$$

We apply one last time (3.7.3) with $\beta = 1$ and get

$$\|v_1\|_{C^{\alpha}(\overline{B}_{1/2})} \leq C\|u\|_{C^{0,1}(\overline{B}_{3/4})} \leq CK,$$

for every $|e| = 1$ and $0 < h < 1/8$. Finally by (3) in Lemma 3.7.3 we complete the proof. □

3.8 $C^{1,\alpha}$ regularity for solutions of $F(D^2u,x) = f$

In this section we will prove interior $C^{1,\alpha}$ regularity for viscosity solutions of the nonhomogeneous, fully nonlinear, uniformly elliptic equation with variable coefficient

$$F(D^2u,x) = f(x), \quad \text{for } x \in B_1. \tag{3.8.1}$$

The argument consists primarily of the following steps:

(1) First we prove that we can assume, without loss of generality, that both $\|f\|_{L^\infty(B_1)}$ and the oscillation of $F(M,x)$ with respect to x are arbitrarily small;

(2) Step (1) implies that for every $x_0 \in B_1$ and $r > 0$ sufficiently small, the equation behaves similarly to the homogeneous equation with constant coefficients

$$F(D^2u,x_0) = 0, \quad \text{for } x \in B_r(x_0); \tag{3.8.2}$$

(3) We then establish an approximation argument which consists in proving that solutions of equations which are similar, have to be close in L^∞ norm. More precisely, because of step (1), solutions of equations (3.8.1) and (3.8.2) have to be arbitrarily close in the L^∞ norm;

(4) Because of Corollary 3.7.4, solutions of (3.8.2) belong to $C^{1,\alpha}(\overline{B}_{1/2})$ with universal estimates;

(5) The final steps consists of importing the information from steps (3) and (4) to get improved regularity of solutions of (3.8.1).

Start by defining the oscillation function of an elliptic operator F by $\beta_{F,x_0} : B_1 \to \mathbb{R}$ given by

$$\beta_{F,x_0}(x) := \sup_{M \in S(d)\setminus\{0\}} \frac{|F(M,x) - F(M,x_0)|}{\|M\|}.$$

We aim at establishing the following regularity result.

Theorem 3.8.1. *Assume that F is continuous in $x \in B_1$, β_{F,x_0} is Hölder continuous for every $x_0 \in B_1$ and $f \in L^\infty(B_1)$. Let $0 < \overline{\alpha} < 1$ and c_e be the constants given by Corollary 3.7.4 and fix $0 < \alpha < \overline{\alpha}$.*

Then any solution u of $F(D^2u,x)=f(x)$ in B_1 is in $C^{1,\alpha}(B_{1/2})$ and satisfies the interior estimate

$$\|u\|_{C^{1,\alpha}(B_{1/2})}\le C\left(\|u\|_{L^\infty(B_1)}+\|f\|_{L^\infty(B_1)}\right).$$

We start by noting that it suffices to prove the theorem assuming that $F(0,x)\equiv 0$, $\|u\|_{L^\infty(B_1)}\le 1$ and for any arbitrary $\varepsilon>0$ to be fixed, $\|f\|_{L^\infty(B_1)}\le\varepsilon$ and there exists a small $r_0>0$ such that for every $x_0\in B_{1/2}$ and every $r<r_0$,

$$\|\beta_{F,x_0}\|_{L^d(B_r(x_0))}\le\varepsilon r.$$

Indeed, since we can rewrite our problem as $F(D^2u,x)-F(0,x)=f(x)-F(0,x)$, it is no restriction to assume that $F(0,x)\equiv 0$. Let $\varepsilon>0$ to be fixed, call

$$K:=\max\left\{\|u\|_{L^\infty(B_1)}+\frac{\|f\|_{L^\infty(B_1)}}{\varepsilon},1\right\}$$

and define $v(x):=K^{-1}u(x)$. Then v is a viscosity solution of $G(D^2v,x)=g(x)$, where $G(M,x)=K^{-1}F(KM,x)$ and $g(x)=K^{-1}f(x)$. Note that G is still (λ,Λ)-elliptic and

$$\begin{aligned}\beta_{G,x_0}&=\sup_{M\in S(d)\setminus\{0\}}\frac{|K^{-1}F(KM,x)-K^{-1}F(KM,x_0)|}{\|M\|}\\&=\sup_{N\in S(d)\setminus\{0\}}\frac{|F(N,x)-F(N,x_0)|}{\|N\|}\\&=\beta_{F,x_0}.\end{aligned}$$

Since $\beta_{F,x_0}\in C^\gamma(B_1)$ for some $0<\gamma<1$ and $\beta_{F,x_0}(x_0)=0$, then for every $x\in B_r(x_0)$ it holds $\beta_{F,x_0}(x)\le Cr^\gamma$ where $C=\|\beta_{F,x_0}\|_{C^\gamma(B_1)}$. Therefore, for r_0 small enough and for every $r<r_0$ we have

$$\|\beta_{F,x_0}\|_{L^d(B_r(x_0))}\le\varepsilon r.$$

Finally we note that $\|v\|_{L^\infty(B_1)}\le 1$ and $\|g\|_{L^\infty(B_1)}\le\varepsilon$. Therefore, if we are able to prove Theorem 3.8.1 under this smallness regime, we get $v\in C^{1,\alpha}(B_{1/2})$ and $\|v\|_{C^{1,\alpha}(B_{1/2})}\le C$ which immediately implies $u\in C^{1,\alpha}(B_{1/2})$ with

$$\|u\|_{C^{1,\alpha}(B_{1/2})}\le C\left(\|u\|_{L^\infty(B_1)}+\|f\|_{L^\infty(B_1)}\right).$$

Next we present a consequence of this smallness regime. It states that if the equation $F(D^2u,x)=f(x)$ is sufficiently close to the equation $F(D^2h,x_0)=0$ then the corresponding solutions are very close, that is, $\|u-h\|_{L^\infty(B_{1/2})}\ll 1$. In what follows, for simplicity of notation, we will always assume that $x_0=0$. We also call $\beta_{F,0}=\beta$.

Lemma 3.8.2. *Assume that F is continuous in $x\in B_1$, β is Hölder continuous and $f\in L^\infty(B_1)$. Let $0<\overline{\alpha}<1$ and c_e be the constants given by Corollary 3.7.4.*

For every $0<\delta<1$, there exists $\varepsilon_0>0$ such that, if $u\in C(B_1)$ is a viscosity solution to $F(D^2u,x)=f(x)$ in B_1 satisfying $\|u\|_{L^\infty(B_1)}\le 1$, $\|f\|_{L^\infty(B_1)}\le\varepsilon_0$ and $\|\beta\|_{L^d(B_r)}\le\varepsilon_0 r$, then one can find a function *h which is a viscosity solution to* $F(D^2h,x_0)=0$ *such that*

$$\|u-h\|_{L^\infty(B_{1/2})}\le\delta.$$

Such a function h satisfies

$$\|h\|_{C^{1,\alpha_0}(B_{1/2})}\le C\,\|h\|_{L^\infty(B_{3/4})}.$$

Proof. We argue by contradiction. For simplicity, we split the proof in 3 steps.

Step 1 – Assume that there exist $\delta_0>0$ and sequences $(f_n)_n$, $(F_n)_n$ and $(u_n)_n$ satisfying

1. $\|f_n\|_{L^\infty(B_1)}\le 1/n$;
2. $\|u_n\|_{L^\infty(B_1)}\le 1$;
3. if we call $\beta_n:=\beta_{F_n}$, then $\|\beta_n\|_{L^d(B_r)}\le r/n$,

linked together by the equations

$$F_n(D^2u_n,x)=f_n;$$

however, for every viscosity solution h of $F(D^2h,0)=0$ it holds

$$\|u_n-h\|_{L^\infty(B_{1/2})}>\delta_0. \tag{3.8.3}$$

By Proposition 3.5.11, there exists a universal β such that $(u_n)_n\subset C^\beta_{loc}(B_1)$ with universal estimates, therefore there exists $u_\infty\in C(B_1)$ such that $u_n\to u_\infty$ locally uniformly in B_1. Clearly $\|u_\infty\|_{L^\infty(B_1)}\le 1$.

At this point, we relate u_∞ and the sequence $(F_n(\,\cdot\,,0))_n$. Notice that for every $n\in\mathbb{N}$, $F_n(\,\cdot\,,0)$ is (λ,Λ)-elliptic and therefore uniformly Lipschitz continuous. Consequently, there exists a (λ,Λ)-elliptic operator $F_\infty(\,\cdot\,,0)$ such that $F_n(\,\cdot\,,0)\to F_\infty(\,\cdot\,,0)$ locally uniformly.

Step 2 – We claim that

$$F_\infty(D^2u_\infty,0)=0\quad\text{in } B_{9/10}. \tag{3.8.4}$$

To establish (3.8.4), let $p(x)$ be a paraboloid touching u_∞ locally from above at a point $x^*\in B_1$. We need to verify that

$$F_\infty(D^2p,0)\le 0. \tag{3.8.5}$$

If (3.8.5) fails to hold, then there exists $\zeta>0$ such that $F_\infty(D^2p,0)=\zeta$. Since $F_n(\,\cdot\,,0)$ converges locally uniformly to $F_\infty(\,\cdot\,,0)$, we may suppose, without loss of generality, that

$$F_n(D^2p,0)-F_\infty(D^2p,0)\ge-\frac{1}{n}.$$

Fix $\lambda^*>0$ as to satisfy

$$\frac{\lambda}{d\lambda^*}\ge\max\left\{1,\Lambda,\left|D^2p\right|\right\},$$

and let $\varphi_n \in C(B_1)$ be the unique viscosity solution to

$$\begin{cases} \mathscr{P}^-_{\lambda^*,1}(D^2\varphi_n) = |\beta_n| + \frac{1}{n} & \text{in } B_1 \\ \varphi_n = 0 & \text{on } \partial B_1. \end{cases} \tag{3.8.6}$$

First, notice that $\varphi_n \to 0$ uniformly. Furthermore

$$\begin{aligned} \mathscr{P}^-_{\lambda^*,1}(D^2\varphi_n) &= -\operatorname{Tr}(D^2\varphi_n)^+ - \lambda^* \operatorname{Tr}(D^2\varphi_n)^- \\ &\leq -\|(D^2\varphi_n)^+\| + d\lambda^*\|(D^2\varphi_n)^-\| \end{aligned}$$

Hence

$$\frac{\lambda}{d\lambda^*}\left(|\beta_n| + \frac{1}{n}\right) \leq -\Lambda\|(D^2\varphi_n)^+\| + \lambda\|(D^2\varphi_n)^-\|.$$

Then, we estimate

$$\begin{aligned} F_n(D^2p + D^2\varphi_n, x) &\geq F_n(D^2p, x) - \Lambda\|(D^2\varphi_n)^+\| + \lambda\|(D^2\varphi_n)^-\| \\ &\geq F_n(D^2p, 0) - \left|D^2p\right|\beta_n(x) - \Lambda\|(D^2\varphi_n)^+\| + \lambda\|(D^2\varphi_n)^-\| \\ &\geq F_\infty(D^2p, 0) - \frac{1}{n} - \left|D^2p\right|\beta_n(x) \\ &\quad + \frac{\lambda}{d\lambda^*}\left(|\beta_n| + \frac{1}{n}\right) \\ &\geq F_\infty(D^2p, 0) \\ &= \zeta. \end{aligned} \tag{3.8.7}$$

Now, because $p(x)$ touches u_∞ from above at x^*, the function

$$\psi_n(x) := p(x) + \varphi_n(x) + \frac{\zeta|x - x^*|^2}{4\Lambda} + C$$

touches u_n from above at some point x_n in a vicinity of x^*, for $n \gg 1$ and some constant $C > 0$. As a consequence, we obtain

$$\begin{aligned} \frac{1}{n} \geq f_n(x_n) &\geq F_n(D^2\psi(x_n), x_n) \\ &\geq F_n(D^2p + D^2\varphi(x_n), x_n) - \Lambda\frac{\zeta}{2\Lambda}, \end{aligned}$$

therefore

$$F_n(D^2p + D^2\varphi(x_n), x_n) \leq \frac{\zeta}{2} + \frac{1}{n},$$

which produces a contradiction, for n large enough, if (3.8.7) is evaluated at x_n. Hence (3.8.4) holds true.

Step 3 – Notice that (3.8.4) falls within the scope of Corollary 3.7.4, which implies that $u_\infty \in C^{1,\overline{\alpha}}_{loc}(B_{9/10})$, with universal estimates. By taking $h := u_\infty$, we reach a contradiction with (3.8.3) and complete the proof. □

Once an approximation lemma is available, we control the oscillation of the solutions in balls of a fixed, universal, radius. This is the content of the next proposition.

Proposition 3.8.3. *Suppose the assumptions of Lemma 3.8.2 are in force. Then, given $\alpha \in (0,\overline{\alpha})$, there exists a universal constant $0<\rho \ll 1$ and an affine function*

$$\ell(x) := a + b\cdot x$$

with $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$, satisfying

$$\|u-\ell\|_{L^\infty(B_\rho)} \le \rho^{1+\alpha}.$$

Proof. Take $\delta_0 > 0$ to be fixed later. It follows from Proposition 3.8.2 that, for ε_0 sufficiently small, there exists $h \in C^{1,\overline{\alpha}}_{loc}(B_1)$ such that

$$\|u-h\|_{L^\infty(B_{9/10})} \le \delta_0.$$

Consider the affine function

$$\ell(x) := h(0) + Dh(0)\cdot x$$

and compute

$$\begin{aligned}\sup_{B_\rho}|u(x)-\ell(x)| &\le \sup_{B_\rho}|h(x)-h(0)-Dh(0)\cdot x| \\ &\quad + \sup_{B_\rho}|u(x)-h(x)| \\ &\le C\rho^{1+\overline{\alpha}} + \delta_0,\end{aligned}$$

where $C>1$ is a universal constant. Making the universal choices

$$\rho := \left(\frac{1}{2C}\right)^{\frac{1}{\overline{\alpha}-\alpha}} \qquad \text{and} \qquad \delta_0 := \frac{\rho^{1+\alpha}}{2}, \tag{3.8.8}$$

the proof is completed. □

Proposition 3.8.4. *Suppose the assumptions of Lemma 3.8.2 are in force. Then, given $\alpha \in (0,\overline{\alpha})$, there exists a sequence $(\ell_n)_{n\in\mathbb{N}}$ of affine functions of the form*

$$\ell_n(x) = a_n + b_n \cdot x,$$

satisfying

$$\|u-\ell_n\|_{L^\infty(B_{\rho^n})} \le \rho^{n(1+\alpha)}, \tag{3.8.9}$$

and

$$|a_n - a_{n-1}| + \rho^{n-1}|b_n - b_{n-1}| \le C\rho^{(2+\alpha)(n-1)}, \tag{3.8.10}$$

for every $n \in \mathbb{N}$.

