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UNIVERSIDADE Ð COIMBRA

Sandra Cruz Caçador

ROBUST OPTIMIZATION – APPLICATION TO THE FIELD OF PORTFOLIO SELECTION

Tese no âmbito do doutoramento em Gestão – Ciência Aplicada à Decisão orientada pelo Professor Doutor Pedro Manuel Cortesão Godinho e pela Professora Doutora Joana Maria Pina Matos Dias apresentada à Faculdade de Economia da Universidade de Coimbra.

Março de 2020

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Resumo

As estratégias clássicas de seleção de carteiras de investimento são frequentemente utilizadas em contexto real. No entanto, é largamente reconhecido que a abordagem da média-variância apresenta graves limitações relativamente à sensibilidade a erros de estimação e ao efeito da incerteza dos parâmetros de entrada na solução ótima do modelo de otimização.

São diversas as metodologias que permitem minimizar as consequências destas limitações e, portanto, garantir que a solução ótima é relativamente imune à incerteza dos parâmetros de entrada e estável face a possíveis erros de estimação. Uma destas metodologias é a otimização robusta, cuja origem remete à teoria do controlo robusto. Reconhecida como uma alternativa computacionalmente atrativa, a otimização robusta não requer a satisfação de todos os pressupostos relativos à distribuição de probabilidades dos parâmetros incertos que são considerados em metodologias afins, tais como a programação estocástica e a programação dinâmica.

Motivados pela necessidade de ajustar a abordagem clássica, de forma a ultrapassar as suas limitações, e pela atratividade, poder de modelação e extensa aplicabilidade da metodologia de otimização robusta, apresentamos um estudo que irá contribuir, assim o esperamos, para disseminar esta metodologia entre os gestores de carteiras de investimento e sua utilização por parte dos decisores. Neste estudo, são propostos novos modelos de otimização robusta de carteiras de investimento e novas medidas de robustez, desenvolvidos com base em estratégias reconhecidas nesta área do conhecimento. São avaliados os benefícios, para o investidor, das soluções robustas propostas de forma a perceber se a metodologia de otimização robusta permite criar valor ao processo de tomada de decisão de investimento e permite mitigar os efeitos dos erros de estimação no cálculo da solução ótima. São, ainda, comparadas as abordagens de robustez absoluta e relativa através da análise da performance das respetivas carteiras ótimas, destacando-se as suas principais vantagens e limitações.

De um modo geral, os resultados deste estudo sustentam o potencial da metodologia de otimização robusta na mitigação dos erros de estimação no cálculo da carteira ótima de investimento. Os resultados mostram que o desempenho das carteiras propostas supera o desempenho das carteiras de referência implementadas neste estudo, à exceção da carteira de variância mínima global. Adicionalmente, as carteiras propostas afiguram-se mais robustas e apresentam resultados mais coerentes comparativamente às carteiras de referência, robustas e não robustas, utilizadas. A análise dos resultados obtidos para diferentes níveis de aversão ao risco confirma que as carteiras propostas se afirmam como alternativas válidas para os investidores que poderão ser mais lesados em consequência das limitações apontadas à estratégia clássica da média-variância. Igualmente, os resultados mostram que a redução da amplitude do período dentro da amostra não tem um efeito substancial nem na exposição das carteiras propostas aos títulos que as constituem nem na coerência dos resultados obtidos fora da amostra, realçando assim a utilidade dos modelos propostos na presença de informação limitada. Espera-se, portanto, que os contributos deste estudo constituam novas ferramentas de apoio à tomada de decisão no âmbito da teoria da carteira em contexto de incerteza.

Palavras-chave: Seleção de carteiras de investimento; incerteza; otimização robusta; robustez relativa; robustez absoluta.

Abstract

Classical portfolio selection strategies are frequently applied in real-life. Nonetheless, it is widely acknowledged that the mean-variance based approach presents critical shortcomings concerning the sensitivity of the optimal solution to estimation error and the effects of the input uncertainty on the outputs of the optimization model.

There are several methodologies that try to mitigate the impact of the estimation errors and try to guarantee that the optimal solution of an optimization problem is assured against some worst-case model misspecification, i.e., it is a robust optimal solution. An example is the robust optimization methodology, whose roots can be found in the field of robust control theory. Recognized as a computationally attractive alternative, the robust optimization methodology does not need to satisfy all the assumptions about the probability distributions of the uncertain parameters that have to be considered in other methodologies, like stochastic programming or dynamic programming.

Motivated by the need to adjust the mean-variance based approach in order to overcome its shortcomings and the computational attractiveness, modelling power and broad applicability of the robust optimization methodology, we present a study that, we hope, will contribute to enhance the dissemination of the robust optimization methodology among quantitative portfolio managers and its use by general decision makers. We developed new robust portfolio optimization models and new robustness measures by extending and combining established methodologies in this field of research. The real benefits of the robust portfolios from the investor perspective were assessed by examining whether the robust optimization methodology adds value to the investment decision problem and mitigates the impact of the estimation errors on the computation of the optimal solution. The relative robust and the absolute robust approaches were also compared by analyzing the performance of the optimal portfolios, emphasizing their main advantages and limitations.

Overall, the empirical evidences found in our study support the potential of the robust optimization methodology in the mitigation of the estimation errors on the computation of the optimal portfolio. The results show that the proposed robust portfolios generally outperform the non-robust benchmarks implemented in our study, with the exception of the global minimum variance solution. Furthermore, the proposed robust portfolios are generally more robust and provide more consistent results than the non-robust benchmarks and other robust solutions already described in the literature. The analysis of the results obtained for different levels of the investor's risk preference confirmed that the proposed robust portfolios are valid alternatives for those investors who can be more affected by the methodological weakness of the classical mean-variance strategy. Furthermore, the empirical evidences show that reducing the in-sample period length seems to have no substantial effect either in the exposure of the proposed robust portfolios to individual assets or in the consistency of their out-of-sample results, highlighting the utility of the proposed robust models in the presence of limited data. It is our hope that the results of our study will constitute new tools to support the investment decision making process under uncertainty.

Keywords: Portfolio selection; uncertainty; robust optimization; relative robustness; absolute robustness.

List of Acronyms

AR	**A**bsolute **R**obust
ARA	**A**bsolute **R**obust model **A**
ARB	**A**bsolute **R**obust model **B**
ARC	**A**bsolute **R**obust model **C**
ARD	**A**bsolute **R**obust model **D**
APT	**A**rbitrage **P**ricing **T**heory
ARCH	**A**uto**r**egressive **C**onditional **H**eteroskedastic
BTM	**B**ook-**T**o-**M**arket
CAPM	**C**apital **A**sset **P**ricing **M**odel
CML	**C**apital **M**arket **L**ine
CEP	**C**ash Earnings to **P**rice
CVaR	**C**onditional **V**alue-**a**t-**R**isk
CRRA	**C**onstant **R**elative **R**isk **A**version
DARA	**D**ecreasing **A**bsolute **R**isk **A**version
DY	**D**ividend **Y**ield
EP	**E**arnings-**P**rice
EF	**E**fficient **F**rontier
EW	**E**qually **W**eighted
GA	**G**enetic **A**lgorithm
GMV	**G**lobal **M**inimum **V**ariance
GP	**G**rowth **P**ortfolio
HML	**H**igh **M**inus **L**ow
i.i.d.	**i**ndependent and **i**dentically **d**istributed
IS	**I**n-**S**ample

ME	**M**arket **E**quity
MRC	**M**aximum **R**isk-Adjusted **R**eturn
MV	**M**ean-**V**ariance
MOM	**M**omentum
MCDA	**M**ulti-**C**riteria **D**ecision **A**iding
NP-hard	**N**on-deterministic **P**olynomial-time **hard**
OS	**O**ut-of-**S**ample
PPP	**P**arametric **P**ortfolio **P**olicy
RR	**R**elative **R**obust
RRA	**R**elative **R**obust model **A**
RRB	**R**elative **R**obust model **B**
RRC	**R**elative **R**obust model **C**
RRD	**R**elative **R**obust model **D**
WS	Relative Robust **W**eighted-**S**um
RO	**R**obust **O**ptimization
SOCP	**S**econd **O**rder **C**one **P**rogramming Problem
SDP	**S**emi**d**efinite **P**rogramming Problem
SMB	**S**mall **M**inus **B**ig
SV	**S**tochastic **V**olatility
VP	**V**alue **P**ortfolio
VaR	**V**alue-**a**t-**R**isk
WCVaR	**W**orst-case **C**onditional **V**alue-**a**t-**R**isk
WVaR	**W**orst-case **V**alue-**a**t-**R**isk

List of Notation

J_α	Abnormal return (Jensen's alpha)
a_A	Absolute risk aversion coefficient
$abs(.)$	Absolute value function
$q_n, n = 1, \dots, N$	Binary decision variables (defining portfolio cardinality)
λ, c_1^g, c_2^g	Coefficients representing the weights of individual objective functions
γ	Constant relative risk aversion parameter
ρ_n, ϱ_n, ϱ	Constants defining the size (confidence level) of the uncertainty sets
D	Covariance diagonal matrix of the residual returns
Σ_t	Covariance matrix of asset returns at time t
F	Covariance matrix of the factor returns
$\sigma_{t,nm}$	Covariance of returns between asset n and asset m at time t
v, v^*	Elements of the sample space Ω
$\|x\|_G$	Elliptic norm of vector x with respect to a symmetric, positive definite matrix G
μ_t^p	Expected return of the portfolio
μ_t^C	Expected return of the tangency portfolio
$\mu_{t,n}$	Expected value, at time t, of the time $t+1$ return of asset n
$V \in \mathbb{R}^{M \times N}$	Factor loading matrix
$\mathbb{z}\,(s)$	First observation randomly selected from the estimation sample for computing scenario s
β, θ, c	General coefficients used in different models
L, H	General lower and upper limits
x	Generic function parameter

p	Generic portfolio
f	Generic (profit) objective function
$u(.)$	Generic Von Neumann and Morgenstern utility function
f_1, f_2	Individual objective functions
Y	In-sample period length
i	i-th sample of historical assets returns
V_0	Least squares estimate of the factor loading matrix V
Ψ	Linear space of random variables
μ_L^p	Lower limit of the portfolio expected return
$R^s \in \mathbb{R}^{J\times N}$	Matrix of sample returns of the N assets defining scenario s
μ_{max}^p	Maximum expected return
$\underline{d}_n$, $\overline{d}_n, n = 1, \dots, N$	Minimum and maximum elements of the main diagonal of the covariance diagonal matrix of the residual returns D, respectively
μ_{min}^p	Minimum expected return
z	Minimum value of the portfolio's risk-adjusted return
N	Number of assets available for constructing the portfolio
E	Number of estimation subsamples
m	Number of factors that drive the market
J	Number of observations in the estimation sample
$\mathcal{O}$	Number of observations in the in-sample period
I	Number of realizations of the uncertain parameter included in the uncertainty set U
Z_n	n-th column of $Z \in \mathbb{R}^{M\times N}$
$f\left(w_t^{s*}, s\right)$	Optimal objective function value for scenario s
μ_t^{p*}	Optimal portfolio's expected return
σ_t^{p*}	Optimal portfolio's standard deviation
w_t^*	Optimal portfolio's vector of weights

z_s^g	Optimal value of the weighted sum of the objective functions for scenario s and weight combination g
$w_{t,n}$	Percentage of the n-asset held in the portfolio at time t
V_t	Portfolio turnover at time t
$x \sim N(\mu, \Sigma)$	Random variable with multivariate normal distribution
A, Z	Random variables in the linear space Ψ
P^s	Regret associated with scenario s
a_R	Relative risk aversion coefficient
$r_{t,n}$	Return of asset n at time t
φ	Risk measure
r_t^f	Risk-free asset's return from time t to $t+1$
R	Sample regret used as a performance measure
Ω	Sample space of a given probability space
k	Scaling parameter that guarantees the normalization of the optimal solution
s	Scenario
X	Set of feasible solutions
$B = \{b_1, \ldots, b_M\}$	Set of M different benchmarks
$\hat{S}$	Sharpe ratio estimator
$S(w_t)$	Sharpe ratio of the portfolio characterized by the vector of weights w_t
C	Tangency portfolio
U	Uncertainty set
Σ^L, Σ^H	Upper and lower bound of the uncertainty set U_Σ
μ^L, μ^H	Upper and lower bound of the uncertainty set U_μ
w_{max}, w_{min}	Upper and lower bounds, respectively, of the assets' weights
P	Upper bound on the true regret
v_U^p	Upper limit of the portfolio variance

$\hat{S}_{VaR}$	Value-at-Risk adjusted Sharpe ratio estimator
υ_t^p	Variance of the portfolio
$\hat{\sigma}^2(\hat{S})$	Variance of the Sharpe ratio estimator
ς	Variance sacrifice (regret) measure
μ_t	Vector of expected returns computed at time t
$\mathbf{1}$	Vector of ones
w_t^{s*}	Vector of optimal weights for scenario s at time t
π	Vector of random factor returns
ε	Vector of residual returns
w_t	Vector of weights at time t
$\mathbf{0}$	Vector of zeros
W	Wealth available for investment
$w_{n,t}^h$	Weight of asset n at time t before the rebalancing

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Chapter 1

Introduction

Faced with the scarcity of resources (wealth), every investor is called upon to make a decision that should provide the maximum possible return from the investment of the available resources. The investment decision should determine how the available wealth is going to be distributed among the chosen set of assets. The complexity of the investment decision arises from the fact that it is made before the future returns of the assets are known.

Portfolio selection models are developed with the aim of supporting and assisting the investment decision making process. These models attempt to deal with the complexity derived from the uncertain evolution of the asset's prices while capturing the features of the decision-making process. The first formulation of the decision-making problem concerning the optimal allocation of an investor's wealth among the possible investment choices was formally presented by Harry Markowitz (1952, 1959). Markowitz considered that the investment decision should be made on the basis of a trade-off between risk and expected return: for any given level of expected return, a rational investor would choose the portfolio with minimum risk from the set of all the possible portfolios. His formulation of the portfolio selection problem was revolutionary not only because it was the first time that this problem was clearly formulated and solved, but also because it represented a paradigm shift (Constantinides & Malliaris, 1995).

Around the same time, Roy (1952) presented a similar approach, considering also that choices should be made on the basis of the mean and variance of the portfolio as a whole. Comparatively to Markowitz approach, Roy did not consider the non-negativity constraint, i.e. that the weights of the assets in the portfolio must be all non-negative, and recommended the choice of a specific portfolio (regardless of the risk-return preferences of the decision maker). Markowitz considers that although he is often called the father of modern portfolio theory, Roy can claim an equal share of this honor (Markowitz, 1999).

The classical portfolio selection problem, as presented by Markowitz, completely disregards the uncertainty of the expected returns and of the covariance matrix of asset's returns. It is assumed that these parameters are capable of representing the inherent uncertainty associated with the investment returns. Actually, as these parameters are, most of the times, calculated from past data, they are themselves subject to uncertainty. It is now widely accepted that not acknowledging the uncertainty in the models' parameters substantially degrades the performance of the optimal solution calculated using mean-variance based models (Best & Grauer, 1991a, 1991b; Chopra & Ziemba, 1993; Michaud, 1989).

This observation has motivated the development of several strategies that try to mitigate the impact of the estimation errors and to guarantee that the optimal solution of an optimization problem is assured against some worst-case model misspecification, i.e., it is a robust optimal solution. Robustness can be taken into consideration in portfolio optimization by using robust estimators for the input parameters (Fabozzi, Kolm, Pachamanova, & Focardi, 2007). Robust estimators are less sensitive to extreme events and sampling errors. Robustness can also be guaranteed by constraining portfolio weights, by applying resampling techniques or by incorporating uncertainty into the portfolio selection model itself (Fabozzi et al., 2007). Fabozzi et al. (2007) highlight three methods that incorporate uncertainty directly into the computation of the optimal solution, overcoming the modern portfolio theory shortcomings concerning the sensitivity to estimation error and the effects of the input uncertainty on the outputs of the optimization model. These methodologies are stochastic programming, dynamic programming and robust methodology (RO).

Motivated by the need to adjust the classical Mean-Variance (MV) framework in order to overcome its shortcomings, we incorporate the uncertainty into the portfolio selection model itself using the RO methodology. We have chosen to apply RO since this methodology has been recognized as a computationally attractive alternative. In fact, it does not need to satisfy all the assumptions regarding the probability distributions of the uncertain parameters that have to be considered in other methodologies, like stochastic programming or dynamic programming (Fabozzi et al., 2007).

The main objectives of our research can be briefly described as:

1. To develop new robust portfolio optimization models and new robustness measures by extending and combining established methodologies in this field of research.
2. To examine the main contribution of the RO methodology when applied to portfolio optimization models subject to estimations errors of different magnitudes.
3. To assess the real benefits of the robust portfolios, from the investor perspective, by examining whether the RO methodology adds value to the investment decision problem and mitigates the impact of the estimation errors on the computation of the optimal solution.
4. To compare the relative robust and the absolute robust approaches, by analyzing the performance of relative robust and absolute robust portfolios, emphasizing their main advantages and limitations.

The objectives of our research were defined in order to fill the gaps identified in the literature. We believe that the RO methodology has not yet been used to its full potential in the field of portfolio selection, mainly because there is a lack of empirical studies that deeply explore the characteristics of robust portfolios. With these objectives in mind, we propose new robust portfolio optimization models and perform a detailed analysis of the robust solutions. We start by focusing our attention on the relative robust approach since this is the least studied approach. The lack of studies in the portfolio selection field that simultaneously explore both relative robust and absolute robust approaches, led us to compare the performance of the relative robust solution and the corresponding absolute robust solution. We also develop robust formulations of classical portfolio selection models subject to estimation errors of different magnitudes. We consider portfolio selection models with different model inputs and different uncertain parameters: mean-variance based models that admit uncertainty in both vector of expected returns and covariance matrix of asset returns; minimum variance models that admit uncertainty only in the covariance matrix of asset returns; and parametric models that admit uncertainty in the parameters defining the portfolio’s weights and in the assets returns.

The research presented in this thesis resulted in four robust portfolio optimization models that are compiled and organized in the following way. Chapter 2 presents basic notation and relevant concepts that will be used in the remaining chapters. The classical formulation of the portfolio selection problem is described, and its main extensions are briefly presented. We focus on the main extensions that are more relevant to our study and, thus, are essential to fully understand it. A discussion of the limitations and misconceptions around the mean-variance framework, addressing the motivation underlying our research, is also carried out.

Chapter 3 starts by introducing the RO methodology and by defining the absolute robust and relative robust approaches. In this chapter, we also present a literature review on the robust formulation of classical portfolio optimization problems. We focus on the application of the RO methodology to the mean-variance portfolio, the maximum Sharpe ratio portfolio, the minimum variance portfolio and multi-objective portfolio problems.

Chapter 4 presents an overview of the proposed models and describes a set of methodological elements that are common to all the empirical applications implemented. In this chapter we present the main contributions of our study, the strategy applied in the empirical applications that were implemented, the general decisions concerning model settings definition and data selection, solvers used to compute the robust solutions, benchmarks and performance measures used to compare and assess their performance.

Chapter 5 presents a new minmax regret portfolio optimization model (model A). Regret is defined as the utility loss for the investor resulting from choosing a given portfolio instead of choosing the optimal portfolio of the realized scenario for the uncertain parameters. An extension to this first model is then presented (model B), which builds on the definition of regret and suggests a new way of defining the relative robust solution. In this latter case, validation subsamples randomly generated from the in-sample data are introduced in the sampling procedure and used to evaluate the performance of the minmax regret portfolios in order to define the relative robust solution. Furthermore, we implement the corresponding absolute robust expected utility optimization model and compare the performances of the relative robust and absolute robust solutions. Both approaches were applied assuming constant relative risk aversion preferences and considering uncertainty in both the vector of assets' returns and in the covariance matrix of asset returns. For the empirical analysis, historical daily data of the stocks of the DAX index was collected from Thomson Reuters Datastream. A first presentation of these works can be found in Caçador, Godinho, and Dias (2020a) and in Caçador, Dias, and Godinho (2019).

Motivated by the stability of the optimal solutions computed by optimizing only the second moment, we developed a method for computing relative robust and absolute robust minimum variance portfolios, which is presented in Chapter 6. In this robust optimization model (model C), uncertainty was allowed in the covariance matrix of asset returns only. For the relative robust strategy, where the maximum regret is minimized, regret is defined as the increase in the investment risk. The absolute robust strategy, which minimizes the maximum portfolio variance, was applied assuming the worst-case scenario over the whole uncertainty set. For the empirical analysis, historical daily data of the stocks of the EURO STOXX 50 index was collected from Thomson Reuters Datastream. An initial version of this work is presented in Caçador, Dias, and Godinho (2020b).

The last robust optimization model (model D) developed is presented in Chapter 7. In this work, new relative robust and absolute robust formulations of the parametric portfolio policies presented by Brandt, Santa-Clara, and Valkanov (2009) are proposed. These authors describe a model that parameterizes the portfolio weight of each stock as a function of the firm's characteristics and estimates the coefficients of the portfolio policy by maximizing the average utility the investor would have obtained by implementing the policy over the historical sample period. To develop the robust versions of this model, uncertainty is considered in the parameters defining portfolio's weights and in the asset returns. The empirical analysis was conducted on the dataset with historical daily data regarding the stocks of the EURO STOXX 50 index (also collected using the Thomson Reuters Datastream).

Finally, the main conclusions and limitations of this thesis are presented in Chapter 8.

Chapter 2

Portfolio selection theory

The aim of this chapter is to contextualize the research problem and motivations of our study. We start by presenting the classical formulation of the portfolio selection problem as first described by Harry Markowitz in 1952. The main extensions of the modern portfolio theory are also briefly presented in section 2.1. We give special attention to the extensions that are more relevant to the full comprehension of our study. Then, a discussion of the limitations and misconceptions around the mean-variance framework is carried out in section 2.2. In both sections, basic notation and relevant concepts that will be used in the remaining chapters of this thesis are introduced.

2.1 Modern portfolio theory

The modern portfolio theory is based on the work presented by Markowitz (1952, 1959). Markowitz's work considers two essential concepts: first, the jointly quantitative assessment of return and risk by considering the expected return, variances and covariances of the securities, and second, the formulation of the portfolio selection problem as an optimization problem (Fabozzi et al., 2007). Assuming that the returns of the assets are random variables, Markowitz defines the portfolio expected return as the weighted average of the expected returns of individual assets and the portfolio risk as a function of the variances of, and the covariances between, assets and their weights in the portfolio. He formulated the mean-variance portfolio model as follows. Let N denote the number of assets available for constructing the portfolio at time t. The investor's decision of how to optimally distribute his wealth, W, among the N assets is described by the vector of weights, $w_t = \left[w_{t,1}\ w_{t,2} \dots w_{t,N}\right]'$, where each $w_{t,n}, n = 1, \dots, N$, represents the wealth percentage assigned to the n-asset held in the portfolio at time t, and

$$\sum_{n=1}^{N} w_{t,n} = 1 \tag{2.1}$$

with $w_{t,n} \geq 0, n = 1, \ldots, N$. Hence, all the wealth available for investment is allocated (completeness constraint) and short sales are not allowed (non-negativity constraint).

Let $r_{t,n}$ represent the return of asset n at time t, $\mu_{t,n} = E_t[r_{t+1,n}]$ be the expected value, at time t, of the return of asset n at time $t+1$ and $\mu_t = [\mu_{t,1}\ \mu_{t,2} \ldots \mu_{t,N}]'$ be the vector of expected returns. Then, the expected return of the portfolio, μ_t^p, is given by

$$\mu_t^p = E_t[r_{t+1}^p] = E_t\left[\sum_{n=1}^{N} w_{t,n} r_{t+1,n}\right] = \sum_{n=1}^{N} w_{t,n}\mu_{t,n} = \mu_t' w_t. \tag{2.2}$$

Let $\sigma_{t,nm}$ denote the covariance of returns between asset n and asset m at time t, such that $\sigma_{t,nn} = \sigma_{t,n}^2$ and $\sigma_{t,nm} = E_t[(r_{t,n} - \mu_{t,n})(r_{t,m} - \mu_{t,m})]$, for $n \neq m$. The variance of the portfolio is given by

$$\begin{aligned} \upsilon_t^p &= E_t\left[r_{t+1}^p - E_t[r_{t+1}^p]\right]^2 \\ &= E_t\left[\sum_{n=1}^{N} w_{t,n} r_{t+1,n} - E_t\left[\sum_{n=1}^{N} w_{t,n} r_{t+1,n}\right]\right]^2 \\ &= \sum_{n=1}^{N}\sum_{m=1}^{N} \sigma_{t,nm} w_{t,n}\, w_{t,m} \\ &= w_t' \Sigma_t w_t \end{aligned} \tag{2.3}$$

where Σ_t represents the (symmetric) covariance matrix of asset returns. Since variance is always nonnegative, it follows that $w_t' \Sigma_t w_t \geq 0$ and, thus, Σ_t is a positive semidefinite matrix. Furthermore, Σ_t is generally assumed to be positive definite, which is essentially equivalent to assuming that no redundant assets are considered in the construction of the portfolio (Cornuejols & Tütüncü, 2006).

For a given lower limit on the portfolio expected return, μ_L^p, Markowitz's mean-variance portfolio model can be formulated as:

$$\begin{aligned} \min_{w_t \in \mathbb{R}^N} \quad & w_t{}' \Sigma_t w_t \\ \text{subject to} \quad & \mu_t' w_t \geq \mu_L^p \\ & \mathbf{1}' w_t = 1 \\ & w_t \geq \mathbf{0} \end{aligned} \tag{2.4}$$

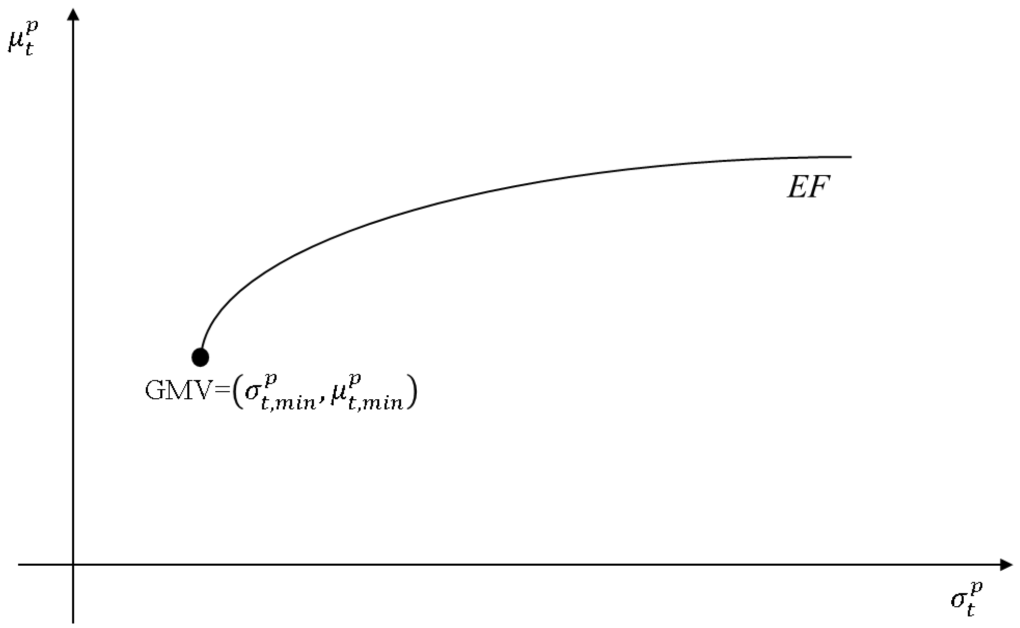
where $\mathbf{1}$ represents the *N*-column vector of ones and $\mathbf{0}$ the *N*-column vector of zeros. A solution is feasible if and only if it satisfies all the constraints, i.e., if it presents an expected return equal to or higher than μ_L^p (first constraint), it guarantees that all the investor's wealth is invested (second constraint) and it is a long-only portfolio (third constraint). With a strictly convex objective function and a convex (non-empty) set of feasible solutions, the quadratic programming problem (2.4), for which the first-order conditions are both necessary and sufficient for optimality, has a unique optimal solution (Cornuejols & Tütüncü, 2006).

The set of solutions that minimize the variance for given desired levels of expect return was designated by Markowitz (1952) as the set of efficient mean-variance combinations and is nowadays known as the Markowitz's Efficient Frontier (EF). For a given covariance matrix estimator, the computation of the EF can be made as follows (Cornuejols & Tütüncü, 2006). First, one has to identify the feasible maximum and minimum values of μ_L^p. Let μ_{max}^p represent the maximum expected return of an efficient portfolio, which is the maximum expected return of the individual assets. Let μ_{min}^p represent the minimum expected return of an efficient portfolio, corresponding to the expected return of the optimal solution of the quadratic problem:

$$\begin{aligned} \min_{w_t \in \mathbb{R}^N} \quad & w_t{}' \Sigma_t w_t \\ \text{subject to} \quad & \mathbf{1}' w_t = 1 \cdot \\ & w_t \geq \mathbf{0} \end{aligned} \tag{2.5}$$

The optimal solution of problem (2.5) is usually designated by Global Minimum Variance (GMV) portfolio, since it corresponds to the efficient solution with the lowest variance. Then, for a representative finite set of values of μ_L^p, with $\mu_{min}^p \leq \mu_L^p \leq \mu_{max}^p$, problem (2.4) is solved and the optimal solution, w_t^*, is identified. Let μ_t^{p*} and σ_t^{p*} represent the optimal portfolio's expected return and standard deviation, respectively. Then, the EF corresponds to the set defined as $\left\{(\sigma_t^{p*}, \mu_t^{p*}) : \mu_t^{p*} = w_t^{*\prime} \mu_t, \sigma_t^{p*} = \sqrt{w_t^{*\prime} \Sigma_t w_t^*}, \forall w_t^* \right\}$. A typical representation of the EF is depicted in Figure 2-1. Once the EF is identified, the investor has then to select the optimal portfolio from the set of efficient solutions according to his risk-return preferences.

Figure 2-1: Illustration of the set of efficient mean-variance combinations (EF).



The standard deviation of the portfolio (σ_t^p) is represented in the horizontal axis, while the vertical axis corresponds to the expected return (μ_t^p). The efficient portfolio located on left-most side of the plot corresponds to the efficient solution with lowest variance and is designated by GMV portfolio.

Problem (2.3) has equivalent formulations, i.e. it can be formulated in different ways, all leading to the same EF. Instead of defining a lower limit for the expect return, one could also begin by defining an upper limit on the portfolio variance, which will be represented by υ_U^p, and then maximize the expected return of the portfolio:

$$\begin{aligned} \max_{w_t \in \mathbb{R}^N} \quad & \mu_t' w_t \\ \text{subject to} \quad & w_t{}' \Sigma_t w_t \leq \upsilon_U^p. \\ & \mathbf{1}' w_t = 1 \\ & w_t \geq \mathbf{0} \end{aligned} \tag{2.6}$$

Another alternative is to maximize the risk-adjusted expected return of the portfolio:

$$\begin{aligned} \max_{w_t \in \mathbb{R}^N} \quad & \mu_t' w_t - \lambda w_t{}' \Sigma_t w_t \\ \text{subject to} \quad & \mathbf{1}' w_t = 1 \\ & w_t \geq \mathbf{0} \end{aligned} \tag{2.7}$$

where $\lambda \in \mathbb{R}$ corresponds to the weight given to the portfolio variance representing the investor's risk appetite.

The mean-variance portfolio theory relies on the following set of theoretical basic assumptions: i) the investment opportunity is completely characterized by the probability distribution of asset returns (measured over the investment horizon); ii) the investment risk is a function of the variability of the asset returns (measured by the standard deviation, or equivalently, the variance of returns); iii) the investor bases his decision on the first two moments of the probability distribution of asset returns only; and iv) for a given value of the return, the investor chooses the portfolio with the lowest variance of returns (Francis & Kim, 2013).

Markowitz established two essential concepts that became the basis of the modern portfolio theory. The first one was the formulation of the portfolio selection problem as an optimization problem: the selection of the securities must guarantee the maximization of the expected return of the portfolio while holding constant the portfolio risk, or equivalently, guarantee the minimization of the portfolio risk while holding constant the portfolio expected return. The second one was the joint quantitative assessment of return and risk by considering the expected return and covariance of the securities. Before Markowitz, the concepts of expected return and risk had not been analyzed in an integrated manner (Constantinides & Malliaris, 1995). Main (separated) contributions in this field were made by Bernoulli (1738), who presented a new theory to define and measure risk, and Fisher (1906), who described asset returns in terms of a probability distribution. An early attempt to analyze expected return and risk in an integrated way was made by Marschak (1938) who expressed preferences as indifference curves in the mean-variance space. The methodological breakthrough was only achieved by the axiomatic theory of choice under uncertainty presented by Von Neumann and Morgenstern (1947), which represented a critical contribution to the development of Markowitz's portfolio selection theory.

The joint quantitative assessment of return and risk laid the foundations for the diversification principle. Diversification involves combining assets with less-than-perfect positive correlations in order to reduce the portfolio's risk without sacrificing its return (Francis & Kim, 2013). Markowitz was the first to draw attention to the fact that a given security which is risky or conservative, appropriate or inappropriate, for one portfolio may be the opposite for another. Therefore, the investor must think of selecting a portfolio as a whole, and not each security per se (Markowitz, 1959). Prior to Markowitz's work, the diversification of the investments was based on the premise of "not putting all your eggs in one basket". Hence, there was still lacking "(…) an adequate theory of investment that covered the effects of diversification when risks are correlated, distinguished between efficient and inefficient portfolios, and analyzed risk-return trade-offs on the portfolio as a whole" (Markowitz, 1999, p. 5).

Further works extended Markowitz's mean-variance model to incorporate additional moments of asset returns like skewness (Chunhachinda, Dandapani, Hamid, & Prakash, 1997; Yu, Wang, & Lai, 2008) and kurtosis (Jurczenko, Maillet, & Merlin, 2005; Lai, Yu, & Wang, 2006). Other works also consider the number of assets in the portfolio (cardinality), bounds in the asset weights, dividends, turnover, transaction costs or growth in sales. See Aouni, Colapinto, and La Torre (2014), Kolm, Tütüncü, and Fabozzi (2014), and Steuer, Qi, and Hirschberger (2008) for further readings.

While Markowitz analyzed the investment decision problem considering only risky assets, Tobin (1958) showed that including a risk-free asset in the investment universe allows to expand the EF and to build portfolios that are more (or equally) efficient than the Markowitz's efficient mean-variance combinations. Tobin considered the possible investment combinations of a risk-free asset and a risky mean-variance portfolio, which can be represented by straight lines in the risk-return space. For a given risky efficient portfolio, the straight line lies above all the others, since the corresponding portfolios will have the lowest standard deviation for any given value of expected return. This risky efficient portfolio corresponds to the tangency point of the line that passes through the point representing the risk-free asset and is tangent to the EF. The optimal combinations of the risk-free asset and the tangency portfolio are identified by the Capital Market Line (CML). Geometrically, the tangency portfolio corresponds to the efficient portfolio that maximizes the slope of all the linear combinations of the risk-free asset and risky efficient portfolios, i.e., that maximizes the quantity

$$S(w_t) = \frac{\mu_t' w_t - \mathrm{r_t^f}}{\sqrt{w_t' \Sigma_t w_t}} \tag{2.8}$$

where w_t represents a portfolio in the EF and r_t^f represents the time $t+1$ return of the risk-free asset. The quantity defined in (2.8) was introduced by Sharpe (1966) to measure the performance of mutual funds and is nowadays commonly known as the Sharpe ratio. An illustration of the CML is depicted in Figure 2-2.

The investment decision, in the presence of a risk-free asset, is now reduced to selecting a portfolio that is a linear combination of the risk-free asset and the tangency portfolio and will depend on the risk preference of the investor. The more (less) risk averse the investor is, the more to the left (right) will be the optimal portfolio selected by him/her (always in CML). Notice that the weights of the risky assets in the tangency portfolio do not depend on the investor's risk preferences.

The works developed by Markowitz (1952, 1959) and Tobin (1958) served as starting points for the numerous contributions that appeared in the following decades. Based on their results, Lintner (1965) and Sharpe (1964) independently developed the Capital Asset Pricing Model (CAPM), which became a pillar for modern financial theory. Under the assumptions of market efficiency and homogeneity of the investors' expectations, the tangency portfolio corresponds to the market portfolio, i.e. the portfolio where all assets available to investors are held in the proportion to their market value relative to the total market value of all assets. Following these assumptions, the authors showed how a risky asset should be priced in equilibrium and proved that the expected return of an individual risky asset is a positive linear function of its systematic risk[1] relative to the excess return of the market portfolio. Ross (1976) proposed an asset pricing model, alternative to the CAPM, called the Arbitrage Pricing Theory (APT). This model asserts that the expected return of an asset is influenced not only by the market risk (as suggested by the CAPM), but also by a variety of risk factors. For additional readings regarding assumptions, formulation and empirical testing of the asset pricing models see Elton, Gruber, Brown, and Goetzmann (2009), Fabozzi et al. (2007) and Fama and French (2004).

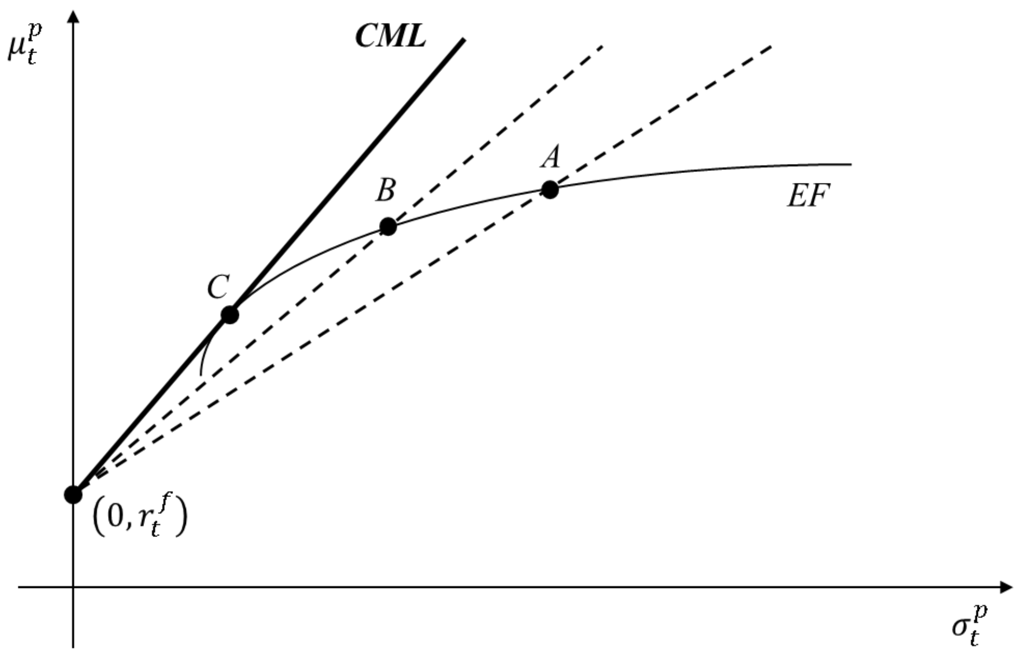
The APT does not specify which are the risk factors that influence asset returns. The identification of the factors driving asset returns has been studied by several authors in the attempt to predict future returns (Carhart, 1997; Chan, Karceski, & Lakonishok, 1999; Chen, Roll, & Ross, 1986; Fama & French, 1993, 1996; Ferson & Harvey, 1991). Chen et al.(1986) show that five economic variables are significant in explaining expected stock returns: industrial production, changes in the risk premium, changes in the yield curve, unanticipated inflation and changes in expected inflation. Fama and French (1993) propose a fundamental three-factor model which defines excess market return, size and book-to-market ratio, as the driving forces of asset returns' variation. Based on the Fama-French three-factor model and motivated by its inability to explain cross-sectional variation in momentum-sorted portfolio returns, Carhart (1997) suggests as an additional factor, the one-year return momentum.

In the field of portfolio selection, these economic and financial state variables and/or firm accounting variables, which influence and explain asset returns, have been used in order to estimate the optimal portfolio weights directly. This can be done by parameterizing the portfolio weights as functions of these observable quantities and then solve the portfolio selection model for the parameters that maximize the investor's expected utility (Brandt, 2010). According to Brandt (2010), parameterizing the portfolio weights as functions of observable quantities has conceptual advantages comparatively to the traditional portfolio optimization approach. The estimation errors

[1] Systematic risk is the risk that derives from general market and economic conditions that cannot be diversified away, while unsystematic risk corresponds to the portion of an asset's variability that can be diversified (Fabozzi et al., 2007).

associated with the estimation of the model inputs are avoided and the dimensionality of the optimization problem is reduced since the parameter space only increases with the number of stock characteristics, rather than with the number of stocks. Furthermore, by framing the portfolio optimization as a statistical estimation problem with an expected utility objective function, it is possible to test individual and joint hypotheses about the optimal portfolio weights.

Figure 2-2: Illustration of the Capital Market Line.



The risk-free asset is represented by the point $(0, r_t^f)$, where r_t^f corresponds to the risk-free asset's return from time t to $t+1$. Point C represents the tangency portfolio and corresponds to the mean-variance portfolio in the EF with maximum Sharpe ratio. The points within the CML between $(0, r_t^f)$ and C correspond to investment combinations where the weights of the risk-free asset and of the tangency portfolio are both non-negative and lower than 1. The points within the CML with higher expected return than the expected return of the tangency portfolio (μ_t^C), correspond to investment combinations where the weight of the risk-free asset is negative (lending at the risk-free rate) and the weight of the tangency portfolio is higher than 1. Linear combinations of the risk-free asset and efficient portfolios A and B, represented in dashed lines, are not optimal since the corresponding portfolios have higher standard deviation comparatively to the portfolios in the CML with the same expected return.

Brandt et al. (2009) propose a parametric portfolio policy that optimizes the portfolio based on firm characteristics (market capitalization, book-to-market ratio and lagged returns). The authors describe a model that parameterizes the portfolio weight of each stock as a function of the firm's characteristics and estimates the coefficients of the portfolio policy by maximizing the average utility the investor would have obtained by implementing the policy over the historical sample period. Let p represent the portfolio and let N be the number of stocks in the investable universe at each period t. Each stock n has a return of $r_{n,t+1}$ from period t to $t+1$ and it is associated with a vector of firm characteristics $x_{n,t}$ observed at period t. In their application example, Brandt et al. (2009) consider three different firm's characteristics: the market capitalization (*ME*), the book-to-market ratio (*BTM*), and the lagged twelve-month return (*MOM*). They present a portfolio selection model that consists of choosing the portfolio weights, $w_{n,t}$, that maximize the conditional expected utility of the portfolio's return r_{t+1}^{p}, given by

$$\max_{w_t \in \mathbb{R}^N} E_t\left[u\left(r_{t+1}^{p}\right)\right] = \max_{w_t \in \mathbb{R}^N} E_t\left[u\left(\sum_{n=1}^{N} w_{n,t} r_{n,t+1}\right)\right]. \tag{2.9}$$

The portfolio weights $w_{n,t}$ are a function of the firm characteristics $x_{n,t}$ and of a vector α of coefficients to be estimated: $w_{n,t} = f\left(x_{n,t}\,;\alpha\right)$.

The authors suggest a linear portfolio weight function:

$$w_{n,t} = \bar{w}_{n,t} + \frac{1}{N}\alpha^T \hat{x}_{n,t}, \tag{2.10}$$

where $\bar{w}_{n,t}$ is the weight of stock n at date t in a benchmark portfolio, such as the value-weighted market portfolio, α is a vector of coefficients to be estimated, and $\hat{x}_{n,t}$ are the characteristics of stock n, standardized cross-sectionally to have zero mean and unit standard deviation across all stocks at time t. Thus, the portfolio weights $w_{n,t}$ are not calculated directly but result instead from the determination of α. This vector can thus be interpreted as a vector of new decision variables.

In the linear policy case (2.10), Brandt et al. (2009) optimization problem can be defined as

$$\max_{\alpha} \frac{1}{T} \sum_{t=0}^{T-1} u\left(\sum_{n=1}^{N} \left(\overline{w}_{n,t} + \frac{1}{N} \alpha^T \hat{x}_{n,t}\right) r_{n,t+1}\right). \tag{2.11}$$

Their empirical results show that the proposed parametric portfolio policies produce sensible portfolio weights and offer robust performance in and out-of-sample.

Behr, Guettler, and Truebenbach (2012) propose an approach that builds on Brandt et al. (2009) where industry momentum characteristics are used instead of using single stock return characteristics. Their empirical results show that the portfolio policies outperform a broad selection of established portfolio strategies in terms of Sharpe ratio and certainty equivalent returns. Hjalmarsson and Manchev (2012) study mean-variance optimization when the portfolio weights are assumed to have a certain functional relationship with the underlying stock characteristics such as value and momentum. The authors have found that the direct approach for estimating portfolio weights clearly beats a naïve regression-based approach that models the conditional mean. Chavez-Bedoya and Birge (2014) formulate an index tracking and enhanced indexation model using a parametric approach, based on the work of Brandt et al. (2009), where the portfolio weights are modeled as functions of assets characteristics and similarity measures of the assets with the index to track. According to the authors, this approach allows handling non-linear and nonconvex objective functions that are difficult to incorporate in existing index tracking and enhanced indexation models. The results of their empirical application reveal that the optimal solution presents a consistent performance, i.e., in-sample and out-of-sample performances very close to each other.

2.2 Limitations of the mean-variance based approach

Markowitz supported his mean-variance analysis of the investment decision problem on mean-variance approximations to the investor's expected utility, which he described as the "MVapproximate" approach: after generating the mean-variance EF, one must choose the efficient portfolio that maximizes the mean-variance function, which approximates the expected value of the investor's utility (Markowitz, 2014). In his empirical support of the "MVapproximate" approach, the author assumed a quadratic utility function and expected returns and covariance matrix of returns as estimators of the input parameters (Markowitz, 1959). In the following years, these two assumptions were subject to a great deal of criticism and pointed out as the main limitations of Markowitz's mean-variance framework. In the next paragraphs, we will contextualize and explain why these assumptions were pointed out as limitations of the modern portfolio theory and address some misconceptions around them.

Following the maximization of the investor's utility approach, the portfolio selection problem can be formulated as the maximization of the investor's wealth utility function. A wealth utility function represents the utility (or happiness) a person derives from different levels of wealth. Consider the notation presented in section 2.1, and let $u(.)$ represent a generic Von Neumann and Morgenstern utility function. The expected utility portfolio maximization problem can be defined as

$$\max_{w_t \in X} E_t\left[u\left(W\left(1+r_{t+1}^p\right)\right)\right] \tag{2.12}$$

where X represents the set of feasible solutions.

Since utility preference orderings are invariant under a positive linear transformation of the utility function and a person's wealth is a function of the rate of return measuring the rate at which wealth is accumulated, the investor's utility of wealth and utility of return functions yield identical preference orderings (Francis & Kim, 2013). Hence, problem (2.12) can be written as:

$$\max_{w_t \in X} E_t\left[u\left(1+r_{t+1}^p\right)\right]. \tag{2.13}$$

Although preferences are not equal for all investors and might change along their life time (Halek & Eisenhauer, 2001), general assumptions regarding the investor's preferences are frequently used. It is usually assumed that an investor prefers always more to less wealth but the marginal utility decreases with increasing wealth (i.e. for generic utility function u and parameter x, $u'(x) > 0$ and $u''(x) \leq 0$) (Fabozzi et al., 2007). The marginal utility of wealth is defined as the additional utility a person gets from a small change in his/her wealth. Thus, the marginal utility of wealth of every rational investor will always be positive. Additionally, investors are assumed to be risk-averse since, when faced with choosing between two investments with the same expected return but two different levels of risk, they prefer the one with the lower risk. Hence, positive but diminishing marginal utility is one of the characteristics that are expected to find in a realistic economic model (Francis & Kim, 2013).