Proof. We reason through an induction argument. For clarity, we choose to split the proof into three steps.

Step 1 – We start with the case $n = 1$. Let $\ell_0 := 0$ and set

$$\ell_1(x) := h(0) + Dh(0) \cdot x + \frac{1}{2} x \cdot D^2 h(0) x,$$

where h is the approximating function whose existence is ensured by Lemma 3.8.2. Owing to Proposition 3.8.3, (3.8.9) is immediately verified. To produce (3.8.10), one resorts to the universal estimates for h, in $C^{1,\overline{\alpha}}_{loc}(B_1)$-spaces, available in Lemma 3.8.2.

Step 2 – We then formulate the induction hypothesis; that is, we suppose (3.8.9)-(3.8.10) have been verified for $n = 1, \ldots, k$ and establish those conditions in the case $n = k+1$. To that end, we introduce

$$v_k(x) := \frac{(u - \ell_k)}{\rho^{k(2+\alpha)}} \left(\rho^k x \right).$$

The function v_k solves

$$\rho^{k(1-\alpha)} F(\rho^{k(\alpha-1)} D^2 v_k, \rho^k x) = \rho^{k(1-\alpha)} f(\rho^k x) \quad \text{in } B_1,$$

which we rewrite as

$$F_k(D^2 v_k, x) = f_k(x) \quad \text{in } B_1,$$

and we will now verify that F_k, v_k and f_k fall under the assumptions of Lemma 3.8.2.

Firstly, it is clear that $F_k(\cdot, 0) \equiv 0$ and F_k is still (λ, Λ)-elliptic. Now we examine $\beta_k := \beta_{F_k}$. As before,

$$\begin{aligned} \beta_k &= \sup_{M \in S(d) \setminus \{0\}} \frac{|\rho^{k(1-\alpha)} F(\rho^{k(\alpha-1)} M, x) - \rho^{k(1-\alpha)} F(\rho^{k(\alpha-1)} M, x_0)|}{\|M\|} \\ &= \sup_{N \in S(d) \setminus \{0\}} \frac{|F(N,x) - F(N,x_0)|}{\|N\|} \\ &= \beta. \end{aligned}$$

Finally, $\|f_k\|_{L^\infty(B_1)} \leq \|f\|_{L^\infty(B_1)} \leq \varepsilon_0$ and by the induction assumption (3.8.9), we have $\|v\|_{L^\infty(B_1)} \leq 1$.

Therefore, v_k is entitled to the conclusion of Lemma 3.8.2, which guarantees the existence of a function $\overline{h} \in C^{1,\overline{\alpha}}_{loc}(B_1)$ such that

$$\left\| v_k - \overline{h} \right\|_{L^\infty(B_{9/10})} \leq \delta.$$

Reasoning as in the proof of Proposition 3.8.3, we conclude

$$\sup_{x \in B_\rho} |v_k(x) - \overline{\ell}_k(x)| \leq \rho^{1+\alpha}, \tag{3.8.11}$$

where

$$\overline{\ell}_k(x) := \overline{h}(0) + D\overline{h}(0) \cdot x.$$

Rescaling (3.8.11) back to the unit ball, we get

$$\sup_{B_{\rho^{k+1}}} |u(x) - \ell_{k+1}(x)| \leq \rho^{(1+\alpha)(k+1)}, \tag{3.8.12}$$

where

$$\ell_{k+1}(x) := a_{k+1} + b_{k+1} \cdot x,$$

with

$$a_{k+1} := a_k + \rho^{k(1+\alpha)}\overline{h}(0), \tag{3.8.13}$$

and

$$b_{k+1} := b_k + \rho^{k(\alpha)} D\overline{h}(0), \tag{3.8.14}$$

Step 3 – From (3.8.12) we establish the inequality in (3.8.9) for $n = k+1$. To verify that (3.8.10) is also satisfied for $n = k+1$, it suffices to rearrange the terms in (3.8.13) and (3.8.14) and combine them with the uniform estimates for $\overline{h}$. □

Theorem 3.8.1 follows immediately from Proposition 3.8.4 combined with Proposition 2.3.3.

Chapter 4

A degenerate fully nonlinear free transmission problem

4.1 Introduction and main results

In this chapter, we will study the following equation

$$|Du|^{\beta(x,u,Du)}F\left(D^2u\right) = f(x) \qquad \text{in } B_1, \tag{4.1.1}$$

where $\beta \geq 0$, F is uniformly elliptic and f is bounded and continuous. The model (4.1.1) accounts for a diffusion process degenerating as a variable power of the gradient.

We will present two main results. First, we obtain a local regularity result under very general assumptions on $\beta(x,u,Du)$. Indeed, we only require β to be well-defined, non-negative and bounded from above.

We emphasize that the dependence of β on u and Du can be nonlocal and therefore we need to adapt the definition of viscosity solution, since we can not write $\beta(x_0,u(x_0),D\varphi(x_0))$ for a test function $\varphi \in C^2(B_1)$.

Definition 4.1.1 (Viscosity solution)**.** *We say that $u \in C(B_1)$ is a viscosity subsolution (resp. supersolution) of* (4.1.1) *if the following condition holds:*

If $x_0 \in B_1$, $\varphi \in C^2(B_1)$ and $u-\varphi$ has a local maximum (resp. minimum) at x_0, then

$$|D\varphi(x_0)|^{\beta(x_0,u,D\varphi)}F\left(D^2\varphi(x_0)\right) \leq f(x_0), (\text{ resp. } \geq f(x_0)).$$

We say u is a viscosity solution, if it is both a subsolution and a supersolution.

The first result is the following.

Theorem 4.1.2 (Local $C^{1,\alpha}$ regularity)**.** *Let $u \in C(B_1)$ be a viscosity solution to* (4.1.1)*. Assume that $0 \leq \beta(x,t,p) \leq \beta_M$ for fixed β_M; assume also that F is uniformly (λ,Λ)-elliptic, $F(0) = 0$ and f is continuous and bounded in B_1. Let finally $\alpha_0 \in (0,1)$ be given in Remark 12 below.*

Then, there exist $\alpha > 0$ *and* $C > 0$ *such that any viscosity solution* u *of* (4.1.1) *is in* $C^{1,\alpha}(B_{1/2})$ *and*

$$[u]_{C^{1,\alpha}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_{9/10})} + \|f\|_{L^\infty(B_1)}\right),$$

where

$$\alpha = \min\left\{\alpha_0^-, \frac{1}{1+\beta_M}\right\},$$

and $C = C(\lambda, \Lambda, d, \beta_M)$.

We now state some remarks concerning this result.

Remark 4.1.3. Theorem 4.1.2 includes the following examples

- $\beta(x,u,Du) = \beta(x)\chi_{G(u)}$, where $G(u) = B_1 \setminus \{u = |Du| = 0\}$;
- $\beta(x,u,Du) = \theta(|Du|)$, where $\theta(t) \to 2$ as $t \to 0$ and $\theta(t) \to 1$ as $t \to \infty$. This equation was considered for the first time in [10].

Remark 4.1.4. The regularity class is interpreted in the following sense: If $\frac{1}{1+\beta_M} < \alpha_0$, then solutions are $C^{1,\alpha}(B_{1/2})$ with $\alpha = \frac{1}{1+\beta_M}$; if, alternatively, $\alpha_0 \leq \frac{1}{1+\beta_M}$, then solutions are $C^{1,\alpha}(B_{1/2})$ for every $\alpha < \alpha_0$.

Remark 4.1.5. By the classical Krylov-Safanov and Trudinger theory (see Corollary 3.7.4), every viscosity solution of

$$F(D^2u) = 0, \quad \text{in } B_1,$$

belongs to $C^{1,\alpha_0}(B_{1/2})$ for a universal $\alpha_0 \in (0,1)$.

As one can see from the previous result, there is an intrinsic dependence between the regularity obtained and the degeneracy rate. Hence, if this rate is variable over the domain, it seems natural to obtain regularity results which also vary over the domain. This idea, put forward in Lemma 4.6.2 which corresponds to the geometric iterations, is the novelty in this chapter. By considering variable exponents in each iteration, we are able to better capture the pointwise degenerate behaviour of the equation.

To obtain this improved regularity, we consider the following explicit expression for the exponent. Let $G_i(u,Du) \subset B_1$, $i = 1,\ldots,N$ be disjoint sets which depend on the solution u and its gradient Du, and define $G_0(u,Du) := B_1 \setminus \bigcup_{i=1}^N G_i(u,Du)$. Assume the exponent β has the form

$$\beta(x,u,Du) = \sum_{i=0}^{N} \beta_i(x)\chi_{G_i(u,Du)}. \tag{4.1.2}$$

An example to keep in mind is the following. Let $N = 2$, $G_1(u,Du) = \{u > 0\}$, $G_2(u,Du) = \{u < 0\}$ and $G_0(u,Du) = \{u = 0\}$. If $\beta_0 = 0$, β_1 and β_2 are constants, then we recover the result from [32]. Our result is thus more refined, not only in the sense that it includes a much broader class of degeneracies, but also because we obtain an improved pointwise regularity.

The second main result in this chapter is the following.

Theorem 4.1.6 (Pointwise $C^{1,\alpha}$ regularity)**.** *Let $u \in C(B_1)$ be a viscosity solution to* (4.1.1)*. Assume that β is given by* (4.1.2) *with $\beta_i(\cdot) \in [0,\beta_M]$ for fixed $\beta_M \geq 0$ and have modulus of continuity ω satisfying*

$$\limsup_{t\to 0} \ln\left(\frac{1}{t}\right)\omega(t) = 0; \tag{4.1.3}$$

assume also that F is uniformly (λ,Λ)-elliptic, $F(0)=0$ and f is continuous and bounded in B_1. Let finally $\alpha_0 \in (0,1)$ be given in Remark 12.

Then any viscosity solution u of (4.1.1) *is $C^{1,\alpha(\cdot)}(B_{1/2})$ and*

$$[u]_{C^{1,\alpha(\cdot)}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_{9/10})} + \|f\|_{L^\infty(B_1)}\right),$$

where

$$\alpha(x) = \min_{i=0,\dots,N}\left\{\alpha_0^-, \frac{1}{1+\beta_i(x)}\right\},$$

and $C = C(\lambda,\Lambda,d,\beta_M,\omega)$.

Assumptions of the type (4.1.3) are typical when obtaining higher regularity of solutions to equations with variable exponents. For example, in [1] the authors are able to prove improved regularity to a class of variational problems with variable exponents, under the assumption above.

Pointwise regularity has been the subject of various papers, see for example [15] and [46]. These are useful when a certain property is not verified locally but instead only at a point. Obtaining such a result as in Theorem 4.1.3 instead of a local regularity result as in Theorem 4.1.2, comes with a cost, since we must assume stronger uniform continuity of the functions β_i. However, more information is gathered. For example, consider $N = 0$,

$$\beta_0(x) = 1000e^{-\frac{1}{2}|1000x|^2}$$

and assume F is convex, so that $\alpha_0 = 1$ (see [15, Chapter 6]). Then a local result would yield $C^{1,\alpha}$ regularity, with $\alpha = \frac{1}{1001}$. The problem with this result is that $\beta_0 \approx 0$ except in a small neighborhood of 0. On the other hand, Theorem 4.1.3 would immediately yield $C^{1,\alpha}$ regularity with $\alpha \approx 1$ for points away from the origin.

Another advantage of having such a sharp pointwise regularity comes when studying the free boundary of the problem, where a finer analysis is required.

This chapter is organized as follows. Section 4.2 introduces the assumptions to hold throughout this chapter, some basic notation and a characterization of Hölder spaces. We also obtain a simple proof for Theorem 4.1.2. In Section 4.3, we simplify the equation, rewriting it as viscosity inequalities and removing the dependence of the exponents on the solution. We then obtain an important smallness assumption, which provides a tangential path between our equation and the homogeneous one.

Hölder continuity of a perturbed equation is the topic of Section 4.4. In Section 4.5 we derive an approximation lemma. Finally, Section 4.6 consists of the geometric iterations with variable exponents which combined with the characterization of Hölder spaces put forward in Section 4.2, provides the improved, pointwise Hölder continuity of the gradient.

4.2 Preliminary material and main assumptions

In this introductory section, we present some basic results that will be instrumental in the sequel and detail our main assumptions. We start with some notation.

Next, we introduce the uniform ellipticity assumption, assumed to hold throughout this chapter.

[A1] (Uniform ellipticity)**.** *The operator* $F : \mathscr{S}(d) \to \mathbb{R}$ *is* (λ, Λ)*-elliptic, i.e, there exist* $0 < \lambda \leq \Lambda$ *such that*

$$\lambda|N| \leq F(M) - F(M+N) \leq \Lambda|N|,$$

for every $M, N \in \mathscr{S}(d)$*, with* $N \geq 0$.

A well-known consequence of [A1] is the uniform Lipschitz regularity of F (see for example [15, Chapter 2]).

Next, let $G_i(u,Du) \subset B_1$, $i = 1,...,N$ be disjoint sets and define $G_0(u,Du) := B_1 \setminus \bigcup_{i=0}^{N} G_i(u,Du)$. Assume the exponent β has the form

$$\beta(x,u,Du) = \sum_{i=0}^{N} \beta_i(x) \chi_{G_i(u,Du)}.$$

We now make some assumptions on β_i. First, assume they have a modulus of continuity which decays at the origin as $o(\ln(1/t)^{-1})$.

[A2] (Uniform continuity of the exponents)**.** *The exponents* $\beta_i : B_1 \to \mathbb{R}$ *have modulus of continuity satisfying*

$$\limsup_{t\to 0} \ln\left(\frac{1}{t}\right) \omega(t) = 0.$$

Note that [A2] is equivalent to the following statement. For every $0 < r < e^{-1}$, the following holds

$$\limsup_{k\to\infty} k\omega(r^k) = 0.$$

Hence, by definition, for every $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if $\rho \leq \delta_1$, then for every $k \in \mathbb{N}$,

$$k \ln\left(\frac{1}{\rho}\right) \omega(\rho^k) \leq \varepsilon.$$

Since ρ will be chosen to be small, we can assume $\rho < e^{-1}$, and it follows that

$$k\omega(\rho^k) \leq \varepsilon.$$

Now, by defining $\varepsilon = \frac{\alpha_0-\alpha}{2}$ (it will become clear later why we make this choice) we also fix δ_1 such that if $\rho \le \delta_1$, then for every $k \in \mathbb{N}$,

$$k\omega(\rho^k) \le \frac{\alpha_0 - \alpha}{2}. \tag{4.2.1}$$

This δ_1 depends only on the continuity modulus ω, the universal exponent α_0 introduced in Remark 12, and the exponent α which is defined in Theorem 4.1.3.