An investor's aversion to risk is measured by the absolute risk aversion (a_A) and the relative risk aversion (a_R) coefficients, defined by Pratt (1964) and given by:

$$a_A(x) = -\frac{u''(x)}{u'(x)} \tag{2.14}$$

and

$$a_R(x) = -\frac{xu''(x)}{u'(x)}. \tag{2.15}$$

The absolute risk aversion is a measure of how the allocation of wealth in risky assets changes with a change in wealth while the relative risk aversion refers to the change in the percentage investment in risky assets as wealth changes (Elton et al., 2009). Further assumptions associated with the behavior of rational risk-averse investors include decreasing absolute risk aversion (DARA) and constant relative risk aversion (CRRA): "Although there is much less consensus regarding relative risk aversion than absolute risk aversion, it is often assumed that constant relative risk aversion [CRRA] is consistent with most investors' behaviour" while "(…) there is general consensus that most investors exhibit decreasing absolute risk aversion [DARA]" (Francis & Kim, 2013, p. 61). If the investor wants to allocate more wealth in risky investments as his wealth increases, it is said that the investor exhibits DARA. So, as the investor gets richer, he or she becomes less risk averse. An investor exhibiting CRRA does not change the percentage of wealth allocated to risky investments even when his wealth changes.

The criticism around the assumption that investors have risk-return preferences described by a quadratic utility function resulted from the fact that the quadratic utility function does not satisfy all the above considerations concerning the behavior of rational risk-averse investors. Consider the following quadratic utility function of wealth:

$$u(x) = \beta x - \theta x^2 \tag{2.16}$$

where β and θ are positive coefficients. The coefficients β and θ must be positive in order for the utility function to have positive but diminishing marginal utility. The absence of a constant term is unimportant because any positive linear transformation of a utility function does not affect portfolio rankings (Francis & Kim, 2013). By replacing the generic parameter x by $\left(1 + r_{t+1}^p\right)$ in equation (2.16), the utility can be expressed as a quadratic function in r_{t+1}^p:

$$u\left(1 + r_{t+1}^p\right) = (\beta - \theta) + (\beta - 2\theta) r_{t+1}^p - \theta\left(r_{t+1}^p\right)^2. \tag{2.17}$$

The marginal utility of returns is given by

$$\frac{\partial u\left(1 + r_{t+1}^p\right)}{\partial r_{t+1}^p} = (\beta - 2\theta) - 2\theta r_{t+1}^p. \tag{2.18}$$

Hence, the marginal utility from additional returns is positive only for $r_{t+1}^p < \frac{\beta - 2\theta}{2\theta}$ (i.e., the quadratic function's marginal utility becomes negative for large wealth levels). Additionally, and as shown by Pratt (1964), the quadratic utility function does not present a decreasing absolute risk aversion parameter. In fact, the absolute risk aversion parameter of a quadratic function, given by

$$a_A\left(r_{t+1}^p\right) = \frac{2\theta}{(\beta - 2\theta) - 2\theta r_{t+1}^p} \quad (2.19)$$

is an increasing function, since $\frac{\partial a_A(r_{t+1}^p)}{\partial r_{t+1}^p} > 0$, for $(\beta - 2\theta), \theta > 0$ and $r_{t+1}^p \in \mathbb{R}$.

These particularities of the quadratic utility function did not represent any drawback for Markowitz's portfolio selection theory since quadratic utility functions are not necessary conditions for its applicability. As proved by Markowitz (1959) and Levy and Markowitz (1979), the expected utility function can be approximated by a function of mean and variance for various utility functions and historical distributions of returns.

Let us now address the second assumption regarding the use of expected returns and covariance matrix of asset returns as model inputs. These parameters are usually estimated from historical data or based on some theoretic assumptions about the data generating process. The estimation process is extremely important since the use of forecasts that do not reflect future expect returns and covariances of assets returns may degrade the out-of-sample performance of the mean-variance optimal solution (Chopra & Ziemba, 1993).

The most commonly used approach for estimating expected returns and covariances of asset returns is to calculate the sample first and second-order moments from historical data (Fabozzi, Focardi, & Kolm, 2006). The problem of using historical data to estimate expected returns is that the historical expected returns are generally poor forecasts of future performances. Additionally, Best and Grauer (1991a, 1991b) proved that the composition of mean-variance portfolios is very sensitive to asset expected returns meaning that subtle deviations from the asset expected returns result in substantially different optimal solutions. Concerning the composition of MV portfolios, Black and Litterman (1992) showed that using historical expected returns leads to optimal solutions with extreme negative weights on assets that have performed poorly and extreme positive weights on assets that have performed well in the particular historical period. Furthermore, when short sales are not allowed, these optimal portfolios present extremely low cardinality (are highly concentrated in a reduced number of assets) and, thus, are not well-diversified. Hence, the MV portfolios computed using sample expected returns are generally highly exposed to individual assets and to the extreme events that might negatively affect their performances, often revealing poor out-of-sample performance. Regarding the sample second-order moment, the errors in the variances and covariances are of less importance in terms of their influence on portfolio optimality (Chopra & Ziemba, 1993). Consequently, the GMV portfolio, which relies only on estimates of variances and covariances of the asset returns, is less vulnerable to estimation error comparatively to other MV portfolios. This outcome helps to explain the empirical evidences reported in several studies supporting the outperformance of the GMV portfolio, in out-of-sample data, comparatively to other MV portfolios (Chan et al., 1999; Jagannathan & Ma, 2003; Jorion, 1985). Furthermore, imposing nonnegativity constraints reduces the sampling error associated with the estimation of the covariance matrix, enhancing (even more) the out-of-sample performance of the GMV portfolio (Jagannathan & Ma, 2003). The absence of estimation errors is also used as an argument to support the outperformance of the Equally Weighted (EW) portfolio, which equally allocates the wealth by the assets and, hence, requires no input estimation (DeMiguel, Garlappi, & Uppal, 2009; Jobson & Korkie, 1981).

The reason why the sample first and second-order moments fail to forecast future expected returns and covariance matrix of asset returns, even when they are estimated from large sample periods, is related with the return generating process. For a random variable with normal (Gaussian) distribution, its probability density function is completely characterized by the first and second-order moments. The use of first and second order moments as model inputs by Markowitz, led to the misconception that the author assumed the normal distribution of asset returns. But, as Markowitz (2014, p. 348) states "(…) the formulas relating the expected return and variance of a portfolio to the expected values, variances and covariances of return of securities do not depend on the form of the probability distribution". The expected return and variance of a portfolio can be computed as long as the expected values, variances and covariances of assets returns exist and are finite. Hence, normal distribution of asset returns is not a necessary condition of the mean-variance optimization problem.

Although many financial concepts and models of Modern Finance rest upon the assumption that asset returns follow a normal distribution, it is widely acknowledge that asset returns are not normally distributed, and, thus, the expected returns and the covariances of asset returns are not accurate forecasts of the true first and second order moments of the joint assets return distribution. The first empirical evidences showing significant deviations of the distribution of asset returns from the normal distribution were presented by Mandelbrot (1963) and Fama (1965). Mandelbrot (1963) analysed the distribution of cotton price changes and found what he described as extraordinarily long tails and an erratic variation of the sample second moment, which did not seem to tend to any limit. Fama (1965) analysed the differences in the logarithm of prices of thirty stocks of the Dow Jones Industrial Average index and confirmed the presence of a higher peak, caused by too many values near the mean, and heavier tails than the normal distribution (characteristics of a leptokurtic distribution). Previous studies also found empirical evidences of leptokurtosis (Kendall & Hill, 1953; Moore, 1962), but these were generally neglected or treated incorrectly as outliers.

Studies on the empirical properties of return time series, independently developed by different researchers, identified a set of properties that are common across several assets, markets and time periods, designated as ‘stylized facts’. Besides presenting too many values around the mean (high peaks) and fat tails, time series of asset returns are characterized by asymmetric tails caused by large downward movements of asset prices and not equally large upward movements. Furthermore, the thickness of the tails decreases, i.e. the distribution approaches the normal distribution, as one increases the time scale used to compute asset returns, a fact known as the aggregational Gaussianity. Additionally, there is no correlation between successive returns which supports the assumption that returns are independent random variables with prices following a random walk. However, for the independence assumption to hold, non-linear functions of returns will also have to be uncorrelated. The analysis of simple non-linear functions of returns, such as absolute or squared returns, show a significant positive autocorrelation, which disputes the random walk argument. Mandelbrot was the first to identify this stylized fact, know as the volatility clustering, which he described as: “(…) large changes tend to be followed by large changes – of either sign – and small changes tend to be followed by small changes (…)” (Mandelbrot, 1963, p. 418). An extended list of the empirical stylized facts of financial time series is presented in the reviews performed by Chakraborti, Toke, Patriarca, and Abergel (2011) and Cont (2001).

In order to successfully fit the distribution of asset returns and explain most of its stylized empirical facts, a parametric model must have at least four parameters: i) a location parameter; ii) a scale parameter; iii) a parameter describing the decay of the tails; and iv) an asymmetry parameter allowing the left and right tails to have different behaviours (Cont, 2001). Since the acknowledgement of the inadequacy of the normal distribution to model the distribution of asset returns, alternative parametric models, presenting such parameters, have been proposed and empirically tested. An example is the stable Paretian or Stable-Lévy distribution, which includes the normal distribution as a special case. Several studies tested the stable Paretian distribution for different financial assets and found that the financial time series under analysis conform better to the stable Paretian distribution than to the normal distribution (Curto, Reis, & Esperança, 2003; Fama, 1965; Mandelbrot, 1963). The main limitation of the stable Paretian distribution results from the fact that, with the exception of three special cases (Gaussian, Cauchy and Lévy distributions), stable Paretian distribution does not have closed form expressions for its probability density function and cumulative distribution function, increasing the computational time and estimation errors when performing numerical approximations (Borak, Misiorek, & Weron, 2011). Another popular approach for explaining and fitting leptokurtic distributions is assuming that the distribution of asset returns corresponds to a mixture of several normal distributions with possibly the same mean but substantially different variances (Fama, 1965). Empirical evidences supporting the ability of mixture of normal distributions in explaining the return generating process were presented by several researchers (Ball & Torous, 1983; Boness, Chen, & Jatusipitak, 1974; Kon, 1984; Ritchey, 1990). Further approaches included Student’s *t* distribution (Blattberg & Gonedes, 1974) and hyperbolic distributions (Eberlein & Keller, 1995; Küchler, Neumann, Sørensen, & Streller, 1999), among others (see Borak et al., 2011, Peiró, 1994, and Rachev, 2003).

The assumption of a mixture of normal distributions implies that time series of returns are non-stationary processes. The dynamic economic environment and the shifts in the characteristics of the firm, like capital structure, financial and operational leverage and net financial income, affect asset prices and, consequently, the expected returns and covariances of assets returns change over time (Boness et al., 1974; Pagan & Schwert, 1990; Patell & Wolfson, 1981). Non-stationarity of asset returns is the main reason why any forecast of future performance based on historical data does not necessarily reflect future performance.

Although the non-stationarity of asset returns is well documented, a basic requirement of any statistical analysis of market data is the existence of some invariance on the return generating process, otherwise it would be worthless trying to forecast future asset returns (Cont, 2001). Resorting to non-stationarity arguments, different volatility models were proposed in order to capture both volatility clustering and leptokurtosis characteristics of financial time series. Autoregressive conditional heteroskedastic (ARCH) models and stochastic volatility (SV) models have received considerable interest in the field of financial econometrics (see Bollerslev, Chou, and Kroner, 1992, Gouriéroux, 2012, Satchell and Knight, 2011, and Taylor, 2008, for further readings).

In conclusion, quantitative portfolio managers, as well as decision makers, in general, have to be aware of the need to adjust the classical mean-variance framework in order to overcome its shortcomings, arising from the uncertainty in which the investment decisions are made. As previously outlined, the main sources of uncertainty are related to the fact that the decision maker has to decide what to do today, without knowing what the future will bring regarding the different investment alternatives. Uncertainty may also derive from the fact that models are flawed approximations of the true data generating process. The impact of data uncertainty on the optimal solution can only be reduced, and some worst-case model misspecification avoided, if uncertainty is acknowledged and explicitly included in the decision-making process.

Chapter 3

RO methodology and portfolio selection

In this chapter we begin by introducing the RO methodology. A literature review on robust portfolio optimization follows. Considering the vast literature associated with robust portfolio selection, we focus our literature review in the robust formulation of classical portfolio optimization problems, namely the MV, the GMV and the maximum Sharpe ratio portfolio optimization strategies. We also consider the robust formulation of multi-objective portfolio optimization problems.

3.1 The RO methodology

With its roots in the robust control theory, the RO methodology has evolved as a distinct research field and its importance has increased in a wide spectrum of domains, including finance, statistics and various areas of engineering (Bertsimas, Brown, & Caramanis, 2011). Bertsimas et al. (2011) presented a survey of the primary research in the area of RO, emphasizing the computational attractiveness, the modelling power and the broad applicability of this methodology.

RO mitigates the effects of uncertainty, leading to an optimal solution that is guaranteed to perform reasonably well for all the realizations of the uncertain input parameters considered in the uncertainty set (Hauser, Krishnamurthy, & Tütüncü, 2013). Different concepts of robust solution emerged in the literature since the decision maker may be interested in guaranteeing that the solution will perform reasonably well relatively to its feasibility, or its optimality, or both its feasibility and its optimality (Gabrel et al., 2014). When the uncertainty affects the feasibility of a solution, which happens when the uncertain parameter is included in the constraints of the optimization model, the RO methodology aims to ensure that the solution will be feasible for all, or at least most of, the realizations of the uncertain parameters. When the uncertain parameter is included in the objective function of the optimization model, affecting the optimality of a solution, the RO methodology aims to ensure that the feasible solution achieves exactly or approximately the optimum value of the objective function, regardless of the realizations of the uncertain parameters.

Thus, the RO methodology requires specifying a (deterministic) uncertainty set for the uncertain parameters based on some generally limited information about their values in order to find an optimal solution that remains feasible and also close to the optimum for any realization of the uncertain parameters within the pre-specified (deterministic) uncertainty set. The first to apply this idea was Soyster (1973), who presented a linear optimization model to compute a solution that was feasible for all the possible values of the uncertain parameter belonging to a convex set (Ben-Tal, El Ghaoui, & Nemirovski, 2009). While Soyster's approach achieved the desired outcome of immunizing the optimal solution against parameter uncertainty (guaranteeing its feasibility), it was widely considered too conservative for practical implementation (Bertsimas & Thiele, 2006). More than 20 years later, other authors addressed the overconservatism within Soyster's approach and robust optimization began to establish itself as a valuable methodology:

> "To the best of our knowledge, in the two subsequent decades [to Soyster's work] there were only two publications on the subject (…). The activity in the area was revived circa 1997, independently and essentially simultaneously, in the frameworks of both Integer Programming (Kouvellis and Yu) and Convex Programming (Ben-Tal and Nemirovski, El Ghaoui and Lebret). Since 2000, the RO area is witnessing a burst of research activity in both theory and applications, with numerous researchers involved worlwide" (Ben-Tal et al., 2009, p. xvii)

In order to characterize the uncertainty set, the structure as well as the scale must be defined. The structure of the uncertainty set refers to its geometry or shape, while the scale is related to the magnitude of the deviations of the uncertain parameters from their nominal values and can be thought of as its size (Gregory, Darby-Dowman, & Mitra, 2011).

Robust optimization overconservatism has been analyzed through the scale of the uncertainty set. The uncertainty set can include every possible realization or just the most likely values of the uncertain parameters, ranging from too conservative to less conservative approaches, respectively. When a less conservative approach is undertaken, the true value of the uncertain parameter may occur outside the bounds of the uncertainty set. In this case, the real optimality will never be achieved since the true realization of the uncertain parameter was not included in the computation of the optimal solution.

Within the linear optimization framework, Bertsimas and Sim (2004, p. 38) investigated ways to adjust the robustness of the proposed method against the level of conservatism of the solution, by introducing an integer parameter that controls the number of realizations of the uncertain parameter that are not considered in the construction of the interval uncertainty set, in terms of probabilistic bounds of constraint violations: "The parameter (…) controls the trade-off between the probability of violation and the effect to the objective function of the nominal problem, which is what we call the price of robustness". Following Bertsimas and Sim (2004), Gregory et al. (2011) suggest two measures for the cost of robustness that measure the difference between the optimal objective of the nominal problem and the objective function value of the nominal problem evaluated at the robust optimal solutions, in an absolute and relative way respectively. The authors define a first measure that assesses the deviation between the value of the non-robust and the robust solutions, and a second measure that gives the deviation as a ratio between the return of the asset and the largest expected return, which can be thought of as a cost-to-maximum potential reward ratio.

Other robustness measures, not directly related to the scale of the uncertainty set, were suggested. Roy (2010) proposed robustness measures based on two parameters, b e w, defined by the investor. Parameter w is a guaranteed value under which the investor refuses to go, regardless of the realization of the uncertain parameter, while parameter b serves to characterize a value boundary that the investor asks to exceed (or not to exceed, accordingly to the definition of the robustness measure) in the greatest possible number of realizations of the uncertain parameter. Kaläı, Lamboray, and Vanderpooten (2012, p. 727) suggested a robustness measure, the lexicographic α-robustness, which, according to the authors and considering discrete and finite uncertainty sets, mitigates some limitations of the absolute robustness approach: "(…) it takes into account the worst scenario (while reducing its weight compared with min–max criteria) as well as the concept of quasi-optimality by considering all the scenarios and introducing a tolerance threshold α".

A different way to surpass overconservatism was suggested by Lu (2006), who presented an alternative for defining the uncertainty sets. Lu (2006) stated that the traditional approach for defining uncertainty sets, where one uncertainty set was defined for each type of uncertain parameter, can lead to robust portfolios highly non-diversified and, thus, too conservative. Therefore, instead of defining independent uncertainty sets for first and second moment estimators for the uncertain parameters, the author proposed a 'joint' uncertainty set that can be constructed as a confidence region associated with a statistical procedure applied in the estimation of both model parameters.

Three different structures of uncertainty sets are prevalent in the initial contributions within the field of robust portfolio selection: interval uncertainty sets, based on confidence intervals defined for a nominal value of the uncertain parameter; ellipsoidal uncertainty sets, which allow the inclusion of second moment information about the distributions of the uncertain parameters; and polyhedral, defined as an intersection of half-spaces. A different way to model uncertainty is to consider discrete uncertainty sets. In this case, the uncertainty set corresponds to a set of scenarios, each representing a possible value of the uncertain parameters. This technique could lead to a considerable number of constraints, which could result into an intractable optimization problem. As a result, richer sets ranging from polytopes to more advanced conic-representable sets derived from statistical procedures are more frequently applied in the robust portfolio selection field (Fabozzi et al., 2007).

The formal relationship between the structure of the uncertainty set and the risk measure selected was independently analyzed by Bertsimas and Brown (2009) and Natarajan et al. (2009). Bertsimas and Brown (2009) consider that the uncertainty set is defined by the particular risk measure that the decision maker selects, while Natarajan et al. (2009) interpret uncertainty sets as being a starting point for arriving at risk measures in finance. At this point, it is necessary to introduce some important definitions and properties regarding risk measures. We will follow the notation presented by Bertsimas e Brown (2009). Consider Ψ a linear space of random variables of a given probability distribution. A function $\varphi\colon \Psi \rightarrow \mathbb{R}$, which satisfies, for all variables $A, Z \in \Psi$:

$$\text{Monotonicity: If } A \geq Z, \text{ then } \varphi(A) \leq \varphi(Z), \quad (3.1)$$

and

$$\text{Translation invariance: If } c \in \mathbb{R}, \text{ then } \varphi(A + c) = \varphi(A) - c, \tag{3.2}$$

is called a risk measure. The property of monotonicity guarantees that if a portfolio A never performs worse than another portfolio Z, then A cannot be riskier than Z. The property of translation invariance ensures that if we augment our investment by a guaranteed amount c, then the investment risk is reduced correspondingly by c.

A risk measure is called a coherent risk measure if, in addition to the monotonicity and translation invariance, it satisfies, for all variables $A, Z \in \Psi$:

$$\text{Positive homogeneity: If } \beta \geq 0, \text{ then } \varphi(\beta A) = \beta\varphi(A), \tag{3.3}$$

and

$$\text{Subadditivity: } \varphi(A + Z) \leq \varphi(A) + \varphi(Z). \tag{3.4}$$

The positive homogeneity property states that risk scales linearly with the size of a position, while the subadditivity axiom ensures that the diversification of positions can never increase the risk of the investment.

Additional properties are defined for comonotonic random variables. Let $A, Z \in \Psi$ and $v, v^* \in \Omega$, where Ω represents the sample space of the given probability space. Two random variables $A, Z \in \Psi$ that satisfy, for all $v, v^* \in \Omega$,

$$\big(A(v) - A(v^*)\big)\big(Z(v) - Z(v^*)\big) \geq 0 \tag{3.5}$$

are called comonotone. And a risk measure that satisfies, for all comonotone $A, Z \in \Psi$,

$$\varphi(A+Z) = \varphi(A) + \varphi(Z), \tag{3.6}$$

is called comonotonic. Finally, a risk measure φ that satisfies

$$\varphi(A) = \varphi(Z), \tag{3.7}$$

for all $A, Z \in \Psi$, such that A and Z have the same probability distribution, is called law invariant. The comonotonic axiom refers to positions that are correlated in a way that the risk of their sum corresponds to the sum of their individual risks. The law invariant property is related with the possibility of estimating the risk measure from historical data. A coherent risk measure that is also comonotonic and law invariant is called a distortion risk measure.

Within the framework of robust optimization for linear problems, Bertsimas and Brown (2009) showed that distortion risk measures lead to a tractable optimization problem with polyhedral uncertainty sets of a special structure. The authors have also showed that the class of all the distortion risk measures and the corresponding polyhedral sets are generated by a finite number of conditional value-at-risk measures, when discrete probability distributions are used. Natarajan et al. (2009) address ellipsoidal, moment cone and moment-generating function uncertainty sets. They found that ellipsoidal uncertainty sets map to the mean-standard deviation portfolio risk measure, or to the Worst-case Value-at-Risk (WVaR), when the exact distribution is unknown; moment cone uncertainty set is equivalent to the Worst-case Conditional Value-at-Risk (WCVaR) measure; moment-generating function uncertainty set is equivalent to a specific risk measure, corresponding to an upper bound of the Conditional Value-at-Risk (CVaR) measure, that, although being a weaker approximation to the Value-at-Risk (VaR) measure when compared to the CVaR, is convex and computationally tractable.

The selection of the structure of the uncertainty set is crucial since it influences the nature of the risk measure and the computational tractability of the resulting formulations. A drawback of the robust modeling framework with ellipsoidal uncertainty sets is that it increases the difficulty to solve the problem considered, i.e., the robust counterpart of a linear programming problem is a second order cone programming problem (SOCP), the robust counterpart of an SOCP becames an semidefinite programming problem (SDP), while the robust counterpart of an SDP is non-deterministic polynomial-time hard (NP-hard) to solve (Bertsimas, Pachamanova, & Sim, 2004). Furthermore, ellipsoidal uncertainty sets and other uncertainty sets defined by general norms that include second moment information about the distribution of the uncertain parameter can be inappropriate or too conservative when the uncertainty distributions are asymmetric (Fabozzi et al., 2007).

Most of the studies in the robust portfolio optimization field aim to optimize the worst-case realization of the objective function, where the optimal solution is computed assuming the worst possible realization within the uncertainty set for the uncertain parameters. This constitutes the absolute robust optimization approach (Hauser et al., 2013). The notion of absolute robustness can be explained resorting to a generic portfolio optimization model, which includes the uncertain parameters in the objective function. Let $w_t \in \mathbb{R}^N$ be the weight combination vector defining the investor's portfolio, X the set of feasible solutions, s the vector defining the input parameters (scenario) and U the uncertainty set, i.e. the set of possible scenarios/realizations for the vector of realized parameters s. Then it is possible to define the following portfolio (non-robust) optimization model for a given scenario s:

$$\max_{w_t \in X} f(w_t, s), \tag{3.8}$$

where f represents a generic (profit) objective function that depends on w_t and s. Then, the absolute robust portfolio (maxmin solution) corresponds to the weight combination vector $w_t \in \mathbb{R}^N$ that solves the optimization model defined by:

$$\max_{w_t \in X} \min_{s \in U} f(w_t, s). \tag{3.9}$$

The absolute robust approach might not be adequate for all investors. One example is when decision makers' performances are judged relative to their peers' performance. They might prefer to make decisions that avoid falling severely behind their competitors under all scenarios, rather than protecting themselves against the worst-case scenarios. Following this idea, Kouvelis and Yu (1997) explored the concept of relative robustness by analyzing the worst-case in a relative manner, considering the best possible solution under each scenario.

Let us now introduce the notion of relative robustness as presented by Cornuejols and Tütüncü (2006). For a given scenario s, let $f(w_t^{s^*}, s)$ and $w_t^{s^*}$ denote, respectively, the optimal objective function value and the vector of optimal weights of problem (3.8). If w_t is chosen as the decision vector when s is the vector of realized parameter values, then the regret associated with having chosen w_t rather than $w_t^{s^*}$ is defined as follows:

$$P^s(w_t) = f(w_t^{s^*}, s) - f(w_t, s) \tag{3.10}$$

Since regret cannot be measured before the realization of vector s, it is possible to consider the maximum regret function instead, which provides an upper bound on the true regret:

$$P(w_t) = \max_{s \in U} P^s(w_t) = \max_{s \in U} \left(f(w_t^{s^*}, s) - f(w_t, s) \right). \tag{3.11}$$

Thus, a relative robust solution w_t corresponds to the weight combination vector that minimizes the maximum regret function and therefore solves the relative robust optimization model:

$$\min_{w_t \in X} \max_{s \in U} (f(w_t^{s^*}, s) - f(w_t, s)). \tag{3.12}$$

Hence, the relative robust optimization approach leads to three-level optimization problems, in contrast with the absolute robust optimization framework which leads to two-level optimization problems (Cornuejols & Tütüncü, 2006; Hauser et al., 2013). Considering the relative robust optimization approach, the first optimization level corresponds to the computation of w_t^{s*}, which is the optimal portfolio for scenario s, for all scenarios $s \in U$. The second optimization level corresponds to the inner maximization problem in (3.12) and allows the computation of the maximum regret for each $s \in U$, providing an upper bound on the true regret for the investor. The third optimization level corresponds to the outer minimization problem in (3.12), which gives the optimal solution that minimizes the maximum regret for all $s \in U$.

Kouvelis and Yu (1997) presented a collection of discrete robust optimization problems based on the relative robustness approach, in the field of operations and production management, while Hauser et al. (2013) proposed a relative robust optimization modelling methodology on the context of continuous portfolio optimization problems, namely under ellipsoidal uncertainty. Both of these works showed that the relative robust formulation resulting from many optimization problems can be reduced to one or a series of single-level deterministic optimization problems that can be solved using deterministic algorithms. These contributions were crucial to the development of specialized algorithms for relative robust optimization applications.

3.2 Robust formulation of classical portfolio optimization models

3.2.1 Robust formulation of the MV portfolio problem

Motivated by Markowitz's mean-variance portfolio selection problem, Halldórsson and Tütüncü (2003) developed a polynomial-time interior-point algorithm to solve nonlinear saddle-point problems and suggested a robust formulation for model (2.4). Considering the uncertainty in the vector of expected returns μ_t and in the covariance matrix of asset returns Σ_t, the authors presented the following robust formulation for the minimization of the variance subject to a lower limit on the expected return:

$$\begin{aligned} \min_{w_t \in \mathbb{R}^N} \quad & \max_{\Sigma_t \in U_\Sigma} w_t{}' \Sigma_t w_t \\ \text{subject to} \quad & \min_{\mu_t \in U_\mu} \mu_t' w_t \geq \mu_0^p, \\ & w_t \in \boldsymbol{X} \end{aligned} \qquad (3.13)$$

where the uncertain parameters μ_t and Σ_t are assumed to lie in the uncertainty sets U_μ and U_Σ, respectively, and X denotes the set of feasible portfolios, which may carry information on short-sale restrictions, sector distribution requirements, etc. These uncertainty sets, bounded respectively by μ^L and μ^H, and Σ^L and Σ^H, are defined as:

$$U_\mu = \{\mu_t : \mu^L \leq \mu_t \leq \mu^H\} \tag{3.14}$$

and

$$U_\Sigma = \{\Sigma_t : \Sigma^L \leq \Sigma_t \leq \Sigma^H, \Sigma_t \succcurlyeq 0\}, \tag{3.15}$$

where $\Sigma_t \succcurlyeq 0$ indicates that Σ_t is a symmetric positive semidefinite matrix. With no empirical examples, the authors suggested different ways of defining the extreme values of the interval uncertainty sets:

> "An extremely cautious modeler may want to use the historical lows and highs of certain input parameters (…). One may generate different estimates using different scenarios on the general economy and then combine the resulting estimates. (…) One may choose a confidence level and then generate estimates of the covariance and return parameters in the form of prediction intervals" (Halldórsson & Tütüncü, 2003, p. 588)

Based on Halldórsson and Tütüncü's framework, Tütüncü and Koenig (2004) presented empirical examples and defined the bounds μ^L, μ^H, Σ^L and Σ^H, using percentiles of bootstrapped samples of historical data as well as the percentiles of moving averages. The authors state that, under the assumptions of nonnegativity weights, allocation of all the available wealth and positive semidefiniteness of Σ_t, the absolute robust asset allocation problem can be reduced to a simple quadratic programming problem: expected returns are realized at their lowest possible values (μ^L) and the covariances are realized at their highest possible values (Σ^H). Also, when these assumptions do not hold, the absolute robust asset allocation problem belongs to a class of nonlinear saddle-point problems that involve semidefiniteness constraints on matrix variables that can be solved using a polynomial-time interior-point algorithm, as the one suggested by Halldórsson and Tütüncü (2003). These results also hold for the robust formulation of the maximization of the risk-adjusted expected return, problem (2.7), also analyzed by Tütüncü and Koenig (2004) and given by:

$$\begin{array}{ll} \min\limits_{w_t \in \mathbb{R}^N} & \max\limits_{\mu_t \in U_\mu, \Sigma_t \in U_\Sigma} \mu_t' w_t - \lambda w_t{}' \Sigma_t w_t \\ \text{subject to} & w_t \in X \end{array}, \qquad (3.16)$$

where the set of feasible solutions is defined as $X = \left\{ w_t \in \mathbb{R}^N : \sum_{n=1}^{N} w_{t,n} = 1, w_{t,n} \geq 0, n = 1, \dots, N \right\}$.

Goldfarb and Iyengar (2003) presented a robust approach for problem (2.4), based on a factor model for the estimation of expected returns and admitting ellipsoidal uncertainty sets. Let $x \sim N(\mu, \Sigma)$ represent a random variable with multivariate normal distribution, where μ and Σ correspond to the vector of expected values and the covariance matrix, respectively. Goldfarb and Iyengar (2003) define the vector of random asset returns, $r \in \mathbb{R}^N$, as

$$r = \mu + V'\pi + \varepsilon, \qquad (3.17)$$

where $r \sim N(\mu, V'FV + D)$, $\mu \in \mathbb{R}^N$ is the vector of mean returns, $V \in \mathbb{R}^{M \times N}$ is the factor loading matrix, $\pi \sim N(0, F)$ is the vector of random returns of the $m(< n)$ factors that drive the market, F is the covariance matrix of the factor returns, $\varepsilon \sim N(0, D)$ is the vector of residual returns and D is the covariance diagonal matrix of the residual returns.

With the exception of the covariance matrix F of the factor returns, the remaining parameters are subject to estimation error. Hence, the uncertain parameters D, V and μ lie within the corresponding uncertainty sets U_D, U_V and U_μ, which the authors define as follows. Consider that individual diagonal elements of the covariance matrix D are assumed to lie in an interval $[\underline{d}_n, \overline{d}_n], n = 1, \dots, N,$ where $\underline{d}_n$ and $\overline{d}_n$ represent the minimum and maximum elements of the main diagonal, respectively. Let V_0 be the least squares estimate of V, Z_n represent the n-th column of $Z \in \mathbb{R}^{M \times N}$ and $\|x\|_G = \sqrt{x'Gx}$

denote the elliptic norm of vector x with respect to a symmetric, positive definite matrix G. The authors defined U_D, U_V and U_μ as:

$$U_D = \{D: D = diag(d), d_n \in [\underline{d}_n, \overline{d}_n], n = 1, \dots, N\}; \tag{3.18}$$

$$U_V = \{V: V = V_0 + Z, \|Z_n\|_G \leq \rho_n, n = 1, \dots, N\}; \tag{3.19}$$

$$U_\mu = \{\mu: \mu = \mu_0 + \xi, |\xi_n| \leq \varrho_n, n = 1, \dots, N\}; \tag{3.20}$$

where the parameters ρ_n and ϱ_n define the confidence level of the uncertainty sets U_V and U_μ.

Considering the previous uncertainty sets and under the hypothesis of normality of the portfolio return, $r^p \sim N(\mu' w, w'(V'FV + D)w)$, Goldfarb and Iyengar (2003) proved that model (2.4) can be formulated as the following robust minimum variance portfolio selection problem:

$$\begin{array}{ll} \min\limits_{w \in \mathbb{R}^N} & \max\limits_{V \in U_V}(w'V'FVw) + \max\limits_{D \in U_D}(w'Dw) \\ \text{subject to} & \min\limits_{\mu \in U_\mu} \mu' w \geq \mu_L^p \\ & \mathbf{1}'w = 1 \end{array} . \tag{3.21}$$

Furthermore, and based on previous research of El Ghaoui and Lebret (1997) and Ben-Tal and Nemirovski (1998), the authors proved that model (3.21) is a SOCP and it can be efficiently solved by interior point algorithms.

Thus, the objective function of the absolute robust minimum variance portfolio selection problem (3.21) is to minimize the worst case variance of the portfolio subject to the constraint that the worst case expected return on the portfolio is at least μ_L^p. Goldfarb and Iyengar also address the maximization of the portfolio's expected return subject to an upper limit on the variance (v_U^p), problem (2.6), and presented its robust formulation as a dual problem of (3.21):

$$
\begin{array}{ll}
\min\limits_{w\in\mathbb{R}^N} & \min\limits_{\mu\in U_\mu} \mu' w \\
\text{subject to} & \max\limits_{V\in U_V}(w'V'FVw) + \max\limits_{D\in U_D}(w'Dw) \leq \upsilon_U^p \\
& \mathbf{1}'w = 1 \\
& w \geq \mathbf{0}
\end{array}
\tag{3.22}
$$

where $\mathbf{0}$ represents the zero vector.

Initial contributions in the field of robust portfolio optimization focused on box (interval) and ellipsoidal uncertainty sets and on how to formulate and solve robust counterparts of classical portfolio selection problems. More recent contributions analyze the characteristics of robust portfolios comparatively to the classical mean-variance portfolio (Kim, Kim, & Fabozzi, 2013a; Kim, Kim, Kim, & Fabozzi, 2014a) and consider different structures of uncertainty sets (Lu, 2006, 2011b). Based on Goldfarb and Iyengar's definition of the uncertainty sets, previously presented, but considering the expected return of the assets as the only uncertain parameters, Kim et al. (2013a) presented an absolute robust formulation for the dual problem of model (2.7):

$$
\begin{array}{ll}
\min\limits_{w\in\mathbb{R}^N} & \max\limits_{\mu\in U_{\hat{\mu}}} w'\Sigma w - \lambda\mu' w \\
\text{subject to} & \mathbf{1}'w = 1
\end{array}
,
\tag{3.23}
$$

where μ is the estimate of the expected returns and $U_{\hat{\mu}}$ represents the uncertainty set. The authors considered two different structures for the uncertainty set, box and ellipsoidal, which they defined, respectively, as:

$$
U_{\hat{\mu}} = \{\mu : |\mu_n - \hat{\mu}_n| \leq \rho_n, n = 1, \ldots, N\}
\tag{3.24}
$$

and

$$
U_{\hat{\mu}} = \{\mu : (\mu - \hat{\mu})'\Sigma^{-1}(\mu - \hat{\mu}) \leq \varrho^2\},
\tag{3.25}
$$

where $\rho_n, n = 1, \dots, N$, and ϱ are constants that define the size of the uncertainty sets.

Allowing short-sales and for specific values of the risk-averse coefficient λ, the authors compared the robust portfolios computed with box and ellipsoidal uncertainty to the MV and GMV portfolios. They analyzed the level of diversification of portfolios by comparing the number of assets with non-zero weights contained in the portfolios, the exposure of the portfolios to individual stocks by comparing the absolute value of the assets' weight, the portfolio's correlation with the market portfolio by comparing portfolios betas, and the correlation between weight and beta of the assets composing the portfolios. Their results revealed that the robust portfolio is less diversified than the non-robust portfolio since it contains fewer stocks with non-zero weights, shows lower exposure to individual stocks since it has lower maximum absolute values of asset weights, it is more exposed to the systematic risk since it has higher portfolio beta and it shows low correlation between weight and beta of the stocks composing the portfolio. The authors have also found that the robust solution based on box uncertainty set has a lower beta and fewer assets with non-zero weights than the robust portfolio based on ellipsoidal uncertainty set, revealing itself as a portfolio less sensitive to systematic risk but less diversified and, thus, more exposed to the individual risk of each asset. On the other hand, the robust portfolio based on an ellipsoidal uncertainty set has higher positive portfolio beta and stronger negative correlation between weight and beta of the stocks composing the portfolio.

Based on the same optimization framework, Kim, Kim, Ahn, and Fabozzi (2013b) showed that, comparatively to the mean-variance portfolio, absolute robust portfolios consistently show stronger correlation with the three fundamental factors used in the Fama-French factor model and that the correlation increases as the robustness of the portfolio is increased. Furthermore, the authors found that the absolute robust portfolio based on ellipsoidal uncertainty set showed stronger correlation with the three fundamental Fama-French factors comparatively to the robust portfolio based on box uncertainty set. The three Fama-French factors are: the market risk premium corresponding to the difference between the expected return of the market portfolio and the risk-free rate; the small minus big (SMB) corresponding to the difference between the return on a portfolio of small stocks and the return on a portfolio of large stocks; and the high minus low (HML) corresponding to the difference between the return on a portfolio of high-book-to-market stocks and the return on a portfolio of low-book-to-market stocks (Fama & French, 1996).

Under ellipsoidal uncertainty set assumed for the expected return of assets, as previously defined, Kim, Kim, and Fabozzi (2014b) showed that, as the robustness of a portfolio increases, its optimal weights approach the portfolio such that variance is maximally explained by the three factors. Based on this result, Kim et al. (2014a) derived new absolute robust models aiming to obtain optimal robust portfolios that match a target factor exposure by adding constraints to the original robust formulation, allowing the adjustment of the robustness of the optimal solution and keeping the factor dependency to a specified level.

As previously mentioned, Lu (2006) presented a different way to construct the uncertainty set. The author constructed a 'joint' uncertainty set as a confidence region associated with a statistical procedure applied to simultaneously estimate both model uncertain parameters (see Lu, 2006, for further details). Following Goldfarb and Iyengar (2003) approach where asset returns are defined by a factor model, Lu (2006) proved that the robust maximum risk-adjusted return problem, i.e., the robust formulation of model (2.7), under a 'joint' ellipsoidal uncertainty set can be reformulated and solved as a cone programming problem. The author claims that this 'joint' ellipsoidal uncertainty set allows to combat the drawbacks, concerning overconservatism and lower diversification of the absolute robust solutions, associated to the separate uncertainty sets defined by Goldfarb and Iyengar (2003). Results showed that the absolute robust portfolio constructed under the 'joint' ellipsoidal uncertainty set outperforms Goldfarb and Iyengar's robust portfolio in terms of wealth growth rate and transaction cost and it also shows a higher level of diversification (Lu, 2006, 2011a, 2011b).

Based on the same optimization problem and under joint uncertainty set, Kim, Fabozzi, Cheridito, and Fox (2014c) proposed an absolute robust mean–variance approach that controls portfolio third and fourth moments, without imposing higher moment terms within the portfolio selection model. The absolute robust counterpart of Markowitz's maximum risk-adjusted return problem was defined by the authors as:

$$\begin{array}{ll} \max\limits_{z \in \mathbb{R}} & z \\ \text{subject to} & z \leq w'\mu_i - \lambda w'\Sigma_i w \,, \quad i = 1, \dots, I \,, \\ & w \in \boldsymbol{X} \end{array} \tag{3.26}$$

where z represents the minimum value of the portfolio's risk-adjusted return, i identifies the i-th independent and identically distributed (i.i.d.) sample of historical assets returns, μ_i and Σ_i are the vector of average returns and the covariance matrix of the i-th sample, respectively, I corresponds to the number of realizations of the uncertain parameter included in the uncertainty set U, and λ represents the weight given to the portfolio variance. The authors defined the 'joint' uncertainty set U as follows. Consider r, the vector of asset returns, and $r_i \in \mathbb{R}^n$ the i-th i.i.d. sample of r, with $i = 1, \dots, I$, and let μ_i and Σ_i represent the sample mean and the sample covariance matrix of the i-th sample, respectively. Also, let J be the number of observations in the i-th sample. The 'joint' uncertainty set U is defined as:

$$U = \{(\mu_i, \Sigma_i), i = 1, \dots, I\}, \tag{3.27}$$

where

$$\mu_i = \frac{1}{J}\sum_{j=1}^{J} r_{ij}\,, i = 1, \dots, I; j = 1, \dots, J \tag{3.28}$$

and

$$\Sigma_i = \frac{1}{J-1}\sum_{j=1}^{J}\left(r_{ij} - \mu_i\right)\left(r_{ij} - \mu_i\right)'\,, i = 1, \dots, I; j = 1, \dots, J. \tag{3.29}$$

Under the previous joint uncertainty set, Kim et al. (2014c) showed that the absolute robust portfolios favor skewness and penalize kurtosis, characteristics preferred by typical investors, since smaller kurtosis are less associated with extreme events and more positively skewed return distributions are less likely to have an extreme left-tail event than one in the right-tail.

3.2.2 Robust formulation of the maximum Sharpe ratio portfolio problem

An alternative approach to the construction of portfolios, which is also based on the optimization of the first and second order moments, consists of maximizing the Sharpe ratio (defined in expression (2.8) and represented by $S(w_t)$). The Sharpe ratio maximization problem can be formulated as:

$$\begin{aligned} \max_{w_t \in \mathbb{R}^N} \quad & \frac{\mu_t' w_t - r_t^f}{\sqrt{w_t' \Sigma_t w_t}}. \\ \text{subject to} \quad & \mathbf{1}' w_t = 1 \end{aligned} \tag{3.30}$$

In order to make the optimization problem (3.30) computationally more tractable, given that it has a nonlinear and nonconcave objective function, Goldfarb and Iyengar (2003) presented an elegant argument that allowed them to transform the previous problem into a convex quadratic problem. Since the components of the weight vector w_t add up to 1, it follows that:

$$\frac{\mu_t' w_t - r_t^f}{\sqrt{w_t' \Sigma_t w_t}} = \frac{\left(\mu_t - r_t^f \mathbf{1}\right)' w_t}{\sqrt{w_t' \Sigma_t w_t}}. \tag{3.31}$$

Function (3.31) is a homogeneous function of the portfolio represented by the weight vector w_t. According to Goldfarb and Iyengar (2003), this implies that the normalization condition $\mathbf{1}' w_t = 1$ can be dropped and the constraint $\min_{\mu_t \in U_\mu} \left(\mu_t - r_t^f \mathbf{1}\right)' w_t = 1$ added to the robust formulation of the Sharpe ratio optimization problem, without any loss of generality. Let $\mathbf{0}$ be the vector of zeros. Maximizing the Sharpe ratio is equivalent to minimizing the worst-case variance of the portfolio, which is a convex quadratic function, and, considering the notation presented in section 3.2.1, concerning Goldfarb and Iyengar's robust formulations of the mean-variance optimization models, the absolute robust maximization Sharpe ratio problem is formulated as:

$$\begin{array}{ll} \min_{w_t \in \mathbb{R}^N} & \max_{V \in U_V} w_t{}'V'FVw_t + w_t{}'\overline{D}w_t \\ \text{subject to} & \left(\mu_0 - \theta - r_t^f \mathbf{1}\right)' w_t \geq 1 \\ & w_t \geq \mathbf{0} \end{array} \tag{3.32}$$

where $\overline{D} = diag(\overline{d})$ and $w_t{}'Dw_t \leq w_t{}'\overline{D}w_t, \forall d_n \in \left[\underline{d}_n, \overline{d}_n\right], n = 1, \dots, N.$ Furthermore, the constraint $\min_{\mu_t \in U_\mu} \left(\mu_t - r_t^f \mathbf{1}\right)' w_t = 1$ was relaxed by recognizing that the relaxed constraint will always be tight at an optimal solution.

Under the same general assumptions and admitting the uncertainty sets defined by Halldórsson and Tütüncü (2003), Tütüncü and Koenig (2004) formulated the following (equivalent) robust version of the Sharpe ratio optimization problem:

$$\begin{array}{ll} \min_{w_t \in \mathbb{R}^N} & \max_{k \in U_\Sigma} w_t' \Sigma_t w_t \\ \text{subject to} & \min_{\mu_t \in U_\mu} \left(\mu_t - r_t^f \mathbf{1}\right)' w_t \geq 1 \\ & (w_t, k) \in X^+ \end{array} \tag{3.33}$$

where X^+ is a cone that lives in a one higher-dimensional space than X, and $w_t \in X$ implies that $\mathbf{1}'w_t = 1$. Furthermore, X^+ is defined by:

$$X^+ = \left\{w_t \in \mathbb{R}^N, k \in \mathbb{R}: k > 0, \frac{w_t}{k} \in X\right\} \cup \{(\mathbf{0}, 0)\}, \tag{3.34}$$

where the vector $(\mathbf{0}, 0)$ was added in order to achieve a closed set and k is a scaling parameter that guarantees the normalization of the optimal solution, defined by $k = \sum w_{t,n}, n = 1, \dots, N;\ \forall w_t \in X.$

The robust counterparts of problems (3.32) and (3.33) can be reduce to SOCP problems (Goldfarb & Iyengar, 2003; Tütüncü & Koenig, 2004).

A different approach was presented by Deng, Dulaney, McCann, and Wang (2013), who defined the uncertainty set directly from the Sharpe ratio estimators, motivated by the fact that these estimators are approximately normally distributed even when asset returns are not. The authors proposed an absolute robust portfolio optimization model based on Value-at-Risk adjusted Sharpe ratio ($\hat{S}_{VaR}$), i.e., a model that selects the portfolio with the largest worse-case-scenario Sharpe ratio within a given confidence interval. The $\hat{S}_{VaR}$ optimization problem is defined by

$$\begin{array}{ll} \max\limits_{w_t \in \mathbb{R}^N} & \min\limits_{S(w_t) \in U_S} S(w_t) \\ \text{subject to} & \mathbf{1}' w_t = 1 \\ & w_{max} \leq w_{t,n} \leq w_{min} \end{array}, \tag{3.35}$$

where U_S is an ellipsoidal uncertainty set containing the unknown true value of $S(w_t)$ and w_{max} and w_{min} are the upper and the lower bounds, respectively, of the assets' weights. The ellipsoidal uncertainty set, U_S, is defined as:

$$U_S = \left\{ S(w_t) \colon \left(S(w_t) - \hat{S}\right) \left(\hat{\sigma}^2(\hat{S})\right)^{-1} \left(S(w_t) - \hat{S}\right) \leq \varrho^2 \right\}, \tag{3.36}$$

where $\hat{S}$ represents the Sharpe ratio estimator, $\hat{\sigma}^2(\hat{S})$ denotes the variance of the Sharpe ratio estimator and ϱ defines the scale of the uncertainty set and corresponds to the confidence level of the VaR estimate. The inner-minimization problem in (3.35) computes the $\hat{S}_{VaR}$, i.e. the minimum possible value of $S(w_t)$ for each given w_t in the uncertainty set U_S, identifying the portfolio with the largest worst-case Sharpe ratio, and is given by:

$$\hat{S}_{VaR}(\tau) = \min_{S(w_t) \in U_S} S(w_t) = \hat{S} - \varrho \hat{\sigma}(\hat{S}). \tag{3.37}$$

The empirical examples carried out by Deng et al. (2013), suggest that the $\hat{S}_{VaR}$ robust portfolio outperforms both the optimal solutions of the classical Sharpe ratio and the probabilistic Sharpe ratio maximization problems, by mitigating realized volatility without sacrificing realized returns.

3.2.3 Robust formulation of GMV portfolio problem

Based on the optimization of the second order moment, Xidonas, Hassapis, Soulis, and Samitas (2017a) presented a relative robust formulation of the GMV problem (the non-robust formulation of the GMV problem is presented in (2.5)). The authors suggest a discrete uncertainty set, U_{Σ}, corresponding to a finite set of covariance matrix scenarios $\Sigma^s, s = 1, \dots, S$. Besides the completeness and non-negativity constraints, Xidonas et al. (2017a) limit the number of assets in the portfolio to be between a lower (L) and an upper bound (H), and define a maximum (w_{max}) and minimum (w_{min}) weight for each asset (applied only if the asset is included in the portfolio). The authors define regret, which they designate by variance sacrifice (ς), as the relative worst variance increase in the robust choice of weights, i.e. the portion of variance that is exchanged for robustness, and formulate the relative robust minimum variance portfolio problem as follows:

$$\begin{array}{ll} \min\limits_{w_t \in \mathbb{R}^N, t \in \mathbb{R}_0^+, \Sigma^s \in U_{\Sigma}} & \varsigma \\ \text{subject to} & w_t{}' \Sigma^s w_t \leq (1 + \varsigma) \nu^{s*} \\ & \mathbf{1}' w_t = 1 \\ & w_t \geq \mathbf{0} \\ & L \leq \sum_{n=1}^{N} q_n \leq H \\ & w_{t,n} - w_{max} q_n \leq 0 \\ & w_{t,n} - w_{min} q_n \geq 0 \end{array} , \qquad (3.38)$$

where $q_n, n = 1, \dots, N$, are binary decision variables and ν^{s*} is the variance of the optimal GMV portfolio under scenario s. In the empirical application, Xidonas et al. (2017a) considered different lower bounds (between 5 and 30) and upper bounds (between 20 and 60) for the number of assets, a minimum share of 0.1% and a maximum share of 20%, and an uncertainty set with three different scenarios: a 3-month scenario for the short-term view, a 6-month scenario for the mid-term view and a 12-month scenario for the long-term view. The results suggest that the proposed relative robust minimum variance optimization produces conservative portfolios, since they include fewer and less risky assets. Moreover, although the relative robust portfolios do not show superior performance, they are in most cases better than the solution computed under the worst performing scenario.

A different definition for regret was presented by Simões, McDonald, Williams, Fenn, and Hauser (2018). Based also on the second order moment optimization and assuming uncertainty of the covariance matrix of asset's returns, the authors define regret as the volatility increase one could be getting against the least volatile (best performing) benchmark. Let $B = \{b_1, \dots, b_M\}$ be the set of M different benchmarks under consideration. Simões et al. (2018) suggested the following relative robust portfolio optimization model:

$$\begin{array}{ll} \min\limits_{w_t \in \mathbb{R}^N} & \max\limits_{\Sigma^s \in U_\Sigma} \left(\sqrt{w_t{}' \Sigma^s w_t} - \min\limits_{b \in B} \sqrt{b' \Sigma^s \mathrm{b}} \right) \\ \text{subject to} & w_t \in X \end{array} \qquad (3.39)$$

where the uncertainty set, U_Σ, is defined as a finite set of scenarios, where each scenario represents a possible realization for the sample covariance matrix.

Furthermore, the authors consider the absolute robust GMV portfolio:

$$\begin{array}{ll} \min\limits_{w_t \in \mathbb{R}^N} & \max\limits_{\Sigma^s \in U_\Sigma} \sqrt{w_t{}' \Sigma^s w_t} \\ \text{subject to} & w_t \in X \end{array}, \qquad (3.40)$$

which is used as a benchmark in order to assess and compare the performance of the robust portfolios. The results show that the relative robust solution is more diversified, less volatile and yields a structurally different portfolio, comparatively to the GMV and the absolute robust GMV portfolios. Simões et al. (2018) present further approaches using the regret measure as a constraint or, in a proportional manner, defining excess volatility not as the difference in volatilities, but instead as the proportion of benchmark volatility that is surpassed.

3.2.4 Robust formulation of multi-objective portfolio problems

The robust methodology has been extended to multi-objective problems only very recently. The analysis of the investment decision problem as an optimization problem that seeks to maximize the expected return of the portfolio while minimizing its risk, i.e. as an optimization problem with multiple conflicting objectives, highlights the multi-objective nature of the portfolio selection problem.

The concept of optimal solution is different in the multi-objective optimization context. Here, it is important to introduce the concept of dominant solution. In the mean-variance based approach, a dominant portfolio is a feasible solution such that no other feasible solution presents, simultaneously, higher return and lower risk. In a multi-objective optimization problem, one needs to compute the Pareto set, i.e. the set of compromise (or non-dominated) solutions that define the best trade-off between the competing objectives, and identify the most desirable solution according to the decision maker's preferences. This corresponds to the optimal solution of the multi-objective problem and can be achieved by applying different multi-criteria decision aiding (MCDA) techniques, like the $\varepsilon-$constraint, scalarization and the goal programming methods, among others. For a detailed coverage of applications of MCDA in financial decision making see Bana e Costa and Soares (2001); Zopounidis and Doumpos (2002); Zopounidis, Galariotis, Doumpos, Sarri, and Andriosopoulos (2015).

Admitting uncertainty in the expected assets' returns and an ellipsoidal uncertainty set, Hasuike and Katagiri (2013) presented a bi-objective model that seeks to simultaneously maximize the portfolio expected return and maximize the scale of the uncertainty set. Let ϱ be the robustness parameter setting the size of the uncertainty set U_μ defined by:

$$U_\mu = \{\mu_t : \mu_t = \mu + \Phi z, \|z\| \leq \varrho\}, \tag{3.41}$$

where μ is the vector of mean returns and Φ is a regular matrix representing the relation between any two assets. Let $\underline{\mu_L^p}$ and μ_L^p be the worst target value and the standard target value for the portfolio expected return, respectively, which are initially defined by the investor. Under the absolute robust approach, the authors presented the following robust optimization model:

$$
\begin{array}{ll}
\max\limits_{\mu_t \in U_\mu} & \varrho \\
\max\limits_{\mu_t \in U_\mu} & \underline{\mu_L^p} \\
\text{subject to} & \mu_t' w_t - \varrho\sqrt{w_t' \Sigma_t w_t} \geq \underline{\mu_L^p}. \\
& \mu_t' w_t \geq \mu_L^p \\
& \mathbf{1}' w_t = 1 \\
& w_t \geq \mathbf{0}
\end{array}
\tag{3.42}
$$

The proposed bi-objective model is transformed into a deterministic equivalent problem by introducing fuzzy goals and applying an interactive fuzzy satisficing method (see Duan and Stahlecker, 2011, and Kato and Sakawa, 2011, for further readings about the interactive fuzzy method). The problem is then solved considering the worst-case realization of the uncertain parameter and applying deterministic algorithms. The numerical application showed that the proposed absolute robust model tends to select well-balanced optimal portfolios between both robustness and maximization of the total return with regard to investor's satisfaction levels. Furthermore, the absolute robust portfolios outperform, in terms of robustness and total return, other widely applied portfolio models used by the authors as benchmarks.

A different approach is proposed by Fliege and Werner (2014). The authors analyze convex parametric multi-objective optimization problems under data uncertainty, admitting a convex structure for the uncertainty set, and introduce for the first time a robust counterpart to the multi-objective programming problem in the style of Ben-Tal and Nemirovski (1998). Their empirical application is based on the bi-objective mean-variance problem which is solved applying scalarization methods. Considering uncertainty in the expected return vector and the covariance matrix of asset returns, the authors define a joint ellipsoidal uncertainty set U, as an ellipsoid around a nominal point (μ, Σ) of size ϱ, in the following way:

$$U = \{(\mu_t, \Sigma_t) \colon \|\mu_t - \mu\| + c\|\Sigma_t - \Sigma\| \leq \varrho\}. \tag{3.43}$$

Then, the absolute robust bi-objective optimization problem is given by:

$$\begin{array}{ll} \min\limits_{w_t \in \mathbb{R}^N} & \max\limits_{(\mu_t, \Sigma_t) \in U} w_t{}' \Sigma_t w_t \\ \min\limits_{w_t \in \mathbb{R}^N} & \max\limits_{(\mu_t, \Sigma_t) \in U} \mu_t{}' w_t \\ \text{subject to} & w_t \in X \end{array} \quad . \tag{3.44}$$

The numerical solution of the robust bi-objective problem boils down to the solution of a family of ordinary scalar robust problems and is computed applying deterministic algorithms, under the worst-case approach. In their empirical example, the authors investigate the relationship between the robust Pareto frontier and the original Pareto frontier and show that the robust frontier lies between the original nominal EF and some corresponding easy-to-determine upper bound.

Xidonas, Mavrotas, Hassapis, and Zopounidis (2017b) presented an relative robust bi-objective optimization problem, that that seeks to simultaneously minimize the mean absolute deviation (f_1) and maximize the expected portfolio return (f_2). The authors applied the weighting method in order to calculate the Pareto optimal solution. We describe this model with greater detail since it is used as a benchmark in chapter 5. Consider that there are g weight combinations, each one characterized by the weight vector $\left(c_1^g, c_2^g\right)$, with c_1^g and c_2^g representing the weights of the objective functions f_1 and f_2, respectively, and $c_1^g + c_2^g = 1, \forall g$. The optimization model corresponding to the maximization of the weighted sum of the two individual objective functions f_1 and f_2, is given by

$$\begin{array}{ll} \min\limits_{w_t \in \mathbb{R}^N} & y_g \\ \text{subject to} & c_1^g \dfrac{f_{1,max}^s - f_1^s(w_t)}{f_{1,max}^s - f_{1,min}^s} + c_2^g \dfrac{f_2^s(w_t) - f_{2,min}^s}{f_{2,max}^s - f_{2,min}^s} \geq \left(1 - y_g\right) z_s^g, \ \ s \in U \\ & w_t \in X \end{array} \tag{3.45}$$

where $f_{1,max}^s$, $f_{1,min}^s$, $f_{2,max}^s$ and $f_{2,min}^s$ are the maximum and the minimum values of the objective functions f_1 and f_2, respectively, for a given scenario $s \in U$, and z_s^g is the optimal value of the weighted sum of the objective functions for scenario s and weight combination g. The empirical test performed by Xidonas et al. (2017b) considered the discretization of the weight space. Eleven different weight combinations and an uncertainty set with five scenarios of return and risk evolution were tested. The results show that the minmax regret portfolio includes more stocks than the optimal portfolios of the individual scenarios, in all the weight combinations, representing a more disperse allocation of the total investment universe. Furthermore, the in-sample performance analysis revealed that the area of the Pareto front that corresponds to minimizing risk against maximizing return (i.e. when minimizing risk is weighted more than maximizing return) provides more robust solutions in terms of the minmax regret criterion, thus lower minmax regret values, where the minmax regret expresses how far one is from the individual optima of each scenario in the worst-case. No out-of-sample performance analysis was presented in this study.

Chapter 4

Overview of the proposed models and methodological elements

In this chapter, we start by presenting an overview of the robust portfolio optimization models that are proposed in this study, highlighting our main contributions. We describe the set of methodological elements that are common to all the empirical applications that were implemented in order to test the proposed models. We present and explain the strategy applied in the empirical applications, the general decisions concerning model settings definition and data selection, the solvers used to compute the robust solutions and the benchmarks and performance measures that were used to compare and assess the proposed methodologies.

4.1 Main contributions of our study

After performing an exploratory literature review on the robust portfolio optimization field, we realized that, as previously mentioned, the relative robust approach is the least studied and applied in the field of robust portfolio selection. This fact helped us set the direction of our research and define its main objectives. We centered our research in the study of the relative robust portfolio optimization approach and its comparative analysis to absolute robust and non-robust portfolio optimization strategies. Hence, the main contribution of this study is to propose new robust portfolio optimization models, given particular attention to relative robust portfolio models and regret measures.

We propose new methods to compute absolute and relative robust portfolios by extending and combining established methodologies. The development of these methods leads to the examination of four other contributions. First, we compare solutions produced by the relative robust and absolute robust formulations of classical and parametric portfolio optimization models. Second, the real benefits, from the investor perspective, of applying the absolute and relative robustness approaches in portfolio selection are analyzed by comparing in-sample and out-of-sample performances of robust and non-robust portfolios. Furthermore, the relevance of the proposed robust models is analysed for different levels of the investor's risk preference. This analysis also allows the determination of the main strengths of the new methodologies proposed since, by locating the computed portfolios in the risk-return space and comparing their in-sample and out-of-sample performances, we analyze whether the robust methodology allows us to enhance the performance of current models available in the literature. Third, by using estimation samples and in-sample sets of different lengths, we investigate the effect of considering different number of scenarios in the uncertainty set as well as long-term historical data over short-term historical data in the definition of the uncertainty set. Finally, the use of an evolutionary algorithm capable of solving the robust portfolio problems allowed to simultaneously tackle two optimization problem levels, namely transforming the three-level relative robust optimization problem into a two-level problem and the two-level absolute robust optimization problem into a one-level problem.

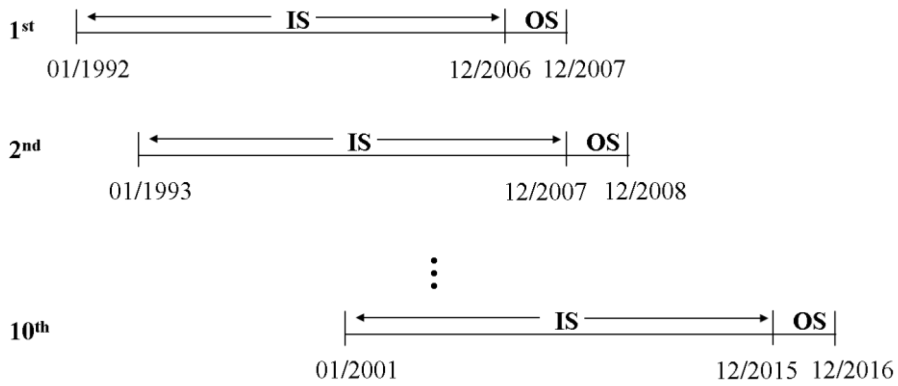
4.2 Rolling window with in-sample and out-of-sample periods

Rolling windows of two different lengths (long *versus* short) is used in the empirical applications. Long and short historical windows are constructed in a way that the last year, defined as the out-of-sample period, is the same in both cases, allowing the comparison of the out-of-sample results. Hence, the data from the last year of the respective window is used to perform the out-of-sample evaluation of the optimal portfolios, while the remaining data is used to perform the in-sample estimations.

The long and short approaches applied in the construction of the rolling windows are depicted in Figure 4-1 and Figure 4-2, respectively. In both cases, a total of 10 historical windows are analysed. The long approach considered a rolling window with a constant length of 16-years: 15-years data to perform in-sample estimations and an out-of-sample evaluation period of 1-year. Thus, for this approach, the first window ranges from January 1992 to December 2007 while the last window ranges from January 2001 to

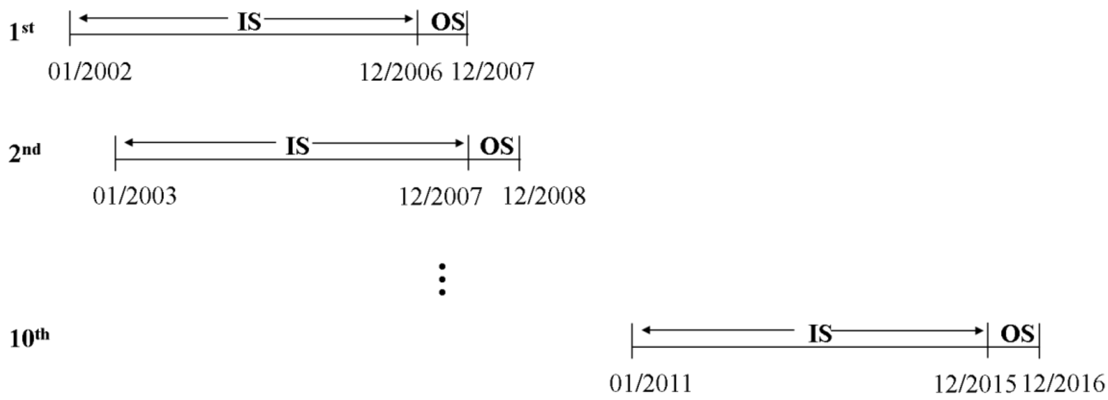
December 2016. The short approach considered a rolling window with a constant length of 5-years: 4-years data to perform in-sample estimations and an out-of-sample evaluation period of 1-year. In this case, the first window ranges from January 2003 to December 2007 (in-sample period from 2003 to 2006 and out-of-sample consisting of 2007) while the last window ranges from January 2012 to December 2016 (in-sample period from 2012 to 2015 and out-of-sample consisting of 2016).

Figure 4-1: Rolling window: long approach



This figure represents the rolling window with a constant length of 16 years. Each one of the 10 historical windows is divided in two periods: the in-sample (IS) period, used for estimating the model inputs, and the out-of-sample (OS) period, used to assess the (out-of-sample) performance of the implemented strategies.

Figure 4-2: Rolling window: short approach



This figure represents the rolling window with a constant length of 5 years. Each one of the 10 historical windows is divided in two periods: the in-sample (IS) period, used for estimating the model inputs, and the out-of-sample (OS) period, used to assess the (out-of-sample) performance of the implemented strategies.

The application of historical windows of different lengths allows to analyze the effect of considering long-term historical data over short-term historical data in the definition of the uncertainty set and, therefore, it allows the investigation of whether the use of long-term historical data affects the predictive accuracy of the models. Previous studies have shown that long-term historical returns (measured over long-term periods) are negatively correlated with future returns, a phenomenon referred to as the long-term reversal effect (Bondt & Thaler, 1985), while short-term historical returns (measured over the last year) are positively correlated with future returns, a phenomenon referred to as the momentum effect (Jegadeesh & Titman, 1993). Additionally, the implementation of rolling windows of different lengths allows an analysis of the usefulness of the proposed models in the presence of limited data.

4.3 Model settings and Data

For all the proposed robust approaches, we defined the feasible set in the same way and considered a discrete and finite uncertainty set. For the definition of the feasible set X, two constraints were considered, namely the completeness constraint and the non-negativity constraint. While the completeness constraint guarantees that all the investor's wealth is invested, the non-negativity constraint was included mainly because short sales are not allowed or are restricted in the majority of stock exchange markets. The set of feasible solutions is, therefore, defined by:

$$X = \left\{ w_t \in \mathbb{R}^N : \sum_{n=1}^{N} w_{t,n} = 1 \,, w_{t,n} \geq 0, \forall n = 1, \dots, N \right\}, \tag{4.1}$$

and is considered in all the portfolio optimization strategies implemented in this study, as a way of preserving the comparability of the results.

The uncertainty set U is defined as a discrete and finite set of scenarios

$$U = \{ s_1, s_2, \dots, s_S \}, \tag{4.2}$$

where S represents the number of scenarios in U. An identical structure of U is presented in Kim et al. (2014c). Each scenario $s \in U$ is built considering the observations associated with different sampling periods randomly gathered from the historical in-sample data.

To define the procedure for generating the sample returns from the in-sample data, consider a universe of N assets for which $\mathcal{O}$ observations regarding consecutive trading days are known. A sample of asset returns on J consecutive trading days is used to define scenario s. Specifically, $\mathbb{z}(s)$ is defined as a random value such that $\mathbb{z}(s) \in \{1, \dots, \mathcal{O} - J + 1\}$, and all observations between $\mathbb{z}(s)$ and $\mathbb{z}(s) + J - 1$ will constitute scenario s. The sample returns of the N assets during this randomly generated time window of length J of scenario s will be represented by matrix $R^s \in \mathbb{R}^{J \times N}$. The vector with the returns of the different assets at time j will be represented by r_{js}. For each one of the proposed models, each scenario $s \in U$ is defined either by a set of input estimators or by a set of sample observations, depending on the model under consideration.

To explore the sensitivity of the results to parameters variation, when computing the scenarios and the uncertainty set, different cardinalities of the uncertainty set (S) are analyzed. Estimations of the model inputs are all performed in R.

The inputs estimation was made using historical daily data from January 1990 to December 2016 of the stocks of two European equity indices: the DAX and the EURO STOXX 50. Data was collected from Thomson Reuters Datastream. While long historical periods are required in order to estimate expected returns more accurately (Elton et al., 2009; Merton, 1980), the use of daily historical data, comparatively to data with lower sampling frequencies, improves the risk estimation process (Clarke, De Silva, & Thorley, 2006).

Adjusted closing prices of the stocks in the constituent list of the indices at the end of the in-sample period were collected and daily continuous returns were calculated. In order to avoid any survivorship bias in the construction of our portfolios, we selected the assets in each of the time windows under analysis in the following way. First, we identified the assets in the constituent list of the stock index at the end of the in-sample period; then, from those assets, we selected the ones that were listed in the stock exchange at the beginning of the in-sample period (regardless of being in the constituent list of the stock index or not). Hence, we are considering a set of assets that does not exactly reflect the assets composing the stock index under consideration.

The evolution of the DAX and the EURO STOXX 50 indexes during the out-of-sample years is depicted in Figure 4-3 and Figure 4-4, respectively. While the years 2008 and 2011 can be characterized by accentuated drops of the price index of both indexes (mainly caused by the subprime mortgage crisis, in the former year, and the European debt crisis, in the latter one), the years 2009, 2012 and 2013 stand out as periods of substantial recoveries. It is also possible to note that the index prices present higher volatility in 2015, experiencing severe drops followed by significant recoveries.

Notice that, from July 2007 to March 2009, the Dax price index fell about 55% (from 8,100 points to 3,660 points). From then until May 2011, it experiences a significant recovery (from 3,660 points to 7,500 points) followed by a severe drop until the end of 2011 (from 7,500 points to 5,070 points). From 2012 onwards, the market portfolio had a remarkable rebound, reaching new historical hights in the beginning of 2015 (from 5,070 points to 12,370 points). Regarding the EURO STOXX 50, from January 2007 to March 2009, the price index experienced a drop of about 56% (from 4,120 points to 1,810 points). From March 2009 to the end of the out-of-sample period, it is possible to notice a significant recovery of the market portfolio (from 1,810 points to 3,291 points), however the price index remains about 20% lower than the value observed in the beginning of the out-of-sample period.

Figure 4-3: Evolution of the DAX index price during the out-of-sample periods.

Figure 4-4: Evolution of the EURO STOXX 50 index price during the out-of-sample periods.

4.4 Solvers used to compute the robust solutions

As described in section 3.1, the relative robust optimization model is a three-level optimization problem. The first optimization process, corresponding to the computation of the optimal solution w_t^{s*} for each of the scenarios admitted in the uncertainty set, is performed using the CPLEX solver. The second and third optimization levels are simultaneously solved using the Genetic Algorithm (GA) toolbox from Matlab R2017a, in order to compute the relative robust optimization solution (minmax regret solution).

There are several deterministic methods available in the literature for solving similar minmax optimization problems (Paç & Pınar, 2018; Pınar, 2016; Pınar & Burak Paç, 2014; Yaman, Karasan, & Pinar, 2007). In this work, a different path was taken. There are two main reasons for resorting to GA: with the proposed measures of regret which are described in the next section, the developed minmax optimization models became nonlinear programming problems; the uncertainty set, as defined in this study, leads to a considerable number of constraints, resulting in highly complex optimization problems. In fact, the application of evolutionary algorithms, such as GA, to optimization problems with non-linear or non-convex objective functions is increasing in the portfolio theory literature (Chang, Yang, & Chang, 2009; Kalayci, Ertenlice, Akyer, & Aygoren, 2017; Soleimani, Golmakani, & Salimi, 2009; Streichert, Ulmer, & Zell, 2004; Zhu, Wang, Wang, & Chen, 2011). Their reasonable computational time to solve more complex and combinatorial problems is pointed out as their main advantage (Soleimani et al., 2009). For the proposed set of robust problems, this approach allows the simultaneous optimization of two of the optimization levels as the method is able to simultaneously solve the inner maximization and outer minimization levels of the proposed robust problems.

Furthermore, the efficiency of the GA as a tool to overcome the difficulties raised by computational complexity when solving (portfolio) optimization problems that address future uncertainty has also been reported by some authors (Jin & Branke, 2005; Yang, 2006). Gen and Cheng (2000) state that the multiple directional and global search performed by the GA where a population of potential solutions is maintained from generation to generation, is useful when exploring Pareto solutions, which is relevant for the investment decision problem since it is, in nature, a multi-objective optimization problem with (multiple) conflicting objectives. According to Shoaf and Foster (1998, p.

358), the "(…) GA can simultaneously minimize risk and maximize expected return (…) This flexibility allows the GA to discover portfolio opportunities which the more traditional [programming] approach misses".

Finally, by applying both deterministic (when computing the optimal solution for each scenario) and evolutionary algorithms, we are conducting a more efficient implementation of an algorithm capable of solving our models, as highlighted by Gen and Cheng (2000, p. 107): "Because GAs, as a kind of meta-heuristics, provide us with great flexibility to incorporate conventional methods into the main framework, we can exploit the advantages of both GAs and conventional methods to establish much more efficient implementations to problems".

Regarding the initial population of the GA, instead of defining a fixed size, as most of the authors do (Kalayci et al., 2017), we defined an initial number of individuals that depends on the dimension of the problem (number of scenarios). In particular, the initial population has twice the size of the uncertainty set and is composed of all the optimal solutions w_t^{s*} (for all scenarios admitted in the uncertainty set) and, additionally, of other feasible randomly generated solutions. When exploratory experiments showed that feeding the initial population of the GA with all the optimal solutions w_t^{s*} caused its premature convergence, deteriorating the out-of-sample performance of the robust solutions, the initial population was only composed by feasible solutions generated randomly (in this case, the initial population has the same size of the uncertainty set).

Concerning the remaining GA specifications, the most usual options were considered (Kalayci et al., 2017). A real valued chromosome representation was used with uniform crossover (with probability rate 0.80) and tournament selection. An elitist strategy is also defined: a fraction of 5% of the best individuals goes directly to the new population, meaning that, on average, 80% of the remaining individuals are generated by the crossover operation. Although there is a wide range of options regarding the mutation type and rate, it was decided to use uniform mutation with rate of 15% for each chromosome (some exploratory experiments indicate that the results show little sensitivity to the mutation rate). Finally, instead of applying a fixed number of iterations as the termination criterion, it was applied a convergence criterion (tolerance of 1e-16 for the average relative change in the best fitness function value from one generation to the next one), in order to avoid unnecessarily long computational times or suboptimal solutions in instances where more computational time is needed. Regarding the individuals' feasibility, all solutions are feasible because of the way they are represented – the weights of each individual are rescaled to one, ensuring its feasibility.

The GA toolbox from Matlab R2017a, the initial population definition and the GA specifications used in order to solve the relative robust three-level optimization problem are equally applied in the computation of the corresponding absolute robust two-level optimization problem.

4.5 Benchmarks and performance measures

In order to investigate the real contribution of robust models within the portfolio optimization field of study, the performance of the proposed robust strategies was analyzed and compared to classical non-robust optimization strategies, considering both in-sample and out-of-sample data. Additionally, and since there is a lack of empirical studies that compare the performance of relative robust and absolute robust portfolios, the performance of the proposed relative robust solutions is compared to the corresponding absolute robust solutions, for some of the developed works.

The non-robust benchmarks used were the mean-variance and the global minimum variance strategies, defined by problems (2.4) and (2.5), respectively. For solving problem (2.4), the Markowitz's efficient frontier was first constructed considering a large set of possible values for μ_L (lower limit on the expected return); then, the MV portfolio was identified by selecting the efficient portfolio that maximizes the investor's expected utility according to the value of the relative risk aversion parameter under consideration, in each of the developed works. Inputs for the classic models were estimated within the entire in-sample window, according to the approach (long and/or short in-sample length) under consideration. Optimal solutions were computed using CPLEX. While selecting Markowitz's MV portfolio as a benchmark to assess the performance of the proposed robust portfolios is almost mandatory, the selection of the GMV portfolio as a benchmark is straightforward due to its well established out-of-sample performance in the portfolio literature. Previous studies have shown that the GMV portfolio with non-negativity constraints outperforms the EW portfolio (Chan et al., 1999; Jagannathan & Ma, 2003), while it performs as well as those GMV portfolios constructed with covariance matrices estimated using factor models and shrinkage methods (Jagannathan & Ma, 2003).

The EW portfolio, also know as the naïve $1/N$ benchmark, which equally allocates the wealth among the assets that were included in each of the windows under analysis, was constructed. The EW portfolio is also used in this study as a benchmark because decision makers continue to use it for allocating their wealth across assets (DeMiguel et al., 2009). Additionally, DeMiguel et al. (2009) compared the out-of-sample performance of the EW portfolio to the performances of the sample-based mean-variance model and its extensions designed to reduce estimation error, using different performance metrics, and found that no single strategy always dominates the equally-weighted strategy. The authors pointed out that the $1/N$ strategy is more likely to outperform when N is large because this improves the potential for diversification.

After determining the optimal solutions of the strategies implemented, in-sample and out-of-sample performances were compared by analyzing, in annualized terms, the portfolio return, given by

$$r_t^p = \mu_t{}' w_t, \tag{4.3}$$

the portfolio variance

$$\upsilon^p = d w_t{}' \Sigma_t w_t, \tag{4.4}$$

where d corresponds to the number of observations (trading days) in a year and the Israelsen's modified Sharpe ratio (Israelsen, 2005) defined by

$$S_I = \frac{\mu_t{}' w_t - r_t^f}{\sqrt{d w_t{}' \Sigma_t w_t}^{\left(\left(\mu_t{}' w_t - r_t^f\right)/abs\left(\mu_t{}' w_t - r_t^f\right)\right)}}, \tag{4.5}$$

where $\mu_t{}' w_t - r_t^f$ represents the annualized excess return of the portfolio comparatively to the return of the risk-free asset (r_t^f), and $abs(.)$ is the absolute value function. The risk-free asset selected for the computation of the modified Sharpe ratio was the 1-year maturity government triple A bond for the Euro area accessible at https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html. This indicator was only computed for the out-of-sample analysis since data on the risk-free asset is only available from September 2004 onwards. Although there is a wide range of performance measures available in the literature (Cogneau & Hübner, 2009), the Israelsen's modified Sharpe ratio (S_I) was selected for two main reasons: 1) the S_I is equal to the standard Sharpe ratio when excess return is positive while providing correct rankings regardless of the excess return being positive or negative; and 2) comparatively to other performance measures it does not require the definition of subjective parameters. For instance, comparatively to Sortino ratio, it does not require the definition of the minimum acceptable return, whose value depends on the investor's preferences.

In addition, the portfolio's regret is computed, based on the definition of regret proposed in each of the developed works, and compared for the in-sample and out-of-sample periods. Composition, cardinality, turnover, abnormal return and beta of the robust and non-robust portfolios are also compared in some of the works developed. The computation of the portfolio turnover followed the definition presented by Brandt et al. (2009), considering the evolution of the assets returns from $t-1$ to t. Hence, the portfolio turnover at time t (V_t) is defined by:

$$V_t = \sum_{n=1}^{N} abs(w_{n,t} - w_{n,t}^h), \tag{4.6}$$

where $abs(.)$ is the absolute value function, $w_{n,t}$ is the weight of asset n at time t after the rebalancing, and $w_{n,t}^h$ is the weight of asset n at time t before the rebalancing, which is given by:

$$w_{n,t}^h = w_{n,t-1}\frac{1 + r_{n,t}}{1 + r_t^p}. \tag{4.7}$$

For the computation of the abnormal return, we applied the CAPM and performed a linear regression considering the stock index as the proxy for the market portfolio. A total of 2540 daily returns (corresponding to the 10 out-of-sample years) were used in the regression analysis. The abnormal return (J_α) corresponds to the Jensen's alpha in the CAPM framework. We also present the portfolios' beta, which is a measure of their systematic risk relative to the excess return of the market portfolio. It is important to notice that the comparability to the market portfolio has some limitations since the assets used to construct the portfolios vary along the 10 time windows under analysis and are not the same as the ones in the constituent list of the stock index during this period (as previously explained).

Results are presented for a given combination of the model parameters. Results for the remaining parameterizations can be made available upon request.

Chapter 5

A relative robust expected utility optimization approach (Models A and B)

5.1 Introduction

The out-of-sample performance of relative robust portfolio optimization portfolios is still little explored in portfolio literature. Further empirical studies are needed in order to unravel their real benefits for the investors. In this study, we present minmax regret portfolio optimization models and compare their performance to robust and non-robust strategies available in the literature. We define regret as the utility loss for the investor resulting from choosing a given portfolio instead of choosing the optimal portfolio of the realized scenario for the uncertain parameters. The proposed approach is implemented assuming that the investor has CRRA preferences.

The empirical analysis was conducted on a dataset with historical daily data regarding the stocks of the DAX index, from January 1992 to December 2016. Different portfolios are computed, corresponding to the application of different methodologies: the proposed relative robust portfolio (hereafter represented by RRA, where the last letter, A, identifies the model under consideration), the classical MV portfolio, the GMV portfolio, the EW portfolio, the relative robust weighted-sum (WS) portfolio presented by Xidonas et al. (2017b) and the absolute robust approach developed by Kim et al. (2014c). In particular, we extended the works of Xidonas et al. (2017b) and Kim et al. (2014c) by analyzing the out-of-sample performance of the robust solutions proposed in these studies (which was not undertaken by their authors), allowing us to see if these approaches can improve the investment performance. The chosen methodological approaches are compared considering both in-sample and out-of-sample results, where return, risk, modified Sharpe ratio and regret are considered.

The results of this study suggest that the proposed relative robust model has more value for risk-taking investors, i.e. for those who can be more affected by the methodological weaknesses of the classical mean-variance model, standing out as a valid alternative. Moreover, the proposed RRA portfolio outperforms the MV portfolio in many of the time windows under analysis. The RRA portfolio also exhibits better out-of-sample performance in many of the considered time windows, relatively to the EW portfolio. This clearly reinforces the relevance of the proposed methodology since, according to DeMiguel et al. (2009), the EW portfolio is difficult to outperform. It is also possible to conclude that the GMV portfolio presents a very good performance, sometimes outperforming the proposed RRA portfolio, which is in accordance with the results of other authors (Chan et al., 1999; Jagannathan & Ma, 2003). Comparatively to the minmax regret model presented by Xidonas et al. (2017b) and the absolute robust approach developed by Kim et al. (2014c), the proposed relative robust approach offers more consistent results since it generates portfolios that present greater stability concerning in-sample and out-of-sample performances.

The remainder of this chapter proceeds as follows. In Section 5.2, the proposed model is described. The empirical analysis and major results are presented in Section 5.3. An extension to model A is presented in section 5.4. Finally, in Section 5.5, main conclusions are highlighted.

5.2 Methodology

5.2.1 The relative robust expected utility optimization model

For describing the investor's preferences, let $f(.)$ be a general utility function of the investor. The expected utility of the portfolio, represented by $E\left[f\left(r_{t+1}^{p}\right)\right]$, can be approximated by the second order Taylor expansion of the expected utility function around the expected return of the portfolio, $E\left[r_{t+1}^{p}\right]$, as follows:

$$E\left[f\left(r_{t+1}^{p}\right)\right] = \frac{\left(1 + E\left[r_{t+1}^{p}\right]\right)^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2}\left(1 + E\left[r_{t+1}^{p}\right]\right)^{-\gamma-1} V\left[r_{t+1}^{p}\right], \tag{5.1}$$

where $V\left[r_{t+1}^{p}\right]$ is the variance of the portfolio returns.

Let S represents the number of elements of the uncertainty set and $s_i \in U, i = 1, \ldots, S$, represent a given scenario. To avoid cluttering the notation, the i index can be dropped from the notation, with s representing a given scenario. As previously mentioned, the proposed relative robust minmax regret approach defines *regret* as the utility loss for the investor resulting from choosing a portfolio characterized by the weight combination vector w_t instead of choosing w_t^{s*} which corresponds to the optimal solution under scenario s.

Uncertainty is admitted not only in the vector of assets' returns but also in the covariance matrix of returns. A joint uncertainty set is defined for both uncertain parameters as in Kim et al. (2014c), based on samples of the asset returns for consecutive trading days. Each scenario of the uncertainty set is described by the vector of expected asset returns and the covariance matrix of returns for each sample randomly gathered from the in-sample data.

Recall the notation and the general procedure for generating the sample returns from the in-sample data, previously described in section 4.3. Each scenario s is defined in the following way:

$$s = (\mu^s, \Sigma^s), \tag{5.2}$$

$$\mu^s = \begin{bmatrix} \frac{1}{J}\sum_{j=\mathbb{z}(s)}^{\mathbb{z}(s)+J-1} r_{j1} \\ \vdots \\ \frac{1}{J}\sum_{j=\mathbb{z}(s)}^{\mathbb{z}(s)+J-1} r_{jN} \end{bmatrix}, \tag{5.3}$$

and

$$\Sigma^s = \frac{1}{J-1} \sum_{j=\mathbb{z}(s)}^{\mathbb{z}(s)+J-1} \left(r_{jn} - \mu^s\right)\left(r_{jn} - \mu^s\right)', n = 1, \dots, N. \tag{5.4}$$

Let $r_{t+1}^{ps*} = r_{t+1}(w_t^{s*})$ be the realized return of the optimal portfolio w_t^{s*}, under scenario s, given by $r_{t+1}^{ps*} = \sum_{n=1}^{N} w_{n,t}^{s*} r_{n,t+1}^{s}$, where $w_{n,t}^{s*}$ represents the weight of asset n in w_t^{s*} and $r_{n,t+1}^{s}$ the respective return, under scenario s. Equivalently, $r_{t+1}^{ps}(w_t)$ corresponds to the expected return of the portfolio characterized by the weight combination vector w_t, under scenario s, which is given by $r_{t+1}^{ps}(w_t) = \sum_{n=1}^{N} w_{n,t} r_{n,t+1}^{s}$, where $w_{n,t}$ represents the weight of asset n and $r_{n,t+1}^{s}$ the respective return, under scenario s. The regret associated to choosing portfolio w_t in scenario s, $P^s(w_t)$, is defined by

$$P^s(w_t) = E\left[f\left(r_{t+1}^{ps*}\right)\right] - E\left[f\left(r_{t+1}^{ps}(w_t)\right)\right] \tag{5.5}$$

and the maximum regret function, $P(w_t)$, is defined by

$$P(w_t) = \max_{s \in U} \left\{E\left[f\left(r_{t+1}^{ps*}\right)\right] - E\left[f\left(r_{t+1}^{ps}(w_t)\right)\right]\right\}. \tag{5.6}$$

The objective is to determine the relative robust solution w_t corresponding to the weight combination vector that minimizes the maximum regret function and therefore solves the relative robust optimization model:

$$\min_{w_t \in X} \max_{s \in U} \left\{E\left[f\left(r_{t+1}^{ps*}\right)\right] - E\left[f\left(r_{t+1}^{ps}(w_t)\right)\right]\right\} \tag{5.7}$$

where the set of feasible solutions is defined as in (4.1).

This approach was applied assuming the following utility function of the investor with constant relative risk aversion (CRRA) preferences over wealth and constant relative risk aversion parameter γ ($\gamma \in IR^{+}\backslash\{1\}$):

$$f\left(r_{t+1}^{p}\right)=\frac{\left(1+r_{t+1}^{p}\right)^{1-\gamma}}{1-\gamma}. \tag{5.8}$$

Hence, assuming CRRA preferences, the optimization problem (5.7) can be defined by:

$$\begin{aligned}\min_{w_t\in X}\max_{s\in U}\Bigg[&\left(\frac{\left(1+\mu^{s\prime}w_t^{s*}\right)^{1-\gamma}}{1-\gamma}-\frac{\gamma}{2}\left(1+\mu^{s\prime}w_t^{s*}\right)^{-\gamma-1}w_t^{s*\prime}\Sigma^s w_t^{s*}\right)\\&-\left(\frac{\left(1+\mu^{s\prime}w_t\right)^{1-\gamma}}{1-\gamma}-\frac{\gamma}{2}\left(1+\mu^{s\prime}w_t\right)^{-\gamma-1}w_t{}^{\prime}\Sigma^s w_t\right)\Bigg].\end{aligned} \tag{5.9}$$

5.2.2 Computing the relative robust portfolio

The process of computing the relative robust portfolio starts with the construction of the uncertainty set. An uncertainty set is constructed by calculating the *S* scenarios from the in-sample period. An estimation window is randomly selected within the in-sample period and the sample mean and the sample covariance matrix are computed.

Then, for each scenario $s \in U$, we solve the following problem:

$$\max_{w_t\in X}\left(\frac{\left(1+\mu^{s\prime}w_t\right)^{1-\gamma}}{1-\gamma}-\frac{\gamma}{2}\left(1+\mu^{s\prime}w_t\right)^{-\gamma-1}w_t{}^{\prime}\Sigma^s w_t\right), \tag{5.10}$$

in order to determine the optimal solution w_t^{s*}, which represents the portfolio on the Markowitz's efficient frontier that maximizes the investor's expected utility when scenario s occurs. This constitutes the first optimization process of our three-level optimization problem.

After computing the optimal solutions for each scenario $s \in U$, the relative robust optimization problem (5.9) is solved by a GA. A fitness function that maximizes the regret as presented in (5.6) was defined and an initial population composed of all the optimal solutions w_t^{s*} and, additionally, by other feasible solutions randomly generated was used.

5.3 Empirical analysis

5.3.1 Data and model settings

For the empirical analysis, historical daily data from January 1992 to December 2016 (25 years) of the stocks of the DAX index was used. Stocks in the constituent list of the DAX index at the end of each in-sample period were selected and adjusted closing prices of the selected stocks were considered in order to calculate daily continuous returns.

In this chapter, the empirical application considers a rolling window with a constant length of 16-years only: 15-years data to perform in-sample estimations and an out-of-sample evaluation period of 1-year. The description of the rolling windows was previously presented in section 4.2. The steps for computing the relative robust solution, previously described in section 5.2.2, are iteratively repeated for each of the 10 time windows defined.

To explore the sensitivity of the results to parameters variation, when computing the scenarios and the uncertainty set, besides considering different number of scenarios in the uncertainty set ($S \in \{100,200\}$), different estimation window lengths ($J \in \{60,120,252\}$) were also analyzed. Furthermore, different relative risk aversion values ($\gamma \in \{2,5,10\}$) were explored. The different values of J, S and γ yielded in 18 different parameters combinations, namely $\big\{(J,S,\gamma): J \in \{60,120,252\}, S \in \{100,200\}, \gamma \in \{2,5,10\}\big\}$, and the relative robust model was solved for each one of these combinations. Due to the long computational times required to solve each model instance, these 18 parameter combinations were the only ones investigated.

5.3.2 In-sample and out-of-sample performances

The performance of the relative robust strategy was analyzed both in-sample and out-of-sample and compared to classical non-robust optimization strategies, to the relative robust optimization approach presented by Xidonas et al. (2017b) and to the absolute robust approach suggested by Kim et al. (2014c). While the comparison with non-robust optimization methodologies allows the recognition of the real contribution of robust models within the portfolio optimization field of study, the comparison with absolute and relative robust strategies allows the appraisal of the current contribution within the field of robust portfolio optimization.

In order to replicate the relative robust strategy presented by Xidonas et al. (2017b), 5 scenarios corresponding to the average return and mean absolute deviation from the last 80, 60, 40, 20 and 10-weeks historical data of the in-sample period were computed. Eleven different weight combinations were used, namely $(0,1)$, $(0.1,0.9)$, ..., $(0.9,0.1)$, $(1,0)$, and, for each window under analysis, eleven relative robust portfolios (hereby designated by WS1, WS2, ..., WS10 and WS11, respectively) were computed using the CPLEX solver.

Finally, an absolute robust approach was also implemented and used as a benchmark. We followed the absolute robust counterpart of the maximum risk-adjusted return classical problem proposed by Kim et al. (2014c), which applies a discrete uncertainty set corresponding to a finite set of scenarios. The absolute robust counterpart of Markowitz's maximum risk-adjusted return (MRC) problem was defined by the authors as:

$$\begin{array}{ll} \max\limits_{z \in \mathbb{R}} & z \\ \text{subject to} & z \leq w^T \mu_i - \lambda w^T \Sigma_i w\,, \quad i = 1, \dots, I\,, \\ & w \in \boldsymbol{X} \end{array} \tag{5.11}$$

where z represents the minimum value of the portfolio's risk-adjusted return, i identifies the i-th sample, μ_i and Σ_i are the vector of average returns and the covariance matrix of the i-th sample, respectively, I corresponds to the number of scenarios included in the uncertainty set U, which is defined as $U = \{(\mu_i, \Sigma_i) | i = 1, \dots, I\}$, and λ represents the weight given to the portfolio variance. In order to avoid the nonlinearity of the constraints in Kim et al.'s model and driven by the computationally more tractable equivalent formulations of problem (5.11) presented by Goldfarb and Iyengar (2003) and Tütüncü and Koenig (2004), we calculated the robust efficient frontier in the following way. We started by constructing the uncertainty set, considering the base case used by Kim et al. (2014c) and collecting 100 i.i.d. samples, each sample consisting of 1000 daily returns randomly selected from the in-sample data. For each sample, the vector of average returns and the covariance matrix were computed and the scenario was defined. Then, we solved problem (2.4) for each lower limit on the expected return (μ_L^p) and each scenario $(\mu_i, \Sigma_i) \in U$. For each μ_L^p, we defined the absolute robust (maxmin) solution as the optimal portfolio (among the I optimal solutions) that presents the maximum risk. This allowed us to identify the robust efficient frontier. The MRC portfolio corresponds to the optimal solution in the robust efficient frontier that maximizes the investor's expected utility. Computations were performed using CPLEX solver.

After determining RRA, MV, GMV, EW, MRC and WS1 to WS11 portfolios, in-sample and out-of-sample performances were compared by analyzing portfolio return, portfolio variance, Israelsen's modified Sharpe ratio, S_I, and regret, defined by:

$$R = \left(\frac{\left(1 + \mu^{s\prime} w_t^{s*}\right)^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2}\left(1 + \mu^{s\prime} w_t^{s*}\right)^{-\gamma-1} w_t^{s*\prime} \Sigma^s w_t^{s*} \right) - \left(\frac{\left(1 + \mu^{s\prime} w_t\right)^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2}\left(1 + \mu^{s\prime} w_t\right)^{-\gamma-1} w_t{}' \Sigma^s w_t \right) \tag{5.12}$$

where w_t^{s*} represents the optimal portfolio (feasible solution with maximum utility) within the sample period under consideration. Portfolio cardinality, turnover and abnormal return were also analyzed and compared among the investment strategies that were implemented. For the computation of the abnormal return, we applied the CAPM and performed a linear regression considering the DAX as the proxy for the market portfolio. A total of 120 monthly returns (corresponding to the 10 out-of-sample years) were used in the regression analysis.

Finally, we analyzed portfolio style and portfolio exposure to risk factors of the relative robust solution. We collected international research returns data from the Fama-French data library, available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. For the portfolio style analysis, we followed the methodology presented by Sharpe (1992) and designated as strong style analysis by Horst et al. (2004). The goal of return-based style analysis is to determine the (positively) weighted style portfolio that is closest to the proposed RRA solution in a least squares sense. We solved a quadratic model that minimizes the variance of the difference between the return of the RRA portfolio and that of a passive (mimicking) portfolio with the same style, subject to portfolio and positivity constraints (i.e., factor loadings must be positive and sum up to 1). In order to construct the mimicking portfolio, we considered the returns of value and growth portfolios available for the German market and based on four different ratios: Book-To-Market (BTM), Earnings-Price (EP), Cash Earnings to Price (CEP), and Dividend Yield (DY). The value portfolios contain firms in the top 30% of the respective ratio and the growth portfolios contain firms in the bottom 30%. As for the analysis of the exposure to risk factors, and similar to Carhart (1997), we used the Fama-French three factors and Momentum (MOM), for the European market, as style indices. Once again, a total of 120 monthly returns (corresponding to the 10 out-of-sample years) were used for performing the time series regressions regarding the analyses of portfolio style and portfolio exposure to risk factors.