Remark 4.2.1. To emphasize this idea, let's consider some concrete examples. Suppose $\omega(t) = t^{1/2}$ and choose α such that $\varepsilon = \frac{\alpha_0-\alpha}{2} = 1/100$. Then one can calculate that (4.2.1) holds for $\rho \le 4.7 \times 10^{-7}$. If $\varepsilon = 1/1000$, then we need $\rho \le 2.55 \times 10^{-9}$. These are numbers that depend only on these quantities and can be calculated, provided we know the expression of ω explicitly.

An example of a modulus of continuity satisfying [A2] is

$$\omega(t) = \ln\left(\frac{1}{t}\right)^{-p},$$

with $p > 1$.

Finally, we assume that the exponents are non-negative and bounded uniformly from above.

[A3] (Boundedness of the exponents)**.** *There exists a constant β_M such that*

$$0 \le \beta_i(\cdot) \le \beta_M < 1.$$

To conclude this introductory section, we present a simple proof of Theorem 4.1.2, using the results from [32].

Lemma 4.2.2. *Let $u \in C(B_1)$ be a viscosity solution to the equation*

$$|Du|^{\beta(x,u,Du)} F\left(D^2u\right) = f(x), \tag{4.2.2}$$

with $\beta \in [0, \beta_M]$ and $\beta_M \ge 0$.

Then u is a viscosity subsolution to the equation

$$\min\left\{F\left(D^2u\right), |Du|^{\beta_M} F\left(D^2u\right)\right\} \le \|f\|_{L^\infty(B_1)}, \tag{4.2.3}$$

and a viscosity supersolution to the equation

$$\max\left\{F\left(D^2u\right), |Du|^{\beta_M} F\left(D^2u\right)\right\} \ge -\|f\|_{L^\infty(B_1)}. \tag{4.2.4}$$

Proof. We prove only that if u is a viscosity subsolution to (4.2.2), then it is a subsolution to (4.2.3), noting that the remaining case follows similarly.

Let $\varphi \in C^2(B_1)$ be such that $u - \varphi$ has a local maximum at x_0. Then

$$|D\varphi(x_0)|^{\beta(x_0,u,D\varphi)} F\left(D^2\varphi(x_0)\right) \le f(x_0).$$

Thus, depending on whether $|D\varphi(x_0)| \geq 1$ or $|D\varphi(x_0)| < 1$ one of the following must hold, respectively

$$F\left(D^2\varphi(x_0)\right) \leq \|f\|_{L^\infty(B_1)},$$
$$|D\varphi(x_0)|^{\beta_M} F\left(D^2\varphi(x_0)\right) \leq \|f\|_{L^\infty(B_1)},$$

provided $F\left(D^2\varphi(x_0)\right) \geq 0$ (clearly if this is not the case, both inequalities are trivially verified). In either case, we have

$$\min\left\{F\left(D^2\varphi(x_0)\right), |D\varphi(x_0)|^{\beta_M} F\left(D^2\varphi(x_0)\right)\right\} \leq \|f\|_{L^\infty(B_1)}.$$

Hence, we have proved that u is a subsolution of (4.2.3).

□

This simple result places the equation (4.2.2) in the framework of [32] with $\theta_1 = 0$ and $\theta_2 = \beta_M$ (see Proposition 1 therein). A direct application of [32, Theorem 2] yields local regularity $u \in C^{1,\alpha}(B_{1/2})$ with

$$\alpha = \min\left\{\alpha_0^-, \frac{1}{1+\beta_M}\right\},$$

together with the estimate

$$[u]_{C^{1,\alpha}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_{9/10})} + \|f\|_{L^\infty(B_1)}\right),$$

which implies Theorem 4.1.2.

The remaining of this chapter is devoted to proving Theorem 4.1.3. In the next section, we begin our analysis.

4.3 Scaling properties

The following result disconnects the dependence of the exponents on the solution, by separating the possible cases.

Lemma 4.3.1. *Let $u \in C(B_1)$ be a viscosity solution to the perturbed equation*

$$|Du+p|^{\beta(x,u,Du)} F\left(D^2u\right) = f(x), \tag{4.3.1}$$

with β given by (4.1.2)*. Assume that assumptions [A1], [A2] and [A3] are in force.*

Then u is a viscosity subsolution to the equation

$$\min_{i=0,\dots,N}\left\{|Du+p|^{\beta_i(x)} F\left(D^2u\right)\right\} \leq \|f\|_{L^\infty(B_1)}, \tag{4.3.2}$$

and a viscosity supersolution to the equation

$$\max_{i=0,\dots,N}\left\{|Du+p|^{\beta_i(x)} F\left(D^2u\right)\right\} \geq -\|f\|_{L^\infty(B_1)}. \tag{4.3.3}$$

Proof. We prove only that if u is a viscosity subsolution to (4.3.1), then it is a subsolution to (4.3.2), noting that the remaining case follows similarly.

Let $\varphi \in C^2(B_1)$ be such that $u-\varphi$ has a local maximum at x_0. Then

$$|D\varphi(x_0)+p|^{\beta(x_0,u,D\varphi)} F\left(D^2\varphi(x_0)\right) \le f(x_0),$$

where

$$\beta(x_0,u,D\varphi)=\sum_{i=0}^{N}\beta_i(x_0)\chi_{G_i(u,D\varphi)}(x_0).$$

Since G_i, $i=0,...,N$, form a disjoint partition of B_1, there is a unique $i_0 \in \{0,...,N\}$ such that $x_0 \in G_{i_0}(u,D\varphi)$. Thus

$$|D\varphi(x_0)+p|^{\beta_{i_0}(x_0)} F\left(D^2\varphi(x_0)\right) \le f(x_0).$$

In particular, we have

$$\min_{i=0,...,N}\left\{|D\varphi(x_0)+p|^{\beta_i(x_0)} F\left(D^2\varphi(x_0)\right)\right\} \le \|f\|_{L^\infty(B_1)}.$$

Hence, we have proved that u is a subsolution of (4.3.2).

□

The following result states that to prove Theorem 4.1.3, we can assume a smallness regime, without loss of generality. It provides a tangential path between our equation and the homogeneous one.

Proposition 4.3.2 (Smallness regime)**.** *Let u be a subsolution to the equation*

$$\min_{i=0,...,N}\left\{|Du|^{\beta_i(x)} F\left(D^2u\right)\right\} \le \|f\|_{L^\infty(B_1)}, \tag{4.3.4}$$

and a supersolution to the equation

$$\max_{i=0,...,N}\left\{|Du|^{\beta_i(x)} F\left(D^2u\right)\right\} \ge -\|f\|_{L^\infty(B_1)}. \tag{4.3.5}$$

satisfying

$$[u]_{C^{1,\alpha}(x_0)} \le C,$$

under the assumption that $\|u\|_{L^\infty(B_{9/10})} \le 1$ *and* $\|f\|_{L^\infty(B_1)} \le \varepsilon_0$ *, where* C *and* ε_0 *are universal constants. Then, Theorem 4.1.3 holds.*

Proof. Let $\overline{u}(x)=Ku(x)$ where

$$K := \left(\|u\|_{L^\infty(B_{9/10})}+\frac{\|f\|_{L^\infty(B_1)}}{\varepsilon_0}\right)^{-1}.$$

Note that we can assume $K \leq 1$, since otherwise we are already in the smallness regime and we can just take $K = 1$.

The function $\overline{u}$ is a viscosity subsolution to

$$\min_{i=0,\dots,N} \left\{ K^{-\beta_i(x)} |D\overline{u}|^{\beta_i(x)} KF\left(K^{-1}D^2\overline{u}\right) \right\} \leq K \|f\|_{L^\infty(B_1)},$$

which implies

$$\min_{i=0,\dots,N} \left\{ |D\overline{u}|^{\beta_i(x)} \overline{F}(D^2\overline{u}) \right\} \leq \max_{i=0,\dots,N} \left\{ K^{1+\beta_i(x)} \right\} \|f\|_{L^\infty(B_1)},$$

where $\overline{F}(M) := KF\left(K^{-1}M\right)$ still satisfies [A1]. Since $K \leq 1$, we immediately get

$$\min_{i=0,\dots,N} \left\{ |D\overline{u}|^{\beta_i(x)} \overline{F}(D^2\overline{u}) \right\} \leq \varepsilon_0.$$

Similarly, we get

$$\max_{i=0,\dots,N} \left\{ |D\overline{u}|^{\beta_i(x)} \overline{F}(D^2\overline{u}) \right\} \geq -\varepsilon_0.$$

Since $\|\overline{u}\|_{L^\infty(B_{9/10})} \leq 1$, we note that the smallness assumptions are now satisfied. Hence, if we verify that

$$[\overline{u}]_{C^{1,\alpha}(B_1)} \leq C,$$

we can infer that

$$[u]_{C^{1,\alpha}(B_1)} \leq C_1 \left(\|u\|_{L^\infty(B_{9/10})} + \|f\|_{L^\infty(B_1)} \right), \tag{4.3.6}$$

where C_1 depends on ε_0, which will be fixed universally.

□

Remark 4.3.3. The choice of K in the previous proof differs from the literature (see for example [32]). The observation that $K \leq 1$ allows us to obtain the simple estimate (4.3.6).

In the following section we obtain improved regularity.

4.4 Hölder continuity

In this section, we obtain a compactness result for solutions. This result is essential when studying stability since it will allow us to obtain convergence of sequences of solutions.

We start by stating the maximum principle for viscosity solutions, Theorem 3.2 of [19].

Proposition 4.4.1 (Maximum principle). *Let Ω be a bounded domain and $G, H \in C\left(\mathscr{S}(d) \times \mathbb{R}^d \times B_1\right)$ be degenerate elliptic. Let u_1 be a viscosity subsolution of $G\left(D^2u_1, Du_1, x\right) = 0$ and u_2 be a viscosity* *supersolution of* $H\left(D^2u_2, Du_2, x\right) = 0$ *in* Ω. *Let* $\varphi \in C^2(\Omega \times \Omega)$. *Define* $v : \Omega \times \Omega \to \mathbb{R}$ *by*

$$v(x,y) := u_1(x) - u_2(y).$$

Suppose further that $(\bar{x}, \bar{y}) \in \Omega \times \Omega$ *is a local maximum of* $v - \varphi$ *in* $\Omega \times \Omega$. *Then, for every* $\iota > 0$, *there exist matrices* X *and* Y *in* $\mathscr{S}(d)$ *such that*

$$G(X, D_x\varphi(\bar{x},\bar{y}), \bar{x}) \le 0 \le H(Y, -D_y\varphi(\bar{x},\bar{y}), \bar{y}),$$

and the matrix inequality

$$-\left(\frac{1}{\iota} + \|A\|\right) I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \iota A^2$$

holds true, where $A := D^2\varphi(\bar{x},\bar{y})$.

We proceed by stating a result from [35], which we present in the following simplified form.

Proposition 4.4.2. *Let* $u \in C(B_1)$ *be a bounded viscosity subsolution to equation*

$$\mathscr{P}^-_{\lambda,\Lambda}\left(D^2u\right) - |Du| = 0, \text{ in } \{|Du| > \gamma\}$$

and a viscosity supersolution to equation

$$\mathscr{P}^+_{\lambda,\Lambda}\left(D^2u\right) + |Du| = 0, \text{ in } \{|Du| > \gamma\}.$$

Then $u \in C^\theta_{loc}(B_1)$ *and, for every* $0 < \tau < 1$, *there exists* $C > 0$ *such that*

$$\|u\|_{C^\theta(B_\tau)} \le C.$$

The constant θ *depends only on* d, λ, Λ *and* C *depends only on* $d, \lambda, \Lambda, \gamma, \|u\|_{L^\infty(B_{9/10})}, \tau$.

Intuitively, in the set where the gradient of a function u is bounded, the function is already Lipschitz. The idea behind the previous result is that if u is a solution of an elliptic equation in the set where its gradient is very large, then we are able to obtain improved regularity.

This proposition will imply Hölder continuity of solutions to (4.3.2) and (4.3.3) in the case where $|p|$ is sufficiently small. More precisely, let $A_0 > 1$ (to be fixed) be such that $|p| < A_0$. We claim that u is a viscosity subsolution to

$$F\left(D^2u\right) - |Du| = 0, \text{ in } \{|Du| > 2A_0\}. \tag{4.4.1}$$

Indeed, take $\varphi \in C^2$ such that $u - \varphi$ has a local maximum at $x_0 \in \{|Du| > 2A_0\}$. Then $|D\varphi(x_0)| > 2A_0$ and therefore $|D\varphi(x_0) + p| \ge A_0 > 1$. From (4.3.2) we have

$$\min_{i=0,\dots,N}\left\{|D\varphi(x_0) + p|^{\beta_i(x_0)} F\left(D^2\varphi(x_0)\right)\right\} \le \|f\|_{L^\infty(B_1)},$$

which implies

$$F\left(D^2\varphi(x_0)\right) \leq \|f\|_{L^\infty(B_1)} \leq |D\varphi(x_0)|$$

since we are under the assumption $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0$ and ε_0 will be chosen very small. Hence, from uniform ellipticity and recalling that $F(0) = 0$,

$$\mathscr{P}^-_{\lambda,\Lambda}\left(D^2\varphi(x_0)\right) - |D\varphi(x_0)| \leq F\left(D^2\varphi(x_0)\right) - |D\varphi(x_0)| \leq 0.$$

We verified that u is a viscosity subsolution to (4.4.1). In a similar way, we prove that u is a viscosity supersolution to

$$\mathscr{P}^+_{\lambda,\Lambda}\left(D^2\varphi(x_0)\right) + |Du| = 0,$$

in $\{|Du| > 2A_0\}$. Hence, we proved the following corollary.

Corollary 4.4.3. *Let $u \in C(B_1)$ be a bounded viscosity subsolution to* (4.3.2) *and a supersolution to* (4.3.3)*. Assume [A1], [A2] and [A3] are in force, let $\|u\|_{L^\infty(B_{9/10})} \leq 1$, $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0$ and assume further that $|p| < A_0$. The constants ε_0 and A_0 will be fixed in the sequel.*

Then $u \in C^\theta_{loc}(B_{9/10})$ for some $\theta \in (0,1)$, depending only on d, λ, Λ. In addition, for every $0 < \tau < 9/10$, there exists $C > 0$ such that

$$\|u\|_{C^\theta(B_\tau)} \leq C,$$

where $C = C(d, \lambda, \Lambda, A_0, \tau)$.

In the following lemma we obtain Hölder continuity for arbitrary $p \in \mathbb{R}^d$, which concludes this section.

Lemma 4.4.4 (C^θ regularity)**.** *Let $u \in C(B_1)$ be a bounded viscosity subsolution to* (4.3.2) *and a supersolution to* (4.3.3)*. Assume [A1], [A2] and [A3] are in force and let $\|u\|_{L^\infty(B_{9/10})} \leq 1$ and $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0$, to be fixed universally.*

Then $u \in C^\theta_{loc}(B_{9/10})$ for some $\theta \in (0,1)$, depending only on d, λ, Λ. In addition, for every $0 < \tau < 9/10$, there exists $C > 0$ such that

$$\|u\|_{C^\theta(B_\tau)} \leq C,$$

where $C = C(d, \lambda, \Lambda)$.

Proof. We begin by using Proposition 4.4.1 to obtain a subjet and a superjet satisfying the estimate (4.4.5) below. This was done in [32, Proposition 7] but for completion we replicate the proof.

Fix $0 < r < \frac{1-\tau}{2}$ and define

$$\omega(t) = t - \frac{t^2}{2}.$$

For constants $L_1, L_2 > 0$ and $x_0 \in B_r$, we set

$$L := \sup_{x,y \in B_r(x_0)} \left[u(x) - u(y) - L_1 \omega\left(|x-y|\right) - L_2 \left(|x-x_0|^2 + |y-x_0|^2 \right) \right]$$

Set $A_0 = 4L_1$ and assume $|p| \geq A_0$.

We aim at establishing that there exist constants L_1 and L_2, independent of x_0, for which $L \leq 0$. This immediately implies that u is Lipschitz continuous in B_τ by taking $x_0 = x$.

We argue by contradiction. Suppose there exists $x_0 \in B_\tau$ for which $L > 0$, regardless of the choices of L_1 and L_2. Consider the auxiliary functions $\psi, \phi : \overline{B}_1 \times \overline{B}_1 \to \mathbb{R}$ given by

$$\psi(x,y) := L_1 \omega\left(|x-y|\right) + L_2 \left(|x-x_0|^2 + |y-x_0|^2 \right)$$

and

$$\phi(x,y) := u(x) - u(y) - \psi(x,y).$$

Let $(\overline{x}, \overline{y})$ be a point where ϕ attains its maximum. Then

$$\phi(\overline{x}, \overline{y}) = L > 0$$

and

$$L_1 \omega\left(|\overline{x}-\overline{y}|\right) + L_2 \left(|\overline{x}-x_0|^2 + |\overline{y}-x_0|^2 \right) \leq 2.$$

Set

$$L_2 := \left(\frac{4\sqrt{2}}{r} \right)^2.$$

Then

$$|\overline{x}-x_0| + |\overline{y}-x_0| \leq \frac{r}{2},$$

which implies that $\overline{x}, \overline{y} \in B_r(x_0)$. In addition, $\overline{x} \neq \overline{y}$, since if this isn't the case we would conclude that $L \leq 0$.

We now use Proposition 4.4.1 to ensure the existence of a subjet (ξ_x, X) of u at $\overline{x}$ and a superjet (ξ_y, Y) of u at $\overline{y}$ with

$$\xi_x := D_x \psi(\overline{x}, \overline{y}) = L_1 \omega'\left(|\overline{x}-\overline{y}|\right) \sigma + 2L_2 (\overline{x} - x_0),$$
$$\xi_y := -D_y \psi(\overline{x}, \overline{y}) = L_1 \omega'\left(|\overline{x}-\overline{y}|\right) \sigma - 2L_2 (\overline{x} - x_0),$$

where

$$\sigma := \frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|}.$$

Since $\omega'(|\bar{x}-\bar{y}|) \leq 1$,

$$|\xi_x| \leq L_1 + \frac{L_2}{2} \leq 2L_1, \tag{4.4.2}$$

and

$$|\xi_y| \leq L_1 + \frac{L_2}{2} \leq 2L_1 \tag{4.4.3}$$

for L_1 large enough.

In addition, the matrices X and Y satisfy the inequality

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + (2L_2 + \iota) I, \tag{4.4.4}$$

for

$$Z := L_1 \omega''(|\bar{x}-\bar{y}|)\, \sigma \otimes \sigma + L_1 \frac{\omega'(|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|} (I - \sigma \otimes \sigma),$$

where $0 < \iota \ll 1$ depends solely on the norm of Z.

Next we apply the matrix inequality (4.4.4) to special vectors as to obtain information about the eigenvalues of $X - Y$. First, apply it to vectors of the form $(z,z) \in \mathbb{R}^{2d}$ to get

$$z \cdot (X - Y) z \leq (4L_2 + 2\iota) |z|^2$$

which implies that all eigenvalues of $X - Y$ are less than or equal to $4L_2 + 2\iota$.

Now we apply (4.4.4) to the vector $\bar{z} = (\sigma, -\sigma)$ to obtain

$$\begin{aligned} \sigma \cdot (X - Y) \sigma \leq & 4L_2 + 2\iota + 4L_1 \omega''(|\bar{x}-\bar{y}|) \\ = & 4L_2 + 2\iota - 4L_1. \end{aligned}$$

We thus conclude that at least one eigenvalue of $X - Y$ is below $4L_2 + 2\iota - 4L_1$, which will be a negative number, provided we choose L_1 large enough.

Evaluating the minimal Pucci operator on $X - Y$, we get

$$\begin{aligned} \mathscr{P}^-_{\lambda,\Lambda}(X - Y) \geq & 4\lambda L_1 - (\lambda (d-1) \Lambda)(4L_2 + 2\iota) \\ \geq & 3\lambda L_1 \end{aligned} \tag{4.4.5}$$

for L_1 even larger, if necessary. Furthermore, these jets satisfy the viscosity estimates

$$\min_{i=0,\ldots,N} \left\{ |p + \xi_x|^{\beta_i(\bar{x})} F(X) \right\} \leq \varepsilon_0, \tag{4.4.6}$$

and

$$\max_{i=0,\ldots,N} \left\{ |p + \xi_y|^{\beta_i(\bar{x})} F(Y) \right\} \geq -\varepsilon_0. \tag{4.4.7}$$

Since we fixed $A_0 = 4L_1$ and assumed $|p| \geq A_0$, this together with (4.4.2) and (4.4.3) imply

$$|p + \xi_x| \geq 2L_1 > 1,$$
$$|p + \xi_y| \geq 2L_1 > 1.$$

Hence, (4.4.6) and (4.4.7) imply, respectively,

$$F(X) \leq \varepsilon_0$$

and

$$F(Y) \geq -\varepsilon_0.$$

Combining these inequalities with (4.4.5) by means of uniform ellipticity, we get

$$3\lambda L_1 \leq 2\varepsilon_0,$$

which is clearly a contradiction, provided we choose L_1 large enough.

This concludes the proof for the case $|p| \geq A_0$, which combined with Corollary 4.4.3 completes the proof. □

With compactness available, we proceed with a key step in our tangential analysis.

4.5 Approximation lemma

We present an approximation lemma for the perturbed equation.

Lemma 4.5.1 (Approximation lemma)**.** *For every* $0 < \delta < 1$*, there exists* $\varepsilon_0 > 0$ *such that, if* $u \in C(B_1)$ *is a viscosity subsolution to* (4.3.2) *and a viscosity supersolution to* (4.3.3)*, satisfying* $\|u\|_{L^\infty(B_{9/10})} \leq 1$ *and* $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0$*, then one can find a function h which is a viscosity solution to* $\overline{F}(D^2h) = 0$ *for some* $\overline{F}$ *satisfying assumption [A1], such that*

$$\|u - h\|_{L^\infty(B_{1/2})} \leq \delta.$$

Such a function h satisfies

$$\|h\|_{C^{1,\alpha_0}(B_{1/2})} \leq C \|h\|_{L^\infty(B_{3/4})}.$$

Proof. We argue by contradiction. For simplicity, we split the proof in steps.

Step 1 – Assume that there exist $\delta_0 > 0$ and sequences $(u_n)_n$, $(F_n)_n$ and $(\beta_i^n)_n$ such that

1. $\|u_n\|_\infty \leq 1$;
2. F_n satisfy [A1];
3. β_i^n satisfy [A2] and [A3];

linked together by the equations

$$\min_{i=0,\dots,N}\left\{ |Du_n+p_n|^{\beta_i^n(x)} F_n\left(D^2u_n\right)\right\}\le \frac{1}{n}$$

and

$$\max_{i=0,\dots,N}\left\{ |Du_n+p_n|^{\beta_i^n(x)} F_n\left(D^2u_n\right)\right\}\ge -\frac{1}{n},$$

in the viscosity sense, in B_1; however, for every function $h\in C^{1,\alpha_0}$, it holds

$$\|u_n-h\|_{L^\infty(B_{1/2})}>\delta_0. \tag{4.5.1}$$

Step 2 – Since u_n are equibounded in $C^\theta(B_{9/10})$, by the Arzelà-Ascoli Theorem they will converge, up to a subsequence, locally uniformly to $u_\infty\in C(B_1)$.

Since F_n are (λ,Λ)-elliptic, they are also Lipschitz continuous. Therefore, again by the Arzelà-Ascoli Theorem they will converge locally uniformly to an (λ,Λ)-elliptic operator F_∞.

Finally, since β_i^n satisfy assumptions [A2] and [A3] they will converge locally uniformly to continuous functions β_i^∞, respectively.

Our goal is to prove that the limiting function u_∞ is a viscosity solution to the equation $F_\infty(D^2u_\infty)=0$. We only prove that it is a subsolution, since the proof for supersolution is analogous. We will consider two cases, depending on the limit behaviour of $(p_n)_n$.

Step 3 – Assume that $(p_n)_n$ does not admit a convergent subsequence. Then $|p_n|\to\infty$. Let $\varphi\in C^2(B_1)$ and assume that $u_\infty-\varphi$ attains a local strict maximum at $x_0\in B_1$. By contradiction, assume that

$$F_\infty\left(D^2\varphi(x_0)\right)>0. \tag{4.5.2}$$

There exists a sequence $x_n\to x_0$ such that $u_n-\varphi$ has a local maximum at x_n. Notice that $D\varphi(x_n)\to D\varphi(x_0)$ and $D^2\varphi(x_n)\to D^2\varphi(x_0)$. Also, by the equation satisfied by u_n in the viscosity sense, we have

$$\min_{i=0,\dots,N}\left\{ |D\varphi(x_n)+p_n|^{\beta_i^n(x_n)} F_n\left(D^2\varphi(x_n)\right)\right\}\le \frac{1}{n}.$$

Taking n large enough, we have $|D\varphi(x_n)+p_n|>1$ and since $\beta_i^n\ge 0$, we get

$$F_n\left(D^2\varphi(x_n)\right)\le\frac{1}{n},$$

which is inconsistent with (4.5.2), when we take the limit $n\to\infty$. Therefore,

$$F_\infty(D^2\varphi(x_0))\le 0,$$

concluding the proof for the case $|p_n|\to\infty$.

Step 4 – Suppose now that we can extract a subsequence $p_n \to p_\infty$. Resorting to standard stability results (see for example [19, Remarks 6.2 and 6.3]), we conclude that u_∞ is a viscosity subsolution to

$$\min_{i=0,\dots,N} \left\{ |p_\infty + Du_\infty|^{\beta_i^\infty(x)} F_\infty(D^2 u_\infty) \right\} \le 0,$$

and a viscosity supersolution to

$$\max_{i=0,\dots,N} \left\{ |p_\infty + Du_\infty|^{\beta_i^\infty(x)} F_\infty(D^2 u_\infty) \right\} \ge 0.$$

We can assume without loss of generality that $p_\infty = 0$, i.e., assume that u_∞ is a viscosity subsolution to

$$\min_{i=0,\dots,N} \left\{ |Du_\infty|^{\beta_i^\infty(x)} F_\infty(D^2 u_\infty) \right\} \le 0,$$

and a viscosity supersolution to

$$\max_{i=0,\dots,N} \left\{ |Du_\infty|^{\beta_i^\infty(x)} F_\infty(D^2 u_\infty) \right\} \ge 0.$$

We now claim that these inequalities imply that $F_\infty(D^2 u_\infty) = 0$. This is proved in Lemma 4.5.2 below.
Step 5 – Since $F_\infty(D^2 u_\infty) = 0$, by Remark 12 we get that $u_\infty \in C^{1,\alpha_0}(B_{1/2})$. This, together with the uniform convergence $u_n \to u_\infty$ produces a contradiction with (4.5.1), which completes the proof.

□

We present a homogeneous division lemma which concludes the proof of Lemma 4.5.1. We follow closely the proof of [34, Lemma 6].

Lemma 4.5.2. *Let $u \in C(B_1)$ be a bounded viscosity subsolution to*

$$\min_{i=0,\dots,N} \left\{ |Du|^{\beta_i(x)} F\left(D^2 u\right) \right\} \le 0, \tag{4.5.3}$$

and a viscosity supersolution to

$$\max_{i=0,\dots,N} \left\{ |Du|^{\beta_i(x)} F\left(D^2 u\right) \right\} \ge 0. \tag{4.5.4}$$

Then u is a viscosity solution to

$$F\left(D^2 u\right) = 0.$$

Proof. We prove that (4.5.4) implies $F\left(D^2 u\right) \ge 0$, noting that $F\left(D^2 u\right) \le 0$ follows similarly from (4.5.3) in a similar way.

Let $P(x) = \frac{1}{2}(x-y)\cdot N(x-y) + b\cdot(x-y) + u(y)$ be a polynomial touching u strictly from below at a point $y \in B_{3/4}$. We shall assume, without loss of generality, that $y = 0$ and $u(0) = 0$. Then we have the estimate

$$\min_{i=0,\dots,N} \left\{ |b|^{\beta_i(0)} F(N) \right\} \le 0.$$

If $b \neq 0$ then the result is trivial, so assume that $P(x) = \frac{1}{2} x \cdot Nx$. We argue by contradiction, assuming that $F(N) < 0$. By ellipticity, this implies that N has at least one positive eigenvalue. Let S be the subspace generated by the eigenvectors corresponding to the positive eigenvalues and consider the projection P_S to this subspace. We consider the following perturbed test function

$$\psi(x) = P(x) + \varepsilon |P_S x|, \text{ for } x \in \overline{B}_r.$$

For ε large enough, $u - \psi$ attains a negative minimum at $B_r - B_{\frac{9}{10}r}$ (since S is not empty and u is continuous). Indeed, let $m = \max_{\overline{B}_r} |u - P|$ and $\varepsilon = \frac{21m}{r}$. Then

$$\min_{x \in \overline{B}_{\frac{9}{10}r}} (u(x) - P(x) - \varepsilon |P_S x|) \geq -m - \frac{21M}{r} \frac{9}{10} r = -\frac{199}{10} m;$$

on the other hand, for $x \in \partial B_r$,

$$u(x) - P(x) - \varepsilon |P_S x| \leq m - \frac{21m}{r} r = -20m,$$

which concludes that the minimum is negative and attained at $x_0 \in B_r \setminus B_{\frac{9}{10}r}$.