5.3.3 Results

The effect of changing parameters J, S and γ on the computed portfolios, both in-sample and out-of-sample, was analyzed. It is important to notice that, while the different values of γ define different MV and MRC portfolios, since for each value a different portfolio is selected from the Markowitz's efficient frontier (and from its absolute robust counterpart), they do not influence the GMV, EW and WS portfolios (only the measure of regret corresponding to them). For each of the 18 parameters combinations considered, a different RRA portfolio is obtained.

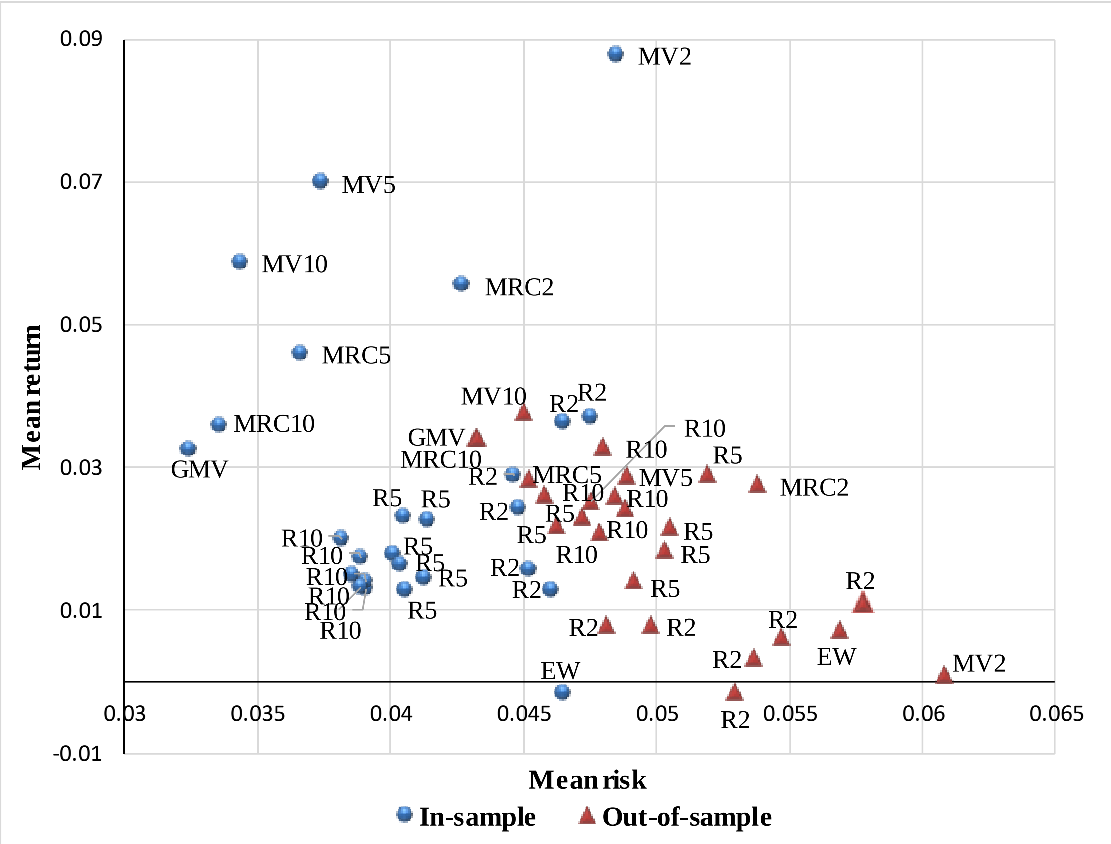
The analysis starts with the portfolios that have been described in the literature namely MV, GMV, EW, MRC and WS portfolios. Afterwards, the effect of the variation of parameters on the 18 RRA portfolios is analyzed. For each computed portfolio, the mean of the portfolio returns (mean return), obtained over the 10 windows, is calculated. Additionally, the mean of the portfolio variances (mean risk) and the mean of the portfolio regrets (mean regret) are also calculated for each portfolio. Then, in-sample and out-of-sample portfolio performances are analyzed by comparing mean return, mean risk and mean regret. Finally, for each of the 10 windows, in-sample and out-of-sample performances of relative robust and non-robust portfolios are compared for the parameter combination $\gamma = 2, S = 100$ and $J = 120$. For simplification purposes, the RRA portfolio will be represented by 'R' or 'RR' in the figures presented in the Results section and the corresponding subsections.

5.3.3.1 Parameters variation effect

Figure 5-1 presents in-sample and out-of-sample mean risk and mean return of the MV, GMV, EW, MRC and RRA portfolios. It can be observed that, when increasing γ, both in-sample mean return and mean risk of the MV portfolio decrease. This is explained by the fact that higher levels of γ result in the selection of an efficient portfolio, from Markowitz's efficient frontier, that is closer to the GMV portfolio. Concerning out-of-sample results, when γ increases, mean return increases while mean risk decreases. It can also be observed that, when γ increases, the out-of-sample portfolio return becomes closer to the expected return (defined with in-sample data), while the out-of-sample portfolio risk is always quite close to the expected risk (defined with in-sample data). This clearly indicates a better out-of-sample performance of the classical MV portfolio when higher levels of the relative constant risk aversion are considered. It is also quite evident that MV portfolios show better in-sample performances comparatively to their correspondent out-of-sample performances.

Analyzing the performances of GMV and EW portfolios, it is evident that while the GMV portfolio shows lower in-sample mean risk and mean return comparatively to MV portfolios, the EW portfolio presents the worst in-sample performance among non-robust portfolios. In fact, the EW portfolio is the only one that presents a negative in-sample mean return. On the other hand, the deviations of these portfolios to their expected risk and expected return are not so substantial as those verified by MV portfolios. Furthermore, these portfolios outperform MV portfolios when considering out of sample mean returns. These results confirm the shortcomings of the MV portfolio concerning its sensitivity to estimation error and the effects of the input uncertainty in the optimization process (Best & Grauer, 1991a; Chopra & Ziemba, 1993; DeMiguel et al., 2009; Jagannathan & Ma, 2003). Notice that, since the computation of the GMV portfolio relies only on estimates of variances and covariances of the asset returns, this portfolio is less vulnerable to estimation error comparatively to the MV portfolio.

Figure 5-1: In-sample and out-of-sample mean risk and mean return of the RRA, MRC, MV, EW and GMV portfolios.



The optimal portfolios were represented according to the value of the risk aversion parameter used in their computation. For instance, 'MRC2', 'MV2' and 'R2' corresponds, respectively, to the MRC, MV and RRA portfolios computed using a value of 2 for the risk aversion parameter. To avoid cluttering the Figure, the values of S and J are not represented for the RRA portfolios, but they will be represented in Figure 5-3.

Assuming that the in-sample results are a measure of the expected performance of the portfolios, and the out-of-sample results are a measure of performance that would be achieved if the portfolios were chosen in that specific time period, then the ones that present the least deviations are the most consistent ones. Analyzing the MRC and the RRA portfolios, it can be confirmed that the robust strategies are the ones that generate solutions with smallest distances between in-sample and out-of-sample mean return and mean risk and, therefore, their performances are more consistent. Regarding the in-sample mean results of the MRC portfolio, it is quite evident that the robust efficient frontier is completely dominated by the non-robust efficient frontier (suggested by the MV portfolios), as it would be expected. Out-of-sample mean results indicate that increasing the level of the relative constant risk aversion significantly decreases the portfolio's risk, while the changes observed in the portfolio's return are not substantial. Furthermore, the absolute robust strategy seems to dominate the relative robust strategy, with some exceptions verified for the lowest level of the relative risk aversion parameter.

The RRA portfolios are generally located at the left side of the scatter-plot, indicating low levels of mean risk, while only one of these portfolios presents negative out-of-sample mean return (namely -0.00134, which is very close to 0). For higher levels of γ, RRA portfolios are more concentrated, presenting similar mean risks and mean returns. It is also possible to observe that out-of-sample mean returns are higher than in-sample mean returns, except for two of the RRA portfolios.

It is important to point out that this tendency, of higher out-of-sample returns comparatively to in-sample returns, is observed among the majority of the computed portfolios and can be explained by the evolution of the stock market, in particular, of the DAX index, during the out-of-sample periods (see Figure 4-3). It is also important to highlight that the MV and the MRC portfolios are the only ones that do not present the tendency of higher out-of-sample returns comparatively to in-sample returns, which reinforces its underperformance even in the presence of favorable market conditions.

The more divergent results observed among all RRA portfolios, are verified, both in-sample and out-of-sample, for $\gamma = 2$. While some of the portfolios RRA portfolios (with $\gamma = 2$) present the highest in-sample mean returns, others are clearly (in-sample) dominated solutions. A similar situation is observed when analyzing the out-of-sample mean results but, in this case, all RRA portfolios (with $\gamma = 2$) are dominated solutions. On the other hand, for $\gamma = 2$, RRA portfolios generally outperform the MV portfolio, both in terms of mean risk and mean return. These results suggest that, while the proposed relative robust methodology provides better solutions when higher levels of the relative constant risk aversion are considered, the RRA portfolio stand out as a valuable alternative to the MV portfolio for investors with smaller risk aversion.

Additionally, for a relative risk aversion parameter of 2, the majority of the RRA portfolios presents lower (mean) out-of-sample risk comparatively to the MRC portfolio.

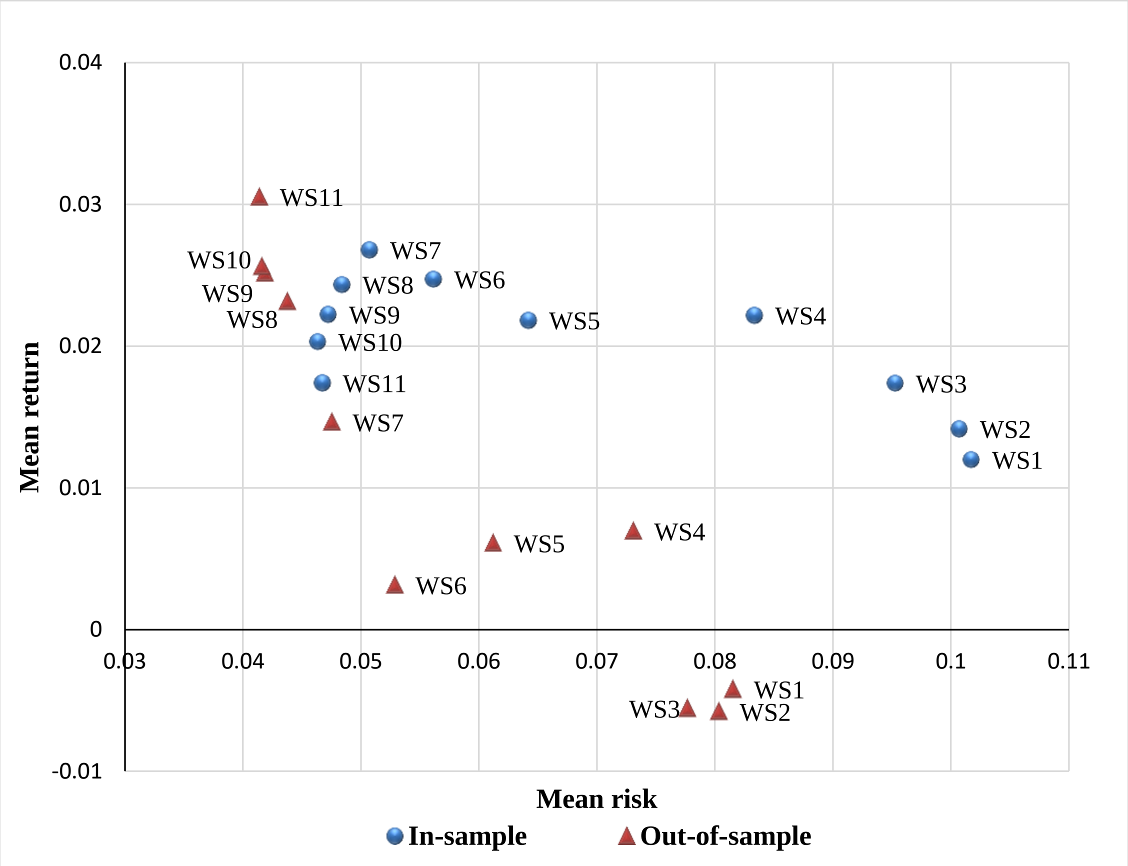
Furthermore, and concerning the out-of-sample results, it can be observed that the majority of the RRA portfolios outperform the EW portfolio, which has been pointed out by some authors as a benchmark difficult to outperform (DeMiguel et al., 2009). Additionally, the outperformance of the GMV portfolio is verified, even outperforming the RRA portfolios, in accordance with the results of previous studies (Chan et al., 1999; Jagannathan & Ma, 2003).

In-sample and out-of-sample mean risk and mean return of the WS portfolios are presented in Figure 5-2. Concerning WS portfolios, an evident trend is observed, both in-sample and out-of-sample. WS portfolios corresponding to lower c_1^g weights (when maximizing return is weighted more than minimizing risk), namely WS1, WS2, WS3 e WS4, underperform comparatively to WS portfolios corresponding to higher c_1^g weights (when minimizing risk is weighted more than maximizing return), namely WS7, WS8, WS9, WS10 and WS11. Comparing in-sample and out-of-sample WS portfolio location (in the mean-variance space), it can be observed that the out-of-sample mean risk is lower than the in-sample mean risk for all WS portfolios. Moreover out-of-sample mean return is higher than the in-sample mean return for WS portfolios corresponding to higher c_1^g weights, which can be explained by the evolution of the DAX index during the out-of-sample periods, as previously pointed out. The deviations of these portfolios to their mean expected risk and mean expected return are more pronounced among those with lower c_1^g weights. These results confirm that WS portfolios corresponding to higher c_1^g weights are more robust, in terms of the deviation to the expected performance, which is in accordance with the (in-sample) empirical results presented by Xidonas et al. (2017b).

Furthermore, the WS portfolios corresponding to lower c_1^g weights reveal the highest levels of (in-sample and out-of-sample) mean risk, while presenting negative out-of-sample mean returns. WS11 portfolio (corresponding to the portfolio that minimizes mean absolute deviation) underperforms out-of-sample comparatively to GMV portfolio while WS1, WS2 and WS3 are dominated solutions relative to MV portfolios (even when $\gamma = 2$).

Regarding the RRA portfolios, it can be observed that their out-of-sample mean risks are always higher than their in-sample mean risks. Additionally, the deviations from their mean expected risks and mean expected returns are not as substantial as the deviations observed for MV (Figure 5-1) or WS portfolios (Figure 5-2).

Figure 5-2: In-sample and out-of-sample mean risk and mean return of the WS portfolios.



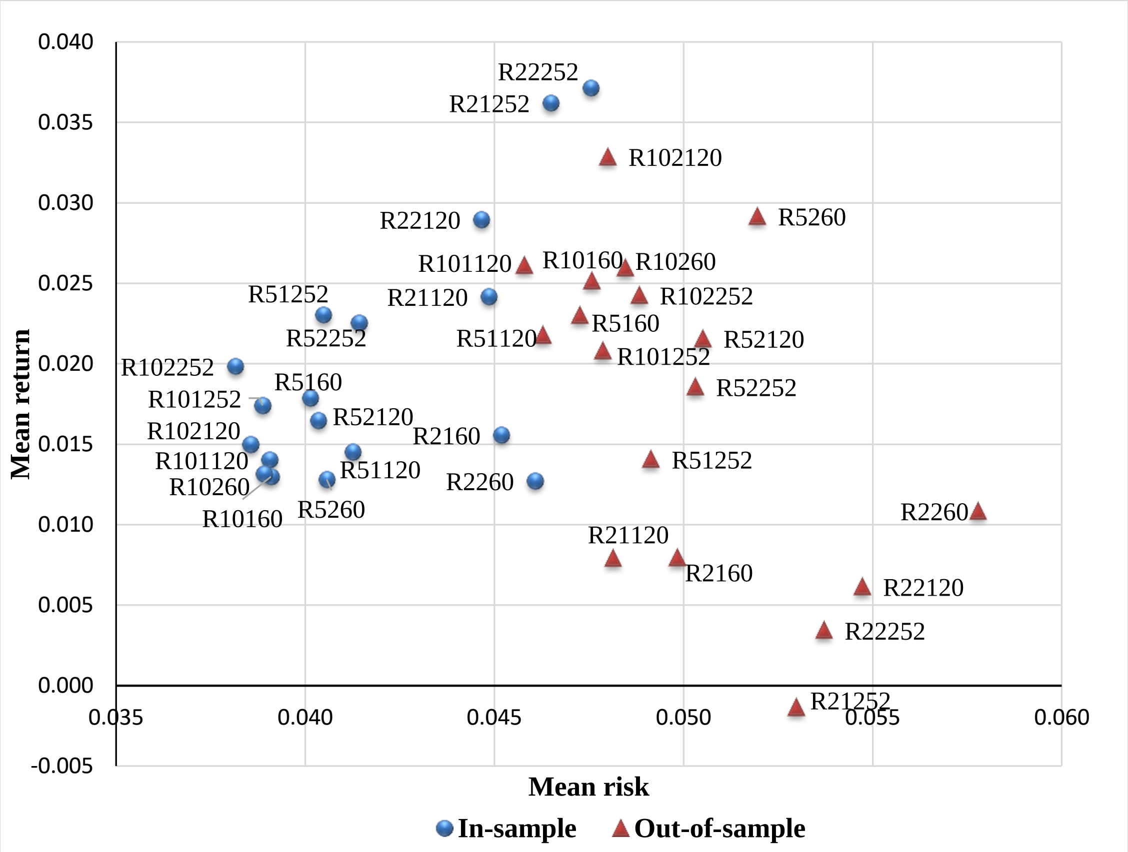
The optimal portfolios were represented according to the weight combination of the individual objective function used in their computation. For instance, 'WS1', 'WS2' and 'WS11' corresponds, respectively, to the WS portfolios computed using the weight vector $(1,0)$, $(0.1,0.9)$, *and* $(0,1)$, *respectively.*

In order to analyze the effect of the variation of S and J, in-sample and out-of-sample mean risk and mean return of the RRA portfolios are presented in Figure 5-3, according to the values of the parameters used in their computation. For simplification purposes, the parameter S was represented by '1' when $S = 100$ and by '2' when $S = 200$. For instance, 'R2260' corresponds to the RRA portfolio computed for $\gamma = 2, S = 200$ and $J = 60$. Analyzing the effect of the variation of S and J, it seems that their highest values are associated with dominant in-sample solutions (for the same level of γ) while no substantial benefits are identified out-of-sample. Notice that, in-sample, R102252 is the dominant solution among all the RRA portfolios when $\gamma = 10$, R51252 is the dominant solution among all the RRA portfolios when $\gamma = 5$ and R22120 is the dominant solution among most of the RRA portfolios when $\gamma = 2$. The respective out-of-sample dominant solutions are R101120 (except for R102120), R51120 (except for R5260 or R5160) and R21120 (except for R2260). These exceptions correspond to solutions that reveal higher mean returns and substantially higher mean risks. Hence, the proposed relative robust methodology seems to lead to solutions that consistently perform better out-of-sample when $S = 100$ and $J = 120$.

In-sample and out-of-sample mean regrets were also analyzed and are presented in Table 5-1. Concerning mean regret, in-sample results show that increasing γ leads to higher levels of mean regret in all of the portfolios computed, which was expected due to the better performance of the MV portfolio for lower levels of γ. Out-of-sample results show that when γ is increased, the mean regret of the MV portfolio decreases, i.e. the robustness of classical mean-variance methodology, in terms of utility loss for the investor, seems to improve, as previously outlined. This outcome generally prevails for the GMV portfolio, but the effect of the variation of γ is not so straightforward.

Concerning WS portfolios, when γ increases, the robustness, in terms of utility loss for the investor, of the WS portfolios corresponding to lower c_1^g weights declines while the robustness of the WS portfolios corresponding to higher c_1^g weights improves. In-sample and out-of-sample mean regrets seem to support Xidonas et al. (2017b) results, since the WS portfolios corresponding to higher c_1^g weights are more robust, relative to the WS portfolios corresponding to lower c_1^g weights, regardless of γ. It is also worth noticing that the lowest mean regrets are usually obtained for γ=5, both for the RRA and for the MRC portfolios. Mean regrets of RRA portfolios are also similar for portfolios computed with the same level of γ. Moreover, while the variation of S and J seems to have no in-sample substantial effect, the lowest out-of-sample mean regret is obtained when $S = 100$ and $J = 120$, regardless of γ.

Figure 5-3: In-sample and out-of-sample mean risk and mean return of the RRA portfolios.



The optimal portfolios were represented according to the values of the parameters γ, S and J, used in their computation. For simplification purposes, the parameter S was represented by 1 when $S = 100$ and 2 when $S = 200$. For instance, 'R2260' corresponds to the RRA portfolio computed admitting $\gamma = 2, S = 200$ and $J = 60$.

Comparing the mean regret of all the computed portfolios, the GMV portfolio is always among the portfolios with lowest mean regrets, regardless of γ. In fact, this portfolio is outperformed only by WS9 when $\gamma = 5$, and by MV, WS7, WS8, WS9, WS10 and WS11, when $\gamma = 10$, while it presents the lowest mean regret when $\gamma = 2$. When $\gamma = 10$, the EW, WS1, WS2, WS3 and WS4 reveal mean regrets which are similar and substantially higher comparatively to the other portfolios.

In the next subsection, the analysis of the in-sample and the out-of-sample performances of relative robust and non-robust portfolios, for each of the 10 windows, is presented. Results are described for the parameters combination $\gamma = 2, S = 100$ and $J = 120$.

Table 5-1: In-sample and out-of-sample mean regret of the RRA, MV, EW, GMVand WS portfolios.

Portfolio	**In-sample mean regret**			**Out-of-sample mean regret**		
	γ=2	**γ=5**	**γ=10**	**γ=2**	**γ=5**	**γ=10**
Rγ160	0.072131	0.071750	0.121632	0.327735	0.249598	0.469121
Rγ1120	0.061977	0.080818	0.116721	0.309436	0.209987	0.254383
Rγ1252	0.050515	0.065315	0.106382	0.335656	0.256156	0.351439
Rγ260	0.076777	0.081479	0.117535	0.319018	0.228061	0.444041
Rγ2120	0.056501	0.074776	0.111041	0.315629	0.230508	0.277701
Rγ2252	0.050466	0.068675	0.095739	0.340163	0.241787	0.352217
MRC	0.026875	0.027173	0.043666	0.296671	0.190008	0.210157
MV	0.000000	0.000000	0.000000	0.343710	0.199943	0.179465
EW	0.092742	0.120679	0.226055	0.328676	0.323939	0.998670
GMV	0.041458	0.034643	0.043965	0.278558	0.181663	0.191338
WS1	0.137664	0.266424	0.615438	0.370089	0.404113	0.943632
WS2	0.133456	0.255477	0.574035	0.372119	0.408781	0.965980
WS3	0.121300	0.218535	0.437290	0.371640	0.407170	0.972191
WS4	0.101714	0.168661	0.302138	0.353249	0.376462	0.901031
WS5	0.083856	0.125191	0.216912	0.330803	0.292991	0.549744
WS6	0.072971	0.102125	0.172536	0.316365	0.234051	0.295680
WS7	0.065309	0.085723	0.139971	0.299449	0.198295	0.183118
WS8	0.065721	0.084155	0.137531	0.289680	0.182118	0.146488
WS9	0.067205	0.086052	0.144526	0.287739	0.179060	0.139712
WS10	0.068550	0.087503	0.149174	0.292429	0.190286	0.171188
WS11	0.072094	0.092945	0.160530	0.291728	0.193357	0.185669

The RRA portfolios were represented according to the values of the parameters S and J used in their computation. For simplification purposes, the parameter S was represented by 1 when $S = 100$ *and by 2 when* $S = 200$*. For instance, 'Rγ160' corresponds to the RRA portfolio* computed *for a given value of* γ *and for the parameters combination* $S = 100$ *and* $J = 60$.

5.3.3.2 Performance of relative robust and non-robust portfolios

The composition of the portfolios is analyzed by comparing the maximum weight of an asset and the sum of the 3 largest weights, in order to assess the exposure to individual stocks, and by comparing the cardinality, measured as the number of assets with weights higher than 0.1% in each computed portfolio. The obtained results are shown in

Table 5-2. The R21120 and MV2 portfolios are hereby designated by RRA and MV portfolios, respectively.

The MV portfolio shows a similar low cardinality among all windows, between 4 and 8, resulting in a low-diversified portfolio, with a maximum asset weight varying between 20% and 53% and, for most of the windows, the allocation of 70% or more of the investors' wealth to 3 assets only. As expected, the GMV portfolio is more diversified, since it presents a minimum cardinality of 12 and a maximum of 15, and a lower exposure to individual stocks, since the maximum weight of an asset varies between 14% and 21% and the sum of the 3 largest weights varies between 37% and 53%.

Concerning WS portfolios, it can be observed that when the c_1^g weights are increased, the cardinality of the portfolios increases and the exposure to individual stocks decreases, resulting in more diversified portfolios. These results were expected since highest levels of the c_1^g weights correspond to portfolios that mainly aim to maximize the expected return while the lowest levels of the c_1^g weights correspond to portfolios that mainly aim to minimize the mean absolute deviation. Thus, the WS1, WS2, WS3 and WS4 portfolios are highly concentrated (cardinality between 2 and 5), with exposure to a single asset higher than 50% and a cumulative exposure to three assets equal to 100%, in most of the windows, while the WS portfolios with the highest c_1^g weights are more diversified (cardinality between 6 and 15), and reveal lower exposure to individual stocks, since the maximum weight of an asset varies between 16% and 62% and the sum of the 3 largest weights varies between 45% and 86%.

The analysis of the portfolio's composition of the robust solutions revealed dissimilar results. The MRC portfolio shows a similar low cardinality among all windows, between 3 and 10, resulting in a low-diversified portfolio. Furthermore, it presents a maximum asset weight varying between 23% and 69% and, for most of the windows, the allocation of 60% or more of the investors' wealth to only 3 assets. While the RRA portfolio is more diversified than the rest of the portfolios, it reveals dissimilar results concerning cardinality, since it varies between 14 and 24, among the windows. It has also the lowest maximum asset weight (varying between 8% and 17%) and the lowest largest investors' wealth allocation to 3 assets (varying between 22% to 45%).

Table 5-2: Composition of the RRA, MRC, MV, EW, GMV and WS portfolios by time window

		RRA	MRC	MV	EW	GMV	WS 1	WS 2	WS 3	WS 4	WS 5	WS 6	WS 7	WS 8	WS 9	WS 10	WS 11
1992-2007	Max%	0.14	0.44	0.25	0.05	0.19	0.65	0.71	0.77	0.77	0.69	0.47	0.26	0.28	0.33	0.34	0.34
	Sum3Max%	0.38	0.75	0.70	0.15	0.49	1	1	1	1	1	0.74	0.62	0.61	0.63	0.59	0.59
	Card.	17	5	6	20	13	2	2	3	2	3	7	9	11	10	11	11
1993-2008	Max%	0.16	0.18	0.44	0.05	0.21	0.52	0.42	0.36	0.42	0.49	0.49	0.42	0.36	0.33	0.28	0.25
	Sum3Max%	0.38	0.49	0.93	0.16	0.52	1	0.87	0.90	0.91	0.82	0.74	0.70	0.69	0.65	0.66	0.67
	Card.	15	10	4	19	13	3	4	4	5	6	8	9	10	11	12	11
1994-2009	Max%	0.12	0.69	0.51	0.05	0.19	0.42	0.43	0.44	0.49	0.51	0.49	0.45	0.41	0.38	0.33	0.28
	Sum3Max%	0.35	1	0.91	0.15	0.51	1	1	1	1	0.99	0.97	0.91	0.90	0.84	0.76	0.73
	Card.	19	3	5	20	12	3	3	3	3	4	5	6	6	6	8	8
1995-2010	Max%	0.13	0.37	0.33	0.05	0.18	0.78	0.73	0.62	0.57	0.41	0.34	0.27	0.25	0.28	0.31	0.32
	Sum3Max%	0.32	0.73	0.73	0.14	0.49	1	1	1	0.99	0.79	0.70	0.69	0.62	0.66	0.71	0.73
	Card.	18	6	5	21	13	2	3	3	4	6	6	8	8	10	10	8
1996-2011	Max%	0.17	0.34	0.42	0.04	0.15	0.78	0.76	0.75	0.74	0.34	0.39	0.52	0.44	0.32	0.28	0.27
	Sum3Max%	0.45	0.76	0.83	0.13	0.42	1	1	1	0.99	0.75	0.70	0.76	0.76	0.83	0.63	0.56
	Card.	14	5	5	23	12	2	2	2	4	6	8	9	8	7	9	10

		RRA	MRC	MV	EW	GMV	WS 1	WS 2	WS 3	WS 4	WS 5	WS 6	WS 7	WS 8	WS 9	WS 10	WS 11
1997-2012	Max%	0.16	0.26	0.53	0.04	0.14	0.78	0.82	0.77	0.73	0.64	0.48	0.42	0.35	0.33	0.32	0.31
	Sum3Max%	0.36	0.58	0.89	0.12	0.37	1	1	1	1	0.93	0.94	0.87	0.78	0.80	0.81	0.76
	Card.	18	10	6	26	13	2	2	3	3	4	4	6	6	7	6	7
1998-2013	Max%	0.14	0.28	0.49	0.04	0.15	0.59	0.60	0.61	0.45	0.35	0.35	0.32	0.27	0.21	0.26	0.28
	Sum3Max%	0.42	0.62	0.85	0.13	0.42	1	1	1	0.87	0.81	0.82	0.76	0.67	0.55	0.60	0.65
	Card.	18	10	7	24	12	3	3	3	5	5	6	8	8	8	10	8
1999-2014	Max%	0.08	0.23	0.20	0.04	0.16	0.73	0.76	0.76	0.68	0.37	0.28	0.20	0.17	0.16	0.20	0.26
	Sum3Max%	0.22	0.57	0.56	0.12	0.46	1	1	1	0.98	0.74	0.68	0.54	0.45	0.46	0.53	0.56
	Card.	24	10	8	26	14	2	2	3	4	6	8	11	15	15	15	13
2000-2015	Max%	0.09	0.30	0.37	0.04	0.17	0.95	0.97	0.96	0.82	0.66	0.59	0.53	0.46	0.32	0.29	0.27
	Sum3Max%	0.25	0.71	0.77	0.12	0.50	1	1	1	1	0.90	0.86	0.81	0.80	0.70	0.68	0.68
	Card.	22	5	6	26	14	2	2	3	3	6	6	7	8	9	9	9
2001-2016	Max%	0.11	0.27	0.38	0.04	0.21	0.69	0.67	0.56	0.43	0.23	0.39	0.51	0.54	0.60	0.60	0.62
	Sum3Max%	0.31	0.68	0.81	0.11	0.53	1	1	1	0.94	0.66	0.72	0.84	0.83	0.84	0.86	0.83
	Card.	19	8	6	28	15	2	2	3	4	5	5	6	6	7	6	7

This table presents the characteristics of the optimal portfolios. The composition of the portfolios regarding the maximum (Max%) and minimum (Min%) weights of an asset, the sum of the 3 largest weights in the portfolio (Sum3Max%), and the number of assets with non-zero weights in each computed portfolio (Cardinality) is described. As explained before, for measuring the cardinality, only those assets with weights higher than 0.1% are considered.

Portfolio turnover was also analyzed and results are displayed in Table 5-3. Yearly portfolio turnover is defined as the sum of the absolute value of the trades (changes in weights) across the available assets, in that year. The portfolio turnover is then defined as the average, over all out-of-sample periods (10 years), of the yearly portfolio turnover. It can be observed that the EW and RRA portfolios (the portfolios that present higher cardinality) present low levels of turnover. In fact, among the non-robust solutions the portfolio with higher turnover is the MV portfolio while the portfolio with lower turnover is the EW portfolio. Regarding the robust solutions, the RRA portfolio is the solution with lower turnover while all the WS portfolios and the MRC portfolio present substantially higher turnover. This result suggests that, in a real investment, the implementation of the RRA strategy would entail lower rebalancing costs comparatively to the strategies proposed by Xidonas et al. (2017b) and Kim et al. (2014c). A further analysis of the WS portfolios shows that, the ones that present the lowest cardinality (between 2 and 5) are the ones that reveal the highest levels of turnover, which was an unexpected result. This can be explained by the fact that these portfolios select different assets along the different windows under analysis. In fact, the majority of the WS portfolios present higher turnover than the absolute robust strategy under analysis. Finally, it is important to point out that, comparatively to the non-robust solutions, the RRA portfolio presents higher turnover, while similar to the one presented by the MV portfolio.

Results concerning in-sample and out-of-sample risks and returns of all the portfolios are presented for all windows in Figure 5-4. Out-of-sample modified Sharpe ratios (S_I) are shown in Table 5-4. While the analysis of the results for each window supports the majority of the outcomes presented in the previous section, some interesting differences stand out.

Analyzing the overall out-of-sample results for the non-robust portfolios, it can be confirmed that the classical mean-variance strategy reveals inconsistent results since for some windows (1992-2007; 1995-2010; 2000-2015; and 2001-2016) the MV portfolio is among those with the highest S_I, while for others (1993-2008; 1996-2011; 1997-2012) it is among those with the lowest S_I. Furthermore, the MV portfolio stands out in some windows (1998-2013 and 1999-2014) as one of the few portfolios with negative return. Regarding the EW portfolio, a similar conclusion can be drawn, since it is among the portfolios with the highest S_I in some windows (2003-2007; 2005-2009;

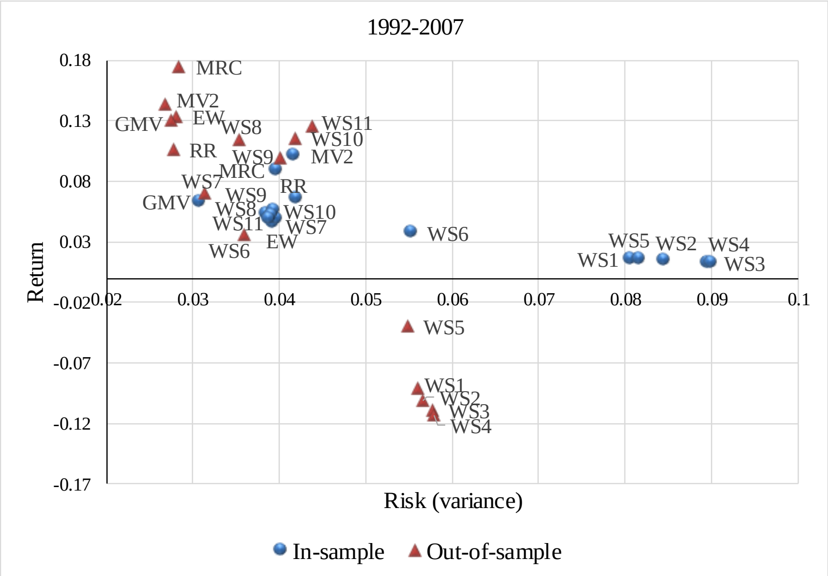
2006-2010; 2008-2012) while it presents poor out-of-sample performance in others (2007-2011; 2010-2014; 2011-2015; 2012-2016). It is also important to highlight that the EW portfolio outperforms the MV portfolio, in terms of risk and/or in terms of return, in most of the time windows under analysis. Concerning the GMV portfolio, although this portfolio shows worse in-sample mean return comparatively to the MV portfolio, a different trend can be observed when comparing out-of-sample performances. In fact, the GMV portfolio is a dominant solution comparatively to the MV and the EW portfolios in the majority of the time windows. Furthermore, this portfolio is among the solutions with the best S_I in some windows (2004-2008; 2005-2009; 2007-2011; 2008-2012; 2009-2013; and 2011-2015) while it is located at the left side of the scatter-plot in all windows, indicating low levels of risk.

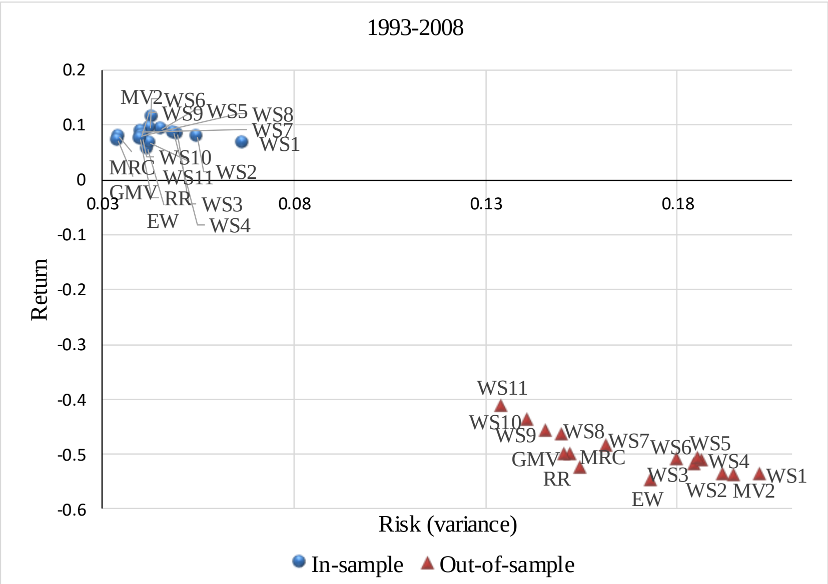
Table 5-3: Turnover of the RRA, MRC, MV, EW, GMV and WS portfolios

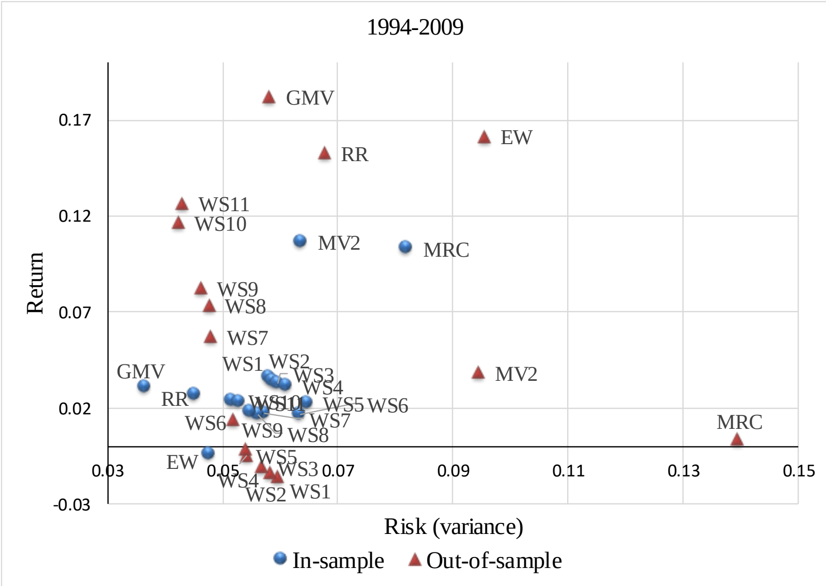
Portfolio	**Turnover**
RRA	0.799760
MRC	1.271759
MV	0.687393
EW	0.291742
GMV	0.392018
WS1	1.939064
WS2	1.953327
WS3	1.980119
WS4	1.952692
WS5	1.821421
WS6	1.705526
WS7	1.514796
WS8	1.389555
WS9	1.261189
WS10	1.205050
WS11	1.230448

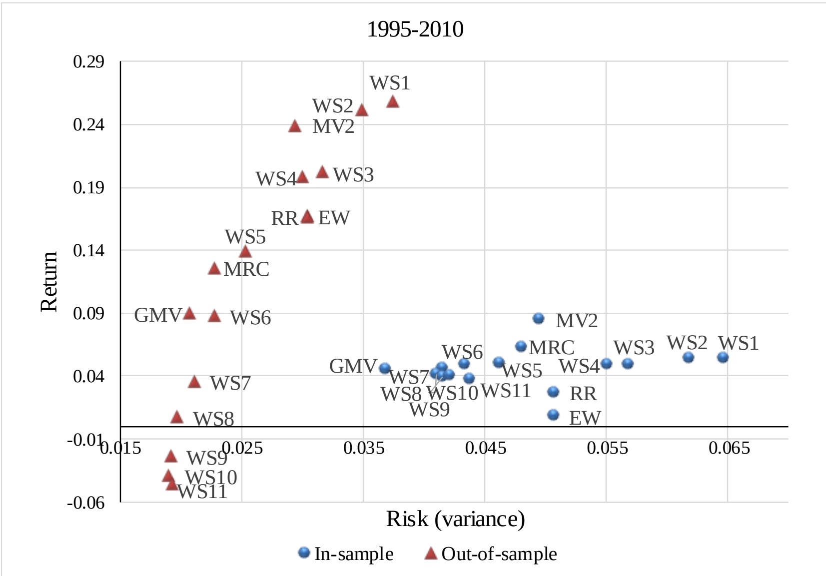
Portfolio turnover corresponds to the average, over all out-of-sample periods (10 years), of the yearly portfolio turnover, with this yearly turnover defined as the sum, for the considered year, of the absolute value of the trades (changes in weights) across the available assets.

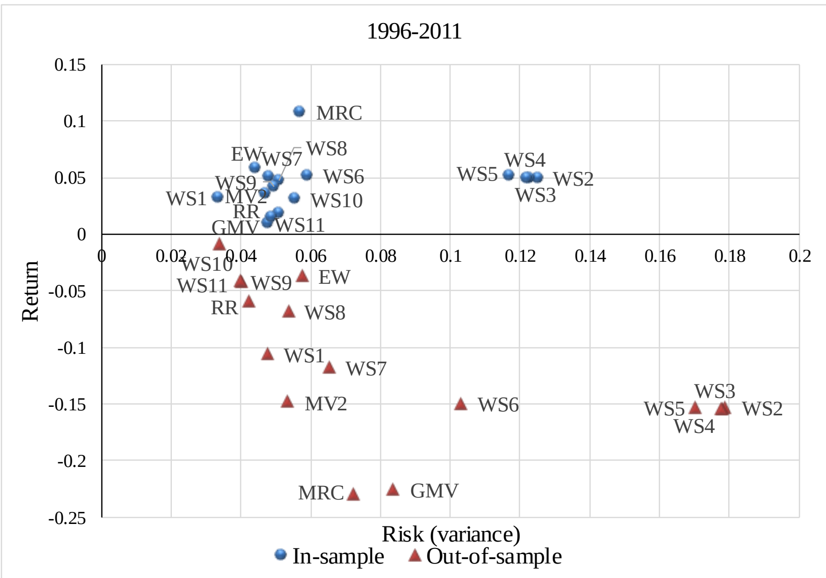
Figure 5-4: In-sample and out-of-sample risks and returns of the RRA, MRC, MV, EW, GMV and WS portfolios computed for each time window

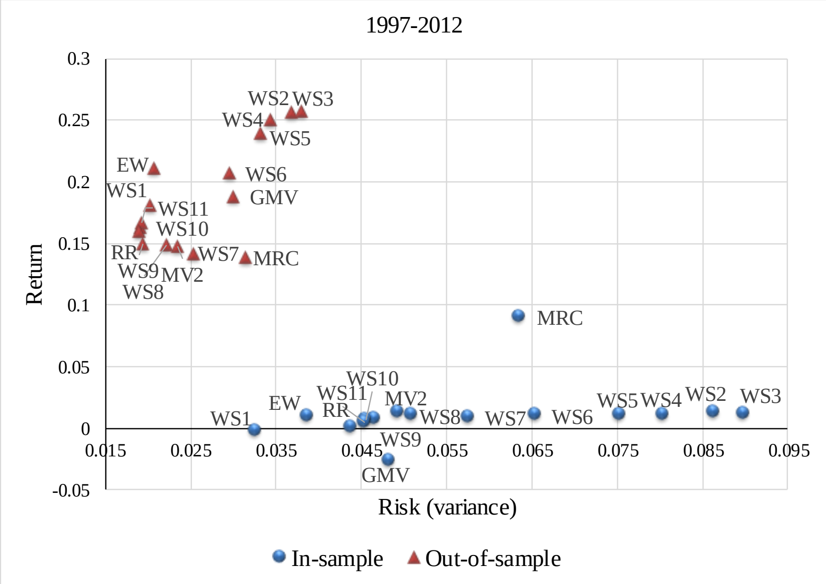


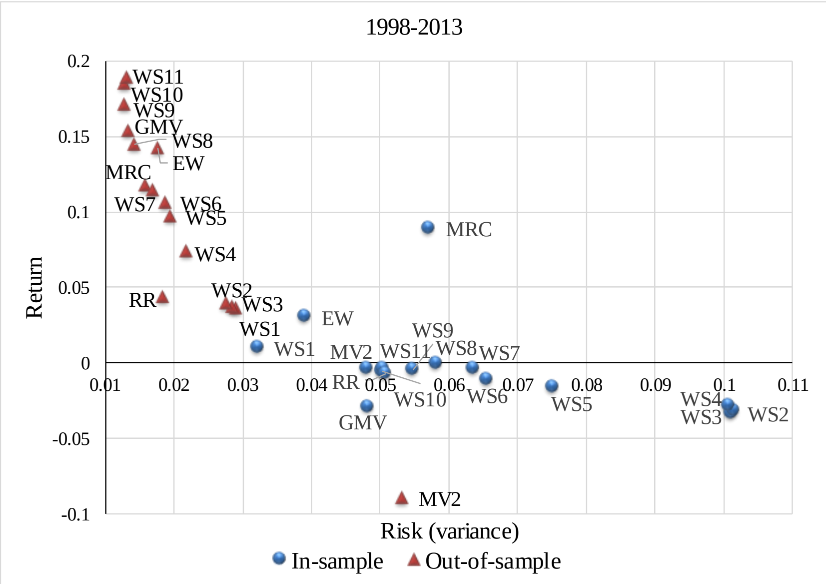


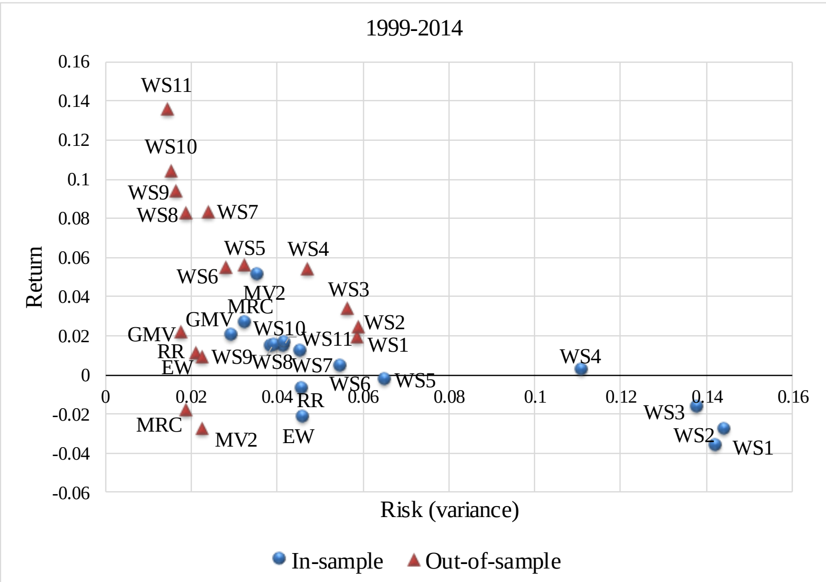

1994-2009
Return
Risk (variance)
0.17
0.12
0.07
0.02
-0.03
0.03
0.05
0.07
0.09
0.11
0.13
0.15
GMV
EW
RR
WS11
WS10
MV2
MRC
WS9
WS8
WS7
WS1
WS2
WS3
WS4
WS5
WS6
WS7
WS8
WS9
WS10
MV2
MRC
EW
WS4
WS2
WS1
In-sample
Out-of-sample

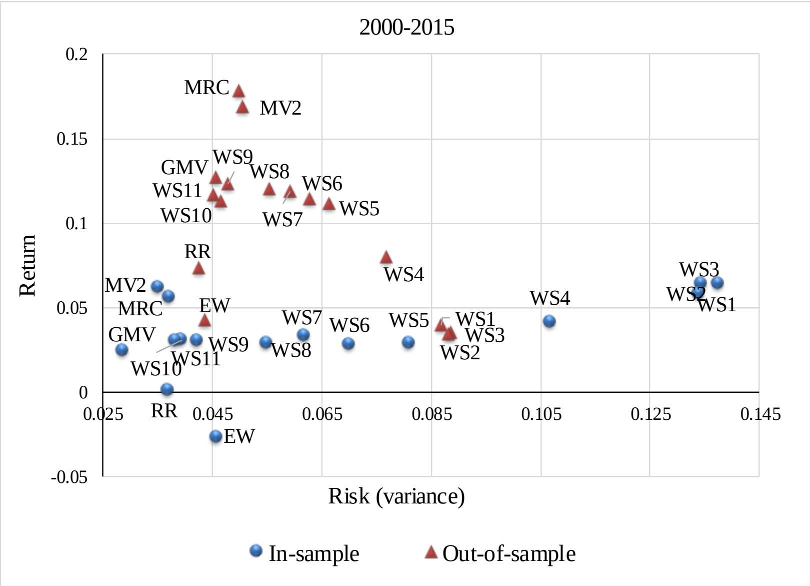

1995-2010
Return
Risk (variance)
0.29
0.24
0.19
0.14
0.09
0.04
-0.01
-0.06
0.015
0.025
0.035
0.045
0.055
0.065
WS1
WS2
MV2
WS4
WS3
RR
EW
WS5
MRC
GMV
WS6
WS7
WS8
WS9
WS10
WS11
MV2
MRC
WS3
WS2
WS1
WS6
GMV
WS7
WS5
WS4
WS8
WS10
WS11
RR
WS9
EW
In-sample
Out-of-sample

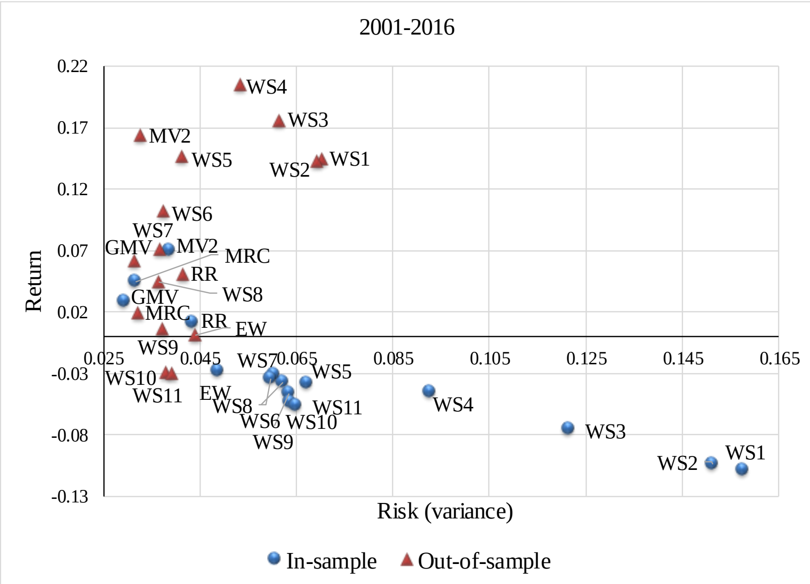

1996-2011
Return
Risk (variance)
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
MRC
EW
WS7
WS8
WS6
WS9
WS1
MV2
WS10
RR
GMV
WS11
WS5
WS4
WS2
WS3
WS10
WS11
WS9
EW
RR
WS8
WS1
WS7
MV2
WS6
WS3
WS5
WS2
WS4
MRC
GMV
In-sample
Out-of-sample


1997-2012
Return
Risk (variance)
In-sample
Out-of-sample
WS2
WS3
WS4
WS5
EW
WS6
WS1
GMV
WS11
WS10
RR
WS7
MRC
WS9
MV2
WS8
MRC
WS10
WS11
EW
RR
MV2
WS1
WS8
WS7
WS6
WS5
WS4
WS2
WS3
WS9
GMV
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0.015
0.025
0.035
0.045
0.055
0.065
0.075
0.085
0.095


1998-2013
Return
Risk (variance)
In-sample
Out-of-sample
WS11
WS10
WS9
GMV
WS8
EW
MRC
WS7
WS6
WS5
WS4
MRC
RR
WS2
WS3
EW
WS1
WS1
WS9
MV2
WS11
WS8
WS7
RR
WS10
WS6
WS5
GMV
WS4
WS3
WS2
MV2
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
0.11


1999-2014
Return
Risk (variance)
In-sample
Out-of-sample
WS11
WS10
WS9
WS8
WS7
WS5
WS4
WS6
MV2
WS3
MRC
GMV
WS10
WS2
GMV
WS11
WS1
RR
WS9
WS8
WS7
WS4
EW
WS6
WS5
RR
WS3
MRC
WS2
MV2
EW
WS1
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16





The optimal portfolios were computed considering the parameters combination γ=2, S=100 and J=120.