Using this ε, we claim that $P_S x_0 \neq 0$. In fact, since $(u - \psi)(x_0) \leq (u - \psi)(x)$,

$$(u - P)(x_0) - \varepsilon |P_S x_0| \leq (u - \psi)(x)$$

if $P_S x_0 = 0$, we take $x = 0$ and get

$$(u - P)(x_0) \leq (u - \psi)(0) = (u - P)(0) = 0.$$

But P touches u strictly from below at $y = 0$ which implies $(u - P)(x_0) \geq 0$ with equality only if $x_0 = 0$, therefore $(u - \psi)(x_0) = 0$, contradicting the fact that the minimum is negative and that $x_0 \neq 0$.

We proved that $|P_S x_0| \neq 0$ which implies that ψ is smooth in a neighbourhood of x_0. Hence, for an appropriate translation of ψ, call it $\tilde{\psi}$, $u - \tilde{\psi}$ has a local minimum in B_r at x_0. Let B be the Hessian of $|P_S x|$ at $x = x_0$. Note that since $|P_S x|$ is a convex function, $B \geq 0$. We also have the viscosity inequality

$$\max_{i=0,\ldots,N} \left\{ |Nx_0 + \varepsilon e_0|^{\beta_i(x_0)} F(N + \varepsilon B) \right\} \geq 0,$$

for $e_0 = P_S x_0 / |P_S x_0|$. Note $(Nx_0 + \varepsilon e_0) \cdot P_S x_0 > 0$, since P_S is the projection into the subspace generated by the eigenvalues of N associated with its positive eigenvalues. Then, by ellipticity we obtain

$$F(N) \geq F(N + \varepsilon B) \geq 0,$$

which is a contradiction. Hence $F\left(D^2 u\right) \geq 0$ which concludes the proof. □

In the next and final section of this chapter, we provide an iterative scheme to control the oscillation of the gradient.

4.6 Hölder continuity of the gradient

In the following lemmas, we proceed with the geometric iteration argument in a sequence of concentric, shrinking balls. The first geometric iteration follows immediately from the approximation lemma.

Lemma 4.6.1. *Let $u \in C(B_1)$ be a viscosity subsolution to* (4.3.2) *and a viscosity supersolution to* (4.3.3). *Under the assumptions of Lemma 4.5.1, for every $\theta < \alpha_0$, there exists a polynomial $\ell(x) = a + b \cdot x$ and a constant $0 < \rho < 1$, depending on ω and universal constants, such that*

$$\|u - \ell\|_{L^\infty(B_\rho)} \leq \rho^{1+\theta}.$$

Furthermore, there exists a universal $C > 0$ such that $|a| \leq C$ and $|b| \leq C$.

Proof. By Lemma 4.5.1, there exists $h \in C^{1,\alpha_0}(B_{1/2})$ such that

$$\|u - h\|_{L^\infty(B_{1/2})} \leq \delta,$$

with the uniform estimate

$$\|h\|_{C^{1,\alpha_0}(B_{1/2})} \leq C \|h\|_{L^\infty(B_{3/4})}.$$

This implies that, for every $0 < r \ll 1$ and for the polynomial $\ell(x) = h(x_0) + Dh(x_0) \cdot (x - x_0)$,

$$\|h - \ell\|_{L^\infty(B_r(x_0))} \leq C r^{1+\alpha_0},$$

with $|h(x_0)| \leq C$ and $|Dh(x_0)| \leq C$ where C is universal.

Let $\theta < \alpha_0$ be arbitrary and take $r = \rho$ given by

$$\rho := \min\left\{\delta_1, (2C)^{\frac{1}{\theta - \alpha_0}}\right\},$$

where δ_1 is given implicitly in (4.2.1). Finally fix $\delta = \frac{\rho^{1+\theta}}{2}$, which also fixes ε_0 via Lemma 4.5.1. Then

$$\begin{aligned}\|u - \ell\|_{L^\infty(B_\rho(x_0))} &\leq \|u - h\|_{L^\infty(B_\rho(x_0))} + \|h - \ell\|_{L^\infty(B_\rho(x_0))} \\ &\leq \delta + C\rho^{1+\alpha_0} \leq \rho^{1+\theta}.\end{aligned}$$

□

Now we iterate the previous result concentric, shrinking balls.

Lemma 4.6.2 (Geometric iterations)**.** *There exist a non-decreasing sequence $(\alpha_k)_k$ and universal constants $\varepsilon_0 > 0$ and $\rho > 0$ such that if u is a viscosity subsolution of* (4.3.2) *and a supersolution of* (4.3.3) *with $p = 0$, satisfying $\|u\|_{L^\infty(B_{9/10})} \leq 1$ and $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0$, there exist polynomials $\ell_k(x) =$*

$a_k + b_k \cdot x$ such that

$$\|u-\ell_k\|_{L^\infty(B_{\rho^k}(x_0))} \le \rho^{k(1+\alpha_k)}, \tag{4.6.1}$$

and

$$|a_k - a_{k-1}| + \rho^{k-1}|b_k - b_{k-1}| \le C_e \rho^{(k-1)(1+\alpha_{k-1})}. \tag{4.6.2}$$

Furthermore, the sequence $(\alpha_k)_k$ converges to

$$\alpha := \min_{i=0,\dots,N}\left\{\alpha_0^-, \frac{1}{1+\beta_i(x_0)}\right\}$$

and

$$\limsup_{k\to\infty} k\,(\alpha-\alpha_k) = 0. \tag{4.6.3}$$

Proof. Assume without loss of generality that $x_0 = 0$. Take ε_0 and ρ given by Lemma 4.6.1, depending on θ, which will be fixed soon.

Define the nondecreasing sequence

$$\alpha_k := \min_{i=0,\dots,N}\left\{\alpha_0^-, \min_{x\in B_{\rho^k}}\left(\frac{1}{1+\beta_i(x)}\right)\right\},$$

which converges to the number

$$\alpha := \min_{i=0,\dots,N}\left\{\alpha_0^-, \frac{1}{1+\beta_i(0)}\right\}.$$

Note that, by [A2],

$$\begin{aligned} k\left(\frac{1}{1+\beta_i(0)} - \frac{1}{1+\max_{x\in B_{\rho^k}}\beta_i(x)}\right) \le& k\left(\max_{x\in B_{\rho^k}}\beta_i(x) - \beta_i(0)\right) \\ \le& k\omega(\rho^k). \end{aligned}$$

Therefore, considering all possible cases, we can easily check that

$$0 \le k(\alpha-\alpha_k) \le k\omega\left(\rho^k\right),$$

with

$$\limsup_{k\to\infty} k\omega(\rho^k) = 0.$$

To prove (4.6.1) and (4.6.2), we will proceed by induction.

Let $\ell_0 \equiv 0$ and ℓ_1 be given by Lemma 4.6.1. Then (4.6.1) and (4.6.2) hold for $k = 1$ by Lemma 4.6.1.

Assume that (4.6.1) and (4.6.2) hold up to k. Define

$$v_k(x) = \frac{(u-\ell_k)}{\rho^{k(1+\alpha_k)}}(\rho^k x).$$

Then $\|v_k\|_{L^\infty(B_1)} \leq 1$ and v_k is a viscosity subsolution to

$$\min_{i=0,\dots,N}\left\{\rho^{k\alpha_k\beta_i(\rho^k x)}\left|Dv_k(x)+\rho^{-k\alpha_k}b_k\right|^{\beta_i(\rho^k x)} F_k\left(D^2 v_k\right)\right\} \leq \rho^{k(1-\alpha_k)}\varepsilon_0,$$

where

$$F_k(M) := \rho^{k(1-\alpha_k)} F\left(\rho^{k(\alpha_k-1)}M\right).$$

This implies the following estimate

$$\begin{aligned}\min_{i=0,\dots,N}&\left\{\left|Dv_k(x)+\rho^{-k\alpha_k}b_k\right|^{\beta_i(\rho^k x)} F_k\left(D^2 v_k\right)\right\}\\ &\leq \max_{i=0,\dots,N}\left\{\rho^{k(1-\alpha_k)-k\alpha_k\beta_i(\rho^k x)}\right\}\varepsilon_0 \leq \varepsilon_0,\end{aligned}$$

where the last inequality follows from the definition of α_k. Calling $p_k := \rho^{-k\alpha_k}b_k$ and $\beta_i^k(x) := \beta_i(\rho^k x)$, we get that v_k is a subsolution to

$$\min_{i=0,\dots,N}\left\{|Dv_k(x)+p_k|^{\beta_i^k(x)} F_k\left(D^2 v_k\right)\right\} \leq \varepsilon_0.$$

Similarly, we prove that v_k is a viscosity supersolution to

$$\max_{i=0,\dots,N}\left\{|Dv_k(x)+p_k|^{\beta_i^k(x)} F_k\left(D^2 v_k\right)\right\} \geq -\varepsilon_0.$$

Note that β_i^k still satisfy assumption [A2].

Hence, we can use Lemma 4.6.1 to guarantee the existence of a linear function $\overline{\ell}(x) = \overline{a} + \overline{p}\cdot x$ such that

$$\sup_{B_\rho}|v_k - \overline{\ell}| \leq \rho^{1+\theta},$$

where $\theta = \frac{\alpha+\alpha_0}{2} < \alpha_0$ and the coefficients satisfy

$$\begin{aligned}\overline{a} &= h(0),\\ \overline{p} &= Dh(0),\end{aligned}$$

where h is a viscosity solution to $G(D^2h) = 0$ and G has the same ellipticity constants as F. Hence, as a straightforward application of ellipticity, h has interior C^{1,α_0} estimates which imply universal bounds on the coefficients of $\overline{\ell}$.

Rescalling back to the unit ball, we get

$$
\begin{aligned}
&\sup_{x\in B_\rho}\left|\frac{u-\ell_k}{\rho^{k(1+\alpha_k)}}(\rho^k x)-\overline{\ell}(x)\right| \le \rho^{1+\theta}\\
\iff &\sup_{y\in B_{\rho^{k+1}}}\left|u(y)-\ell_k(y)-\rho^{k(1+\alpha_k)}\overline{\ell}(\rho^{-k}y)\right| \le \rho^{1+\theta}\rho^{k(1+\alpha_k)}\\
\iff &\sup_{y\in B_{\rho^{k+1}}}|u(y)-\ell_{k+1}(y)| \le \rho^{1+\theta}\rho^{k(\alpha_k-\alpha_{k+1})}\rho^{k(1+\alpha_{k+1})},
\end{aligned}
$$

where

$$
\begin{aligned}
\ell_{k+1}(y)-\ell_k(y) =& (a_{k+1}-a_k)+(p_{k+1}-p_k)\cdot y\\
=& \rho^{k(1+\alpha_k)}h(0)+\rho^{k\alpha_k}Dh(0)\cdot y.
\end{aligned}
$$

Because of (4.2.1), we have

$$
\rho^{k(\alpha_k-\alpha_{k+1})} \le \rho^{-k\omega(\rho^k)} \le \rho^{\frac{\alpha-\alpha_0}{2}},
$$

hence, we can further estimate

$$
\rho^{1+\theta}\rho^{k(\alpha_k-\alpha_{k+1})}\rho^{k(1+\alpha_{k+1})} \le \rho^{\theta-\alpha_{k+1}}\rho^{\frac{\alpha-\alpha_0}{2}}\rho^{(k+1)(1+\alpha_{k+1})}
$$

and since $\theta = \frac{\alpha+\alpha_0}{2}$, we can write

$$
\theta-\alpha_{k+1}+\frac{\alpha-\alpha_0}{2} = \alpha-\alpha_{k+1} \ge 0.
$$

Combining all these inequalities, we can finally estimate

$$
\sup_{y\in B_{\rho^{k+1}}}|u(y)-\ell_{k+1}(y)| \le \rho^{(k+1)(1+\alpha_{k+1})}
$$

which proves (4.6.1) and since $|h(0)| \le C$, $|Dh(0)| \le C$, estimate (4.6.2) follows immediately aswell. This concludes the proof for the case $x_0 = 0$. A standard translation locates this argument at any point $x_0 \in B_{1/2}$. □

Theorem 4.1.3 follows from Lemma 4.6.2 together with Proposition 2.3.4.

Chapter 5

Fully nonlinear Hamilton-Jacobi equations of degenerate type

5.1 Introduction and main results

In this chapter, we study a fully nonlinear Hamilton-Jacobi equation of the form

$$F(D^2u) + H(Du,x) = f(x) \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \tag{5.1.1}$$

where $F : S(d) \to \mathbb{R}$ is degenerate elliptic, the Hamiltonian $H = H(p,x)$ satisfies natural growth and continuity conditions, and $f \in L^\infty(\Omega)$ is Lipschitz continuous. In the superlinear setting, we prove that viscosity solutions to (5.1.1) are locally Lipschitz-continuous. In addition, we examine a two-phase free boundary problem driven by the operator in (5.1.1). In this context, our findings include the existence of solutions and regularity estimates across the free boundary. The conditions we impose on the structure of the problem are fairly general and cover important models, such as Bellman and Isaacs equations. An example of Hamiltonian falling under our assumptions is

$$H(p,x) := a(x)\left(1+|p|^2\right)^{\frac{m}{2}} + V(x),$$

provided $a, V : \Omega \to \mathbb{R}$ are Lipschitz-continuous and bounded from above and below, and $m > 1$.

Our contribution is two-fold. By developing an intrinsically nonlinear argument, we prove that viscosity solutions to (5.1.1) are Lipschitz continuous, with estimates. Then we examine a consequence of our regularity result to a two-phase free boundary problem and prove the existence of solutions, with estimates in Hölder spaces. Our main result reads as follows.

Theorem 5.1.1 (Improved regularity of solutions). *Let $u \in C(\Omega)$ be a viscosity solution to* (5.1.1) *where $F : S(d) \to \mathbb{R}$ is degenerate elliptic, Lipschitz-continuous and positively homogeneous of degree* 1*, and $f \in L^\infty(\Omega)$ is Lipschitz-continuous. Suppose the Hamiltonian H is superlinear and satisfies natural growth and continuity conditions, detailed in Section 5.2. Then u is locally Lipschitz-continuous in Ω. Moreover, for every $\Omega' \Subset \Omega$, there exists $C > 0$ such that*

$$|u(x) - u(y)| \leq C\left(1 + \|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\right)|x-y|,$$

for every $x,y \in \Omega'$. The constant $C > 0$ depends only on Ω', and the data of the problem.

The proof of Theorem 5.1.1 relies on two building blocks. First, we examine the superquadratic case, i.e., $H(p,x) \sim C + C|p|^m$, with $m > 2$. In this setting, we prove that solutions to (5.1.1) are γ-Hölder continuous for $\gamma := (m-2)/(m-1)$. Here we follow closely the strategy put forward in [2, Section 3], adapting its techniques to the fully nonlinear setting. Then we refine the application of the Ishii-Jensen Lemma to produce Lipschitz regularity in the superlinear setting.

Once Theorem 5.1.1 is available we turn to a free boundary problem driven by a particular instance of the operator in (5.1.1). For constants $\lambda_+, \lambda_- \in \mathbb{R}$, consider the problem

$$\mathrm{Tr}(A(x)D^2u) + H(Du,x) = \lambda_+\chi_{\{u>0\}} + \lambda_-\chi_{\{u<0\}} \quad \text{in} \quad \Omega(u), \tag{5.1.2}$$

where $\Omega(u) := \{u > 0\} \cup \{u < 0\}$. We note that (5.1.2) holds *only* in the region where the solutions do not vanish, and no PDE information is available in $\{u = 0\}$. In many cases, viscosity solutions exist following Perron's method, when a comparison principle is available, and one can build appropriate sub- and supersolutions. Meanwhile, the present setting introduces important difficulties. First, the dependence of (5.1.2) with respect to the solution implies the lack of properness. As a consequence, one should not expect a comparison principle to hold at the level of the equation, which precludes the use of usual arguments. Also, and perhaps even more important, the growth regime of the Hamiltonian H requires further compatibility conditions for the boundary data; see [16].

We argue through a regularization of the right-hand side of (5.1.2), removing the dependence of the equation on zero-order terms. We combine our findings in regularity theory with former results on the existence of solutions for superquadratic Hamilton-Jacobi equations. Then a fixed-point argument ensures the existence of viscosity solutions to the Dirichlet problem associated with (5.1.2). For a similar approach in the context of free transmission problems, see [56].

The remainder of this chapter is organised as follows. Section 5.2 gathers our primary assumptions and recalls a few preliminaries. The proof of Theorem 5.1.1 is the subject of Section 5.3. Finally, in Section 5.4, we prove the existence of viscosity solutions for the Dirichlet problem associated with (5.1.2).

5.2 Preliminary material and main assumptions

Here we detail the main assumptions used in this chapter and collect a few preliminaries. We start with the conditions imposed on the second-order operator F.

Because we rely on the monotonicity of F in the space of symmetric matrices, we equip the latter with a partial order relation. For $M,N \in S(d)$, we say that $M \geq N$ if $M - N$ is positive semi-definite, i.e., for every $\xi \in \mathbb{R}^d$, we have

$$\xi^T(M-N)\xi \geq 0.$$

Our primary condition on F concerns its degenerate ellipticity.

Definition 5.2.1. *We say $F : S(d) \to \mathbb{R}$ is degenerate elliptic if*

$$F(M) \leq F(N)$$

whenever $M, N \in S(d)$ *are such that* $M \geq N$.

We continue with an assumption combining degenerate ellipticity and Lipschitz continuity for F.

[A1] (Monotonicity and Lipschitz continuity)**.** *The operator* $F : S(d) \to \mathbb{R}$ *is monotone non-increasing and Lipschitz continuous. That is, there exists a constant* $C_F > 0$ *such that*

$$F(M) - F(N) \leq C_F \left|(N - M)_+\right|, \tag{5.2.1}$$

for every $M, N \in S(d)$.

The condition in (5.2.1) is equivalent to requiring F to be degenerate elliptic and Lipschitz continuous, with constant $C_F > 0$; see Lemma 5.2.3. The choice for (5.2.1) stems from our argument since we compare $F(M)$ and $F(N)$ in terms of the eigenvalues of M and N. We also require F to be positively homogeneous of degree one.

[A2] (Homogeneity of F)**.** *The operator* F *is positively homogeneous of degree* 1*. That is, for every* $M \in S(d)$ *and every* $s \geq 0$*, we have*

$$F(sM) = sF(M).$$

The typical example of an operator satisfying assumptions [A1] and [A2] is the Bellman operator. Indeed, let $\mathscr{A}$ be a measurable index set and consider a family of matrices $(A_\alpha)_{\alpha \in \mathscr{A}}$ such that

$$0 \leq A_\alpha \leq (C_F d^{-1}) I,$$

for every $\alpha \in \mathscr{A}$. Then the operator

$$F(M) := \inf_{\alpha \in \mathscr{A}} \left(-\operatorname{Tr}(A_\alpha M)\right)$$

satisfies both [A1] and [A2].

Now, we turn to the Hamiltonian H and detail the growth and continuity conditions under which we work.

[A3] (Structural conditions)**.** *We suppose there exist constants* $m > 1$ *and* $C_1, C_2, C_3 > 0$ *such that* $H = H(p, x)$ *satisfies*

$$-C_1 + C_2 |p|^m \leq H(p, x) \leq C_3 \left(1 + |p|^m\right), \tag{5.2.2}$$

for every $p \in \mathbb{R}^d$ *and* $x \in \Omega$*. Also,*

$$|H(p, x) - H(p, y)| \leq \left(C_3 |p|^m + C_1\right) |x - y| \tag{5.2.3}$$

for every $x, y \in \Omega$*, and* $p \in \mathbb{R}^d$*. Finally, we require*

$$|H(p, x) - H(q, x)| \leq C_3 \left(|p| + |q| + 1\right)^{m-1} |p - q|, \tag{5.2.4}$$

for every $p, q \in \mathbb{R}^d$*, and every* $x \in \Omega$.

The typical example of a Hamiltonian satisfying [A3] is

$$H(p,x) = a(x)\left(1+|p|^2\right)^{\frac{m}{2}} + V(x),$$

where $a, V : \Omega \to \mathbb{R}$ are Lipschitz continuous, with

$$0 < C_* \leq a(x) \leq C^* \quad \text{ and } \quad 0 \leq V(x) \leq C^*,$$

for some fixed constants $0 < C_* \leq C^*$.

Next we recall the definition of viscosity solution adapted to this problem.

Definition 5.2.2 (Viscosity solution)**.** *We say $u \in \mathrm{USC}(\Omega)$ is a viscosity sub-solution to* (5.1.1) *if, for every $x_0 \in \Omega$ and every $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0, we have*

$$F(D^2\varphi(x_0)) + H(D\varphi(x_0), x_0) \leq f(x_0).$$

Likewise, we say that $u \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution to (5.1.1) *if, for every $x_0 \in \Omega$ and every $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0, we have*

$$F(D^2\varphi(x_0)) + H(D\varphi(x_0), x_0) \geq f(x_0).$$

If $u \in C(\Omega)$ is simultaneously a viscosity sub-solution and a viscosity supersolution to (5.1.1)*, we say it is a viscosity solution to* (5.1.1).

Recall that we say $u \in \mathrm{USC}(\Omega)$ if for every $x_0 \in \Omega$,

$$\limsup_{x \to x_0} u(x) \leq u(x_0).$$

We say $u \in \mathrm{LSC}(\Omega)$ if $-u \in \mathrm{USC}(\Omega)$.

We use Proposition 4.4.1, corresponding to the Jensen-Ishii Lemma, to prove preliminary regularity results as usual in the literature. Now, the relevance of estimating $F(X_\varepsilon) - F(Y_\varepsilon)$ in terms of the eigenvalues of $X_\varepsilon - Y_\varepsilon$ becomes clear. Hence, we proceed by verifying that [A1] is equivalent to supposing that F is degenerate elliptic and Lipschitz continuous.

Lemma 5.2.3. *Suppose $F : S(d) \to \mathbb{R}$ satisfies* [A1]*. Then F is Lipschitz continuous, with constant C_F, and monotone non-increasing. That is, for every $M, N \in S(d)$, we have*

$$|F(M) - F(N)| \leq C_F\,|N - M|\,, \tag{5.2.5}$$

and

$$F(M) \leq F(N), \tag{5.2.6}$$

provided $N \leq M$. Conversely, suppose F satisfies (5.2.5) *and* (5.2.6)*. Then it also satisfies* [A1].

Proof. We start by proving that [A1] implies (5.2.5) and (5.2.6). Indeed, if $M \geq N$ then $(N-M)_+ = 0$ and we immediately get (5.2.6). Also, since $|(N-M)_+| \leq |N-M|$, we get

$$F(M) - F(N) \leq C_F\,|N - M|\,.$$

Swapping M and N we get

$$F(N) - F(M) \leq C_F\,|N - M|\,,$$

and (5.2.5) follows.

Now we prove that (5.2.5) and (5.2.6) imply [A1]. Fix $M, N \in S(d)$ arbitrarily and recall that $M - N = (M-N)_+ - (M-N)_-$. Then

$$F(M) = F(N + (M-N)) \leq F(N - (M-N)_-) \leq F(N) + C_F|(N-M)_+|,$$

where the first inequality follows from (5.2.6), and the second one is a consequence of (5.2.5). □

5.3 Interior Lipschitz continuity

We reduce the problem posed in $\Omega \subset \mathbb{R}^d$ to an equation prescribed in the unit ball, $B_1 \subset \mathbb{R}^d$. Let $\Omega' \Subset \Omega$. For every $r \in (0,1)$, one can find a natural number $n = n(r) \in \mathbb{N}$ such that there exists a subset $\{x_1, \ldots, x_n\} \subset \Omega'$, with $\overline{B_r(x_i)} \subset \Omega$ for every $i \in \{1, \ldots, n\}$. In addition, the family $(B_{r/2}(x_i))_{i=1}^n$ covers Ω'; that is,

$$\Omega' \subset \bigcup_{i=1}^n B_{r/2}(x_i).$$

As a result, we suppose $\Omega' = B_{r/2}(x_1)$, with $x_1 = 0 \in \Omega'$, and prescribe our problem of interest in open balls. In what follows, we denote with $C(d)$ any constant depending only on the dimension; this notation refers to possibly different constants within our arguments.

The next lemma accounts for the Hölder regularity of sub-solutions to (5.1.1). The strategy of the proof yields a modulus of continuity depending explicitly on the growth regime of H.

Lemma 5.3.1 (Hölder continuity for sub-solutions)**.** *Let $u \in \mathrm{USC}(B_1)$ be a sub-solution to* (5.1.1). *Suppose assumptions* [A1]-[A2] *are in force. Suppose further H satisfies* [A3] *with $m > 2$. Then*

$$|u(x) - u(y)| \leq K|x-y|^\gamma,$$

for every $x, y \in B_{1/4}$, where

$$\gamma = \frac{m-2}{m-1}, \tag{5.3.1}$$

and

$$K := 2^{\frac{1}{m-1}}\left(4^{\frac{m}{m-1}}\left(\frac{C_F C(d)}{C_2\gamma^m}\right)^{\frac{1}{m-1}} + 4\left(\frac{\|f\|_{L^\infty(B_1)} + C_1}{C_2\gamma^m}\right)^{\frac{1}{m}}\right).$$

Proof. Fix $x \in B_{1/2}$ and define the function $\phi : B_{1/2}(x) \to \mathbb{R}$ as

$$\phi(y) := K\left(\frac{1}{4} - |y-x|^2\right)^{-1}|y-x|^\gamma.$$

To establish the lemma, it suffices to show that

$$w(y) := u(y) - u(x) - \phi(y) \leq 0, \tag{5.3.2}$$

for every $y \in B_{1/2}(x)$. Indeed, it would imply that, for every $y \in B_{1/4}(x)$,

$$u(y) - u(x) \leq \phi(y) = K\left(\frac{1}{4} - |y-x|^2\right)^{-1} |y-x|^\gamma \leq \frac{16}{3} K |y-x|^\gamma.$$

We split the remainder of the proof into three steps.

Step 1 – Here, we prove that w, as defined in (5.3.2), has no local maximum in $B_{1/2}(x) \setminus \{x\}$. Suppose otherwise, and let $y^* \in B_{1/2}(x) \setminus \{x\}$ be a point of local maximum for w. Because u is a viscosity sub-solution of (5.1.1), Definition 5.2.2 implies

$$F(D^2\phi(y^*)) + H(D\phi(y^*), y^*) \leq f(y^*).$$

We will produce a contradiction by verifying that ϕ is also a strict supersolution.

Step 2 – Without loss of generality, set $x = 0$ and notice that

$$D\phi(y) = K\left(\frac{\frac{\gamma}{4}|y|^{\gamma-2} + (2-\gamma)|y|^\gamma}{\left(\frac{1}{4} - |y|^2\right)^2}\right) y,$$

and

$$|D\phi(y)|^m \geq K^m \left(\frac{1}{4} - |y|^2\right)^{-2m} |y|^{\gamma-2} \frac{\gamma^m}{4^m}, \tag{5.3.3}$$

where the choice of γ in (5.3.1) is instrumental. Moreover, a dimensional constant $C(d) > 0$ exists such that

$$|D^2\phi(y)| \leq C(d) K \left(\frac{1}{4} - |y|^2\right)^{-3} |y|^{\gamma-2}.$$

Because F is Lipschitz continuous and satisfies $F(0) = 0$, we obtain

$$-F(D^2\phi(y)) \leq C_F |D^2\phi| \leq C_F C(d) K \left(\frac{1}{4} - |y|^2\right)^{-3} |y|^{\gamma-2}. \tag{5.3.4}$$

Combining (5.3.3) with (5.3.4), we get

$$\begin{aligned}
F(D^2\phi(y)) + H(D\phi(y), y) &\geq -C_F C(d) K \left(\frac{1}{4} - |y|^2\right)^{-3} |y|^{\gamma-2} - C_1 \\
&\quad + C_2 K^m \left(\frac{1}{4} - |y|^2\right)^{-2m} |y|^{\gamma-2} \gamma^m \\
&\geq -C_1 + \left(-C_F C(d) K + C_2 \frac{\gamma^m}{4^m} K^m\right),
\end{aligned}$$

where the last inequality holds because $(1/4-t^2)^{-3}t^{\gamma-2} > 1$, for every $t \in (0,1/2)$ and $\gamma \in (0,1)$. Finally, the choice of K ensures that

$$\left(-C_F C(d)K + C_2\frac{\gamma^m}{4^m}K^m\right) > \|f\|_{L^\infty(B_1)} + C_1;$$

hence, ϕ is a supersolution of (5.1.1), and we obtain a contradiction. Therefore, w does not have an interior local maximum point. In the next step, we prove that w cannot attain its local maximum on $\partial B_{1/2}$.

Step 3 – To see that w does not attain a local maximum on $\partial B_{1/2}$, start by noticing that $\phi(y)$ blows up as $y \to \partial B_{1/2}(x)$. As a consequence, the supremum of w in $\overline{B_{1/2}}$ has to be attained in $B_{1/2}$.

Because of Step 2, it cannot be attained in $B_{1/2}(x) \setminus \{x\}$; hence, the supremum of w is attained at x. Because $w(x) = 0$, we conclude $w \leq 0$ in $B_{1/2}(x)$, and the proof is complete. □

In the sequel, we produce a Lipschitz-regularity result for the solutions to (5.1.1). Our argument is based on Proposition 4.4.1, and follows closely the reasoning developed in [2]. The main difference stems from the (fully) nonlinear character of the problem. Here, we resort to assumption [A1] and explore the interplay between the operator F, the eigenvalues of a given matrix, and the Hamiltonian H. In what follows, we include the sub-quadratic case $1 < m \leq 2$.