Regarding WS portfolios, some unexpected outcomes are observed. While the WS portfolios corresponding to the highest c_1^g weights always present better in-sample performances comparatively to the WS portfolios corresponding to the lowest c_1^g weights, this does not always occur out-of-sample. Specifically, in some windows (1995-2010; 1997-2012; and 2001-2016), WS1, WS2, WS3 and WS4 portfolios present higher S_I than WS6, WS7, WS8, WS9, WS10 and WS11 portfolios; moreover, WS10 and WS11 portfolios show negative out-of-sample returns in 1995-2010 and 2001-2016, and are the only portfolios with negative out-of-sample returns in 1995-2010. In other windows (1992-2007 and 1994-2009), WS1, WS2, WS3, WS4 and WS5 portfolios are the only portfolios that present negative out-of-sample returns. These results unveil lack of consistency in the out-of-sample performance of the relative robust methodology presented by Xidonas et al. (2017c).

Some lack of consistency can also be evidenced by the MRC strategy. This absolute robust portfolio is among the solutions with best performance in some of the windows under analysis (1992-2007 and 2000-2015) while it is among those with worst performance in other windows (1994-2009; 1996-2011; 1997-2012; 1999-2014; and 2001-2016). In the particular period of 1999-2014, the MRC portfolio is one of only two solutions that present negative returns.

Table 5-4: Out-of-sample modified Sharpe ratio (S_I) of the RRA, MRC, MV, EW, GMV and WS portfolios.

Portfolio	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
RRA	0.411	-0.222	0.518	0.906	-0.036	0.954	0.325	0.071	0.358	0.270
MRC	0.814	-0.210	-0.005	0.779	-0.010	1.456	0.940	-0.003	0.801	0.130
MV	0.650	-0.255	0.068	1.347	-0.063	0.774	-0.021	-0.004	0.754	0.927
EW	0.571	-0.244	0.465	0.910	-0.067	1.079	1.073	0.053	0.206	0.024
GMV	0.563	-0.209	0.681	0.564	-0.024	1.264	1.341	0.157	0.600	0.373
WS1	-0.030	-0.259	-0.008	1.294	-0.067	1.331	0.214	0.076	0.136	0.559
WS2	-0.033	-0.252	-0.008	1.303	-0.067	1.317	0.220	0.096	0.121	0.557
WS3	-0.035	-0.239	-0.007	1.090	-0.067	1.341	0.240	0.137	0.119	0.724
WS4	-0.036	-0.237	-0.005	1.095	-0.065	1.304	0.505	0.245	0.293	0.905
WS5	-0.018	-0.235	-0.004	0.823	-0.050	1.197	0.700	0.305	0.435	0.743
WS6	0.000	-0.232	-0.001	0.526	-0.032	0.880	0.781	0.319	0.461	0.548
WS7	0.181	-0.210	0.180	0.189	-0.017	0.987	0.884	0.531	0.490	0.388
WS8	0.410	-0.195	0.256	0.000	-0.010	1.064	1.216	0.594	0.514	0.255
WS9	0.310	-0.189	0.300	-0.004	-0.003	1.170	1.518	0.723	0.567	0.054
WS10	0.383	-0.178	0.479	-0.007	-0.009	1.193	1.644	0.831	0.527	-0.005
WS11	0.418	-0.165	0.524	-0.007	-0.013	1.146	1.652	1.115	0.551	-0.005

This table shows the out-of-sample S_I of the optimal portfolios by out-of-sample year. Results are presented for the parameters combination γ=2, S=100 and J=120.

Our relative robust strategy stands out as a more solid relative robust methodology in this sense. First, the RRA portfolio only provides negative out-of-sample returns in periods where all the computed portfolios present negative out-of-sample returns, namely in 1993-2008 and 1996-2011[2]. These windows are characterized by out-of-sample periods of atypically high volatility (see assets risks variation in Figure 5-5) and, simultaneously, atypically low assets mean returns (see assets mean returns variation in Figure 5-6). Out-of-sample risk of the RRA portfolio is lower or similar to its in-sample risk and out-of-sample return of the RRA portfolio is always higher than its in-sample return, except for the windows where all portfolios severely underperform out-of-sample (1993-2008 and 1996-2011) and for the windows where RRA portfolios present higher out-of-sample risks but substantially higher out-of-sample returns (1994-2009 and 2000-2015). As previously pointed out, this tendency of higher out-of-sample returns can be explained by the evolution of the DAX index during the out-of-sample periods. These results generally prevail regardless of the values of the model parameters.

Additionally, the RRA portfolio stands out as a dominant solution (with higher out-of-sample S_I) comparatively to the MV portfolio in 5 of the 10 windows, to the EW portfolio in 6 of the 10 windows, to the MRC portfolio in 4 of the 10 windows, and to some of the WS portfolios in 9 of the 10 windows. Relative to the GMV, and as previously pointed out, RRA portfolio stands out as a dominated solution in 8 of the 10 windows, while it outperforms the GMV in 1995-2010 only, where it shows a higher S_I.

In order to better understand the consequences, from the investor perspective, of applying the proposed relative robustness approach in portfolio selection, we also compared the risk-adjusted out-of-sample performance of the RRA portfolio with the benchmarks used in the study, by analysing the abnormal return over the entire out-of-sample period (10 years). Results are presented in Table 5-5. As previously mentioned, the abnormal return corresponds to the Jensen's alpha in the CAPM framework, with the DAX price index as the proxy for the market portfolio return. We also present the beta of the portfolios, which is a measure of their systematic risk relative to the excess return of the market portfolio. It is important to notice that the comparability to the market portfolio (DAX index) has some limitations since the assets used to construct the portfolios vary along the 10 time windows under analysis and are not the same as the ones in the constituent list of the DAX index during this period. Regarding the results

[2] Notice that the out-of-sample periods of the windows 1993-2008 and 1996-2011 overlapped the subprime mortgage crisis and the European debt crisis, respectively.

presented in Table 5-5, it can be observed that, the GMV portfolio has the best performance, even outperforming the market proxy (presents positive abnormal return), which supports the findings previously described. Although presenting negative abnormal return, the RRA portfolio outperforms the MRC, the EW and the MV portfolios (presents higher abnormal return) and, thus, offers a better performance considering its systematic risk. Finally, it is important to highlight that all portfolios show a lower systematic risk than the market (proxy) with the exception of the EW portfolio, which presents a beta coefficient higher than 1.

Analyzing the distances between in-sample and out-of-sample portfolios location in Figure 5-4, it can be observed that the GMV and the RRA portfolios are more consistent comparatively to the MV and the EW portfolios. Relative to the WS portfolios and although previous results are generally confirmed, an opposite pattern can be observed in the window 1994-2009, since the WS portfolios corresponding to lower c_1^g weights are more consistent than the WS portfolios corresponding to higher c_1^g weights.

Analyzing the robustness in terms of the utility loss for the investor (Table 5-6), it can be observed that the MV portfolio and the WS portfolios with lowest c_1^g weights are the more robust portfolios (present the lowest regrets) in the windows characterized by out-of-sample low volatility and high return, like the sample periods 1992-2007, 1995-2000 and 2001-2016. For out-of-sample periods with atypical out-of-sample volatility and returns, like 1993-2008, 1994-2009 and 1996-2011, the more robust portfolios are the GMV portfolio, the WS portfolios with highest c_1^g weights and the RRA portfolio, since they have the lowest regrets. Following no trend concerning volatility or returns, the MRC portfolio stands out as one of the more robust solutions in some windows (1992-2007 and 2000-2015) while it presents the highest regrets in others (1994-2009, 1999-2014 and 2001-2016).

While in-sample results support the (in-sample) conclusion presented by Xidonas et al. (2017b), in particular the fact that the area of the Pareto front that corresponds to the portfolios with highest c_1^g weights (when minimizing risk is weighted more than maximizing return) provides more robust solutions in terms of the minmax regret criterion, out-of-sample results do not corroborate these conclusions. This fact underlines the relevance of the out-of-sample performance analysis of new portfolio optimization models.

Figure 5-5: Variation of the out-of-sample assets risks (variance) computed from daily returns.

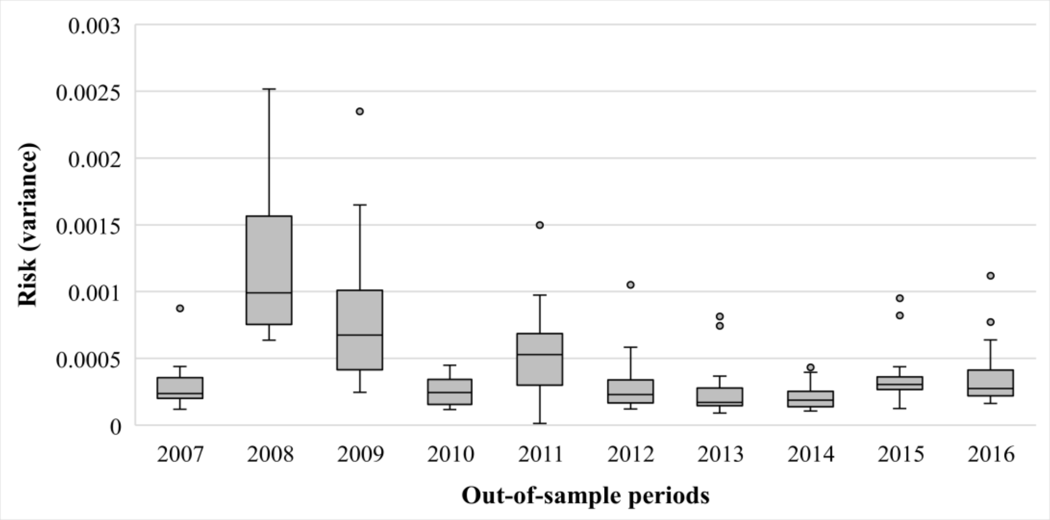
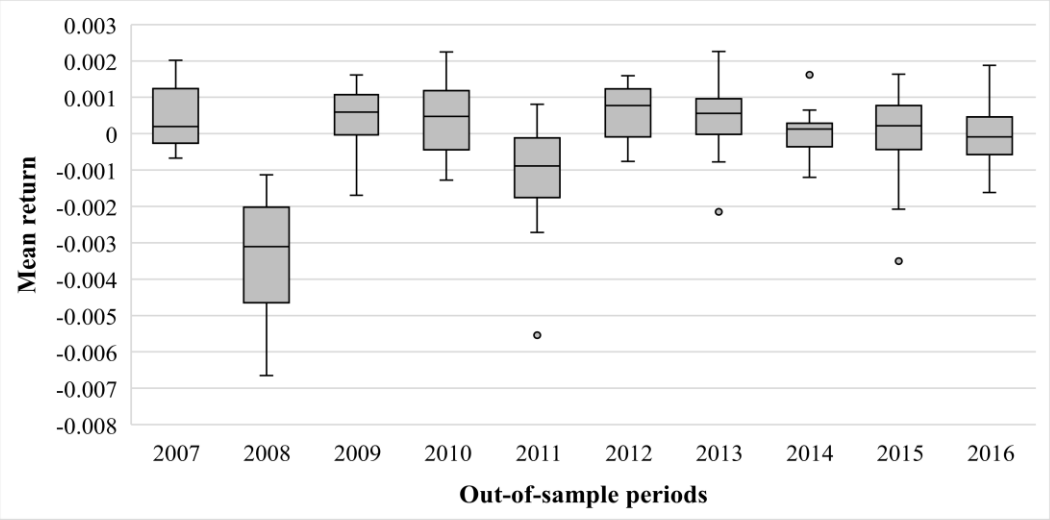


Figure 5-6: Variation of the out-of-sample assets mean returns computed from daily returns.



It is also important to highlight that the GMV and the RRA portfolios are the only ones that are never among the portfolios with the highest regrets. In our opinion, this result validates the proposed relative robust methodology, supporting its robustness.

Table 5-5: Abnormal returns (J_α) and beta of the RRA, MRC, MV, EW and GMV portfolios, computed over the entire out-of-sample period (10 years).

Portfolio	J_α (%)	Beta
RRA	-0.0515	0.8977***
MRC	-0.0584	0.8520***
MV	-0.0933	0.8525***
EW	-0.0714	1.0384***
GMV	0.0306	0.8133***

****Significance level of the regression coefficient (p-value<0.001). The estimation of the abnormal return was based on the CAPM, where the DAX index was used as the proxy for the market portfolio. A total of 120 monthly observations (corresponding to the monthly returns of the 10 out-of-sample years), were used in order to compute monthly abnormal returns.*

Table 5-6: Out-of-sample regret of the RRA, MRC, MV, EW, GMV and WS portfolios.

Portfolio	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
RRA	0.295	1.753	0.221	0.295	0.407	0.197	0.398	0.334	0.285	0.353
MRC	0.240	1.429	0.443	0.323	0.249	0.147	0.330	0.363	0.198	0.377
MV	0.263	2.342	0.356	0.241	0.603	0.209	0.592	0.377	0.205	0.246
EW	0.273	2.281	0.231	0.295	0.617	0.169	0.311	0.338	0.316	0.409
GMV	0.274	1.421	0.190	0.353	0.332	0.169	0.299	0.320	0.237	0.334
WS1	0.546	2.399	0.387	0.232	0.621	0.124	0.415	0.361	0.358	0.287
WS2	0.562	2.302	0.384	0.235	0.622	0.124	0.414	0.356	0.364	0.288
WS3	0.575	1.915	0.378	0.269	0.623	0.128	0.410	0.343	0.365	0.255
WS4	0.582	1.837	0.368	0.271	0.607	0.134	0.373	0.313	0.305	0.226
WS5	0.474	1.782	0.364	0.314	0.490	0.155	0.350	0.299	0.267	0.265
WS6	0.368	1.752	0.345	0.356	0.375	0.203	0.342	0.297	0.261	0.301
WS7	0.331	1.328	0.295	0.403	0.285	0.195	0.333	0.267	0.255	0.330
WS8	0.294	1.053	0.279	0.431	0.236	0.192	0.307	0.263	0.250	0.355
WS9	0.310	0.959	0.269	0.463	0.189	0.182	0.286	0.251	0.242	0.396
WS10	0.297	0.769	0.235	0.481	0.235	0.179	0.275	0.242	0.251	0.438
WS11	0.291	0.568	0.226	0.488	0.261	0.185	0.273	0.215	0.247	0.440

This table shows the out-of-sample regret of the optimal portfolios by out-of-sample year. Results are presented for the parameters combination γ=2, S=100 and J=120.

The portfolio style analysis of the RRA portfolio is presented in Table 5-7. As previously mentioned, a total of 8 value (VP) and growth portfolios (GP) were used.

The calculated coefficients indicate a tilt toward assets with low CEP (cash earnings to price) ratio, and also some weight of assets with extreme (high or low) dividend yields.

Finally, the exposure of the RRA portfolio to risk factors was analysed and results are presented in Table 5-8. As previously described, we considered 4 factors in the regression analysis: the market excess return, SMB, HML, and MOM, for the German market, all retrieved from the Fama-French library. Results show that the coefficients (estimates) of the market excess return and the HML factors are significantly different from zero, and, thus, market excess return and the HML help explain the monthly returns of the RRA portfolio during the entire out-of-sample period. There is a positive relation with the market excess return, and a negative relation with the HML factor.

Table 5-7: Portfolio style analysis of the RRA portfolio

	Estimate	**SE**
BTM VP	0.000000	0.000000
BTM GP	0.000000	0.000000
EP VP	0.000000	0.000000
EP GP	0.000000	0.000000
CEP VP	0.000000	0.000000
CEP GP	0.631030	0.002544
DY VP	0.205116	0.000970
DY GP	0.163854	0.000755

Table 5-8: Exposure to risk factors of the RRA portfolio

	Estimate	**SE**
Intercept	-0.000426	0.001375
Market excess return	0.262292***	0.020748
SMB	-0.037726	0.051802
HML	-0.132032*	0.054058
MOM	-0.033553	0.029848

*Significance levels: *** p-value<0.001; * p-value<0.05.*

5.4 Refining the relative robust solution through the sampling procedure by applying validation subsamples (model B)

Based on the relative robust portfolio model A presented in the beginning of this chapter, we built on the definition of regret and the relative robust expected utility optimization model and propose a new way of defining the relative robust solution. We introduce validation subsamples, which are randomly generated from the in-sample data and used to evaluate the performance of the minmax regret portfolios, in order to define the relative robust solution. Furthermore, and since there is a lack of empirical studies comparing the performance of relative robust and absolute robust portfolios, we present the corresponding absolute robust expected utility optimization model and compare the performances of the relative robust and absolute robust solutions.

For the empirical analysis, we considered the same dataset: a historical period from January 1992 to December 2016 of the stocks of the DAX index, collected from Thomson Reuters Datastream. We applied also the same rolling window procedure but, in this case, we considered long and short rolling windows with constant lengths of 16-years and 5-years, respectively. The resulting historical windows were previously described in section 4.2.

The different methodological strategies considered are the proposed relative robust (RRB) and absolute robust (ARB) portfolios, and some of the benchmarks described in section 5.3.2 used for comparison with the performance of the RRA portfolio, namely the MV portfolio, the GMV portfolio, the EW portfolio and the WS portfolios. Inputs for the MV and the GMV models were estimated within the entire in-sample window, namely the in-sample mean and the in-sample covariance matrix were calculated considering 15-years data or 4-years data, according to the length of the rolling time windows considered. The performances of the different computed portfolios are compared considering both in-sample and out-of-sample data, for return, risk, modified Sharpe ratio and regret.

The results of this model suggest that reducing the in-sample period length increases the exposure of the computed portfolios to individual stocks while it seems to improve the overall out-of-sample performance of the ARB, the RRB and the GMV portfolios and substantially deteriorates the out-of-sample performance of the MV portfolio.

Regardless of the in-sample period length, it can be observed that the RRB portfolio is highly diversified, assigning non-zero weights to the majority of the assets.

The overall results support previous findings concerning the sensitivity of the MV portfolio to the estimation error and the effects of the input uncertainty in the optimization process (Best & Grauer, 1991a; Chopra & Ziemba, 1993; DeMiguel et al., 2009; Jagannathan & Ma, 2003), as well as the outperformance of the GMV portfolio (Chan et al., 1999; Jagannathan & Ma, 2003).

Furthermore, our results suggest that the proposed relative robust model B generates optimal portfolios that consistently present low risk, and (non-negative) attractive returns, standing out as one of the very few optimal portfolios with no poor performances. In fact, the RRB portfolio outperforms the MV portfolio and the EW portfolio in many of the windows under analysis, even outperforming the GMV portfolio in some windows. This highlights the relevance of the proposed methodology among non-robust portfolio optimization methodologies. When compared to the minmax regret model presented by Xidonas et al. (2017b), the proposed relative robust approach seems to provide more consistent results considering the out-of-sample performance of the generated portfolios.

Finally, the comparison of the proposed relative robust and absolute robust models leads to important conclusions concerning the real benefits of the proposed methodology from the investors' perspective. The findings suggest that the proposed RRB portfolio generally outperforms the proposed ARB portfolio, even when higher levels of risk aversion are considered.

The remainder of the section proceeds as follows. In Section 5.4.1, the methodology is presented. Here, a different structure is followed in order to highlight the main differences of the relative robust portfolios proposed in model A and model B. We start by explaining the process for computing the relative robust solution (RRB portfolio) and present and describe the corresponding absolute robust expected utility optimization problem. In Section 5.4.2, we describe the empirical analysis and main results.

5.4.1 Methodology

5.4.1.1 Defining the RRB portfolio

Consider the notation and the CRRA utility function previously presented in section 5.2.1. As already mentioned, we build on the relative robust expected utility maximization problem presented in (5.9), and propose a new definition for the relative robust solution. The new way of defining the relative robust solution is explained presenting the steps followed in its computation.

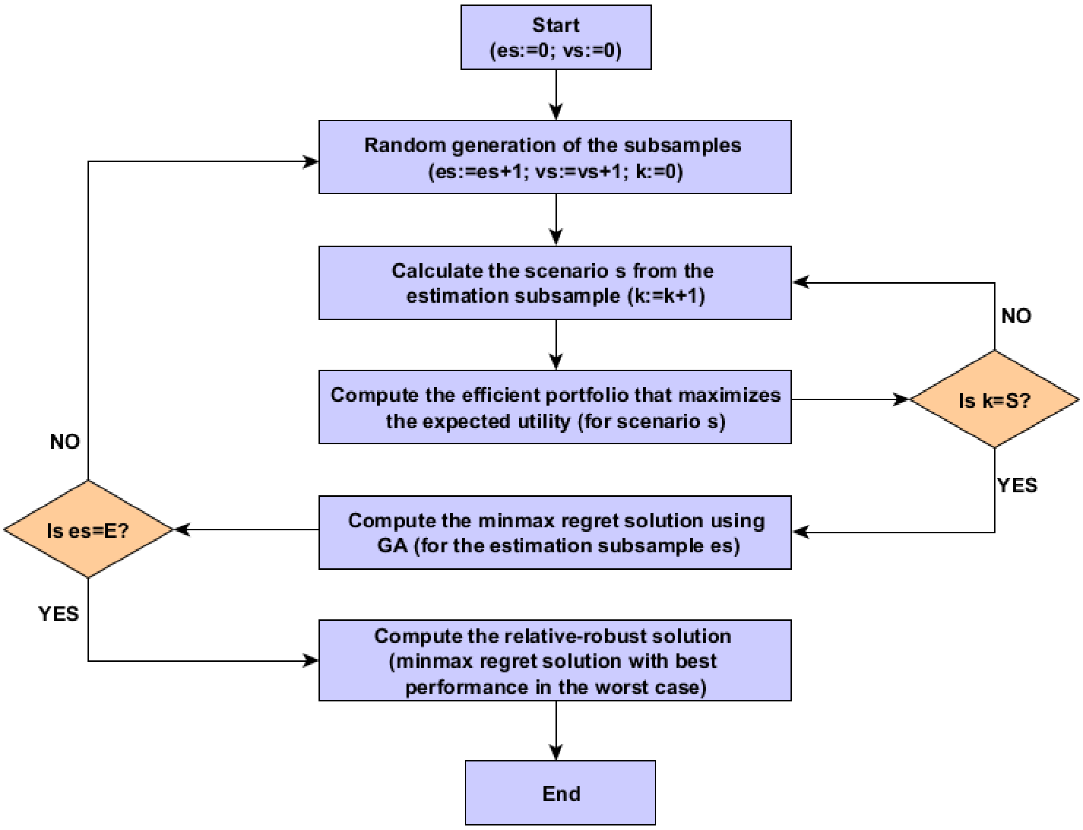
To avoid reaching a solution strongly dependent on the underlying in-sample data, two disjoint subsamples are randomly generated from the in-sample data: an estimation subsample used to estimate the inputs of the model and a validation subsample used to evaluate its performance. From the estimation subsample, an uncertainty set U is constructed by calculating the S scenarios. We followed the same procedure as in model A for building the uncertainty set, which is described in section 5.2.1. Then, for each scenario $s \in U$ the maximum expected utility problem, as defined in (5.10), is solved, in order to determine the optimal solution w_t^{s*}(we followed the steps previously described in section 5.2.2). After computing the optimal solutions for each scenario $s \in U$, the relative robust optimization problem, as presented in (5.9) is solved using GA. Except for the initial population, which in this case was composed only of feasible solutions randomly generated, the GA options used in this application were the same as the ones previously described and applied in model A.

The random generation of both estimation and validation subsamples and the GA optimization is repeated E times, producing different uncertainty sets $(U_e, e = 1, \dots, E)$ and sets of different minmax regret solutions and different validation subsamples. The estimation and validation subsamples are generated at the same time, guaranteeing that they do not overlap. The validation subsamples are used in a later stage, to assess each one of the minmax regret solutions computed. It is, thus, possible to guarantee that each minmax regret solution is tested in at least one validation subsample that does not overlap with its estimation subsample.

For testing each minmax regret solution in each one of the validation subsamples, regret was defined as the difference between the utility of the optimal portfolio (solution on the Markowitz's efficient frontier that maximizes the investor's expected utility) in the validation subsample period and the utility of the minmax regret solution. Then, the corresponding maximum regret is found. By considering the worst-case for each minmax regret solution, all possible bias, deriving from testing these solutions in validation subsamples that could overlap with their corresponding estimation subsamples, is avoided. Finally, the RRB portfolio defined as the minmax regret solution presenting the best performance in the worst-case is identified.

Figure 5-7 illustrates the flowchart of the proposed methodology for computing the relative robust portfolio.

Figure 5-7: Flowchart of the method for computing the relative robust solution.



5.4.1.2 The absolute robust expected utility optimization model

The absolute robust portfolio optimization problem (model B), which admits the worst-case realization within the uncertainty set for the uncertain parameters, is defined as follows:

$$\max_{w_t \in X} \min_{s \in U} E\left[f\left(1 + r_{t+1}^{p}(w_t)\right)\right] = \max_{w_t \in X} \min_{s \in U}\left(\frac{\left(1 + \mu^{s\prime} w_t\right)^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2}\left(1 + \mu^{s\prime} w_t\right)^{-\gamma-1} w_t{}' \Sigma^{s} w_t\right) \tag{5.13}$$

where U is defined as $U = \{U_e, e = 1, \dots, E\}$. Hence, the ARB portfolio (maxmin solution) corresponds to the weight combination vector w_t that solves problem (5.13).

Concerning the computation of the ARB portfolio, a similar process was applied. From each estimation subsample, an uncertainty set U is constructed as previously described. After computing the S scenarios, the maxmin solution is calculated by solving problem (5.13) using the GA. In this case, the fitness function was defined as the minimum portfolio's expected utility for all the scenarios considered in U (inner maximization problem in (5.13)). Hence, the optimization is performed assuming the worst-case performance over the whole uncertainty set. The initial population and the GA options used were the same as the ones defined in section 5.4.1.1. Then, each maxmin solution is tested in each one of the validation subsamples. The expected utility of the maxmin solution in each validation subsample is calculated and the minimum expected utility is considered. Finally, the absolute robust portfolio defined as the maxmin solution presenting the best performance in the worst-case (highest minimum expected utility) is identified.

5.4.2 Empirical analysis

5.4.2.1 Model settings

The steps for computing the absolute robust and relative robust solutions are iteratively repeated for each of the time windows defined. For each of the time windows under analysis, the random generation of the estimation and validation subsamples and the GA optimization are repeated 100 times ($E = 100$). A constant length of 1-year observations was defined for each subsample; hence, the estimation and validation subsamples correspond to a 2-years in-sample period of consecutive daily returns. A total of 100 scenarios ($S = 100$) are computed for each estimation subsample. Each scenario, defined by the sample mean and the sample covariance matrix, is computed considering an estimation window length of 120 consecutive daily returns. Estimations of the model inputs are performed in R.

To explore the sensitivity of the results to the variation of the relative risk aversion parameter, different values ($\gamma \in \{0.5,2,5\}$) were explored and the absolute robust and relative robust models as well as the classical MV model were solved for each one of these values. As explained in section 5.3.1, the different values of γ define different ARB, RRB and MV portfolios while they only influence the corresponding measure of regret of the GMV, EW and WS portfolios.

After determining ARB, RRB, MV, GMV, EW and WS1 to WS11 portfolios, in-sample and out-of-sample performances were compared by analyzing the portfolios annualized return, variance and (modified) Sharpe ratio, as defined in section 4.5. In addition, the regret, as defined in (5.12), was calculated and compared for the in-sample and out-of-sample periods.

5.4.2.2 Results

The effect of the variation of both the in-sample period length and the relative risk aversion parameter is analyzed by comparing the composition and the performance of the RRB, the MV and the GMV portfolios. In particular, the composition of the portfolios concerning the maximum weight of an asset (Max%), the sum of the 3 largest weights in the portfolio (Sum3Max%) and the number of assets with non-zero weights in each computed portfolio (Cardinality) are identified. Mean values obtained over the 10 windows are presented. Since the optimal portfolios have some assets with very small but not necessarily zero weights, we measure cardinality as the number of assets with weights higher than 0.1%. Acknowledging the limitations of using the average return as the sole comparison measure, the portfolios' performances are analyzed, both in-sample and out-of-sample, by comparing the mean of the portfolios' returns (mean return) and the mean of the portfolios' variances (mean risk), obtained over the 10 windows. Additionally, the mean of the portfolios' regrets (mean regret) and the mean of the portfolios' (out-of-sample) modified Sharpe ratio (S_I), obtained over the 10 windows, are also analyzed for all the computed portfolios. The consistency of the portfolios in terms of the proximity to their expected performance is assessed by comparing the in-sample and out-of-sample results. The portfolios' regrets reflect the robustness of the optimal solutions in terms of the utility loss for the investor resulting from choosing a given portfolio instead of choosing the optimal portfolio of the realized scenario for the uncertain parameters.

Then, the composition and the in-sample and out-of-sample performances of relative robust and non-robust portfolios are compared for each of the 10 windows. Finally, the performances of the RRB and the ARB portfolios are compared in each one of the out-of-sample years. Results are presented for the in-sample period length and the values of the risk aversion parameter associated with the best (mean) performances for both in-sample and out-of-sample datasets. For simplification purposes, the RRB and ARB portfolios will be represented by 'RR' and 'AR', respectively, in the figures presented in this Results section.

5.4.2.2.1 Effect of the variation of the in-sample period length and the risk aversion parameter

Table 5-9 presents the composition of the RRB, MV and GMV portfolios by length of the in-sample period and/or the value of the risk aversion parameter considered for their computation. Only average results for the 10 windows are shown. It can be observed that, for higher levels of risk aversion and regardless of the in-sample period length, the exposure of the RRB and MV portfolios to individual stocks decreases since the mean of the maximum values of the assets' weights (Max% and Sum3Max%) decreases and the mean number of assets with non-zero weights (Cardinality) increases.

Furthermore, when the length of the in-sample period is reduced (from 15 to 4 years), the exposure of the portfolios to individual stocks increases in the three optimal portfolios, since the mean of the maximum values of the assets' weights (Max% and Sum3Max%) increases and the mean number of assets with non-zero weights (Cardinality) decreases. This result is verified regardless of the risk aversion parameter's value, except for the RRB portfolio computed using a value of 0.5 for the risk aversion parameter. Although the cardinality slightly decreases for this portfolio, the mean of the maximum values of the assets' weights decreases when the in-sample length is reduced. It is also possible to observe that, regardless of the in-sample period length, the MV portfolio is the less diversified portfolio while the RRB portfolio is the most diversified one.

Table 5-9: Composition of the RRB, MV and GMV portfolios by length of the in-sample period and/or the value of the risk aversion parameter considered for their computation.

Portfolio	Max%	Sum3Max%	Cardinality
RRB40.5	0.323	0.723	18
RRB42	0.215	0.519	19
RRB45	0.144	0.353	22
RRB150.5	0.500	0.794	19
RRB152	0.168	0.422	21
RRB155	0.113	0.278	23
MV40.5	0.893	1.000	2
MV42	0.625	0.925	4
MV45	0.462	0.881	5
MV150.5	0.733	0.956	3
MV152	0.393	0.797	6
MV155	0.251	0.611	8
GMV4	0.263	0.600	10
GMV15	0.175	0.470	13

This table presents the characteristics of the optimal portfolios. Here, the composition of the portfolios regarding the maximum weight of an asset (Max%), the sum of the 3 largest weights in the portfolio (Sum3Max%), and the number of assets with non-zero weights in each computed portfolio (Cardinality) are described. As explained before, for measuring the cardinality, only those assets with weights higher than 0.1% are considered. Only average results for the 10 windows are shown. The optimal portfolios were represented according to the length of the in-sample period and the value of the risk aversion parameter used in their computation. For instance, 'RRB155', 'MV155' and 'GMV15' corresponds, respectively, to the RRB, MV and GMV portfolios computed using 15-years data to perform in-sample estimations and, in the case of the RRB and the MV portfolios, using a value of 5 for the risk aversion parameter.

The in-sample and the out-of-sample mean risk and mean return of the RRB, MV and GMV portfolios are presented in Figure 5-8. The optimal portfolios were represented according to the length of the in-sample and/or the value of the risk aversion parameter considered in their computation. This means that RRB155 represents the relative robust portfolio computed using an in-sample period of 15 years and a value of 5 for the risk aversion parameter while GMV4 represents the global minimum variance portfolio based on an in-sample period of 4 years, for instance.

Analysing the effect of the length of the in-sample period in the portfolios' performance, it can be observed that, when reducing the length of the in-sample period (from 15 years to 4 years), the in-sample mean return and mean risk of the MV portfolio increase. Notice that, while the mean return suffers a substantial increase regardless of the value of the risk aversion parameter, the increment of the mean risk is more accentuated for lower risk aversion. A similar trend is observed among the RRB portfolios. Reducing the length of the in-sample period (from 15 years to 4 years) increases both the in-sample mean return and mean risk of the RRB portfolios. It is also quite evident that the window length seems to have a more substantial effect in the in-sample mean return of the RRB portfolio, since the mean risk is very similar when the risk aversion parameter is 5 or 2; for a value of the risk aversion parameter of 0.5, the overall in-sample performance of the RRB portfolio is improved when reducing the length of the in-sample period since the RR40.5 portfolio presents higher mean return and lower mean risk comparatively to the RR150.5 portfolio. Regarding the GMV portfolio, its overall in-sample performance is improved when reducing the length of the in-sample period as the in-sample mean return substantially increases while the in-sample mean risk decreases.

Except for the parametrization described by an in-sample period of 15 years and a value of 5 for the risk aversion parameter, it can be confirmed that the RRB always reveals lower in-sample mean risk comparatively to the corresponding MV portfolio (for a given parametrization). Additionally, the MV portfolio presents the best (highest) in-sample mean return and the GMV portfolio presents the best (lowest) in-sample mean risk.

Concerning the effect of the in-sample period length in the out-of-sample performance of the optimal solutions, it can be observed that, when the length is reduced, the out-of-sample mean return of the RRB portfolio is improved, regardless of the value of the risk aversion parameter. On the other hand, reducing the length of the in-sample period seems to improve the out-of-sample mean risk when the risk aversion parameter is 5 or 0.5; for a value of 2, the mean risk slightly worsens (from 0.054 to 0.061, approximately) when the in-sample period length is reduced. It is also important to highlight that the RR152 is the only RRB portfolio that presents negative (-0,010%, very close to 0%) out-of-sample mean return.

Analyzing the MV portfolio, its general out-of-sample performance deteriorates when reducing the in-sample period length since the mean return decreases while its mean risk increases, regardless of the value of the risk aversion parameter. It is also quite clear that the overall out-of-sample performance of the MV portfolio is improved when the value of the risk aversion parameter is increased, regardless of the in-sample period length used in its computation. The worst performances are presented by the MV42, MV40.5 and MV150.5 portfolios, which present the lowest (negative) out-of-sample mean returns and the highest out-of-sample mean risks.

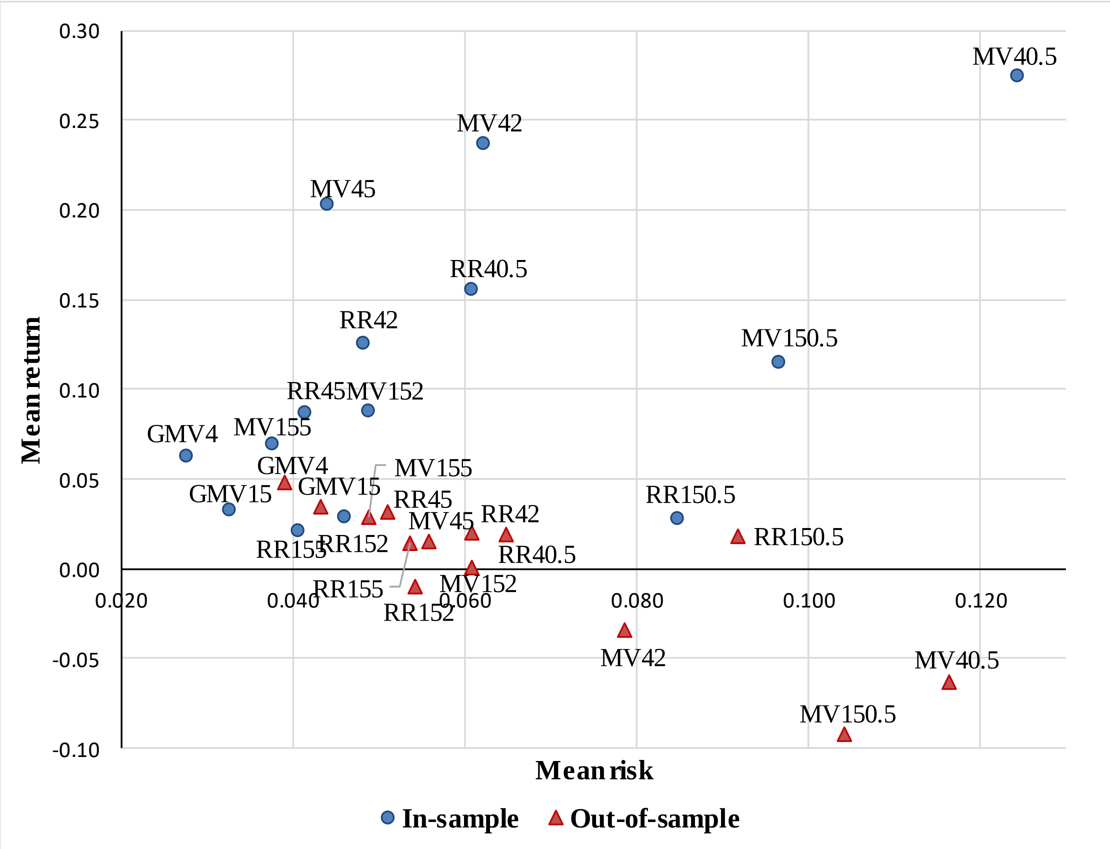
An opposite trend is observed for the GMV portfolio, since its out-of-sample mean return increases while its out-of-sample mean risk decreases, revealing an improvement in its overall out-of-sample performance. Here, an unexpected result is found regarding the GMV portfolio: although the exposure of the portfolio to individual stocks increased when reducing the length of the in-sample period, as previously verified (Table 5-9), both the in-sample and the out-of-sample mean risks of the GMV portfolio have decreased. A similar behaviour can be observed by the RRB portfolios computed using a risk aversion parameter of 5 or 0.5; for the remaining portfolios, the decrease in the diversification level caused, as expected, an increase in their mean risk.

It is important to highlight that, when reducing the length of the in-sample period, the RRB becomes a dominant solution when compared to the corresponding MV portfolio since it presents better out-of-sample performance in all of the performance measures considered for the analysis. Furthermore, regardless of the length of the in-sample period and the value of the risk aversion parameter, the GMV portfolio is a dominant solution when compared to the RRB and the MV portfolio, presenting better out-of-sample performance in all of the performance measures under consideration. This result is in accordance with previous studies (Chan et al., 1999; Jagannathan & Ma, 2003) supporting the outperformance of the GMV portfolio.

The distances between in-sample and out-of-sample portfolios' locations can be observed in Figure 5-8, allowing us to draw some conclusions about the consistency between the expected and the realized performances of the portfolios. The portfolios that present the smallest distances between in-sample and out-of-sample mean return and mean risk and, therefore, are more consistent concerning the proximity to the expected performance, are the GMV portfolio (regardless of the length of the in-sample period) and RRB portfolio computed using the in-sample period length of 15 years (regardless of the risk aversion parameter).

The same cannot be stated for the MV portfolio and the RRB portfolio computed using an in-sample period length of 4 years. The proximity to the expected performance of the optimal portfolio RRB seems to be negatively affected when the length of the in-sample period is reduced, especially when mean returns are considered. It is also important to highlight that the proximity to the expected performance of the RRB portfolio computed using an in-sample period length of 4 years seems to be improved when higher levels of risk aversion are considered.

Figure 5-8: In-sample and out-of-sample mean risk and mean return of the RRB, MV, GMV portfolios.



The optimal portfolios were represented according to the length of the in-sample period and the value of the risk aversion parameter used in their computation. For instance, 'RRB155', 'MV155' and 'GMV15' corresponds, respectively, to the RRB, MV and GMV portfolios computed using 15-years data to perform in-sample estimations and, in the case of the RRB and the MV portfolios, using a value of 5 for the risk aversion parameter.

Considering the MV portfolio, it presents the largest distances between in-sample and out-of-sample portfolio locations, with the higher distance occurring for the MV computed based on an in-sample period of 4 years (regardless of the risk aversion parameter). Thus, reducing the length of the in-sample period from 15 to 4 years seems to negatively affect the proximity to the expected performance of the MV solution. Increasing the value of the risk aversion parameter seems to positively affect the proximity to the expected performance of the MV solution since the distances between in-sample and out-of-sample mean return and mean risk decrease. This is a similar situation as observed for the RRB portfolio computed based on an in-sample period of 4 years.

The out-of-sample (OS) performances of the optimal relative robust and non-robust portfolios concerning regret and modified Sharpe ratio (S_I) are described in Table 5-10. The results are presented according to the in-sample period length and for a risk aversion parameter equal to 5. Even though the results are not presented for the remaining values of the risk aversion parameter, the conclusions are valid for all the parametrizations considered. Notice that, the computation of the EW and WS portfolios is not influenced by the variation of the in-sample period length or the risk aversion parameter. The values presented correspond to the mean of the portfolios' regrets and mean of the portfolios S_I, obtained over the 10 time windows.

It can be observed that reducing the in-sample period length leads to lower levels of out-of-sample mean regret for the RRB and GMV portfolios. In the case of the MV portfolio the mean regret has a slight increase. Concerning the mean S_I, the results confirm that reducing the in-sample period length seems to substantially improve the performance of the RRB and GMV portfolios, while deteriorating the overall out-of-sample performance of the MV portfolio. Regarding the EW and the WS portfolios, which are not influenced by the in-sample period length, the results are already described in section 5.3.3.1.

Finally, it is important to outline some relevant results based on an in-sample period of 4 years. The RRB presents itself as a more robust solution than the MV, the EW and the majority of the WS portfolios, in terms of utility loss for the investor, since it shows a lower out-of-sample mean regret. Comparing the out-of-sample mean S_I of all the computed portfolios, it is possible to observe that the RRB, GMV and WS11 portfolios show similar performances, with the best performance achieved by the RRB portfolio. On the contrary, the worst performances are observed for the EW portfolio and the WS portfolios, corresponding to lower c_1^g weights.

Table 5-10: Out-of-sample regret and modified Sharpe ratio (S_I) of the RRB, MV, GM, EW and WS portfolios.

Portfolio	**OS Regret**		**OS S_I**	
	15Years	**4Years**	**15Years**	**4Years**
RRB	0.25350	0.19833	0.48734	0.57761
MV	0.19994	0.21983	0.48688	0.43926
GMV	0.18166	0.15336	0.53089	0.54749
EW	0.32394		0.40703	
WS1	0.40411		0.32463	
WS2	0.40878		0.32534	
WS3	0.40717		0.33024	
WS4	0.37646		0.40022	
WS5	0.29299		0.38958	
WS6	0.23405		0.32499	
WS7	0.19830		0.36022	
WS8	0.18212		0.41041	
WS9	0.17906		0.44457	
WS10	0.19029		0.48574	
WS11	0.19336		0.52163	

This table presents the OS performances of the optimal portfolios concerning regret and modified Sharpe ratio (S_I) by length of the in-sample period and considering a value of 5 for the risk aversion parameter. The values presented correspond to the mean of the portfolios' regrets and mean of the portfolios S_I, obtained over the 10 time windows. For computing the mean regret of the RRB portfolio, the worst result was not considered since it showed an atypical regret value (twenty-two hundred times higher than the highest of the remaining values) in the out-of-sample year of 2008.

The analysis of the in-sample and the out-of-sample performances of relative robust and non-robust portfolios, for each of the 10 windows, is presented next. Since the RRB portfolio, as well as the GMV portfolio, generally present better performances for the in-sample period length corresponding to 4 years of historical data and/or a value of 5 for the risk aversion parameter, results are described for this particular case. However, we note that the results that will be presented in the next section generally prevail regardless of the length of the in-sample period and the risk aversion parameter.

5.4.2.2.2 Performance of relative robust and non-robust portfolios

Before analyzing the performance of the optimal portfolios, it is important to examine their characteristics concerning composition in order to identify similarities as well as the main differences between robust and non-robust solutions. The results are shown in

Table 5-11. The RRB portfolio reveals a minimum cardinality of 19 and a maximum of 27, while it presents a maximum asset weight varying between 10% and 35%, and a maximum investors' wealth allocation to 3 assets varying between 25% and 61%. The MV portfolio presents a low diversification level since its cardinality is between 3 and 7, it has a maximum asset weight varying between 23% and 68% and, for most of the windows, it allocates 81% or more of the investors' wealth to only 3 assets. Comparatively to the MV portfolio, the GMV portfolio is more diversified, presenting a minimum cardinality of 8 and a maximum of 12, and a lower exposure to individual stocks, since the maximum weight of an asset varies between 18% and 34% while the sum of the 3 largest weights varies between 48% and 70%. Although the results obtained for the EW and the WS portfolios are already presented in Table 5-2 and described in section 5.3.3.2, we decided to report them once again in Table 5-11 in order to facilitate the comparison of the results.

Therefore, and comparatively to all the optimal solutions with the exception of the EW portfolio, the RRB portfolio has the lowest maximum asset weight and the lowest maximum wealth allocation to 3 assets, revealing itself as the optimal solution with lower exposure to individual stocks. Regarding cardinality, similar results can be observed when comparing the RRB and the EW portfolios, suggesting that the proposed methodology tends to assign non-zero weights to the majority of the assets. It is important to highlight that these results are not in accordance with the results concerning the absolute robust portfolios described in the literature (Kim et al., 2013a; Kim et al. 2014b).

Table 5-11: Composition of the RRB, MV, EW, GMV and the WS portfolios by time window.

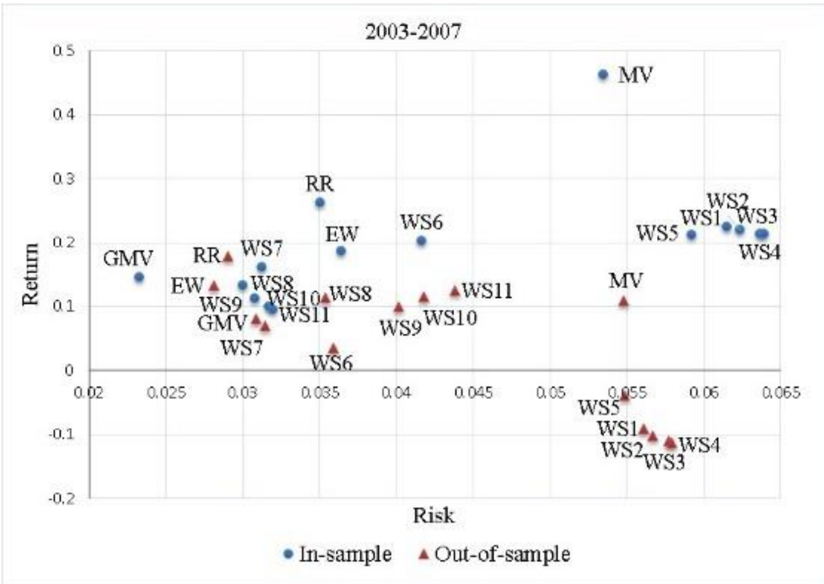
		RRB	**MV**	**EW**	**GMV**	**WS 1**	**WS 2**	**WS 3**	**WS 4**	**WS 5**	**WS 6**	**WS 7**	**WS 8**	**WS 9**	**WS 10**	**WS 11**
2003-2007	Max%	0.16	0.68	0.05	0.34	0.65	0.71	0.77	0.77	0.69	0.47	0.26	0.28	0.33	0.34	0.34
	Sum3Max%	0.41	1	0.15	0.70	1	1	1	1	1	0.74	0.62	0.61	0.63	0.59	0.59
	Card.	19	3	20	10	2	2	3	2	3	7	9	11	10	11	11
2004-2008	Max%	0.35	0.41	0.05	0.21	0.52	0.42	0.36	0.42	0.49	0.49	0.42	0.36	0.33	0.28	0.25
	Sum3Max%	0.61	0.93	0.16	0.48	1	0.87	0.90	0.91	0.82	0.74	0.70	0.69	0.65	0.66	0.67
	Card.	19	5	19	11	3	4	4	5	6	8	9	10	11	12	11
2005-2009	Max%	0.12	0.35	0.05	0.28	0.42	0.43	0.44	0.49	0.51	0.49	0.45	0.41	0.38	0.33	0.28
	Sum3Max%	0.31	0.81	0.15	0.60	1	1	1	1	0.99	0.97	0.91	0.90	0.84	0.76	0.73
	Card.	20	5	20	9	3	3	3	3	4	5	6	6	6	8	8
2006-2010	Max%	0.11	0.34	0.05	0.27	0.78	0.73	0.62	0.57	0.41	0.34	0.27	0.25	0.28	0.31	0.32
	Sum3Max%	0.31	0.75	0.14	0.55	1	1	1	0.99	0.79	0.70	0.69	0.62	0.66	0.71	0.73
	Card.	21	6	21	8	2	3	3	4	6	6	8	8	10	10	8
2007-2011	Max%	0.10	0.23	0.04	0.25	0.78	0.76	0.75	0.74	0.34	0.39	0.52	0.44	0.32	0.28	0.27
	Sum3Max%	0.27	0.63	0.13	0.57	1	1	1	0.99	0.75	0.70	0.76	0.76	0.83	0.63	0.56
	Card.	22	7	23	10	2	2	2	4	6	8	9	8	7	9	10

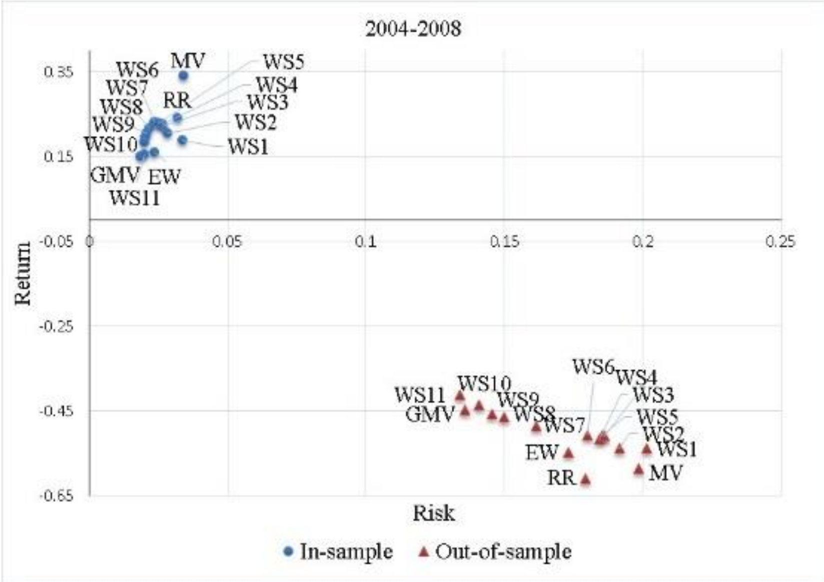
		RRB	MV	EW	GMV	WS 1	WS 2	WS 3	WS 4	WS 5	WS 6	WS 7	WS 8	WS 9	WS 10	WS 11
2008-2012	Max%	0.13	0.64	0.04	0.27	0.78	0.82	0.77	0.73	0.64	0.48	0.42	0.35	0.33	0.32	0.31
	Sum3Max%	0.37	0.90	0.12	0.67	1	1	1	1	0.93	0.94	0.87	0.78	0.80	0.81	0.76
	Card.	26	5	26	11	2	2	3	3	4	4	6	6	7	6	7
2009-2013	Max%	0.12	0.35	0.04	0.27	0.59	0.60	0.61	0.45	0.35	0.35	0.32	0.27	0.21	0.26	0.28
	Sum3Max%	0.36	0.96	0.13	0.63	1	1	1	0.87	0.81	0.82	0.76	0.67	0.55	0.60	0.65
	Card.	22	4	24	9	3	3	3	5	5	6	8	8	8	10	8
2010-2014	Max%	0.12	0.55	0.04	0.26	0.73	0.76	0.76	0.68	0.37	0.28	0.20	0.17	0.16	0.20	0.26
	Sum3Max%	0.31	0.92	0.12	0.62	1	1	1	0.98	0.74	0.68	0.54	0.45	0.46	0.53	0.56
	Card.	23	5	26	9	2	2	3	4	6	8	11	15	15	15	13
2011-2015	Max%	0.12	0.51	0.04	0.31	0.95	0.97	0.96	0.82	0.66	0.59	0.53	0.46	0.32	0.29	0.27
	Sum3Max%	0.32	0.91	0.12	0.68	1	1	1	1	0.90	0.86	0.81	0.80	0.70	0.68	0.68
	Card.	20	5	26	10	2	2	3	3	6	6	7	8	9	9	9
2012-2016	Max%	0.11	0.57	0.04	0.18	0.69	0.67	0.56	0.43	0.23	0.39	0.51	0.54	0.60	0.60	0.62
	Sum3Max%	0.25	1	0.11	0.49	1	1	1	0.94	0.66	0.72	0.84	0.83	0.84	0.86	0.83
	Card.	27	3	28	12	2	2	3	4	5	5	6	6	7	6	7

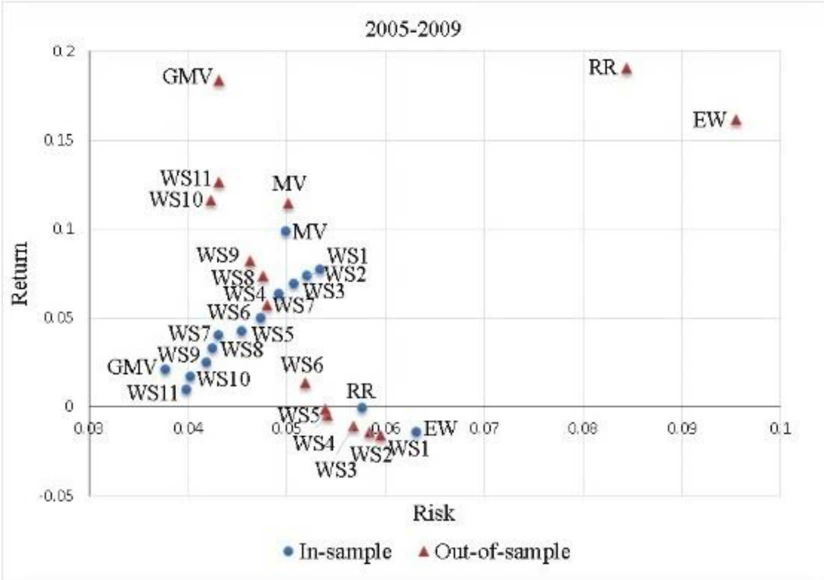
This table presents the characteristics of the optimal portfolios. The composition of the portfolios regarding the maximum weight of an asset (Max%), the sum of the 3 largest weights in the portfolio (Sum3Max%), and the number of assets with non-zero weights in each computed portfolio (Cardinality) is described. As explained before, for measuring the cardinality, only the number of assets with weights higher than 0.1% is considered.

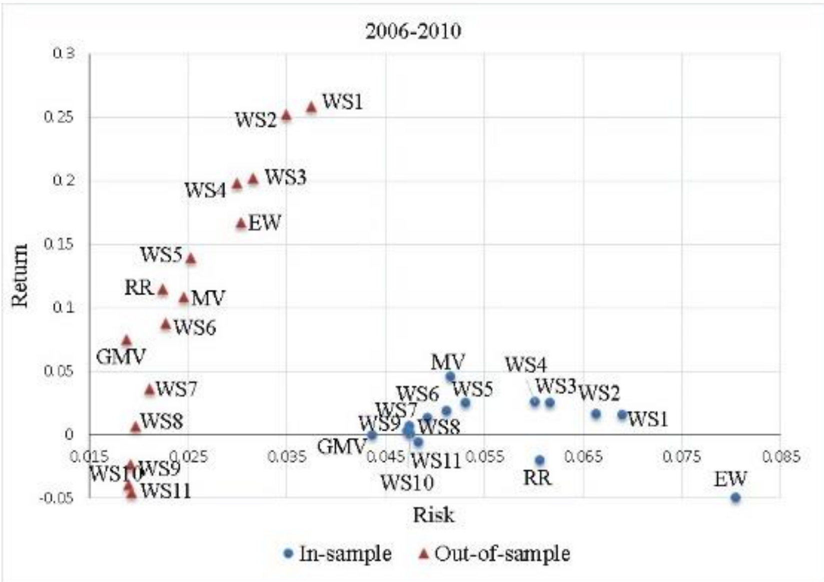
Figure 5-9 shows the in-sample and out-of-sample risks and returns of all the optimal portfolios by time window under analysis. Out-of-sample modified Sharpe ratios (S_I) are presented in Table 5-12. The results of the MV, EW, GMV and WS portfolios are described in section 5.3.3.2. Recall that our findings suggested lack of consistency in the out-of-sample performance of the relative robust methodology presented by Xidonas et al. (2017b). The empirical results show that the proposed relative robust approach stands out as a solid relative robust methodology. We notice that the RRB portfolio provides negative out-of-sample returns in time windows characterized by out-of-sample periods of high volatility (the variance of returns of individual assets for the out-of-sample years is shown in Figure 5-5) and, simultaneously, low mean returns (the mean return of individual assets for the out-of-sample years is shown in Figure 5-6). Specifically, the worst performances of the RRB portfolio occur in 2004-2008, 2007-2011 and 2012-2016, where the majority of the optimal portfolios perform poorly.

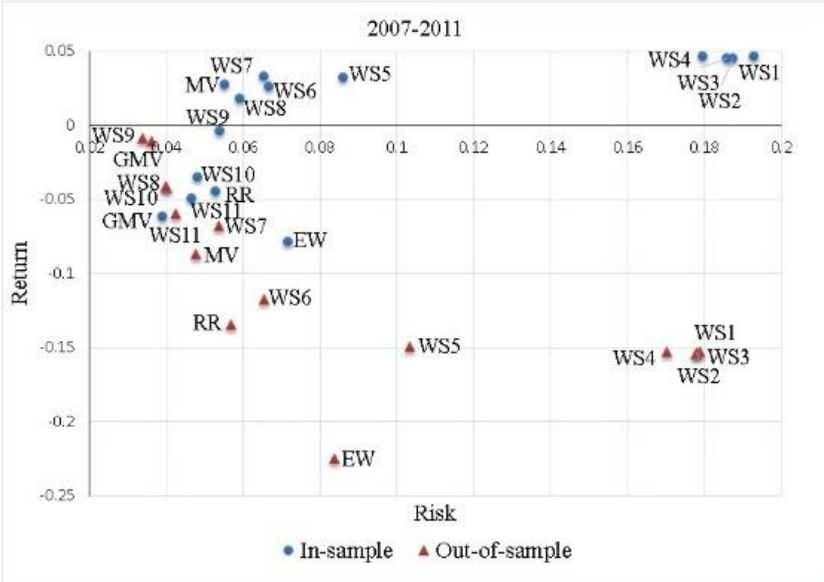
Figure 5-9: In-sample and out-of-sample risks and returns of the RRB, MV, EW, GMV and WS portfolios computed for each time window.

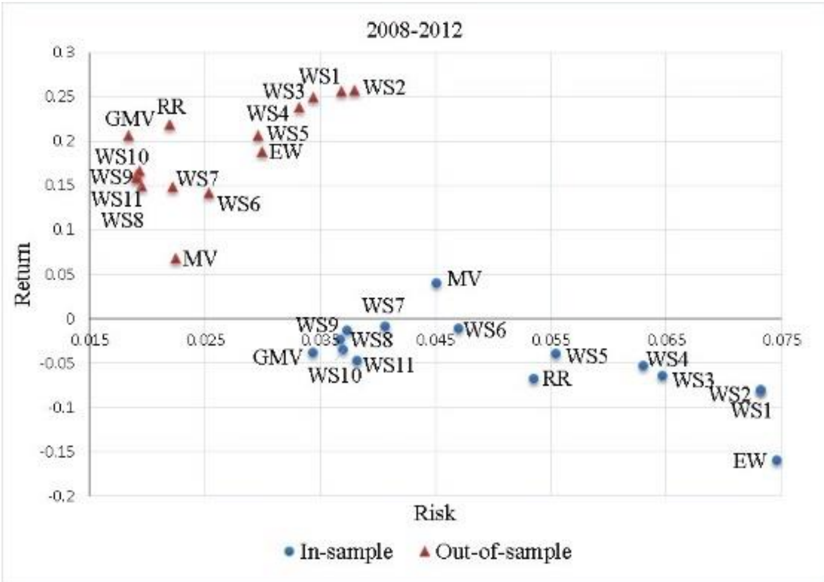


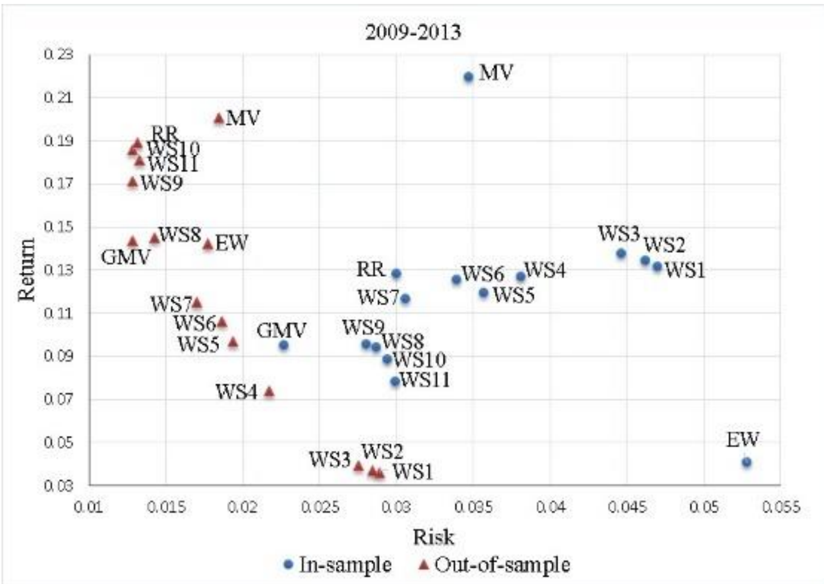

2004-2008
Return
0.35
0.15
-0.05
-0.25
-0.45
-0.65
0
0.05
0.1
0.15
0.2
0.25
Risk
In-sample
Out-of-sample

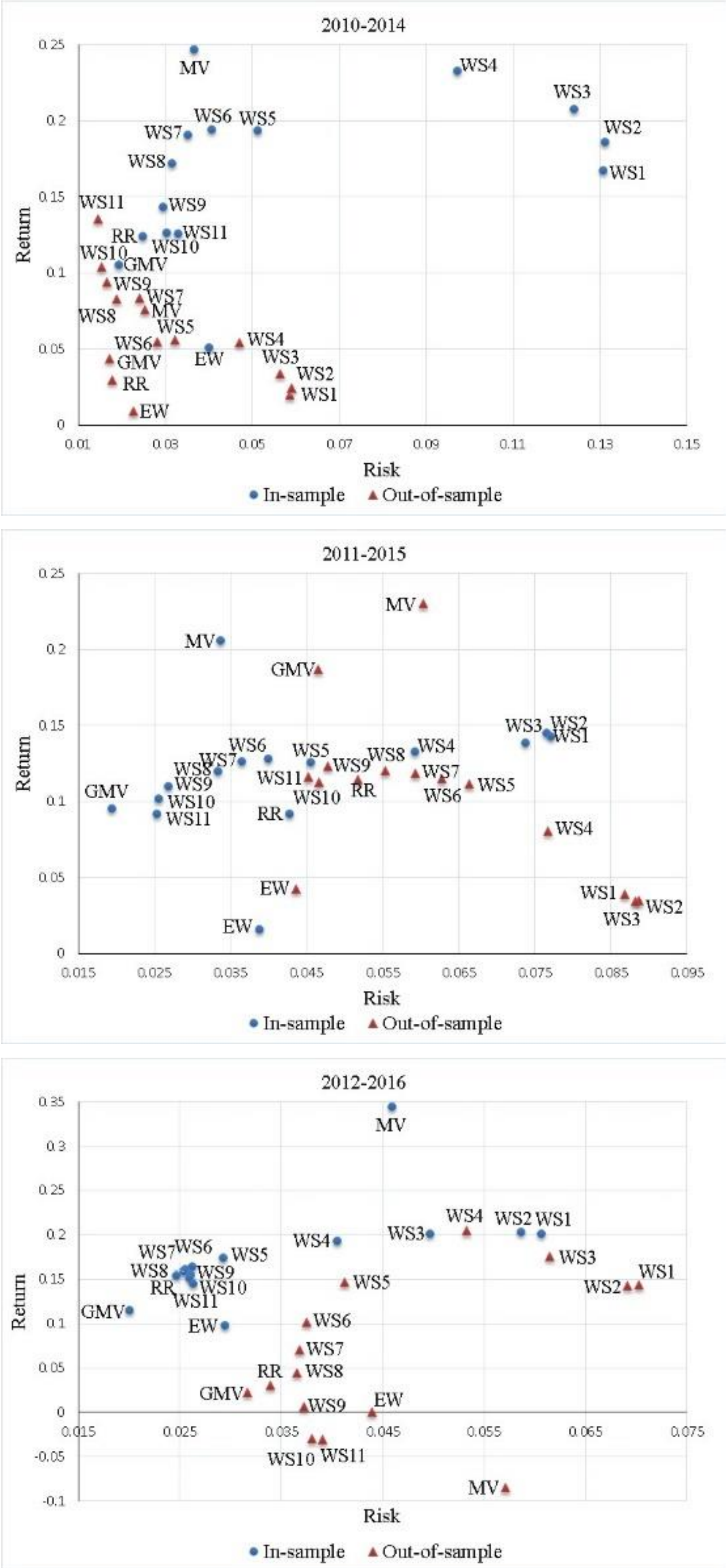

2005-2009
Return
0.2
0.15
0.1
0.05
0
-0.05
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
Risk
In-sample
Out-of-sample


2006-2010
Return
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0.015
0.025
0.035
0.045
0.055
0.065
0.075
0.085
Risk
In-sample
Out-of-sample


2007-2011
Return
Risk
In-sample
Out-of-sample


2008-2012
Return
Risk
In-sample
Out-of-sample


2009-2013
Return
Risk
In-sample
Out-of-sample



The optimal portfolios were computed considering the in-sample period length of 4 years and the parameters combination $\gamma = 5, E = 100, S = 100$ and $J = 120$.

The RRB portfolio is located at the left side of the scatter-plot, among the optimal solutions with the lowest risk and it even shows lower risk in out-of-sample data than in in-sample data, in many of the windows under analysis. Considering the S_I measure and the out-of-sample results, the RRB solution outperforms some of the WS portfolios in 9 of the 10 windows, and it always outperforms the WS portfolios in 3 of the 10 windows. It outperforms the MV portfolio in 7 of the 10 windows, the EW portfolio in 7 of the 10 windows and the GMV portfolio in 3 of the 10 windows. These results clearly reinforce the relevance of the proposed methodology, since previous studies confirmed the good performances of both the EW portfolio (DeMiguel et al., 2009) and the GMV portfolio (Chan et al., 1999; Jagannathan & Ma, 2003).

Furthermore, the out-of-sample return of the RRB portfolio is higher than its in-sample return in several windows under analysis. In fact, the tendency of higher out-of-sample returns comparatively to in-sample returns can be explained by the evolution of the stock market, in particular of the DAX index during the out-of-sample periods, depicted in Figure 4-3 and explained in section 4.3. The MV portfolio and some of the WS portfolios corresponding to the lowest c_1^g weights do not show the propensity to higher out-of-sample returns comparatively to in-sample returns, which reinforces the underperformance of these portfolios even in favorable market conditions.

Figure 5-9 also allows the observation of the proximity to the expected performance of the optimal portfolios by analyzing the distances between in-sample and out-of-sample portfolios' location. Although none of the portfolios systematically reveals greater proximity to the expected performance in all of the windows under analysis, it can be observed that, in many of these windows, the MV and the WS portfolios corresponding to the lowest c_1^g weights (WS1, WS2, WS3 and WS4) stand out as the solutions that exhibit the highest deviations to their expected performances and, thus, are the less consistent solutions as previously outlined. It can also be observed that, while the RRB portfolio exhibits similar deviations to its expected performance comparatively to the other portfolios in many time windows, in those where the deviations from the expected performance are substantially larger, the RRB portfolio stands out as one of the optimal solutions with highest out-of-sample return.

Analyzing the robustness in terms of the utility loss for the investor (Table 5-13), it can be observed that the RRB portfolio is systematically more robust than the EW and the majority of the WS portfolios, revealing lower regrets in most of the windows.

Additionally, the RRB portfolio is more robust than the MV and the GMV portfolios in 5 of the 10 windows.

To conclude, it is also important to highlight that the GMV and the RRB portfolios are among the portfolios with the lowest regrets in the majority of the time windows under analysis, which, in our opinion, validates the proposed relative robust methodology, supporting its robustness.

Table 5-12: Out-of-sample modified Sharpe ratio (S_I) of the RRB, MV, EW, GMV and WS portfolios.

Portfolio	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
RRB	0.825	-0.275	0.594	0.712	-0.033	1.474	1.572	0.212	0.508	0.188
MV	0.306	-0.279	0.432	0.636	-0.020	0.447	1.482	0.468	0.940	-0.019
EW	0.571	-0.244	0.465	0.910	-0.067	1.079	1.073	0.053	0.206	0.024
GMV	0.245	-0.179	0.800	0.484	-0.003	1.517	1.269	0.322	0.872	0.148
WS1	-0.030	-0.259	-0.008	1.294	-0.067	1.331	0.214	0.076	0.136	0.559
WS2	-0.033	-0.252	-0.008	1.303	-0.067	1.317	0.220	0.096	0.121	0.557
WS3	-0.035	-0.239	-0.007	1.090	-0.067	1.341	0.240	0.137	0.119	0.724
WS4	-0.036	-0.237	-0.005	1.095	-0.065	1.304	0.505	0.245	0.293	0.905
WS5	-0.018	-0.235	-0.004	0.823	-0.050	1.197	0.700	0.305	0.435	0.743
WS6	0.000	-0.232	-0.001	0.526	-0.032	0.880	0.781	0.319	0.461	0.548
WS7	0.181	-0.210	0.180	0.189	-0.017	0.987	0.884	0.531	0.490	0.388
WS8	0.410	-0.195	0.256	0.000	-0.010	1.064	1.216	0.594	0.514	0.255
WS9	0.310	-0.189	0.300	-0.004	-0.003	1.170	1.518	0.723	0.567	0.054
WS10	0.383	-0.178	0.479	-0.007	-0.009	1.193	1.644	0.831	0.527	-0.005
WS11	0.418	-0.165	0.524	-0.007	-0.013	1.146	1.652	1.115	0.551	-0.005

This table shows the out-of-sample S_I of the optimal portfolios by out-of-sample year. Results are presented for the in-sample period length of 4 years and the parameters combination $\gamma = 5, E = 100, S = 100$ and $J = 120$.

Table 5-13: Out-of-sample regret of the of the RRB, MV, EW, GMV and WS portfolios.

Portfolio	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**
RR	0.108	132.315	0.126	0.159	0.599	0.066	0.109	0.204	0.168	0.247
MV	0.190	103.397	0.155	0.167	0.383	0.166	0.104	0.171	0.091	0.552
EW	0.136	51.794	0.162	0.133	1.476	0.088	0.135	0.238	0.236	0.313
GMV	0.183	10.449	0.093	0.186	0.174	0.069	0.129	0.188	0.106	0.253
WS1	0.567	52.260	0.357	0.091	1.510	0.060	0.244	0.306	0.325	0.179
WS2	0.605	49.858	0.350	0.092	1.517	0.060	0.242	0.298	0.337	0.179
WS3	0.636	36.598	0.340	0.114	1.519	0.061	0.237	0.278	0.337	0.144
WS4	0.655	33.841	0.320	0.114	1.451	0.065	0.192	0.232	0.243	0.117
WS5	0.419	32.160	0.314	0.145	0.975	0.078	0.169	0.203	0.191	0.144
WS6	0.241	31.699	0.283	0.181	0.574	0.112	0.161	0.197	0.182	0.176
WS7	0.195	20.804	0.213	0.228	0.351	0.104	0.152	0.162	0.174	0.206
WS8	0.159	14.707	0.193	0.258	0.243	0.100	0.130	0.155	0.167	0.235
WS9	0.179	12.848	0.181	0.298	0.163	0.092	0.113	0.142	0.155	0.288
WS10	0.167	9.241	0.143	0.321	0.240	0.090	0.106	0.133	0.163	0.350
WS11	0.161	5.984	0.135	0.333	0.289	0.094	0.105	0.111	0.158	0.355

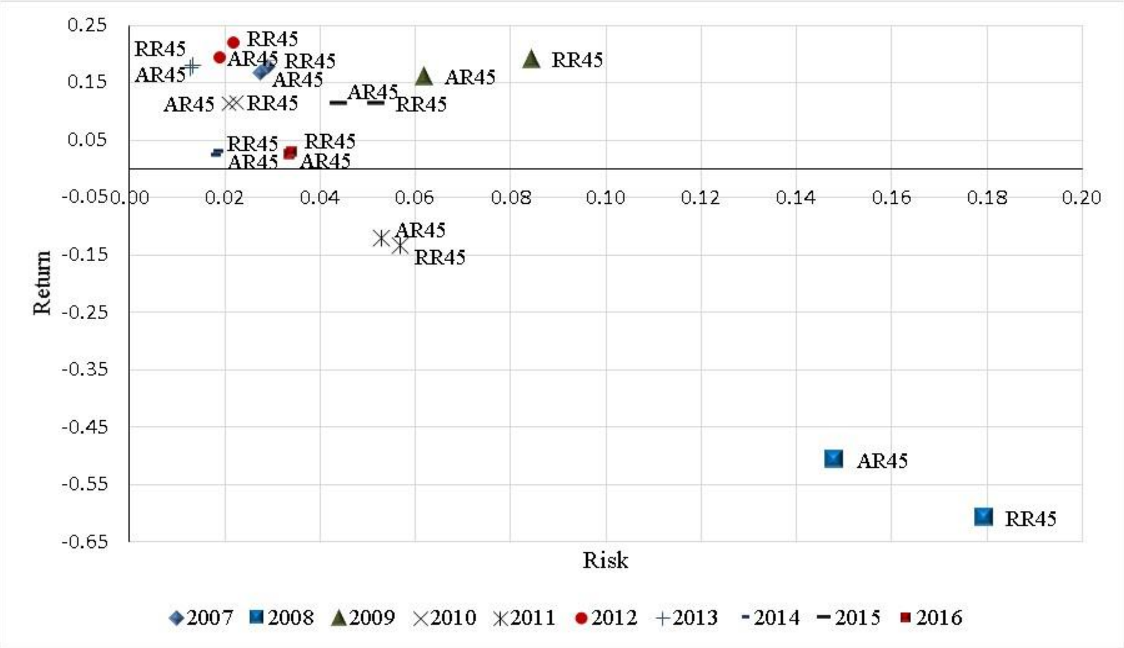
This table shows the out-of-sample regret of the optimal portfolios by out-of-sample year. Results are presented for the in-sample period length of 4 years and the parameters combination $\gamma = 5, E = 100, S = 100$ *and* $J = 120$.

5.4.2.2.3 Performance of relative robust and absolute robust portfolios

As previously mentioned, there is a lack of empirical studies that compare the performance of relative robust and absolute robust portfolio models. For that reason, the performance of the proposed RRB and ARB portfolios were also compared in each of the out-of-sample years and the main results are now presented for the same parametrization (in-sample period length of 4 years and a value of 5 for the risk aversion parameter).

Figure 5-10 shows the out-of-sample risks and returns of the RRB and ARB portfolios by out-of-sample year under analysis. The proximity between the RRB and ARB portfolios' location in the risk/return space shows in a clear way that the RRB and ARB portfolios reveal similar performances in the majority of the out-of-sample years. However, in some cases, substantial differences can be observed.

Figure 5-10: Out-of-sample risks and returns of RRB and ARB portfolios.



Portfolios' risk and return were computed by out-of-sample year and considering an in-sample period length of 4 years and the parameters combination $\gamma = 5, E = 100, S = 100$ *and* $J = 120$.

The ARB portfolio is a dominant solution comparatively to the RRB portfolio only in the out-of-sample years where all the optimal portfolios presented in the empirical application revealed poor performances, namely in the out-of-sample years 2008 and 2011, previously described as periods of high volatility and, simultaneously, low mean returns. For the remaining out-of-sample years, the RRB portfolio generally presents identical risks levels (except in 2009 and 2015) while it always presents higher returns, comparatively to the ARB portfolio.

Out-of-sample regret and S_I support the previous outcome. In fact, the RRB solution outperforms the ARB portfolio when the S_I measure is considered (see Table 5-12 and Table 5-14) while it is systematically more robust, revealing lower regrets than the ARB portfolio, in most of the windows (see Table 5-13 and Table 5-14).

Table 5-14: Out-of-sample regret and Sharpe ratio (S_I) of the ARB portfolio

Measure	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**
$\boldsymbol{S_I}$	0.782	-0.210	0.579	0.732	-0.029	1.402	1.564	0.177	0.544	0.160
Regret	0.113	26.045	0.127	0.157	0.517	0.075	0.111	0.208	0.159	0.253

This table shows the out-of-sample regret and S_I of the ARB portfolios by out-of-sample year. Results are presented for the in-sample period length of 4 years and the parameters combination $\gamma = 5, E = 100, S = 100$ and $J = 120$.

It was also observed that, although reducing the in-sample period length has an improving effect in the overall performance of both the ARB and RRB portfolios, this effect is more substantial in the RRB portfolio. Finally, an important fact concerning the investor's risk preferences was also observed. Since the absolute robust approach is described in the robust optimization theory as a more conservative methodology, a better performance of the ARB portfolio for higher levels of the risk aversion parameter would be expected. While, on the one hand, the ARB portfolio always presents lower (mean) risk comparatively to the RRB portfolio (0.058 versus 0.065, when $\gamma = 0.5$; and 0.044 versus 0.051, when $\gamma = 5$), on the other hand, it reveals worst (mean) risk-adjusted return, measured by the S_I (0.528 versus 0.529 when $\gamma = 0.5$, and 0.570 versus 0.578 when $\gamma = 5$). The values previously presented were verified for an in-sample period of 4 years. These results clearly reinforce the relevance of the relative robust methodology within the robust portfolio theory.

5.5 Conclusions

In this chapter, a new minmax regret portfolio optimization model is presented. Regret is defined as the utility loss for the investor resulting from choosing a given portfolio instead of choosing the optimal portfolio of the realized scenario for the uncertain parameters. Two different ways of defining the relative robust solution are proposed. In the first approach (model A), we construct one uncertainty set for the entire sample period and the relative robust portfolio is defined as the minmax regret solution presenting the best performance in the worst-case scenario. In the second approach (model B), we introduced validation subsamples in the sampling procedure and repeat the computational procedure (implemented and described in model A) for different estimation and validation subsamples, producing different uncertainty sets and sets of different minmax regret solutions. Then, for each one of these sets, the minmax regret solution with maximum regret (corresponding to the difference between the utility of the optimal portfolio in the validation subsample period and the utility of the minmax regret solution) is identified. The relative robust portfolio is defined as the minmax regret solution presenting the best performance in the worst-case (for all the uncertainty sets and all the scenarios under consideration). In both cases, it is assumed that the investor has constant relative risk aversion preferences. The empirical applications presented in this chapter were based on historical daily data of the stocks of the constituent list of the DAX index.

The results regarding the performance of the non-robust portfolios support previous findings concerning the sensitivity of the MV portfolio to estimation error and the effects of the input uncertainty in the optimization process (Best & Grauer, 1991b; Chopra & Ziemba, 1993; DeMiguel et al., 2009; Jagannathan & Ma, 2003), as well as the good performance of the GMV portfolio (Chan et al., 1999; Jagannathan & Ma, 2003). The out-of-sample performance of the non-robust portfolios show that reducing the in-sample period length seems to improve the overall performance of the GMV while substantially deteriorating the out-of-sample performance of the MV portfolio, whose exposure to individual stocks is considerably increased.

Our overall empirical results suggest that the proposed relative robust models have more value for risk-taking investors, i.e. for those who can be more affected by the methodological weakness of the classical mean-variance model, standing out as a valid alternative. The proposed relative robust methodology generates, in both approaches, optimal portfolios that are highly diversified and consistently present low risk, even in out-of-sample periods characterized by atypical high volatility, and (non-negative) attractive returns. Their consistency and in-sample and out-of-sample robustness (concerning utility loss for the investor) were confirmed, standing out as one of the very few optimal portfolios with no poor performances. Furthermore, the RRA and RRB portfolios outperforms the MV portfolio and the EW portfolio, in many of the windows under analysis and for the majority of the performance measures applied in our study, which enhances the relevance of the proposed methodology among non-robust portfolio optimization methodologies.

Our findings suggest that the proposed relative robust strategies generally outperform the relative robust and absolute robust methodologies implemented in this chapter. Analysing the proposed relative robust models individually, it can be observed that the RRA portfolio provides more consistent results comparatively to the robust solutions generated by the minmax regret model presented by Xidonas et al. (2017b) and the absolute robust approach developed by Kim et al. (2014c) concerning their out-of-sample performance. Additionally, it presents lower exposure to individual stocks (higher cardinality) and substantially lower turnover levels than the robust solutions generated by those robust methodologies. The portfolio style analysis suggests a higher representation of assets with low CEP ratio and extreme (either high or low) dividend yield. Concerning the exposure of the RRA portfolio to risk factors, a positive relation with the market excess return and a negative relation with the HML factor were found. Regarding model B, the worst performance of the RRB portfolio occurs in time windows characterized by out-of-sample periods of high volatility and, simultaneously, low assets mean returns, where the majority of the optimal portfolios perform poorly. Furthermore, the consistency of the proposed relative robust model B was confirmed when compared to the minmax regret model presented by Xidonas et al. (2017b). Finally, the findings suggest that the proposed RRB portfolio generally outperforms the proposed ARB portfolio, even when higher levels of risk aversion are considered.

Although the RRA and RRB portfolios were not directly compared, a particular result can be confronted in order to better understand the real contribution of introducing validation subsamples in the sampling procedure. The out-of-sample mean regret of the RRA and the RRB portfolios computed for the parameters combination $\gamma = 5, S = 100$

and $J = 120$, are displayed in Table 5-1 and Table 5-10, respectively. As it can be observed, the RRA portfolio presents a lower mean regret (0.20999) comparatively to the RRB portfolio (0.25350), suggesting that the introduction of the validation sets in the sampling procedure increases the utility loss (regret) of the investor. This result is not surprising since the proposed model B contemplates 100 times more scenarios and, thus, results in a more (and, probably, too) conservative approach. Additionally, we must also bear in mind that the latter methodology increases significantly the number of instances of the relative robust optimization problem, and, thus, the computational time necessary to compute the relative robust solution.

Besides unravelling the real benefits, from the investor perspective, of applying the absolute and relative robustness approaches in portfolio selection, this study clearly contributes to reinforce the relevance of RO methodologies, in particular of relative robust models, within the field of portfolio selection under uncertainty.

Chapter 6

A robust minimum variance optimization approach (model C)

6.1 Introduction

In order to further investigate the real contribution of the RO methodology, in particular of the relative robust approach, in the field of portfolio selection, we develop robust formulations of classical portfolio selection models with different model inputs and different uncertain parameters. In this chapter, we are motivated to extend the literature on the stability of optimal solutions by optimizing only the second moment and applying the RO methodology. Hence, this research presents new methods for computing relative robust and absolute robust minimum variance portfolios. An empirical application is introduced to comparatively assess the performance of the two alternative robust optimization methods against non-robust portfolios already described in the portfolio theory literature.

Comparatively to the relative robust minimum variance approach presented by Xidonas et al. (2017a), and previously introduced in section 3.2.3, our approach considers both absolute robust and relative robust minimum variance solutions, includes a wider uncertainty set (the authors define 3 scenarios while we define a minimum number of scenarios of 100) and examines the effect of the in-sample period length for the estimation of the model parameters, by considering long-term past data over short-term past data in their computation.

The results of our empirical application suggest that increasing the in-sample period length for the estimation of the model parameters has no substantial effect in either the composition or the performance of the robust portfolios, which highlights the utility of the proposed models in the presence of limited data. It is possible to observe that the proposed robust methodologies generate optimal portfolios that consistently present out-of-sample portfolio risk measures that lie between the risk measures of the GMV and the EW portfolios. Out-of-sample portfolio returns are between, or higher, than the portfolio returns of the two benchmarks. Additionally, for most of the windows under analysis, the proposed relative robust (RRC) and absolute robust (ARC) portfolios outperform the EW portfolio. This finding enhances the relevance of the proposed methodology since it has been pointed out by some authors as a benchmark difficult to outperform (DeMiguel et al., 2009). The results also support previous findings concerning the good performance of the GMV portfolio (Chan et al., 1999; Jagannathan & Ma, 2003). Furthermore, robust solutions and non-robust solutions present similar behaviours concerning robustness and negative (loss) versus positive (gain) returns, which suggests that the proposed methodologies are as consistent as the benchmarks used for comparing portfolios' performances.

The remainder of the chapter proceeds as follows. In Section 6.2, the methodology is described. Section 6.3 describes the empirical analysis and presents its main results. Finally, the main conclusions are highlighted in Section 6.4.

6.2 Methodology

6.2.1 The robust minimum variance optimization models

We start by defining the uncertainty set used in this model. The uncertainty set, U, is defined as a finite set of scenarios, where each scenario represents a possible realization of the sample covariance matrix. Each scenario $s_i, i = 1, \dots, S,$ is described by the sample covariance matrix, Σ^s, as defined in section 5.2.1.

For computing the RRC portfolio, regret is defined as the increase in the investment risk resulting from investing in a portfolio characterized by the weight combination vector w_t instead of investing in w_t^{s*}, which corresponds to the optimal solution (global minimum variance portfolio) under scenario s. Let w_t^{s*} be the global minimum variance portfolio for the scenario s. The regret associated to choosing portfolio w_t in scenario s, $P^s(w_t)$, is defined by

$$P^s(w_t) = w_t{}'\Sigma^s w_t - w_t^{s*\prime}\Sigma^s w_t^{s*} \tag{6.1}$$

and the maximum regret function, $P(w_t)$, is defined by

$$P(w_t) = \max_{s \in U} w_t{}'\Sigma^s w_t - w_t^{s*\prime}\Sigma^s w_t^{s*}. \tag{6.2}$$

The relative robust portfolio (RRC) corresponds to the weight combination vector w_t that solves the minmax regret optimization model:

$$\min_{w_t \in X} \max_{s \in U} w_t{}'\Sigma^s w_t - w_t^{s*\prime}\Sigma^s w_t^{s*} \tag{6.3}$$

where X corresponds to the set of feasible solutions.

For computing the ARC portfolio, we solve the absolute robust optimization model defined by:

$$\min_{w_t \in X} \max_{s \in U} w_t{}'\Sigma^s w_t. \tag{6.4}$$

6.2.2 Computing the relative robust and absolute robust portfolio

The computation of the RRC portfolio runs as follows. An uncertainty set U is constructed by calculating the S scenarios. For each scenario $s \in U$, an estimation window is randomly selected from the in-sample period, and the corresponding covariance matrix is computed. Then, for each scenario $s \in U$ the following problem is solved

$$\min_{w_t \in X} w_t{}'\Sigma^s w_t, \tag{6.5}$$

in order to determine the optimal solution w_t^{s*}, which represents the portfolio on the Markowitz's efficient frontier with minimum variance. This constitutes the first optimization process of the proposed thee-level optimization.

After computing the optimal solution for each scenario $s \in U$, the relative robust optimization problem (6.3) is solved using the GA. A fitness function that maximizes the regret as presented in (6.2) and corresponding to the increase in the investment risk resulting from investing in a portfolio characterized by the weight combination vector w_t instead of investing in the optimal solution of the realized scenario, was defined. The initial population is twice the size of the uncertainty set and is comprised of all the optimal solutions w_t^{s*} as well as other feasible randomly generated solutions.

Regarding the absolute robust optimization solution, after computing the S scenarios, as previously described, problem (6.4) is solved. In this case, the fitness function was defined as the maximum portfolio variance function (inner maximization problem in (6.4)). Hence, the optimization is performed assuming the worst-case performance over the whole uncertainty set.

6.3 Empirical analysis

6.3.1 Data and model settings

For the empirical analysis, we used historical daily data from January 1992 to December 2016 of the stocks of the EURO STOXX 50 index. Adjusted closing prices of the stocks in the constituent list of the EURO STOXX 50 index at the end of the in-sample period were collected and daily continuous returns were calculated. We considered rolling windows of two different lengths. As previously described in section 4.2, a long rolling window with a constant length of 16-years and a short rolling window with a constant length of 5-years. Uncertainty sets with different number of scenarios ($S \in \{100,200,500\}$) were analyzed. Each scenario considers an estimation window length of 120 consecutive daily returns. Estimations of the model inputs were performed in R.

The steps for computing the robust solutions, described in section 6.2.2, are iteratively repeated for each of the time windows. Once the RRC and ARC portfolios are computed for each of the time windows under analysis, in-sample and out-of-sample performances are analyzed.

6.3.2 In-sample and out-of-sample performances

The performances of the robust strategies were analyzed and compared to classical non-robust portfolio selection strategies, considering both in-sample and out-of-sample data. The non-robust optimization portfolio that was considered as benchmark was the GMV portfolio. Problem (2.5) was solved and the GMV portfolio was identified. Inputs were estimated for the entire in-sample window, namely the in-sample covariance matrix was calculated considering 15-years data or 4-years data, according to the window length under consideration. The EW portfolio was also created and also used as a benchmark in this chapter.

After determining RRC, ARC, GMV and EW portfolios, in-sample and out-of-sample performances were compared by analyzing the portfolios annualized return, risk and the Israelsen modified Sharpe ratio (S_I). In addition, the regret, defined by

$$R = {w_t}'\Sigma w_t - {w_t^*}'\Sigma w_t^* \qquad (6.6)$$

and representing the increase in the investment risk resulting from investing in a portfolio characterized by the weight combination vector w_t instead of investing in the optimal portfolio w_t^* (feasible solution with minimum variance) of the sample period under consideration, was calculated and compared for the in-sample and out-of-sample periods.

6.3.3 Results

We start by analyzing how the composition and the performance of the optimal solutions are influenced by the in-sample period length. In particular, we analyze the composition of the portfolios regarding the maximum weight of an asset (Max%), minimum weight of an asset (Min%), the sum of the 3 largest weights in the portfolio (Sum3Max%) and the number of assets with non-zero weights in each portfolio

(Cardinality). Mean values obtained over the 10 windows are presented. Cardinality is measured as the number of assets with weights higher than 0.1%, since the optimal portfolios have some assets with very small but not necessarily zero weights. The portfolios' performances are analyzed, both in-sample and out-of-sample, by comparing the mean of the portfolios' returns (mean return) and the mean of the portfolios' variances (mean risk), obtained over the 10 windows. Additionally, the mean of the portfolios' regrets (mean regret) and the mean of the portfolios' out-of-sample modified Sharpe ratio (S_I), obtained over the 10 windows, are also analyzed for all the computed portfolios. The consistency of the portfolios in terms of the deviation from their expected performance is assessed by comparing the in-sample and out-of-sample results. The portfolios' regrets reflect the robustness of the optimal solutions in terms of the increase in the investment risk resulting from choosing a given portfolio instead of choosing the optimal portfolio for the realized scenario.