Because the source term f is Lipschitz-continuous and bounded in Ω, we can absorb it into the Hamiltonian H, at the expense of changing the constants appearing in (5.2.2)-(5.2.4) accordingly. In doing so, we are allowed to examine the homogeneous variant of (5.1.1) given by

$$F(D^2u) + H(Du,x) = 0 \quad \text{in} \quad B_1. \tag{5.3.5}$$

In the sequel we detail the proof of Theorem 5.1.1.

Proof of Theorem 5.1.1. We start by setting L as

$$\begin{aligned} L = \max\Bigg\{ & 2^{\frac{1}{m-1}}\left(\left(\frac{C_F C(d)}{C_2\gamma^m}\right)^{\frac{1}{m-1}} + \left(\frac{C_1}{C_2\gamma^m}\right)^{\frac{1}{m}}\right), \\ & 2\left(3^m C_F C(d)\frac{C_3}{C_2}\right)^{\frac{1}{m-1}}, \left(\frac{C_1}{C_3}\right)^{\frac{1}{m}}\Bigg\}. \end{aligned} \tag{5.3.6}$$

It suffices to prove that, for every $\hat{x} \in B_{1/2}$,

$$\limsup_{x\to\hat{x}} \frac{u(\hat{x}) - u(x)}{|\hat{x} - x|} \leq L.$$

We suppose there exists $\hat{x} \in B_{1/2}$ such that

$$\limsup_{x\to\hat{x}} \frac{u(\hat{x}) - u(x)}{|\hat{x} - x|} > L. \tag{5.3.7}$$

By combining Proposition 4.4.1 and Lemma 5.2.3 with the conditions in assumptions [A1]-[A3], we obtain a contradiction and complete the proof. For ease of presentation, we split the argument into four steps.

Step 1 – Consider first an auxiliary function. Let $\phi : B_{3/4} \to [1,\infty)$ be such that $\phi \equiv 1$ in $B_{1/2}$, with $\phi(x) \to \infty$ as $|x| \to 3/4$. Suppose also

$$|D^j \phi(x)| \leq C(d)\,(\phi(x))^{jm+(1-j)}, \tag{5.3.8}$$

for $j \in \{1,2\}$, $x \in B_{3/4}$, and some dimensional constant $C(d) > 0$. For $\alpha > 0$, denote with $\Psi : B_{3/4} \times B_{3/4} \to \mathbb{R}$ the function

$$\Psi(x,y) := u(x) - u(y) - L\phi(y)\,|x-y| - \frac{1}{2\alpha}\,|x-y|^2.$$

For $0 < \alpha \ll 1$ sufficiently small, we claim that there exist $x_\alpha, y_\alpha \in B_{3/4}$ such that

$$\Psi(x_\alpha, y_\alpha) = \sup_{x,y \in B_{3/4}} \Psi(x,y) > 0. \tag{5.3.9}$$

In addition, the function $\phi(y)$ localizes y_α away from the boundary $\partial B_{3/4}$. Also, because $0 < \alpha \ll 1$, the term $-\frac{1}{2\alpha}\,|x-y|^2$ ensures that x_α is close to y_α and, therefore, also away from $\partial B_{3/4}$. Finally, $x_\alpha \neq y_\alpha$, since otherwise the supremum in (5.3.9) would be zero.

Notice that

$$\frac{1}{2\alpha}\,|x_\alpha - y_\alpha|^2 \leq \operatorname{osc}_{B_{3/4}} u \leq 2\|u\|_{L^\infty(B_{3/4})}.$$

Hence by the continuity of u, we get

$$\begin{aligned}
&\limsup_{\alpha\to 0}\left(L\phi(y_\alpha)\,|x_\alpha - y_\alpha| + \frac{1}{2\alpha}\,|x_\alpha - y_\alpha|^2\right)\\
&\leq \limsup_{\alpha\to 0}\sup\left\{u(y) - u(z) : y,z \in B_{3/4}, |y-z| \leq \left(2\alpha \operatorname{osc}_{B_{3/4}} u\right)^{\frac{1}{2}}\right\}\\
&=0.
\end{aligned} \tag{5.3.10}$$

In the case $m > 2$, Lemma 5.3.1 yields

$$L\phi(y_\alpha)\,|x_\alpha - y_\alpha| + \frac{1}{2\alpha}\,|x_\alpha - y_\alpha|^2 \leq u(x_\alpha) - u(y_\alpha) \leq \tilde{K}\,|x_\alpha - y_\alpha|^\gamma,$$

where $\gamma = \frac{m-2}{m-1}$ and $\tilde{K}$ stands for the constant K in Lemma 5.3.1 in the case $f \equiv 0$. Thus,

$$\phi^{m-1}(y_\alpha)|x_\alpha - y_\alpha| \leq L^{1-m}\tilde{K}^{m-1}. \tag{5.3.11}$$

Next we resort to Proposition 4.4.1.

Step 2 – Because (x_α, y_α) is a maximum point for Ψ and u solves (5.3.5), Proposition 4.4.1 yields symmetric matrices $X_{\varepsilon,\alpha}$ and $Y_{\varepsilon,\alpha}$ such that

$$\begin{pmatrix} X_{\varepsilon,\alpha} & 0 \\ 0 & -Y_{\varepsilon,\alpha} \end{pmatrix} \leq J_\alpha + \varepsilon J_\alpha^2, \tag{5.3.12}$$

for every $\varepsilon > 0$ and $\alpha > 0$, sufficiently small. Moreover,

$$F(X_{\varepsilon,\alpha}) + H(P_\alpha, x_\alpha) \leq 0 \leq F(Y_{\varepsilon,\alpha}) + H(P_\alpha - Q_\alpha, y_\alpha), \tag{5.3.13}$$

where

$$\sigma_\alpha := \frac{x_\alpha - y_\alpha}{|x_\alpha - y_\alpha|}, \quad P_\alpha := \left(L\phi(y_\alpha) + \frac{|x_\alpha - y_\alpha|}{\alpha} \right) \sigma_\alpha,$$

and

$$Q_\alpha := L|x_\alpha - y_\alpha| D\phi(y_\alpha).$$

Finally, we write J_α as

$$J_\alpha = \frac{L\phi(y_\alpha)}{|x_\alpha - y_\alpha|} \begin{pmatrix} Z_1 & -Z_1 \\ -Z_1 & Z_1 \end{pmatrix} + \frac{1}{\alpha} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + L \begin{pmatrix} 0 & Z_2 \\ Z_2^T & Z_3 \end{pmatrix},$$

with $Z_1 := I - \sigma_\alpha \otimes \sigma_\alpha$, $Z_2 := D\phi(y_\alpha) \otimes \sigma_\alpha$, and

$$Z_3 := -(Z_2 + Z_2^T) + D^2\phi(y_\alpha)|x_\alpha - y_\alpha|.$$

In the next step, we estimate $F(Y_{\varepsilon,\alpha}) - sF(X_{\varepsilon,\alpha})$ from above.

Step 3 – It follows from [A1] that

$$F(Y_{\varepsilon,\alpha}) - F(sX_{\varepsilon,\alpha}) \leq C_F \left| (sX_{\varepsilon,\alpha} - Y_{\varepsilon,\alpha})_+ \right|. \tag{5.3.14}$$

For $s > 0$, let A_s be given by

$$A_s = \begin{pmatrix} s^2 I & sI \\ sI & I \end{pmatrix}.$$

Multiply both sides of (5.3.12) by A_s and evaluate the resulting inequality at vectors of the form $(\omega, \omega) \in \mathbb{R}^{2d}$. As a consequence, one obtains

$$\omega^T \left((s^2 + s) X_{\varepsilon,\alpha} - (s+1) Y_{\varepsilon,\alpha} \right) \omega \leq L(s+1) \omega^T \left(Z_2 + sZ_2^T + Z_3 \right) \omega + O(\varepsilon).$$

Set $s := 1+\beta|x_\alpha - y_\alpha|$, with $\beta = \overline{\beta}\phi^{m-1}(y_\alpha)$, for $\overline{\beta}$ yet to be fixed. It follows that

$$\begin{aligned}\omega^T\left(sX_{\varepsilon,\alpha} - Y_{\varepsilon,\alpha}\right)\omega &\leq L\omega^T\left(Z_2 + sZ_2^T + Z_3\right)\omega + O(\varepsilon)\\ &= L\omega^T\left((s-1)\sigma_\alpha \otimes D\phi(y_\alpha)\right)\omega\\ &\quad + L\omega^T\left(D^2\phi(y_\alpha)|x_\alpha - y_\alpha|\right)\omega + O(\varepsilon)\\ &\leq |\omega|^2 L\left((s-1)|D\phi(y_\alpha)||x_\alpha - y_\alpha|\right)\\ &\quad + |\omega|^2 L\left(|D^2\phi(y_\alpha)||x_\alpha - y_\alpha|\right) + O(\varepsilon)\\ &\leq |\omega|^2 LC(d)\overline{\beta}\phi^{2m-1}(y_\alpha)|x_\alpha - y_\alpha| + O(\varepsilon).\end{aligned}$$

In conclusion,

$$\begin{aligned}F(Y_{\varepsilon,\alpha}) - sF(X_{\varepsilon,\alpha}) &\leq C_F\left|(sX_{\varepsilon,\alpha} - Y_{\varepsilon,\alpha})_+\right|\\ &\leq C_F C(d) L\overline{\beta}\phi^{2m-1}(y_\alpha)|x_\alpha - y_\alpha| + O(\varepsilon).\end{aligned} \tag{5.3.15}$$

In what follows, we estimate $F(Y_{\varepsilon,\alpha}) - sF(X_{\varepsilon,\alpha})$ from below.

Step 4 – We start with three auxiliary inequalities. Because of (5.3.8), we have

$$\frac{|Q_\alpha|}{|x_\alpha - y_\alpha|} \leq C(d)\phi^{m-1}(y_\alpha) \leq C(d)L^{1-m}|P_\alpha|^m. \tag{5.3.16}$$

More generally, for any $\theta > 0$,

$$\frac{|Q_\alpha|}{|x_\alpha - y_\alpha|} \leq C(d)L^{1-\theta}\phi^{m-\theta}(y_\alpha)|P_\alpha|^\theta. \tag{5.3.17}$$

Also,

$$L|D^2\phi(y_\alpha)| \leq C(d)L^{1-m}\phi^{m-1}(y_\alpha)|P_\alpha|^m. \tag{5.3.18}$$

In the sub-quadratic case $1 < m \leq 2$, one combines (5.3.17) with $\theta = 1$ and (5.3.10) to get

$$\lim_{\alpha\to 0}\frac{|Q_\alpha|}{|P_\alpha|} \leq \lim_{\alpha\to 0} C(d)|\phi(y_\alpha)|^{m-1}|x_\alpha - y_\alpha| = 0.$$

For $m > 2$, in the superquadratic case, (5.3.16) builds upon (5.3.11) to produce

$$|Q_\alpha| \leq C(d)L^{1-m}\tilde{K}^{m-1}|P_\alpha|.$$

Hence, in either case, we have $|Q_\alpha| \leq |P_\alpha|$, for α small enough, since (5.3.6) implies

$$L \geq 2^{\frac{1}{m-1}}\left(\left(\frac{C_F C(d)}{C_2\gamma^m}\right)^{\frac{1}{m-1}} + \left(\frac{C_1}{C_2\gamma^m}\right)^{\frac{1}{m}}\right). \tag{5.3.19}$$

Now we use (5.3.13) to write

$$\begin{aligned}
F(Y_{\varepsilon,\alpha}) - F(sX_{\varepsilon,\alpha}) &\geq sH(P_\alpha, x_\alpha) - H(P_\alpha - Q_\alpha, y_\alpha) \\
&\geq (s-1)H(P_\alpha, x_\alpha) - C_3(1+2|P_\alpha|)^{m-1}|Q_\alpha| \\
&\quad - (C_3|P_\alpha|^m + C_1)|x_\alpha - y_\alpha| \\
&\geq (s-1)(C_2|P_\alpha|^m - C_1) - C_3(1+2|P_\alpha|)^{m-1}|Q_\alpha| \\
&\quad - (C_3|P_\alpha|^m + C_1)|x_\alpha - y_\alpha|.
\end{aligned}$$

We used [A2] in the first inequality, whereas $|Q_\alpha| \leq |P_\alpha|$ leads to the second one. Since $|P_\alpha| \geq L > 1$, it follows that $1 + 2|P_\alpha| \leq 3|P_\alpha|$, and from the lower bound $L \geq (C_1/C_3)^{1/m}$, we also obtain $C_3|P_\alpha|^m + C_1 \leq 2C_3|P_\alpha|^m$. Thus, we can further estimate

$$\begin{aligned}
&F(Y_{\varepsilon,\alpha}) - F(sX_{\varepsilon,\alpha}) \\
&\geq (s-1)C_2|P_\alpha|^m - C_3(3|P_\alpha|)^{m-1}|Q_\alpha| - 2C_3|P_\alpha|^m|x_\alpha - y_\alpha| \\
&\geq \left(\beta C_2|P_\alpha|^m - C_3(3|P_\alpha|)^{m-1}\frac{|Q_\alpha|}{|x_\alpha - y_\alpha|}\right)|x_\alpha - y_\alpha| \\
&\quad - 2C_3|P_\alpha|^m|x_\alpha - y_\alpha| \\
&\geq \left(\beta C_2|P_\alpha|^m - C_3(3|P_\alpha|)^{m-1}\left(\phi^{m-1}(y_\alpha)|P_\alpha|\right)\right)|x_\alpha - y_\alpha| \\
&\quad - 2C_3|P_\alpha|^m|x_\alpha - y_\alpha| \\
&= |P_\alpha|^m\left(\beta C_2 - C_3 3^{m-1}\phi^{m-1}(y_\alpha) - 2C_3\right)|x_\alpha - y_\alpha|.
\end{aligned}$$

We used (5.3.17) with $\theta = 1$ in the last inequality. Because $\beta = \overline{\beta}\phi^{m-1}(y_\alpha)$, we get

$$\begin{aligned}
F(Y_{\varepsilon,\alpha}) - F(sX_{\varepsilon,\alpha}) &\geq |P_\alpha|^m\phi^{m-1}(y_\alpha)\left(\overline{\beta}C_2 - C_3 3^{m-1} - 2C_3\phi^{1-m}(y_\alpha)\right)|x_\alpha - y_\alpha| \\
&\geq |P_\alpha|^m\phi^{m-1}(y_\alpha)|x_\alpha - y_\alpha|, \qquad (5.3.20)
\end{aligned}$$

provided we set $\overline{\beta} = \frac{C_3}{C_2}(3^{m-1} + 2)$.