Then, the in-sample and out-of-sample performances of robust and non-robust portfolios are compared for each of the 10 windows. Results are presented for the in-sample period length associated with the best mean performances for both in-sample and out-of-sample datasets. For simplification purposes, the RRC and ARC portfolios will be represented by 'RR' and 'AR', respectively, in the figures presented in the Results section and the corresponding subsections.

6.3.3.1 Effect of the variation of the in-sample period length

Table 6-1 presents the composition of the RRC, ARC, GMV and EW portfolios, taking into account the length of the in-sample period considered for their computation. The optimal portfolios were represented according to the length of the in-sample period and the number of scenarios (only the first digit was used to keep the representation simpler) used in their computation. Hence, RR151 represents the relative robust minimum variance portfolio computed using an in-sample period of 15 years and an uncertainty set with 100 scenarios, while AR45 represents the absolute robust minimum variance portfolio based on an in-sample period of 4 years and an uncertainty set with 500 scenarios. The representation of the GMV portfolio was made according to the length of the in-sample period only.

Analyzing the overall results, it is possible to observe that, regardless of the in-sample period length, the GMV portfolio is the less diversified portfolio while the robust and the EW portfolios are the most diversified ones. Concerning the robust portfolios, and although they present very similar compositions, it can be observed that using longer in-sample periods seems to slightly decrease the exposure of these portfolios to individual stocks, since the RRC and ARC computed with an in-sample period of 15 years present lower values in the maximum weight of an asset (Max%) and in the sum of the 3 largest weights in the portfolio (Sum3Max%). Just as the EW portfolio, the robust portfolios assign non-zero weights to all the assets in the dataset, regardless of the in-sample period length and of the number of scenarios in the uncertainty set. A closer examination of the results allows us to confirm that both the robust and the GMV portfolios assign largest weights to the same assets.

Some analysis can be made concerning the portfolios' cardinality results. The GMV portfolio is highly concentrated in a lower number of assets. Previous studies claim that the minimum variance portfolio has a maximum of 40 assets for large samples (Jagannathan & Ma, 2003) and that it usually over-weights stocks with low market beta, underperforming in bull markets and outperforming in bear markets (Chow, Hsu, Kuo, & Li, 2014). The EW portfolio might be more protected against extreme events since it is more diversified than the GMV portfolio (DeMiguel et al., 2009). Therefore, the robust portfolios seem to embrace the potential for diversification of the equally-weighted strategy, while assigning maximum weights to the same assets selected by the minimum variance strategy.

The results presented in Table 6-1 also indicate that there are no substantial differences between the RRC and ARC portfolios computed using the same in-sample period length. In fact, an unexpected result was obtained concerning the optimal solutions yielded by the relative robust and absolute robust formulations of the global minimum variance model, which deserves a closer examination. As suggested in Table 6-1, the RRC and ARC portfolios are identical when the in-sample length is 4 years and number of scenarios in the uncertainty set is 100 or 200, and when the in-sample length is 15 years and the number of scenarios in the uncertainty set is 200. For all the other combinations of these parameters (in-sample length and number of scenarios in the uncertainty set) the computed solutions are different.

Table 6-1: Composition of the RRC, ARC, EW and GMV portfolios.

Portfolios	Max%	Min%	Sum3Max%	Cardinality
RRC151	6.4	1.1	14.3	41
RRC152	6.5	1.1	14.8	41
RRC155	6.4	1.1	14.8	41
ARC151	6.4	0.9	14.9	41
ARC152	6.5	1.1	14.8	41
ARC155	6.6	1.1	15.0	41
RRC41	6.5	0.9	15.1	41
RRC42	7.1	1.2	15.7	41
RRC45	7.0	1.2	15.8	41
ARC41	6.5	0.9	15.1	41
ARC42	7.1	1.2	15.7	41
ARC45	7.0	1.2	15.8	41
EW	2.4	2.4	7.3	41
GMV15	19.6	<0.1	42.0	15
GMV4	21.1	<0.1	41.9	10

This table presents the characteristics of the optimal portfolios. Here, the composition of the portfolios regarding the maximum (Max%) and minimum (Min%) weights of an asset, the sum of the 3 largest weights in the portfolio (Sum3Max%), and the number of assets with non-zero weights in each computed portfolio (Cardinality) are described. As explained before, for measuring the cardinality, only those assets with weights higher than 0.1% are considered. Only average results for the 10 windows are shown. The optimal portfolios were represented according to the length of the in-sample period and the value of the risk aversion parameter used in their computation. For instance, 'RRC155', 'ARC155' and 'GMV15' corresponds, respectively, to the RRC, ARC and GMV portfolios computed using 15-years data to perform in-sample estimations and, in the case of the RRC and the ARC portfolios, using a value of 5 for the risk aversion parameter.

Analyzing the RRC and ARC solutions computed using the same uncertainty set, it is possible to verify that when changes in the variance of the optimal solutions are small (i.e. up to 5.7E-05), the RRC and ARC tend to yield identical solutions. When there is a small subset of scenarios in which the optimal solution (global minimum variance portfolio) of each one of those scenarios presents atypical variance (much higher variance comparatively to the remaining global minimum variance portfolios), the relative robust and absolute robust models yield different solutions. The reason is that, in the former case (similar variance for all scenarios), the second term of (6.3), corresponding to the risk of the optimal portfolio when scenario *s* occurs, becomes quite similar for all scenarios and acts as if it was a constant; thus, problem (6.3) and problem (6.4) instances become equivalent, leading to the same solution, and we end up with identical RRC and ARC portfolios. In the latter case, the second term in (6.3) may become much different for some scenarios, leading to a significantly different problem instance from (6.4).

Table 6-2 presents some statistics regarding the standard deviations of the optimal portfolios' returns associated to all scenarios considered in the uncertainty set, for the in-sample period length of 4 years. Results for the in-sample period length of 15 years are presented in Table 6-3.

Table 6-2: Some statistics regarding the standard deviations of the optimal solutions (GMV portfolios) for the scenarios belonging to the uncertainty set, for the in-sample period length of 4 years.

Statistics	**Number of scenarios in the uncertainty set**		
	100	**200**	**500**
Mean	2.6469E-03	2.6426E-03	2.7962E-03
St.Deviation	6.9540E-03	7.1180E-03	7.5007E-03
1st quartile	6.0840E-07	6.0004E-07	5.3675E-07
2nd quartile	1.3133E-06	1.4928E-06	1.4183E-06
3rd quartile	2.1814E-05	1.8888E-05	1.4342E-05
99th percentile	2.9207E-02	2.9460E-02	3.0376E-02
Maximum	2.9215E-02	3.0064E-02	4.0459E-02
2nd maximum	2.8387E-02	2.9462E-02	3.4491E-02
3rd maximum	2.6780E-02	2.9215E-02	3.4471E-02
Minimum	4.1181E-08	4.1181E-08	3.2147E-08

This table presents measures of centre and dispersion of the standard deviations of the GMV portfolios associated to all scenarios considered in uncertainty set, for different number of scenarios and for an in-sample period length of 4 years.

It can be observed that the standard deviations of the optimal solution corresponding to an uncertainty set with 500 scenarios generally present lower quartiles and higher maximum value, which suggests a wider dispersion of the standard deviations. This is confirmed when the 3 maximum values are analyzed together with the 99th percentile, supporting the wider variation of the standard deviations values for the uncertainty set with 500 scenarios. For the uncertainty sets with 100 and 200 scenarios, in which the models yielded identical solutions, it is possible to observe that the 99th percentile value is between the three largest values and these three largest values are closer and, thus, less dispersed. This result prevails regardless of the in-sample period length, explaining also why we have the same solution for both models when the in-sample period length is 15 years and the number of scenarios in the uncertainty set is 200.

Table 6-3: Some statistics regarding the standard deviations of the optimal solutions (GMV portfolios) for the scenarios belonging to the uncertainty set, for the in-sample period length of 15 years.

Statistics	**Number of scenarios in the uncertainty set**		
	100	**200**	**500**
Mean	1.4464E-04	2.1475E-04	2.1476E-04
St.Deviation	4.2803E-04	7.1953E-04	6.7444E-04
1st quartile	7.0613E-14	1.1360E-13	6.5133E-14
2nd quartile	1.9073E-12	1.9562E-12	1.4937E-12
3rd quartile	8.9307E-06	2.2073E-05	4.9216E-06
99th percentile	2.3935E-03	2.6159E-03	3.2412E-03
Maximum	2.3952E-03	7.9947E-03	6.4770E-03
2nd maximum	2.2255E-03	2.6202E-03	3.6437E-03
3rd maximum	1.4662E-03	2.1900E-03	3.6025E-03
Minimum	1.3027E-15	8.0469E-16	6.8155E-16

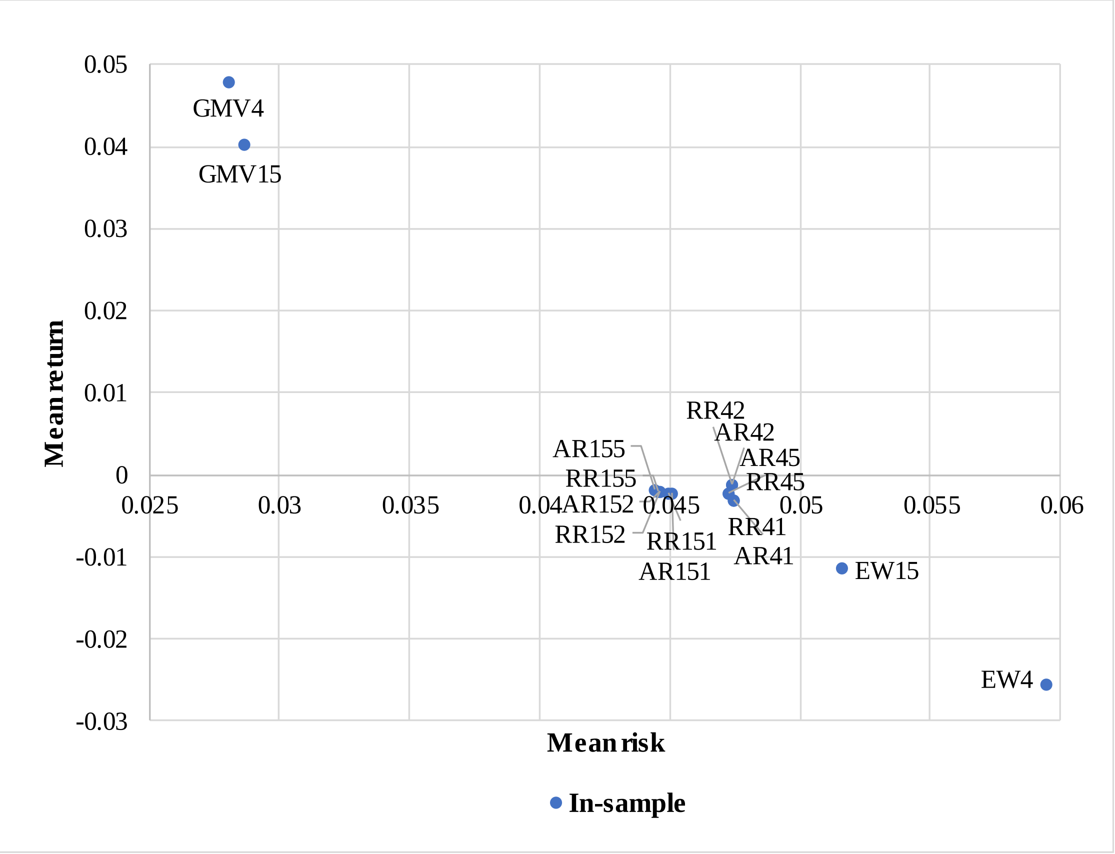
This table presents measures of centre and dispersion of the standard deviations of the GMV portfolios associated to all scenarios considered in uncertainty set, for different number of scenarios and for an in-sample period length of 15 years.

The in-sample and the out-of-sample mean risk and mean return of the RRC, ARC, GMV and EW portfolios are presented in Figure 6-1 (in-sample), Figure 6-2 (out-of-sample) and Figure 6-3 (both in-sample and out-of-sample).

Analyzing the effect of the length of the in-sample period in the in-sample performance (Figure 6-1), it is quite evident that the length seems to have no substantial effect in the in-sample mean return of the robust portfolios, while increasing the in-sample length from 4 to 15 years, seems to improve (decrease) their in-sample mean risk. Comparing RRC and ARC portfolios, we conclude that the worst performances are obtained when the uncertainty set has 100 scenarios, regardless of the in-sample length.

Regarding the GMV portfolio, its overall in-sample performance is improved when reducing the length of the in-sample period since the in-sample mean return increases while the in-sample mean risk slightly decreases. Concerning the EW portfolio, notice that the in-sample length and the number of scenarios in the uncertainty set do not influence its calculation; nevertheless, its in-sample performance, and consequently its location in the risk-return space, is different for different in-sample lengths.

Figure 6-1: In-sample mean risk and mean return of the RRC, ARC, EW and GMV portfolios.



The optimal portfolios were represented according to the length of the in-sample period and the number of scenarios considered in the uncertainty set used in their computation. For instance, RR151 and AR45 correspond, respectively, to the RRC portfolio computed using an in-sample period of 15 years and an uncertainty set with 100 scenarios, and the ARC portfolio based on an in-sample period of 4 years and an uncertainty set with 500 scenarios.

Regardless of the length of the in-sample period, the results obtained from the experiment confirm that the robust portfolios always reveal worse overall in-sample performance comparatively to the non-robust GMV portfolio and better overall in-sample performance comparatively to the EW portfolio. Furthermore, the GMV portfolio is the only one that presents positive in-sample mean returns. Although robust portfolios present negative mean returns, these values are close to 0% and represent substantially smaller losses than those incurred by the EW portfolio.

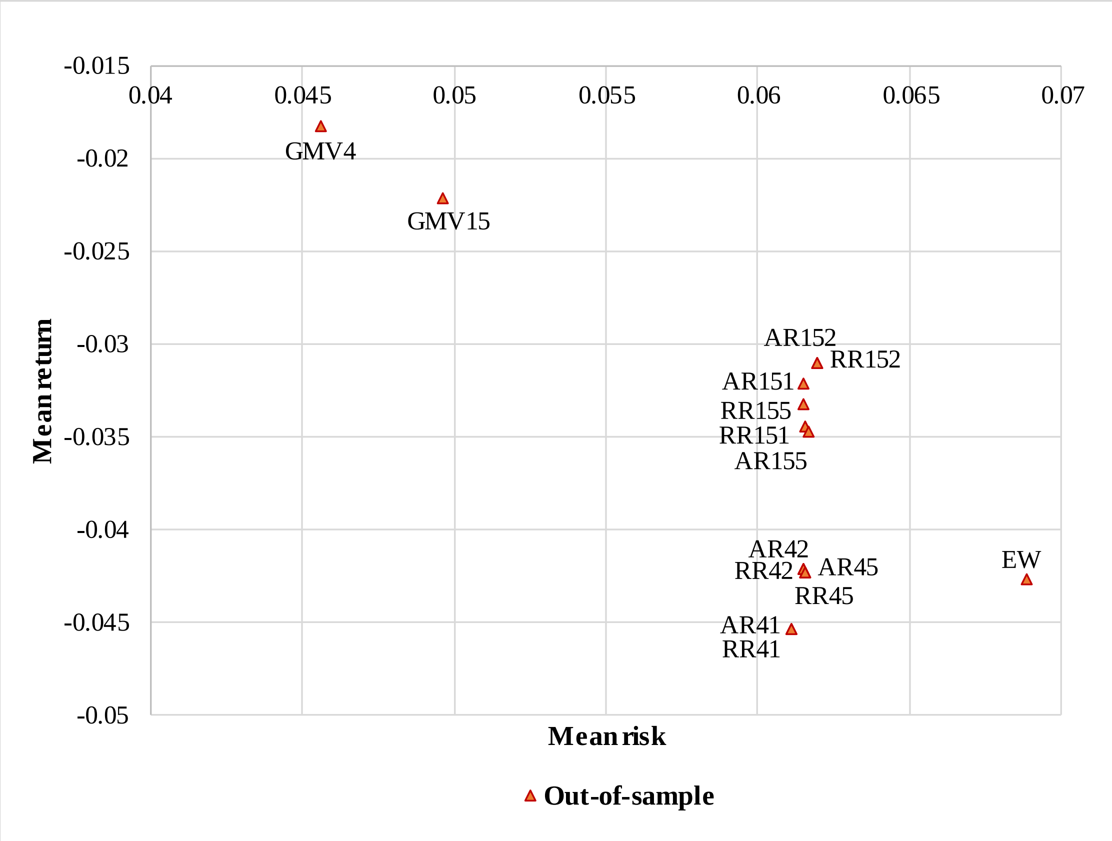
Concerning the effect of the in-sample period length in the out-of-sample performance of the optimal solutions (Figure 6-2), it can be observed that, apart from the fact that all computed portfolios have negative mean returns in the out-of-sample data, the overall results, previously described for the in-sample performance, continue to occur. This significant and important finding provides evidence of how the implemented strategies may produce similar performance stability when applied to new data sets.

Before proceeding with the out-of-sample results, it is important to address the fact that all the implemented portfolio optimization strategies present negative out-of-sample returns, when mean results are analyzed. This outcome can be explained by the evolution of the EURO STOXX 50 index during the out-of-sample period considered in this study (January 2007 to December 2016), depicted in Figure 4-4 and explained in section 4.3.

Returning to the out-of-sample results (Figure 6-2), it can be confirmed that the portfolio with worst overall out-of-sample performance is the EW portfolio and the best is the GMV portfolio (with higher return and lower risk when the in-sample length is 4 years). This latter result is in accordance with previous studies (Chan et al., 1999; Jagannathan & Ma, 2003) supporting the outperformance of the GMV portfolio.

Analyzing the out-of-sample performance of the robust portfolios, the out-of-sample mean return of the robust portfolios is substantially improved while no substantial effect in the out-of-sample mean risk is observed when the in-sample length is increased. In fact, the best out-of-sample performance seems to be achieved when the robust solutions are computed using an uncertainty set with 200 scenarios, regardless of the in-sample length used to calculate the scenarios. Except for the RR41 and AR41 portfolios (RRC and ARC portfolios computed with an in-sample period length of 4 years and an uncertainty set with 100 scenarios), the robust portfolios present a mean risk higher than the GMV portfolio and lower than the EW portfolio, while their mean returns are (again) between the mean returns of these two benchmarks. It is clear that the majority of the robust portfolios are dominant solutions comparatively to the EW portfolio.

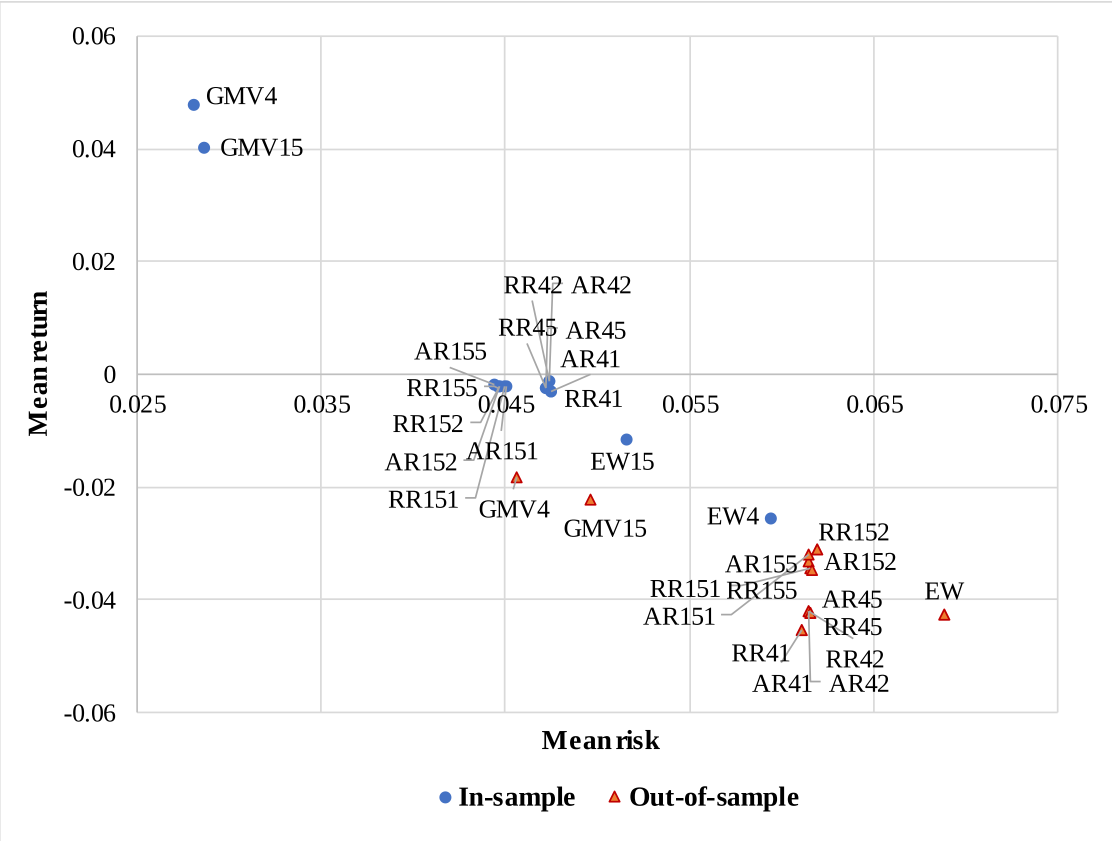
Figure 6-2: Out-of-sample mean risk and mean return of the RRC, ARC, EW and GMV portfolios.



The optimal portfolios were represented according to the length of the in-sample period and the number of scenarios considered in the uncertainty set used in their computation. For instance, RR151 and AR45 correspond, respectively, to the RRC portfolio computed using an in-sample period of 15 years and an uncertainty set with 100 scenarios, and the ARC portfolio based on an in-sample period of 4 years and an uncertainty set with 500 scenarios.

The distances between in-sample and out-of-sample portfolios' locations can be observed in Figure 6-3, allowing to draw some conclusions about the consistency of the portfolios concerning deviation to the expected performance (calculated with in-sample data). The GMV portfolio presents the largest distances between in-sample and out-of-sample portfolio locations, regardless of the length of the in-sample period. The remaining portfolios present very similar distances between in-sample and out-of-sample mean returns and mean risks. The length of the in-sample period seems to have no substantial effect on the performance consistency, in terms of the deviation to the expected performance of the robust portfolios, since RRC and ARC portfolios present similar distances between in-sample and out-of-sample performances. However, it can be observed that the deviations from the expected mean returns are slightly smaller for the robust portfolios computed with an in-sample period length of 15 years while the deviations from the expected mean risks are slightly smaller for the robust portfolios computed with an in-sample period length of 4 years.

Figure 6-3: In-sample and out-of-sample mean risk and mean return of the RR, AR, EW and GMV portfolios.



The optimal portfolios were represented according to the length of the in-sample period and the number of scenarios considered in the uncertainty set used in their computation. For instance, RR151 and AR45 correspond, respectively, to the RRC portfolio computed using an in-sample period of 15 years and an uncertainty set with 100 scenarios, and the ARC portfolio based on an in-sample period of 4 years and an uncertainty set with 500 scenarios.

The analysis of Figure 6-3 also reveals that the out-of-sample mean performance is worse than the in-sample mean performance for all the computed portfolios. Moreover, an interesting result is confirmed both in-sample and out-of-sample: almost all of the robust portfolios are located between the GMV and EW portfolios, in terms of return and risk. Therefore, except for the RR41 and AR41 portfolios, the proposed relative robust and absolute robust portfolios have lower mean returns and higher mean risk than the GMV portfolio and higher mean return and lower mean risk than the EW portfolio.

Table 6-4 presents the in-sample and the out-of-sample performances of the optimal portfolios concerning regret and modified Sharpe ratio (S_I). The values presented correspond to the mean of the portfolios' regrets and mean of the portfolios S_I, obtained over the 10 time windows. It can be observed that increasing the in-sample period length leads to lower levels of in-sample mean regret of the RRC, ARC and EW portfolios. When different, the RRC and ARC portfolios present very similar (in-sample and out-of-sample) mean regrets.

It can also be observed that increasing the in-sample period length has different effects in the out-of-sample mean regret of the robust portfolios since mean regret generally decreases for the RRC portfolios while it slightly increases for the ARC portfolio. Concerning the mean S_I, the results confirm that increasing the in-sample length seems to substantially improve the performance of the RRC and ARC portfolios, which supports the results previously described that associate better performances of the robust portfolios with the longer in-sample period length. Concerning the GMV portfolio, the increase of the in-sample period length leads to higher mean regrets and lower S_I, which also supports the results previously described. Additionally, the GMV portfolios are the optimal portfolios with the lowest mean regrets.

Finally, it is important to outline that although the robust portfolios present similar mean regrets for the different in-sample lengths under analysis, the mean S_I is substantially higher for the in-sample period of 15 years. In fact, for this in-sample length, both RRC and ARC portfolios have higher average S_I than the GMV and the EW portfolios. This result is somewhat unexpected, since the GMV portfolio estimated with an in-sample period of 15 years shows a higher average expected return and a lower average risk than the robust portfolios estimated with the same in-sample period length. However, as we will see in the next sub-section, for the 1998-2013 window, the GMV portfolio has a low out-of-sample return (close to 3%) while the robust portfolios show high returns (between 15% and 20%). Since the out-of-sample portfolio risk measures are low in this window, the use of a ratio between return and risk amplifies the difference in returns and ends up leading to a higher average S_I for the robust portfolios.

The analysis of the in-sample and out-of-sample performances of robust and non-robust portfolios, for each of the 10 windows, is presented next. Since the RRC and the ARC portfolios generally present better performances for the in-sample period length corresponding to 15 years of historical data, results are described for this particular case. The results that will be presented in the next section generally prevail regardless of the length of the in-sample period.

Table 6-4: In-sample and the out-of-sample regret and modified Sharpe ratio (S_I) of the RRC, ARC, EW and GMV portfolios

Portfolios	**IS Regret**	**OS Regret**	**OS S_I**
RR151	1.6326E-02	3.5854E-02	1.7319E-01
RR152	1.6002E-02	3.6250E-02	1.8731E-01
RR155	1.5909E-02	3.5820E-02	1.7815E-01
AR151	1.6410E-02	3.5791E-02	1.8482E-01
AR152	1.6002E-02	3.6250E-02	1.8731E-01
AR155	1.5794E-02	3.6011E-02	1.7548E-01
RR41	1.9436E-02	3.5410E-02	1.3233E-01
RR42	1.9350E-02	3.5836E-02	1.4440E-01
RR45	1.9191E-02	3.5854E-02	1.3969E-01
AR41	1.9436E-02	3.5410E-02	1.3233E-01
AR42	1.9350E-02	3.5836E-02	1.4440E-01
AR45	1.9191E-02	3.5854E-02	1.3969E-01
EW15	2.2957E-02	4.3142E-02	1.5341E-01
EW4	3.1416E-02		
GMV15	0.0000E+00	2.3958E-02	1.6597E-01
GMV4	0.0000E+00	1.9956E-02	2.3685E-01

This table presents the IS and OS performances of the optimal portfolios concerning regret and modified Sharpe ratio (S_I) by length of the in-sample period and by risk aversion parameter used in their computation. The values presented correspond to the mean of the portfolios' regrets and mean of the portfolios S_I, obtained over the 10 time windows.

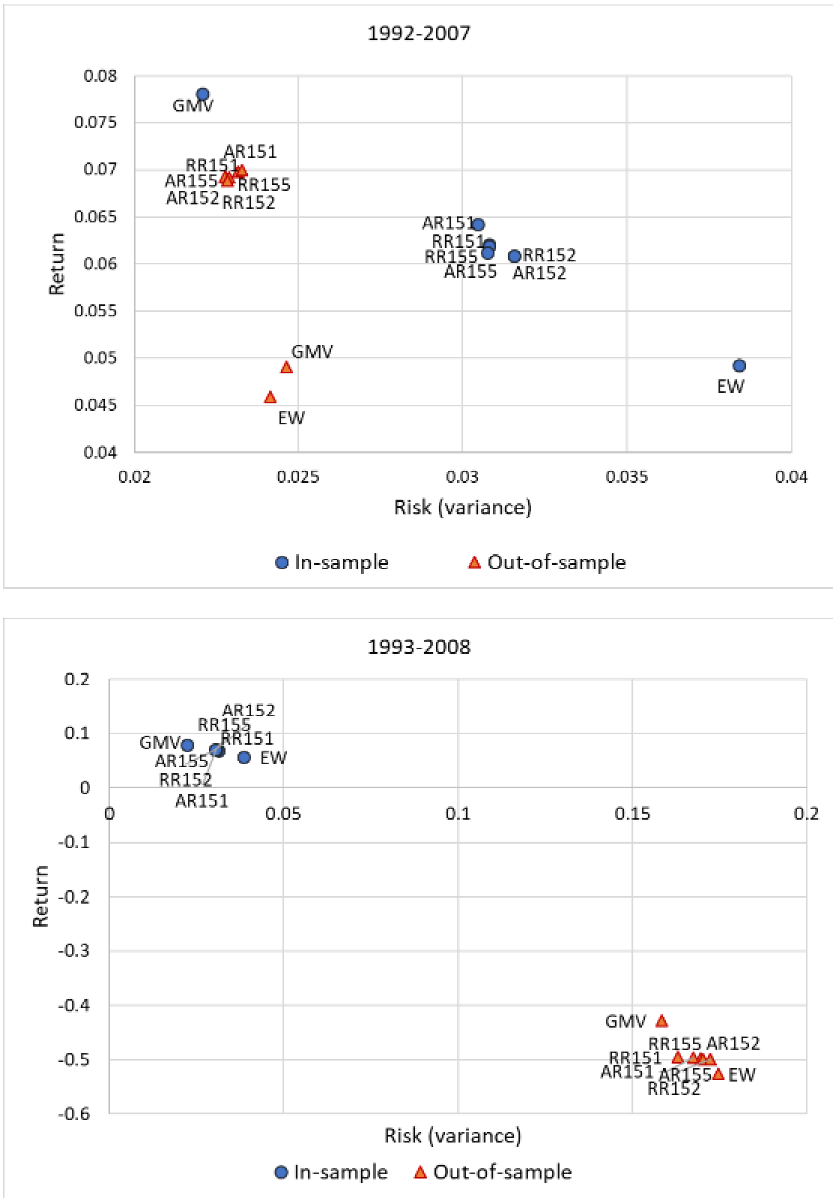
6.3.3.2 Performance of relative robust and non-robust portfolios

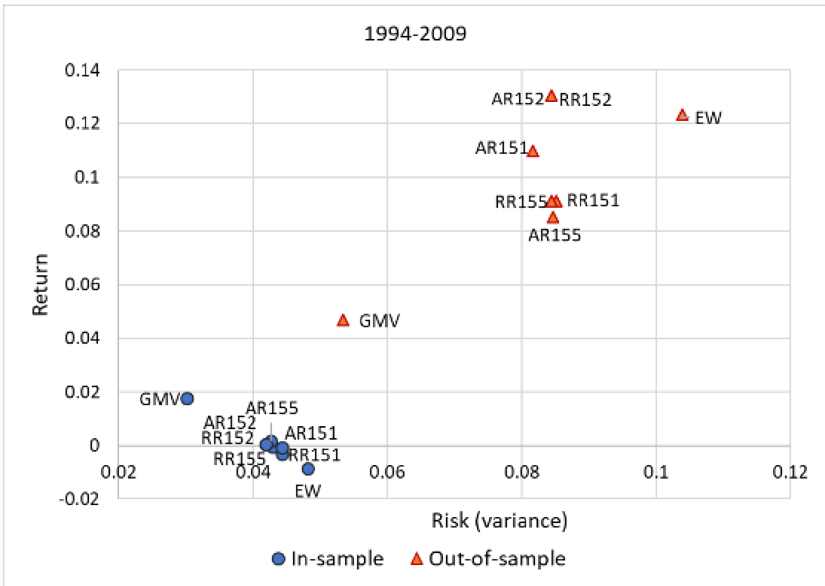
Figure 6-4, Table 6-5 and Table 6-6 shows the performance, in each window, of the portfolios calculated with an in-sample period length of 15 years. We start the analysis with in-sample performance results, where the same conclusions can be drawn for all the windows considered. The GMV portfolio is the dominant solution comparatively to all the other portfolios under analysis. The EW is the worst-performing portfolio, presenting lower returns and higher risk than the remaining portfolios. The ARC and RRC portfolios present very similar performances and are always located between the GMV and the EW portfolios, in terms of both return and risk. Regarding the location of the robust portfolios in the risk-return space, it is also important to point out that they are always closer to the EW portfolio than to the GMV portfolio.

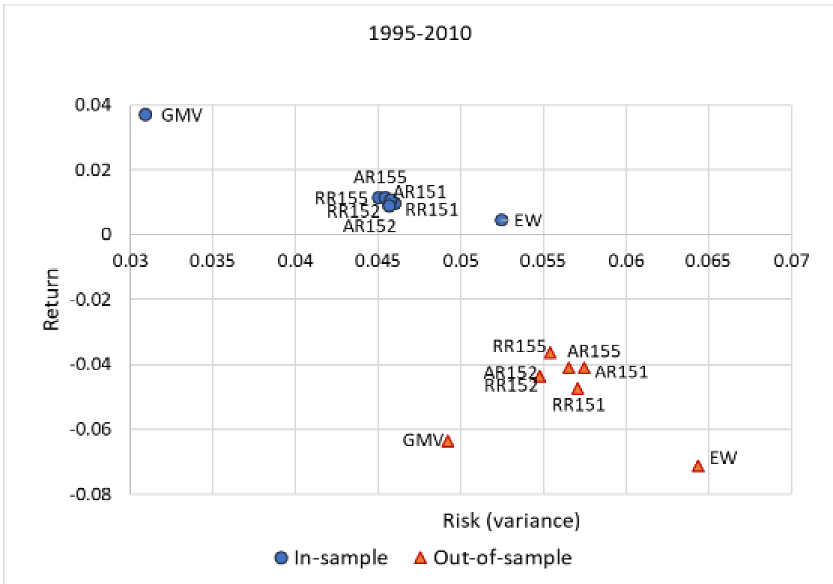
Concerning the out-of-sample performance, some differences can be observed comparatively to the in-sample results previously described. In particular, the GMV portfolio is not a dominant solution in all of the out-of-sample years under analysis. In fact, the outperformance of the GMV portfolio is only confirmed when comparing out-of-sample portfolio risk measures, presenting the lowest risk in all windows, except for the 1992-2007 period where the robust portfolios present themselves as dominant solutions. The GMV portfolio underperforms, comparatively to the other portfolios, when out-of-sample returns are compared in the (out-of-sample) years 2007, 2009, 2010, 2012 and 2013, where this portfolio presents the lowest return or is among those with the lowest returns. Recall that 2009, 2012 and 2013 were previously described as periods where the EURO STOXX 50 index experienced significant recoveries. Additionally, and comparatively to the other portfolios, the GMV portfolio reveals the highest return in the years 2008 and 2011, where all the computed portfolios present negative returns. These results support previous findings concerning the underperformance of the GMV portfolio in bull markets and its outperformance in bear markets (Chow et al., 2014).

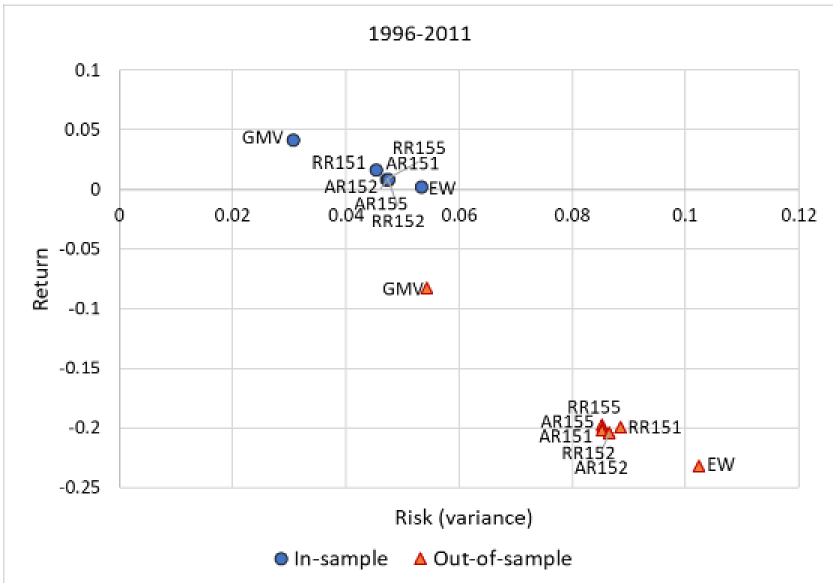
Considering the EW portfolio, the underperformance is generally confirmed both in terms of risk and return. This portfolio always presents the highest risk with the exception of the 1992-2007 period, where it is the GMV portfolio that shows the highest risk. Concerning out-of-sample returns, conflicting results can be observed since the EW portfolio is among those with best performance in some out-of-sample years (2009, 2012 and 2013) while it reveals the worst performance in others (2007, 2008, 2010, 2011 and 2014).

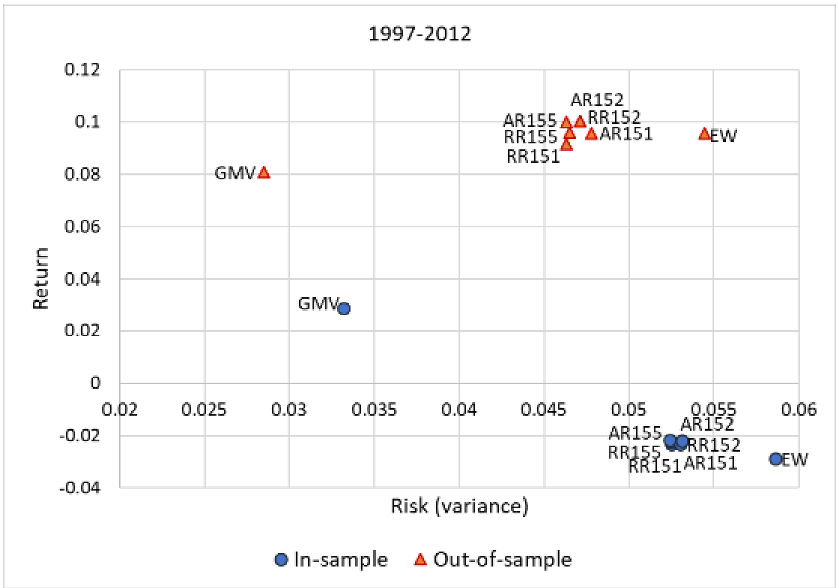
Figure 6-4: In-sample and out-of-sample risks and returns of the RRC, ARC, EW and GMV portfolios computed for each time window.

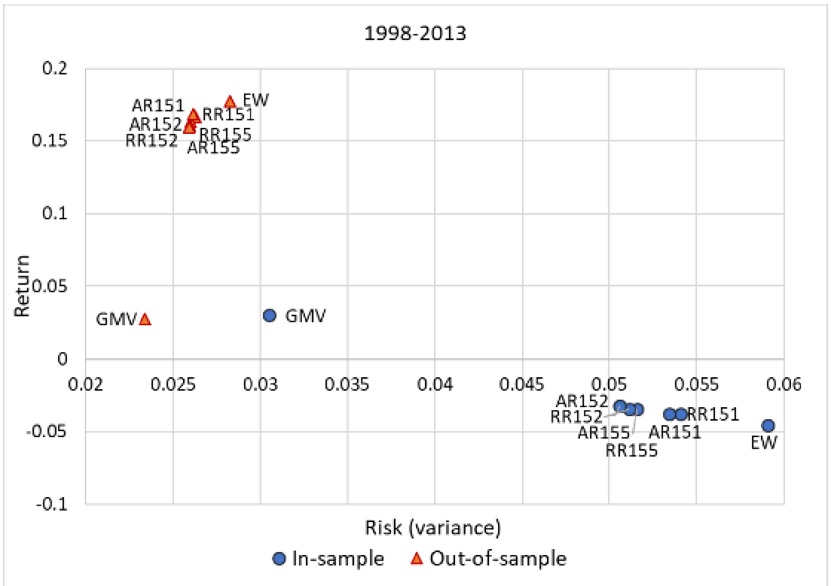


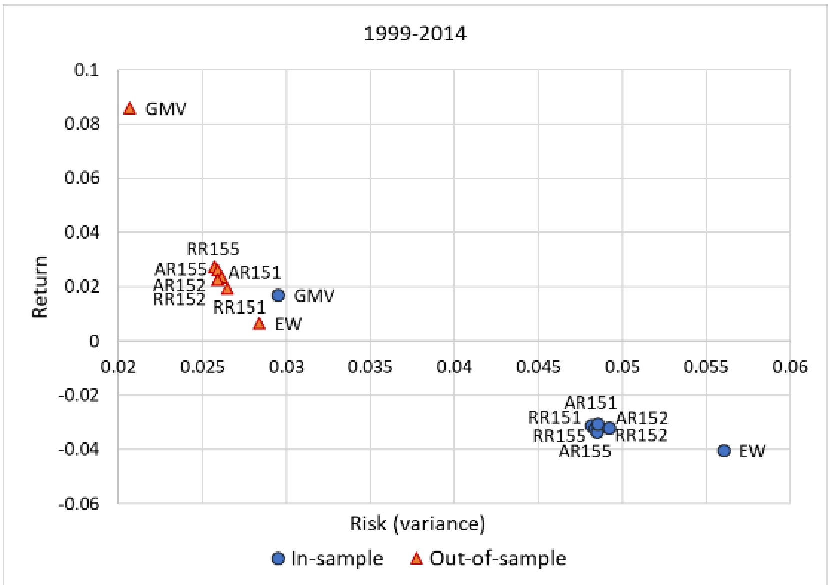

1994-2009
Return
Risk (variance)
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0.02
0.04
0.06
0.08
0.1
0.12
AR152
RR152
EW
AR151
RR155
RR151
AR155
GMV
GMV
AR155
AR152
RR152
AR151
RR155
RR151
EW
In-sample
Out-of-sample

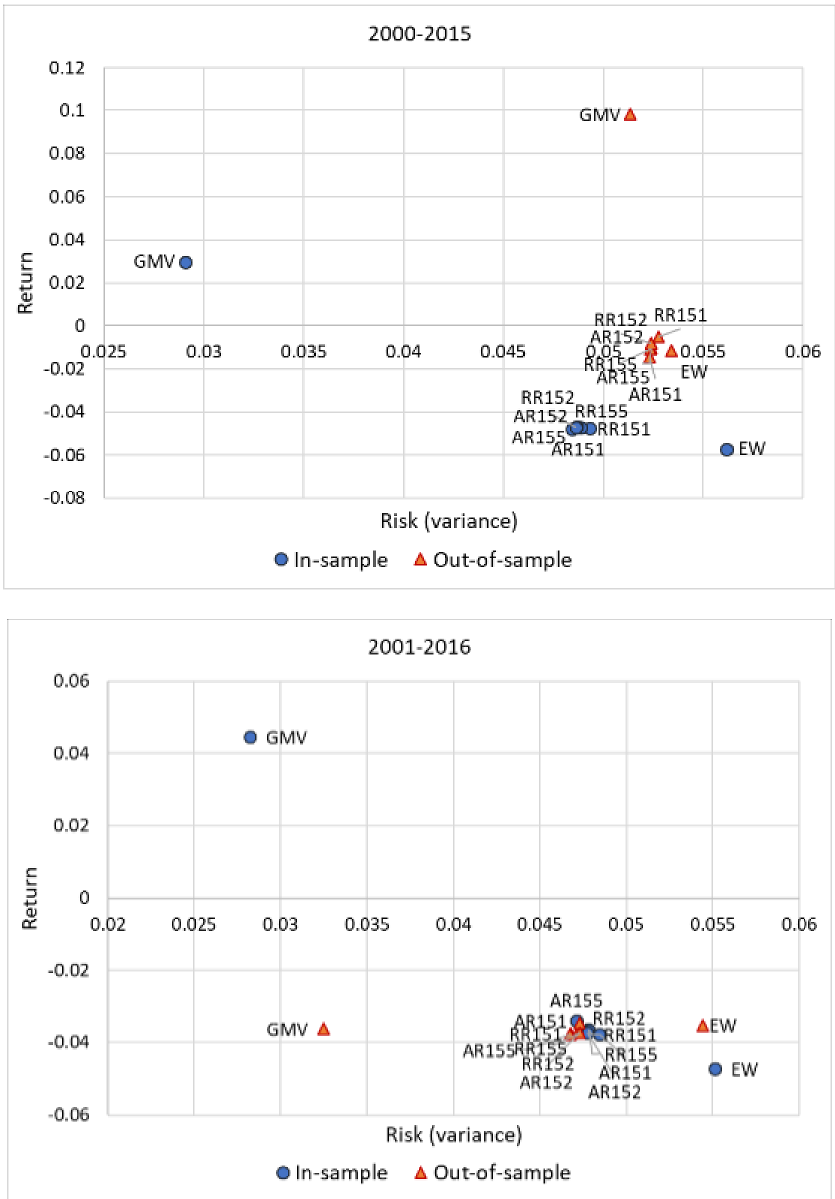

1995-2010
Return
Risk (variance)
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
0.03
0.035
0.04
0.045
0.05
0.055
0.06
0.065
0.07
GMV
AR155
RR155
AR151
RR152
RR151
AR152
EW
RR155
AR155
AR152
RR152
AR151
RR151
GMV
EW
In-sample
Out-of-sample


1996-2011
Return
Risk (variance)
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
0.02
0.04
0.06
0.08
0.1
0.12
GMV
RR155
RR151
AR151
AR152
EW
AR155
RR152
GMV
RR155
AR155
RR151
AR151
RR152
AR152
EW
In-sample
Out-of-sample


1997-2012
Return
Risk (variance)
GMV
AR155
RR155
RR151
AR152
RR152
AR151
EW
AR155
RR155
RR151
AR152
RR152
AR151
EW
In-sample
Out-of-sample


1998-2013
Return
Risk (variance)
AR151
AR152
RR152
EW
RR151
RR155
AR155
GMV
GMV
AR152
RR152
AR155
RR155
RR151
AR151
EW
In-sample
Out-of-sample


1999-2014
Return
Risk (variance)
GMV
RR155
AR155
AR152
RR152
AR151
RR151
GMV
EW
AR151
RR151
RR155
AR155
AR152
RR152
EW
In-sample
Out-of-sample



The optimal portfolios were computed considering the in-sample period length of 15 years and represented according to the value of the risk aversion parameter used in their computation. For instance, 'AR155' and 'RR155' corresponds, respectively, to the ARC and RRC portfolios computed using 15-years data to perform in-sample estimations and a value of 5 for the risk aversion parameter.

Analyzing the out-of-sample performance of the robust portfolios, it can be confirmed that the robust solutions present very similar performances, as previously suggested by the mean results. In fact, there is no ARC or RRC portfolio that systematically stands out as a dominant solution comparatively to the other robust portfolios. Furthermore, the dominance of the robust solutions over the EW portfolio is confirmed for the majority of the RRC and ARC portfolios and for all the windows under analysis, except in the out-of-sample years 2009, 2012, 2013, 2015 and 2016. Although the robust portfolios are not dominant solutions comparatively to the EW portfolios in these periods, they generally outperform when the S_I measure is considered (Table 6-5). In fact, the robust portfolios present higher S_I than the EW portfolio in all of the windows under analysis with the only exception for 2013. Comparatively to the GMV portfolio, the robust portfolios do not (generally) stand out as dominant solutions, but they present higher out-of-sample returns in 5 of the 10 windows (1992-2007, 1994-2009, 1995-2010, 1997-2012 and 1998-2013), namely in periods characterized by significant recoveries of the EURO STOXX 50 index. Moreover, the majority of the robust solutions present higher S_I than the GMV portfolio in 3 of the 10 windows (1992-2007, 1994-2009, 1995-2010).