Combining (5.3.20) and (5.3.15) we get

$$|P_\alpha|^m\phi^{m-1}(y_\alpha)|x_\alpha - y_\alpha| \leq C_F\left(C(d)L\overline{\beta}\phi^{2m-1}(y_\alpha)|x_\alpha - y_\alpha| + O(\varepsilon)\right).$$

Let $\varepsilon \to 0$ and divide both sides of the former inequality by the quantity $|x_\alpha - y_\alpha|\phi^{2m-1}(y_\alpha)$. Then

$$L^m \leq \frac{|P_\alpha|^m}{\phi^m(y_\alpha)} \leq \overline{C}L,$$

where $\overline{C} = C_F C(d)\overline{\beta}$. This is a contradiction since

$$L \geq 2\left(3^m C_F C(d)\frac{C_3}{C_2}\right)^{\frac{1}{m-1}}.$$

Therefore we have proven that u is locally Lipschitz continuous, with constant L given by (5.3.6). □

Remark 5.3.2. The proof of Theorem 5.1.1 provides a constructive way to produce the Lipschitz constant C associated with u. In fact, this is given by $L > 0$, as defined in (5.3.6).

5.4 A two-phase free boundary problem

Now, we explore a consequence of Lemma 5.3.1 in the context of free boundary problems. It concerns the existence of a viscosity solution to

$$\begin{cases} -\mathrm{Tr}(A(x)D^2u) + H(Du,x) = \lambda_+\chi_{\{u>0\}} + \lambda_-\chi_{\{u<0\}} & \text{in} \quad \Omega(u) \\ u = g & \text{on} \quad \partial\Omega \end{cases} \tag{5.4.1}$$

where Ω is a C^2-domain, $A : \Omega \to S(d)$ is degenerate elliptic, $0 < \lambda_- < \lambda_+$ are constants, $g \in C^{0,\frac{m-2}{m-1}}(\partial\Omega)$ is given, and $\Omega(u)$ is given by

$$\Omega(u) := \{x \in \Omega \,|\, u(x) \neq 0\}.$$

We notice the equation holds *only* where the solution does not vanish, and hence, no information is available across the free boundary $\Gamma(u) := \partial\{u > 0\} \cup \partial\{u < 0\}$.

We prove the existence of a locally Hölder-continuous viscosity solution to (5.4.1) with suitable, estimates. To do that, we introduce an assumption on the data $A : \Omega \to S(d)$ and $g : \partial\Omega \to \mathbb{R}$.

[A4]. *We suppose $A : \Omega \to S(d)$ to be degenerate elliptic and bounded from above. In addition, there exists $\lambda > 0$ such that*

$$\nu(x)^T A(x)\nu(x) \geq \lambda$$

for every $x \in \partial\Omega$. Also, we suppose $g \in C^{0,\frac{m-2}{m-1}}(\partial\Omega)$, with

$$|g(x) - g(y)| \leq K|x-y|^{\frac{m-2}{m-1}}$$

for every $x, y \in \partial\Omega$, where $K > 0$ is fixed, though yet to be determined. In addition, suppose

$$0 < \inf_{x\in\partial\Omega} g(x) < 2\lambda_-.$$

The importance of [A4] is in unlocking an intermediate step in our analysis, namely [16, Theorem 2.12]. In fact, the superquadratic character of (5.4.1) introduces a number of subtleties in the arguments leading to the existence of solutions. See the discussion in [16, Section 2.3].

Theorem 5.4.1 (Existence of solutions)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain of class C^2. Suppose assumption* [A4] *is in force. Then there exists a viscosity solution $u \in C(\Omega)$ to the problem* (5.4.1)*. In addition, we have $u \in C^{0,\frac{m-2}{m-1}}_{\text{loc}}(\Omega)$. Finally, for every $\Omega' \Subset \Omega$, there exists a positive constant $C = C\left(m, \|A\|_{L^\infty(\Omega)}, K, \lambda, \mathrm{diam}(\Omega), \mathrm{dist}(\Omega', \partial\Omega)\right)$ such that*

$$\|u\|_{C^{0,\frac{m-2}{m-1}}(\Omega')} \leq C\left(1 + \|u\|_{L^\infty(\Omega)} + \max\{|\lambda_+|, |\lambda_-|\}\right).$$

The proof of Theorem 5.4.1 combines several ingredients. First, we consider a family of auxiliary equations indexed by a parameter $\varepsilon > 0$. For each equation in the family, the existence of a (unique) viscosity solution follows from [16, Theorem 2.12]. Lemma 5.3.1 implies estimates independent of $\varepsilon > 0$ and allows us to apply Schauder's Fixed Point Theorem to conclude the argument. We proceed by introducing an auxiliary problem.

For $v \in C(\overline{\Omega})$ and $0 < \varepsilon < 1$, define $g_\varepsilon^v : \mathbb{R}^d \to \mathbb{R}$ as

$$g_\varepsilon^v(x) := \max\left(\min\left(\frac{v(x)+\varepsilon}{2\varepsilon}, 1\right), 0\right),$$

if $x \in \Omega$, and $g_\varepsilon^v \equiv 0$ in $\mathbb{R}^d \setminus \Omega$. Now, for $x \in \Omega$, let $h_\varepsilon^v(x) := (g_\varepsilon^v * \eta_\varepsilon)(x)$, where η_ε is a standard mollifier. We consider the auxiliary problem

$$\begin{cases} \varepsilon u - \mathrm{Tr}(A(x)D^2u) + H(Du,x) = \lambda_+ h_\varepsilon^v + \lambda_-(1-h_\varepsilon^v) & \text{in} \quad \Omega \subset \mathbb{R}^d \\ u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{5.4.2}$$

Through a fixed-point approach, we show the existence of a solution $u_\varepsilon \in C(\Omega)$ to the Dirichlet problem

$$\begin{cases} \varepsilon u_\varepsilon - \mathrm{Tr}(A(x)D^2u_\varepsilon) + H(Du_\varepsilon,x) = \lambda_+ h_\varepsilon^{u_\varepsilon} + \lambda_-(1-h_\varepsilon^{u_\varepsilon}) & \text{in} \quad \Omega \subset \mathbb{R}^d \\ u_\varepsilon = g & \text{on} \quad \partial\Omega. \end{cases} \tag{5.4.3}$$

This argument relies on a variant of Lemma 5.3.1 applied to (5.4.2). By taking the limit $\varepsilon \to 0$ in (5.4.3) and applying Lemma 5.3.1 once more, we obtain the existence of a viscosity solution to (5.4.1). The first step towards proving Theorem 5.4.1 is the following proposition.

Proposition 5.4.2. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain of class C^2. Suppose assumption* [A4] *is in force. Then, for every $0 < \varepsilon < 1/4$, there exists $u_\varepsilon \in C(\Omega)$ solving* (5.4.3) *in the viscosity sense. In addition, $u_\varepsilon \in C^{0,\frac{m-2}{m-1}}_{\mathrm{loc}}(\Omega)$. Moreover, for every $\Omega' \Subset \Omega$, there exists a positive constant $C = C\left(m, \|A\|_{L^\infty(\Omega)}, K, \lambda, \mathrm{diam}(\Omega), \mathrm{dist}(\Omega', \partial\Omega)\right)$ such that*

$$\|u\|_{C^{0,\frac{m-2}{m-1}}(\Omega')} \leq C\left(1 + \|u\|_{L^\infty(\Omega)} + \max\{|\lambda_+|, |\lambda_-|\}\right). \tag{5.4.4}$$

Proof. For ease of presentation, we split the proof into three steps.

Step 1 – Notice that, given $v \in C(\Omega)$, we have

$$|\lambda_+ h_\varepsilon^v(x) + \lambda_-(1-h_\varepsilon^v(x)) - \lambda_+ h_\varepsilon^v(y) - \lambda_-(1-h_\varepsilon^v(y))| \leq \omega_{v,\varepsilon}(|x-y|),$$

where $\omega_{v,\varepsilon}(\cdot)$ is a modulus of continuity depending on v and $\varepsilon > 0$. Hence, the right-hand side of the equation in (5.4.2) is a continuous function up to the boundary $\partial\Omega$. A straightforward application of [16, Theorem 2.12] ensures the existence of a unique viscosity solution u_ε^v to (5.4.2).

Notice also the proof of Lemma 5.3.1 extends to the case of (5.4.2). As a consequence, $u_\varepsilon^v \in C^{0,\frac{m-2}{m-1}}_{\mathrm{loc}}(\Omega)$ for every $0 < \varepsilon < 1/4$ and every $v \in C(\overline{\Omega})$. Moreover, for every $\Omega' \Subset \Omega$, there exists a constant $C > 0$ depending on the data of the problem and Ω', but not depending on ε or v, such that

$$\|u_\varepsilon^v\|_{C^{0,\frac{m-2}{m-1}}(\Omega')} \le C. \tag{5.4.5}$$

uniformly in $v \in C(\Omega)$ and $\varepsilon \in (0, 1/4)$.

Step 2 – We now define $\mathscr{K} \subset C(\Omega)$ as

$$\mathscr{K} := \left\{ w \in C(\overline{\Omega}) \ : \ \|w\|_{L^\infty(\Omega)} \le C_0 \right\},$$

where $C_0 > 0$ will be chosen later. Notice that $\mathscr{K}$ is closed in $C(\overline{\Omega})$. In the sequel, we define a map $T : \mathscr{K} \to C(\overline{\Omega})$. For fixed $\varepsilon \in (0, 1/4)$, take $v \in \mathscr{K}$ and denote with u_ε^v the unique solution to (5.4.2), whose existence follows from the previous step. Define $Tv := u_\varepsilon^v$ and notice that the existence of a fixed point for T is tantamount to the existence of solutions to (5.4.3).

To prove the existence of a fixed point for T, we start by noticing that it is possible to choose $C_0 > 0$, independent on v and ε, such that $T(\mathscr{K}) \subset \mathscr{K}$. This follows from the construction of sub and supersolutions for the problem; see the proof of [16, Theorem 2.12].

We continue by proving that $T(\mathscr{K})$ is pre-compact. Let $(Tv_n)_{n\in\mathbb{N}}$ be a sequence of elements in $T(\mathscr{K})$. Because $|Tv_n| \le C_0$, for every $n \in \mathbb{N}$, the sequence is equibounded. In addition, (5.4.5) ensures it is also equicontinuous. Hence, $(Tv_n)_{n\in\mathbb{N}}$ converges to an element $v^* \in T(\mathscr{K})$, through a subsequence if necessary.

Finally, we verify that T is sequentially continuous. Suppose $(v_n)_{n\in\mathbb{N}} \subset \mathscr{K}$ converges to some $v \in \mathscr{K}$. We prove that $Tv_n \to Tv$, as $n \to \infty$. Indeed, because $T(\mathscr{K})$ is pre-compact, we infer that $(Tv_n)_{n\in\mathbb{N}}$ converges, through a subsequence if necessary, to some $w \in \mathscr{K}$. The stability of viscosity solutions and the uniqueness available for (5.4.2) ensure that $Tv = w$. To conclude that T is continuous, we must verify the former equality does not depend on the particular subsequence. Indeed, suppose a different subsequence yields $Tv_{n_k} \to w'$, as $k \to \infty$. Since $v_{n_k} \to v$ as $k \to \infty$, we reason as before (resorting to the stability of viscosity solutions and the uniqueness for (5.4.2)) to obtain $Tv = w'$.

Step 3 – The properties of the subset $\mathscr{K}$ and the operator T allow us to apply the Schauder Fixed Point Theorem, as in [31, Corollary 11.2], to conclude the existence of $u_\varepsilon \in \mathscr{K}$ such that $Tu_\varepsilon = u_\varepsilon$. That is, u_ε solves (5.4.3). Because the conclusions of Lemma 5.3.1 apply, we have $u_\varepsilon \in C^{0,\frac{m-2}{m-1}}_{\text{loc}}(\Omega)$ and the estimate in (5.4.4) holds. □

Now, we detail the proof of Theorem 5.4.1.

Proof of Theorem 5.4.1. We take a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and consider the sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to

$$\varepsilon_n u_n - \operatorname{Tr}(A(x)D^2u_n) + H(Du_n, x) = \lambda_+ h_{\varepsilon_n}^{u_n} + \lambda_-(1 - h_{\varepsilon_n}^{u_n}) \quad \text{in} \quad \Omega.$$

Because the estimate in (5.4.4) holds for u_n, for every $n \in \mathbb{N}$, we conclude there exists $u^* \in C^\beta_{\text{loc}}(\Omega)$ such that $u_n \to u^*$ in the C^β-topology, for every $0 < \beta < \frac{m-2}{m-1}$.

Now, let $x \in \{u^* > 0\}$ and write $\tau := u^*(x)$. Suppose $u^* - \varphi$ has a (strict) local maximum at x, for $\varphi \in C^2(\Omega)$. There exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $u_n - \varphi$ has a local maximum at x_n.

On the other hand, there exists $N \in \mathbb{N}$ such that

$$u_n(x_n) > \frac{\tau}{2} > \varepsilon_n$$

provided $n > N$. Hence,

$$\varepsilon_n u_n(x_n) - \mathrm{Tr}(A(x_n)D^2\varphi(x_n)) + H(D\varphi(x_n), x_n) \leq \lambda_+ h_{\varepsilon_n}^{u_n}(x_n) + \lambda_-(1 - h_{\varepsilon_n}^{u_n}(x_n));$$

by taking the limit $n \to \infty$, we obtain

$$-\mathrm{Tr}(A(x)D^2\varphi(x)) + H(D\varphi(x), x) \leq \lambda_+.$$

Conversely, suppose $x \in \{u^* < 0\}$ and write $\sigma := u^*(x)$. If $u^* - \varphi$ has a (strict) local maximum at x, for $\varphi \in C^2(\Omega)$ we reason as before to conclude

$$-\mathrm{Tr}(A(x)D^2\varphi(x)) + H(D\varphi(x), x) \leq \lambda_-.$$

It ensures that u^* is a sub-solution to (5.4.1) in $\Omega(u)$. An analogous argument ensures that u^* is also a supersolution and completes the proof. □

Remark 5.4.3. Our proof of Theorem 5.4.1 yields further information since it produces two viscosity inequalities satisfied by the solution *in the region* $\{u = 0\}$. Indeed, the viscosity solution to (5.4.1), whose existence follows from Theorem 5.4.1, satisfies

$$\lambda_- \leq -\mathrm{Tr}(A(x)D^2u) + H(Du, x) \leq \lambda_+ \quad \text{in} \quad \Omega$$

in the viscosity sense. Besides solving the equation in the positive and negative phases, it also solves a pair of inequalities in the whole domain.

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