A consistent result is the fact that the robust solutions always present risk measures between the risk measures of the GMV and the EW portfolios, with the only exception being the 1992-2007 period where they outperform both benchmarks. In relation to the returns of the robust solutions, it can be confirmed that, for the majority of the windows under analysis and for the majority of the robust solutions, their returns are between the returns of the 2 benchmarks. Only in 2 of the 10 windows some of the robust solutions have the lowest out-of-sample returns (2000-2015 and 2001-2016) while in 4 of the 10 windows some of these robust solutions present the highest returns (1992-2007, 1994-2009, 1995-2010 and 1997-2012). It is also important to highlight that in 5 of the 10 windows the robust portfolios present out-of-sample portfolio risk measures lower than those computed using in-sample data. Finally, except for one of the windows under analysis (2000-2015) in which the GMV portfolio is the only computed solution with positive out-of-sample return, in all the other windows, the computed portfolios seem to behave in the same way, either all presenting gains (positive returns) or all presenting losses (negative returns).

Figure 6-4 also allows the observation of the consistency of the optimal portfolios in terms of the deviation to the expected performance, by analyzing the distances between in-sample and out-of-sample portfolios' location. Although none of the portfolios systematically reveals better consistency in all of the windows under analysis, it can be observed that in 4 of the 10 windows the robust portfolios exhibit the lowest deviations to the expected performances and, thus, can be considered the more consistent solutions.

Analyzing the robustness of the computed portfolios in terms of the regret (measured as the increase in the investment risk), it can be observed, in Table 6-6, that the RRC and ARC portfolios present similar robustness and are systematically more robust than the EW presenting lower regrets in all of the windows under analysis. Additionally, the GMV portfolio stands out as the most robust solution in all of the windows except in 1992-2007, where the robust solutions show a smaller regret.

To conclude, it is also important to highlight that the robust portfolios are never the worst-performing portfolios since they are not simultaneously dominated by both benchmarks used in this study. Furthermore, the robust portfolios reveal the potential for diversification similar to the EW strategy, while assigning maximum weights to the same assets selected by the minimum variance strategy (low-beta assets). In favorable market conditions, we can generalize our findings accordingly: the robust portfolios present lower risk than EW portfolios and higher returns when compared to GMV portfolios. These results clearly reinforce the relevance of the proposed methodology, since previous studies confirmed the good performances of both the EW benchmark (DeMiguel et al., 2009) and the GMV portfolio (Chan et al., 1999; Jagannathan & Ma, 2003).

Table 6-5: Out-of-sample modified Sharpe ratio (S_I) of the RRC, ARC, EW and GMV portfolios.

Portfolio	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
RR151	0.21197	-0.21522	0.26161	-0.01302	-0.06161	0.41860	1.02401	0.11343	-0.00077	-0.00715
RR152	0.20740	-0.22314	0.39865	-0.01189	-0.06224	0.45473	0.98369	0.13454	-0.00146	-0.00714
RR155	0.21077	-0.22102	0.26274	-0.01021	-0.05990	0.43796	1.00640	0.16390	-0.00201	-0.00713
AR151	0.21260	-0.21880	0.33267	-0.01154	-0.06125	0.43010	1.03418	0.13964	-0.00291	-0.00654
AR152	0.20740	-0.22314	0.39865	-0.01189	-0.06224	0.45473	0.98369	0.13454	-0.00146	-0.00714
AR155	0.20929	-0.22185	0.24225	-0.01147	-0.06069	0.45712	0.99285	0.15655	-0.00203	-0.00721
EW	0.05408	-0.23593	0.33675	-0.01990	-0.07670	0.40281	1.04944	0.03305	-0.00230	-0.00717
GMV	0.07392	-0.18573	0.13811	-0.01569	-0.02101	0.46983	0.17611	0.58913	0.44079	-0.00573

This table shows the out-of-sample S_I of the optimal portfolios by out-of-sample year. The optimal portfolios were computed considering the in-sample period length of 15 years and represented according to the value of the risk aversion parameter used in their computation. For instance, 'AR155' and 'RR155' corresponds, respectively, to the ARC and RRC portfolios computed using 15-years data to perform in-sample estimations and a value of 5 for the risk aversion parameter.

Table 6-6: Out-of-sample regret of the RRC, ARC, EW and GMV portfolios.

Portfolio	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
RR151	0.01334	0.09974	0.05467	0.03223	0.06276	0.02833	0.01022	0.01172	0.02604	0.01949
RR152	0.01302	0.10905	0.05396	0.02994	0.06071	0.02915	0.00982	0.01114	0.02567	0.02003
RR155	0.01296	0.10627	0.05392	0.03057	0.05952	0.02851	0.00991	0.01097	0.02567	0.01989
AR151	0.01348	0.10429	0.05115	0.03261	0.05949	0.02982	0.01005	0.01140	0.02562	0.02000
AR152	0.01302	0.10905	0.05396	0.02994	0.06071	0.02915	0.00982	0.01114	0.02567	0.02003
AR155	0.01308	0.10706	0.05417	0.03168	0.05970	0.02832	0.00988	0.01114	0.02563	0.01946
EW	0.01432	0.11141	0.07332	0.03953	0.07668	0.03651	0.01219	0.01362	0.02669	0.02715
GMV	0.01484	0.09525	0.02292	0.02438	0.02855	0.01056	0.00730	0.00592	0.02464	0.00522

This table shows the out-of-sample regret of the optimal portfolios by out-of-sample year. The optimal portfolios were computed considering the in-sample period length of 15 years and represented according to the value of the risk aversion parameter used in their computation. For instance, 'AR155' and 'RR155' corresponds, respectively, to the ARC and RRC portfolios computed using 15-years data to perform in-sample estimations and a value of 5 for the risk aversion parameter.

6.4 Conclusions

This work extends and combines recognized methodologies to develop a method of calculating relative robust and absolute robust portfolios. For the relative robust strategy, where the maximum regret is minimized, regret is defined as the increase in the investment risk resulting from investing in a given portfolio instead of choosing the optimal portfolio for the realized scenario. In the absolute robust strategy, minimization of risk was applied to the worst-case scenario over the whole uncertainty set.

The results suggest that increasing the in-sample period length for the estimation of the model parameters has no substantial effect on the composition and performance of the robust portfolios, which highlights the usefulness of the proposed models in the presence of limited data. ARC and RRC portfolios always assign non-zero weights to all the assets in the dataset, thereby capturing the potential for diversification of the equally-weighted strategy. Moreover, the robust portfolios assign maximum weights to the same assets selected by the minimum variance strategy.

The proposed robust portfolios consistently present out-of-sample portfolio risk measures that lie between the portfolio risk measures of the GMV portfolio and those of the EW portfolio. For the majority of the windows, out-of-sample returns of the robust portfolios are between, or higher, than the portfolio returns of the two benchmarks. Hence, two major conclusions can be drawn. The consistent outperformance (in terms of return, risk or modified Sharpe ratio) of the robust portfolios comparatively to the EW portfolio confirms the benefits of investing in the optimal portfolio instead of simply allocating the investor's wealth equally among the assets. In the presence of favorable market conditions, where the GMV portfolio performs poorly, the robust portfolios exhibit substantially higher returns. These conclusions support the ability of the robust strategies to optimize the first and second moments of portfolio returns.

Additionally, the empirical results suggest that when the distribution of the variances of the optimal portfolios associated with the scenarios belonging to the uncertainty set is less dispersed, the relative robust and absolute robust models may often yield identical solutions. Since the probability of less dispersed values is higher for shorter in-sample period lengths, this is an important outcome to take into consideration in the presence of limited data. Similar behaviours were also observed among the robust solutions and the non-robust solutions when losses (negative returns) versus gains (positive returns) are compared. This finding, together with the outcomes regarding the robustness and the consistency of the proposed portfolios, in terms of increase in the investment risk (regret) and deviations to their expected performances, suggests that the proposed methodologies are as consistent as the benchmarks used for comparing portfolios' out-of-sample performances which, in our opinion, validates the proposed methodologies.

Overall, the results presented in this work reinforce the relevance of robust optimization within the field of portfolio selection under uncertainty.

Chapter 7

A robust parametric portfolio policy (model D)

7.1 Introduction

Traditional portfolio optimization approaches, like the mean-variance optimization problem, are generally a two-step procedure: in a first step, the model inputs are estimated based on some theoretic assumptions about the data generating process; then the optimal portfolio weights, modeled as functions of those estimators, are computed. A different approach consists in parameterizing the portfolio weights as functions of observable quantities, estimating these weights directly, and then solving the portfolio selection model for the parameters that maximize the investor's expected utility (Brandt, 2010).

The aim of our study is to build on the parametric portfolio policies presented by Brandt et al. (2009) by incorporating the uncertainty into the optimization model itself using the RO methodology and assessing the performance of the proposed robust parametric portfolio policies by comparing them to the (non-robust) parametric portfolio policies proposed by Brandt et al. (2009) and other (non-robust) classical portfolio strategies.

The empirical analysis uses historical daily data from the stocks of the EURO STOXX 50 index. Different optimal portfolios are computed and compared, considering both in-sample and out-of-sample data, for turnover, abnormal return, beta (systematic risk), location in the risk-return space, modified Sharpe ratio and regret. The considered portfolios are the proposed relative robust parametric portfolio policy (RRD), the absolute robust parametric portfolio policy (ARD), the MV portfolio, the GMV portfolio, the EW portfolio and the parametric portfolio policies (PPP) proposed by Brandt et al. (2009).

The empirical analysis suggests that the ARD portfolio generally outperforms the remaining investment strategies applied in this study in at least one of the considered performance measures. Although outperforming the PPP portfolio in many of the time windows under analysis, the RRD portfolios do not reveal results as good and as consistent as the ARD portfolio. The analysis of the coefficients of the ARD, RRD and PPP portfolios reveals that, regardless of the value of the risk aversion parameter, the deviation of the optimal solution from the benchmark portfolio decreases with the firm's size. Concerning the book-to-market ratio, the RRD and the PPP portfolios present a similar behavior: both assume larger positions in value firms' stocks for low values of the risk aversion parameter, and larger positions in growth firms' stocks for high values of the risk aversion parameter. Considering the momentum, it is possible to observe that ARD and the PPP portfolios present a similar behavior: both assume larger positions in past losers for lower levels of risk aversion and larger positions in past winners for higher levels of risk aversion. Results also suggest that both absolute robust and relative robust parametric portfolio policies seem to benefit from the use of long-term past data in order to estimate the input parameters, while the non-robust parametric portfolio policy benefits from its reduction. Thus, our findings confirm that the robust methodology promotes the enhancement of the performance of current models available in the literature, specifically of the parametric portfolio policies presented by Brandt et al. (2009), reinforcing the relevance of robust optimization within the field of portfolio selection under uncertainty.

The remainder of this chapter proceeds as follows. In Section 7.2, the proposed robust parametric portfolio policies are presented and the process for computing the robust parametric portfolios is described. The empirical analysis is presented and main results are described in Section 7.3. Finally, in Section 7.4, the main conclusions are highlighted.

7.2 Methodology

7.2.1 The robust parametric portfolio policies

In this work, we propose robust optimization models based on the parametric portfolio policies presented by Brandt et al. (2009) and previously described in section 2.1. To develop a robust version of this model, uncertainty was considered in the asset weights, in the asset characteristics and in the asset returns.

Recall the notation and the general procedure for generating the sample returns from the in-sample data, previously described in section 4.3. The uncertainty set corresponds to a finite set of scenarios, $s_i \in U$, where each s_i is described by a set of asset returns ($r_{n,t}$), a set of asset weights ($\bar{w}_{n,t}$) and a set of asset characteristics ($\hat{x}_{n,t}$), and is defined in the following way (to avoid cluttering the notation, we drop the index i from s_i in this definition):

$$s = \left(R^s, \bar{W}^s, \hat{C}^s\right) \tag{7.1}$$

with $R^s = [r_{nt}^s]$, $\bar{W}^s = [\bar{w}_{nt}^s]$, $\hat{C}^s = [\hat{x}_{nt}^s]$, $n = 1, \dots, N, t = \mathbb{z}(s), \dots, \mathbb{z}(s) + J - 1.$

The proposed relative robust parametric portfolio policy defines *regret* as the utility loss for the investor resulting from choosing a portfolio characterized by the vector of coefficients α instead of choosing α^{s*} which corresponds to the optimal vector of coefficients under scenario s. The vector of coefficients α represents the over or underweighting of each stock, relative to the benchmark portfolio, based on the firm's characteristics. Let $r_{t+1}^{ps\,*}$ be the realized return of the optimal portfolio under scenario s, calculated using α^s as $r_{t+1}^{ps\,*} = \sum_{n=1}^{N} \left(\bar{w}_{n,t}^s + \frac{1}{N} \alpha^{sT} \hat{x}_{n,t}^s\right) r_{n,t+1}^s$. Vector α^{s*} is the optimal solution of the following problem:

$$\max_{\alpha^s} \frac{1}{J} \sum_{t=\mathbb{z}(s)}^{\mathbb{z}(s)+J-1} u\left(\sum_{n=1}^{N} \left(\bar{w}_{n,t}^s + \frac{1}{N} \alpha^{sT} \hat{x}_{n,t}^s\right) r_{n,t+1}^s\right). \tag{7.2}$$

Equivalently, $r_{t+1}^{ps}(\alpha)$ corresponds to the expected return of the portfolio p, under scenario s, calculated using the vector of coefficients α as $r_{t+1}^{ps}(\alpha) = \sum_{n=1}^{N} \left(\bar{w}_{n,t}^s + \frac{1}{N} \alpha^T \hat{x}_{n,t}^s\right) r_{n,t+1}$. The regret associated with scenario s when vector α is considered, $P^s(\alpha)$, is defined by

$$P^s(\alpha) = E\left[u\left(r_{t+1}^{ps\,*}\right)\right] - E\left[u\left(r_{t+1}^{ps}(\alpha)\right)\right] = E\left[u\left(r_{t+1}^{ps}(\alpha^{s*})\right)\right] - E\left[u\left(r_{t+1}^{ps}(\alpha)\right)\right] \tag{7.3}$$

and the maximum regret function, $P(\alpha)$, is defined by

$$P(\alpha) = \max_{s \in U} E\left[u\left(r_{t+1}^{ps\,*}\right)\right] - E\left[u\left(r_{t+1}^{ps}(\alpha)\right)\right]. \tag{7.4}$$

The minmax regret solution α corresponds to the vector of coefficients that optimizes the relative robust optimization model

$$\min_{\alpha} \max_{s \in U} E\left[u\left(r_{t+1}^{ps\,*}\right)\right] - E\left[u\left(r_{t+1}^{ps}(\alpha)\right)\right], \tag{7.5}$$

and satisfies the non-negativity constraint. As suggested by Brandt et al. (2009), we imposed this constraint by truncating and renormalizing the relative robust portfolio weights. Hence, instead of considering the $w_{n,t}$ weights, as defined in (2.10), we considered the non-negative weights ${w_{n,t}}^+$, defined as follows:

$${w_{n,t}}^+ = \frac{max[0, w_{n,t}]}{\sum_{m=1}^{N} max[0, w_{m,t}]}. \tag{7.6}$$

Note that problem (7.5) is a three-level optimization problem. The computation of vectors α^{s*}, $\forall s$, which represent the optimal solution for each scenario s, constitutes the first optimization level. The inner maximization problem in (7.5) constitutes the second optimization level and allows the computation of the maximum regret for each $s \in U$, providing an upper bound on the true utility loss for the investor. Finally, the third optimization level corresponds to the outer minimization problem in (7.5), which gives the optimal solution that minimizes the maximum regret for all $s \in U$.

For the absolute robust parametric portfolio policy, the absolute robust portfolio (maxmin solution) corresponds to the weight combination vector that solves the absolute robust optimization model defined by

$$\max_{\alpha}\min_{s\in U} E\left[u\left(r_{t+1}^{p}(\alpha)\right)\right] = \max_{\alpha}\min_{s\in U}\frac{1}{J}\sum_{\tau=\mathbb{Z}(s)}^{\mathbb{Z}(s)+J-1} u\left(\sum_{n=1}^{N}\left(\bar{w}_{n,\tau}^{s}+\frac{1}{N}\alpha^{T}\hat{x}_{n,\tau}^{s}\right)r_{n,\tau+1}^{s}\right) \qquad (7.7)$$

and satisfies the non-negativity constraint (previously explained and defined in (7.6)).

7.2.2 Computing the robust parametric portfolios

We begin by explaining the computation of the relative robust portfolio. We start by constructing the scenario set U calculating the S scenarios, as already described. The set of stock returns $r_{n,t}^{s}$, weights $\bar{w}_{n,t}^{s}$ and characteristics $\hat{x}_{n,t}^{s}$, defining a scenario s, is used in order to solve problem (7.2) and to determine the optimal solution α^{s*} (which characterizes the portfolio that maximizes the average utility the investor would have obtained by implementing the parametric portfolio policy over the period of the estimation window). This constitutes the first optimization process of the proposed three-level optimization problem.

After computing the optimal solutions for each scenario $s \in U$, the relative robust optimization problem (7.5) is solved using the GA. The fitness function considers the maximization of the regret as presented in (7.4) and corresponding to the utility loss for the investor resulting from choosing, under scenario s, a portfolio calculated using coefficients α instead of α^{s*}. The population considered has twice the size of the uncertainty set, and it is initially composed of all the optimal solutions $\alpha^{s*}, \forall s$, and other feasible solutions randomly generated. The estimated portfolios are assessed by considering out-of-sample data corresponding to the next year.

Concerning the computation of the absolute robust portfolio, a similar process was applied. The uncertainty set U is constructed as previously described. After computing the S scenarios, the maxmin solution is calculated by solving problem (7.7) using the GA. In this case, the fitness function was defined as the minimum portfolio's expected utility for all the scenarios considered in U (inner maximization problem in (7.7). Hence, the optimization is performed assuming the worst-case realization of the uncertain parameters over the whole uncertainty set.

7.3 Empirical analysis

7.3.1 Data and model settings

Stocks in the constituent list of the EURO STOXX 50 index at the end of the in-sample period were identified and historical daily data from January 1990 to December 2016 (27 years) was used. As Brandt et al. (2009), we considered the following three firm characteristics: *ME*, *BTM*, and *MOM*. These characteristics were calculated from the following Thomson Reuters Datastream variables: total assets (WC02999), total liabilities (WC03351), deferred taxes and investment tax credits (WC03263), preferred stock value (WC03451), common shares outstanding (WC05301) and price per share (P). If total assets, liabilities, price or shares outstanding were missing, the observation was not included in the data set. Consider the book equity (*be*), corresponding to the total assets minus total liabilities plus deferred taxes and investment tax credits minus preferred stock value, and the market equity (*me*), corresponding to the price per share times shares outstanding. Then, the *BTM* and *ME* characteristics are defined as follows:

$$BTM = log\left(1 + \frac{be}{me}\right) \tag{7.8}$$

and

$$ME = log(me). \tag{7.9}$$

Daily continuous returns were calculated from the adjusted closing prices of the stocks. Momentum (*MOM*) was defined as the daily compounded return between days $t - 254$ and t. To make sure that only the information from an already published annual report of the firm is reflected in the portfolio determination, we follow Brandt et al. (2009) suggestion and we allow a minimum of six-month lag between the fiscal year end of the accounting variables considered and the returns. The accounting variables were constructed at the end of the fiscal years from 1990 to 2016. Data from the first two years of each in-sample period is exclusively used to calculate the lagged values of the firm's BTM and ME.

We also follow Brandt et al. (2009) regarding the choice of the benchmark portfolio and we consider the value-weighted portfolio. Thus, the weight of stock n at date t in the benchmark portfolio ($\bar{w}_{n,t}$) is defined as:

$$\bar{w}_{n,t} = \frac{me_{n,t}}{\sum_{n=1}^{N} me_{n,t}}. \tag{7.10}$$

The empirical analysis used rolling windows of two different lengths. In one case, a rolling window with a constant length of 18-years is defined: 17-years data to perform in-sample estimations and an out-of-sample evaluation period of 1-year. In this case the first window ranges from January 1990 to December 2007 (in-sample period from 1990 to 2006 and out-of-sample consisting of 2007) while the last window ranges from January 1999 to December 2016 (in-sample period from 1999 to 2015 and out-of-sample consisting of 2016). In the second case, a rolling window with a constant length of 7-years is considered: 6-years data to perform in-sample estimations and an out-of-sample evaluation period of 1-year. The first window now ranges from January 2001 to December 2007 (in-sample period from 2001 to 2006 and out-of-sample consisting of 2007) while the last window ranges from January 2010 to December 2016 (in-sample period from 2010 to 2015 and out-of-sample consisting of 2016).

Similarly to Brandt et al. (2009), we assume constant relative risk aversion (CRRA) preferences over wealth and constant relative risk aversion parameter ($\gamma \in \mathbb{R}^{+} \setminus \{1\}$), for describing the investor's preferences. Different values of the relative risk aversion parameter ($\gamma \in \{0.5,2,5\}$) were used in order to explore the sensitivity of the results to this parameter. The absolute robust and relative robust models, as well as the parametric portfolio policy proposed by Brandt et al. (2009) and the classical mean-variance model, were solved for each one of these values.

The steps for computing the absolute robust and relative robust solutions, described in the previous section, are iteratively repeated for each of the time windows defined. Each scenario s, defined by the set of sample returns $r_{n,t}^{s}$, weights $\bar{w}_{n,t}^{s}$ and characteristics $\hat{x}_{n,t}^{s}$, is computed considering an estimation window length of 120 consecutive daily returns. Estimations of the model inputs are performed in R. Once the robust and non-robust portfolios are computed for each of the time windows under analysis, in-sample and out-of-sample performances are analyzed.

7.3.2 In-sample and out-of-sample performances

In order to conclude whether robust models can be an added-value for portfolio optimization, the in-sample and out-of-sample performances of the proposed relative robust and absolute robust parametric portfolio policies were analyzed and compared to the parametric portfolio policy presented by Brandt et al. (2009) as well as to classical non-robust optimization strategies (benchmarks). The parametric portfolio policy model, described in (2.11), was applied and the PPP optimal solution was computed. The benchmark portfolio considered in its computation was the value-weighted portfolio (defined in (7.10)). The classical non-robust portfolios considered in the performance analysis were the EW, GMV and the MV portfolios.

The performance analysis starts by investigating the parameters' variation effect over the proposed robust solutions. We consider the results over the entire out-of-sample period (10 years) and analyze the variation of the number of scenarios admitted in the uncertainty set (S), the risk aversion parameters (γ) and the length of the in-sample period (Y), by comparing the RRD and ARD portfolios' location in the risk-return space. Then, for a given parameter combination (Y, S), we compare the composition, cardinality, turnover, abnormal return and beta of the RRD, ARD and PPP portfolios. The computation of the portfolio turnover followed the definition presented by Brandt et al. (2009), considering the evolution of the assets returns from $t-1$ to t, as defined in (4.6).

For the computation of the abnormal return (J_{α}), we applied the CAPM and performed a linear regression considering the EURO STOXX 50 index as the proxy for the market portfolio. A total of 2540 daily returns (corresponding to the 10 out-of-sample years) were used in the regression analysis. We also present the portfolios' beta, which is a measure of their systematic risk relative to the excess return of the market portfolio. It is important to notice that the comparability to the market portfolio (EURO STOXX 50 index) has some limitations since the assets used to construct the portfolios vary along the 10 time windows under analysis and are not the same as the ones in the constituent list of the EURO STOXX 50 index during this period.

Finally, in-sample and out-of-sample results are examined for each of the 10 time windows and portfolios' location in the risk-return space and modified Sharpe ratio are compared. For the RRD, ARD and PPP portfolio, we also compare the regret, defined by

$$R = \frac{1}{T}\left[\left(\sum_{t=0}^{T-1} \frac{\left(1 + \sum_{n=1}^{N}\left(\overline{w}_{n,t}^{s} + \frac{1}{N}\alpha^{s*T}\hat{x}_{n,t}^{s}\right) r_{n,t+1}^{s}\right)^{1-\gamma}}{1-\gamma}\right) - \left(\sum_{t=0}^{T-1} \frac{\left(1 + \sum_{n=1}^{N}\left(\overline{w}_{n,t}^{s} + \frac{1}{N}\alpha^{T}\hat{x}_{n,t}^{s}\right) r_{n,t+1}^{s}\right)^{1-\gamma}}{1-\gamma}\right)\right], \tag{7.11}$$

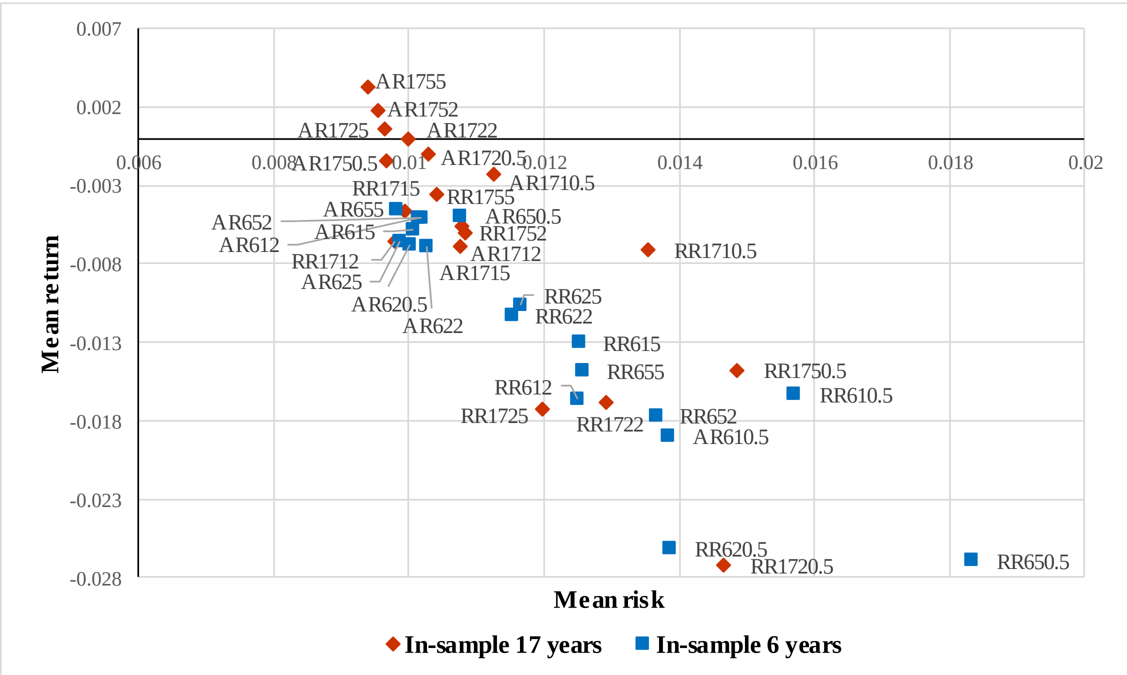
where α^{s*} represents the vector coefficients of the optimal portfolio (feasible solution with maximum utility) within the sample period under consideration, with no portfolio weight constraint. After imposing the no-short-sale constraint, the optimum of the objective function is no longer guaranteed. Thus, in order to define regret, we considered the objective function value of the optimal solution α^{s*} without imposing any portfolio weight constraints.

7.3.3 Results

7.3.3.1 Parameter variation effect

As previously mentioned, a sensitivity analysis regarding the considered parameters has been made. We consider the results over the entire out-of-sample period (10 years) and analyze the variation of the number of scenarios admitted in the uncertainty set (S), the risk aversion parameters (γ) and the length of the in-sample period (Y), by comparing the RRD and ARD portfolios' location in the risk-return space. Acknowledging the limitations of using the average return as the sole comparison measure, the portfolios' performances are analyzed, both in-sample and out-of-sample, by comparing the mean of the portfolios' returns (mean return) and the mean of the portfolios' variances (mean risk), obtained over the 10 windows. Out-of-sample results are presented in Figure 7-1. For simplification purposes, the RRD and ARD portfolios will be represented by 'RR' and 'AR', respectively, in the figures presented along the Results section and corresponding subsections.

Figure 7-1: Out-of-sample mean risk and mean return of the RRD and ARD portfolios.



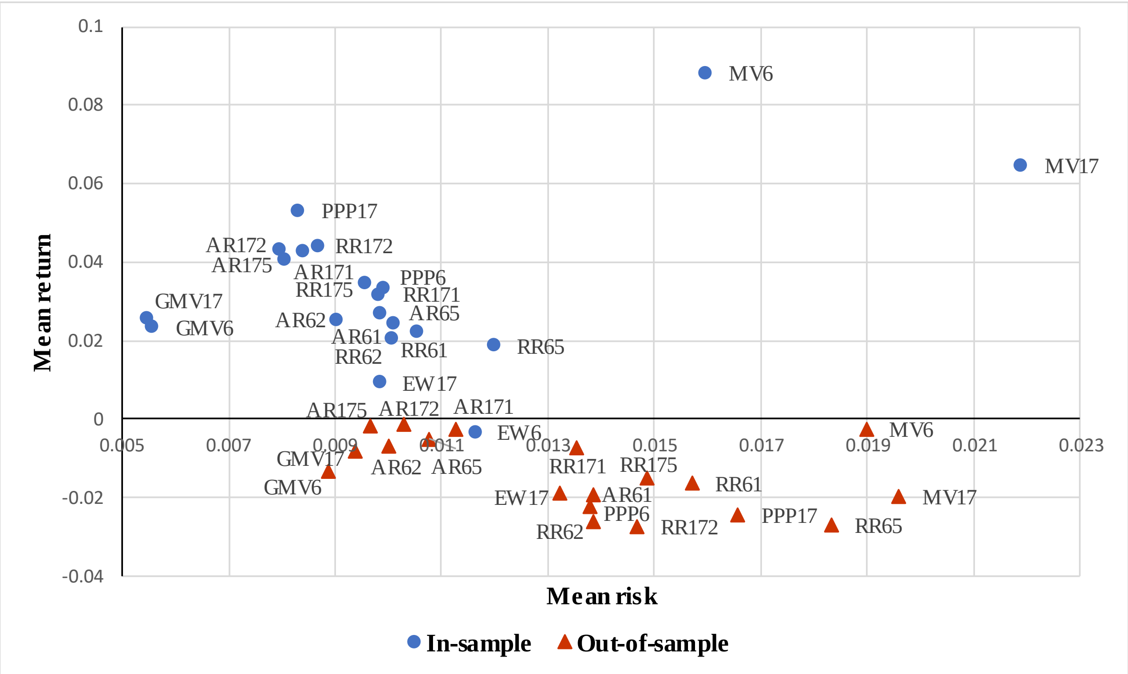
The optimal portfolios were represented according to the in-sample period length (Y), the number of scenarios in the uncertainty set (S) and the risk aversion parameter (γ), used in their computation. For simplification purposes, the parameter S was represented by 1, 2 or 5 when $S = 100$, $S = 200$ or $S = 500$, respectively.For instance, RR1710.5 corresponds to the RRD portfolio computed using an in-sample period of 17 years, an uncertainty set with 100 scenarios and a risk aversion parameter of 0.5.

It can be observed that the ARD portfolios are more concentrated in the upper left corner of Figure 7-1, while the RRD portfolios are more dispersed and always present negative mean returns, regardless of the in-sample period length used in their computation. This clearly suggests a better general performance of the proposed absolute robust approach comparatively to the relative robust approach. It can also be observed that the robust solutions computed with an in-sample period length of 17 years are also more concentrated in the upper left corner comparatively to the robust solutions computed with an in-sample period length of 6 years. Thus, both absolute robust and relative robust methodologies seem to benefit from the use of long-term past values in order to estimate the input parameters. Concerning the number of scenarios, increasing this number generally improves both the return and the risk of the ARD portfolios. The effect of increasing the number of scenarios is not so straightforward for the RRD portfolios. For instance, while the RR1725 portfolio is a dominated solution comparatively to the RR1715 and the RR1755 portfolios, the RR655 portfolio is a dominated solution comparatively to the RR615 and the RR625 portfolios. In fact, a closer analysis allows us to conclude that for $S = 100$ and $S = 500$, the RR17 portfolios (RRD portfolios computed using an in-sample period length of 17 years) are dominant solutions comparatively to the RR6 portfolios (RRD portfolios computed using an in-sample period length of 6 years); when $S = 200$, the opposite occurs, i.e. the RR6 portfolios are dominant solutions comparatively to the RR17 portfolios.

The analysis of the mean results is extended in order to compare the overall performances of robust and non-robust portfolios. The in-sample and the out-of-sample mean risk and mean return of the ARD, RRD, PPP, MV, GMV and EW portfolios are compared and the results are presented in Figure 7-2 and Figure 7-3. The optimal portfolios were represented according to the in-sample period length (Y) and/or the number of scenarios of the uncertainty set (S) used in their computation. Results are presented for two values of the risk aversion parameter: 0.5 and 5.

The analysis of the effect of the length of the in-sample period in the portfolios' performance shows that reducing the length of the in-sample period (from 17 years to 6 years) improves both in-sample and out-of-sample overall performances of the MV portfolio, regardless of the value of the risk aversion parameter. A different behaviour can be observed for the GMV portfolio. Whilst the GMV17 portfolio is a dominant solution in-sample, with out-of-sample data it presents higher return and higher risk, comparatively to the GMV6 portfolio. It is also important to notice that the GMV is the optimal solution with the lowest out-of-sample mean risk. Regarding the PPP portfolio, its overall in-sample performance deteriorates when the length of the in-sample period is reduced, as the in-sample mean return substantially decreases while the in-sample mean risk increases. Out-of-sample, the opposite occurs: the PPP6 portfolio stands out as a dominant solution comparatively to the PPP17 portfolio. Furthermore, the PPP17 solution is among the optimal solutions with best in-sample performances, while it is among the optimal solutions with worst out-of-sample performances. These results are confirmed regardless of the value of the risk aversion parameter.

Figure 7-2: In-sample and out-of-sample mean return and mean risk of the RRD, ARD, PPP, MV, EW and GMV portfolios computed using a risk aversion parameter of 0.5.

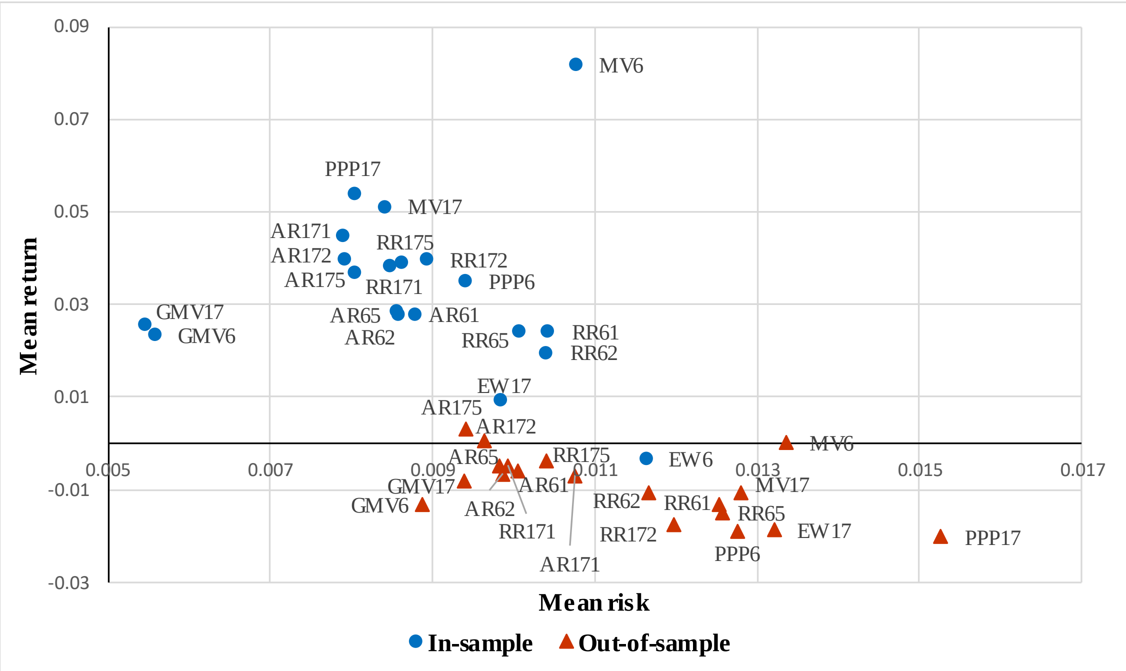


The optimal portfolios are represented according to the in-sample period length (Y) and the number of scenarios in the uncertainty set (S), used in their computation. For simplification purposes, the parameter S was represented by 1, 2 or 5 when $S = 100$, $S = 200$ or $S = 500$, respectively. Results are presented for the risk aversion parameter of 0.5. For instance, RR171 corresponds to the RRD portfolio computed using an in-sample period of 17 years and an uncertainty set with 100 scenarios, and PPP6 corresponds to the parametric portfolio policy proposed by Brandt et al. (2009) computed using an in-sample period length of 6 years.

The analysis of the robust solutions shows that reducing the length of the in-sample period deteriorates the overall in-sample performance of both ARD and RRD portfolios, as aforementioned. The best performance is achieved by the ARD approach for the in-sample period length of 17 years and 500 scenarios. For this parametrization, the ARD portfolio stands out as a dominant solution for all the computed (robust and non-robust) portfolios with the exception of the GMV portfolio. For the RRD approach, the best performance is achieved for the in-sample period length of 17 years and the uncertainty set with 100 scenarios, while the dominance is, once again, confirmed comparatively to the PPP and the MV17 portfolios. Once again, these results are verified regardless of the value of the risk aversion parameter.

It should also be pointed out that all the implemented portfolio optimization strategies present negative out-of-sample returns, when mean results are analyzed. This outcome can be explained by the evolution of the EURO STOXX 50 index during the out-of-sample period considered in this study (January 2007 to December 2016), depicted in Figure 4-4 and explained in section 4.3.

Figure 7-3: In-sample and out-of-sample mean return and mean risk of the RRD, ARD, PPP, MV, EW and GMV portfolios computed using a risk aversion parameter of 5.



The optimal portfolios are represented according to the in-sample period length (Y) and the number of scenarios in the uncertainty set (S), used in their computation. For simplification purposes, the parameter S was represented by 1, 2 or 5 when $S = 100$, $S = 200$ or $S = 500$, respectively. Results are presented for the risk aversion parameter of 5. For instance, RR171 corresponds to the RRD portfolio computed using an in-sample period of 17 years and an uncertainty set with 100 scenarios, and PPP6 corresponds to the parametric portfolio policy proposed by Brandt et al. (2009) computed using an in-sample period length of 6 years.

In what follows, we analyse more closely the proposed robust solutions and the parametric portfolio policy presented by Brandt et al. (2009). Coefficients of the parametric portfolio policies, composition, cardinality, turnover, abnormal return and beta of the ARD, RRD and PPP portfolios, are presented in Table 7-1, according to the risk aversion parameter used in their computation.

The analysis of the coefficients of the parametric portfolio policies presented in Table 7-1 allows us to conclude that, regardless of the value of the risk aversion parameter and for all the investment strategies (ARD, RRD and PPP), the weight of an asset in the optimal solution decreases with the firm's size (present negative market capitalization's coefficient). Additionally, for lower levels of risk aversion, the representation of larger firms' stocks becomes sparser in the optimal solution, since the market capitalization (negative) coefficient decreases. Concerning the BTM ratio, the RRD and the PPP portfolios present a similar behavior: the weight of an asset in the optimal solution increases with the firm's BTM ratio for low levels of risk aversion, while it decreases for higher levels of risk aversion. Thus, RRD and PPP optimal solutions assume larger positions in value firms' stocks for low values of the risk aversion parameter, and larger positions in growth firms' stocks for high values of the risk aversion parameter. The ARD portfolios present larger positions in growth firms' stocks regardless of the value of the risk aversion parameter (they show a consistent BTM coefficient near -3). Furthermore, the weight of an asset in the optimal solution increases with the firms' one-year lagged return for the RRD approach (regardless of the value of the risk aversion parameter), and for the ARD and PPP approaches when considering higher levels of risk aversion. For lower levels of risk aversion, the ARD and PPP portfolios assume larger positions in past losers (present negative momentum coefficient).

Table 7-1 shows that, for all the ARD, RRD and PPP portfolios, the maximum weight of an asset in the portfolio decreases while the cardinality slightly increases for higher levels of risk aversion. Thus, the exposure to individual assets decreases when the risk aversion parameter increases. Although the cardinality is very similar among investment strategies and regardless of the value of the risk aversion parameter, the cardinalities of the ARD portfolios are slightly higher than those of the RRD and the PPP portfolios. Despite presenting the highest cardinality, ARD portfolios reveal the lowest turnover, regardless of the value of the risk aversion parameter, while the highest turnover is exhibited by the RRD portfolios. Furthermore, for all the ARD, RRD and PPP portfolios, the turnover decreases when higher levels of risk aversion are considered. These results suggest that, for higher levels of risk aversion, the daily absolute weight changes decrease over time and, thus, the investment strategies tend to select the same assets over the different periods.

Regarding the results for the abnormal return and the portfolio beta, it can be observed that the ARD portfolios have the best performance, even outperforming the market proxy (they present positive abnormal returns), for all the values of the risk aversion parameter. The RRD portfolios present positive abnormal return except for the risk aversion value of 0.5. Regardless of the value of the risk aversion parameter, both ARD and RRD portfolios outperform the PPP portfolios (that is, they present higher abnormal return) and, thus, offer a better performance considering their systematic risk. Finally, it is important to highlight that the ARD and RRD portfolios show a lower systematic risk than the market (proxy) with the exception of the RRD portfolio computed assuming a risk aversion parameter of 0.5 (which presents a beta coefficient higher than 1). With higher systematic risk (even higher than the market) and negative abnormal return, the PPP portfolios stand out as the worst performing strategy.

To analyze the effect of the variation of the in-sample period length, we considered a risk aversion parameter of 0.5 and an uncertainty set with 500 scenarios. The results are shown in Table 7-2. It is possible to conclude that decreasing the in-sample period length leads to a higher representation (higher absolute coefficients' values) of value firms, small firms and past winners in the PPP portfolio, to a higher representation of growth firms, small firms and past winners in the ARD portfolio, and to a lower representation of value firms and higher representation of small firms and past losers in the RRD portfolio. Moreover, decreasing the in-sample period length seems to slightly reduce the exposure to individual assets and the turnover of the PPP portfolio, while it seems to have no substantial effect over the cardinality. For the ARD and RRD strategies, decreasing the in-sample period length seems to slightly increase the exposure to individual assets, the cardinality and the turnover of the optimal solutions.

Table 7-1: Variation of the risk aversion parameter

	PPP			RRD			ARD		
	$\gamma = 0.5$	$\gamma = 2$	$\gamma = 5$	$\gamma = 0.5$	$\gamma = 2$	$\gamma = 5$	$\gamma = 0.5$	$\gamma = 2$	$\gamma = 5$
α BTM	4.473	0.551	-0.277	16.607	-9.093	-4.484	-2.985	-3.055	-3.003
α ME	-62.628	-17.901	-8.368	-214.486	-21.786	-9.502	-3.361	-3.160	-2.713
α MOM	-0.871	0.032	0.192	13.849	7.336	3.230	-0.199	-0.079	0.059
Max $w_{t,n}$	0.147	0.142	0.135	0.138	0.128	0.125	0.111	0.107	0.107
Min $w_{t,n}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Card.	18.310	18.693	19.344	18.145	17.617	18.435	20.255	20.537	21.064
Turnover	6.01E-03	5.63E-03	5.18E-03	6.75E-03	6.21E-03	5.74E-03	4.20E-03	4.07E-03	3.96E-03
J_α	-4.88E-05	-4.35E-05	-3.53E-05	-1.06E-05	1.32E-05	1.90E-05	2.37E-05	3.58E-05	4.13E-05
Beta	1.133	1.118	1.093	1.076	0.926	0.910	0.883	0.878	0.871

This table shows the out-of-sample results concerning the coefficients of the parametric portfolio policy for the three characteristics (BTM, ME and MOM), the maximum weight of an asset (Max $w_{t,n}$), the minimum weight of an asset (Min $w_{t,n}$), the cardinality (card.), the turnover, the abnormal return (J_α) and the beta of the ARD, RRD and PPP portfolios, according to the value of the risk aversion parameter (γ) used in their computation. Results are presented for the in-sample period length of 17 years and the uncertainty set with 500 scenarios. After computing the portfolio policy's coefficients, out-of-sample daily portfolios are formed using those coefficients in the next year. Values for the portfolio policy's coefficients, the maximum and minimum weights, and the cardinality, are averages, for the 10 out-of-sample periods under study. For measuring the cardinality, only those assets with weights higher than 0.1% are considered. The portfolio turnover corresponds to the average, over all out-of-sample periods (10 years), of the daily portfolio turnover as defined in (4.6). The estimation of J_α was based on the CAPM, where the EURO STOXX 50 index was used as the proxy for the market portfolio. A total of 2540 daily observations (corresponding to the daily returns occurred in the 10 out-of-sample years), were used in order to compute daily abnormal returns.

Table 7-2: Variation of the in-sample period length

	PPP		RRD		ARD	
	$Y = 6$	$Y = 17$	$Y = 6$	$Y = 17$	$Y = 6$	$Y = 17$
α BTM	14.3342	4.4730	8.7883	16.6072	-6.5373	-2.9850
α ME	-89.7758	-62.6282	-216.8923	-214.4863	-123.6033	-3.3610
α MOM	7.8518	-0.8706	-45.1860	13.8489	3.8224	-0.1992
Max $w_{t,n}$	0.1323	0.1467	0.1508	0.1381	0.1223	0.1112
Min $w_{t,n}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Card.	18.3893	18.3100	18.4369	18.1449	21.3057	20.2550
Turnover	5.880E-03	6.008E-03	6.805E-03	6.746E-03	5.066E-03	4.197E-03
J_α	-4.948E-05	-4.882E-05	1.550E-05	-1.064E-05	2.191E-05	2.369E-05
Beta	1.1330	1.1330	1.0100	1.0760	0.9523	0.8831

This table shows the out-of-sample results concerning the coefficients of the parametric portfolio policy for the three characteristics (BTM, ME and MOM), the maximum weight of an asset (Max $w_{t,n}$), the minimum weight of an asset (Min $w_{t,n}$), the cardinality (card.), the turnover, the abnormal return (J_α) and the beta of the ARD, RRD and PPP portfolios, according to the in-sample period length (Y) used in their computation. Results are presented for risk aversion parameter of 0.5 and the uncertainty set with 500 scenarios. After computing the portfolio policy's coefficients, out-of-sample daily portfolios are formed using those coefficients in the next year. Results for the portfolio policy's coefficients, the maximum and minimum weights, and the cardinality, are averages, for the 10 out-of-sample periods under study. For measuring the cardinality, only those assets with weights higher than 0.1% are considered. The portfolio turnover corresponds to the average, over all out-of-sample periods (10 years), of the daily portfolio turnover as defined in (4.6). The estimation of J_α was based on the CAPM, where the EURO STOXX 50 index was used as the proxy for the market portfolio. A total of 2540 daily observations (corresponding to the daily returns occurred in the 10 out-of-sample years), were used in order to compute daily abnormal returns.

Finally, the RRD portfolio is the only optimal solution that seems to benefit from the reduction of the in-sample period length, presenting higher (positive) abnormal return and lower beta when using an in-sample period of 6 years in its computation. For the ARD and PPP solutions, the reduction of the in-sample period generally deteriorates abnormal returns and increases the portfolio's beta.

7.3.3.2 Performance of robust and non-robust portfolio

We will now analyse the in-sample and the out-of-sample performances of robust and non-robust portfolios, for each of the 10 windows. The performance analysis starts with the examination of in-sample and out-of-sample risks and returns of all ARD, RRD, PPP, MV, GMV and EW portfolios by time window. Results are described for the in-sample period length corresponding to 17 years of historical data and a value of 0.5 for the risk aversion parameter and are presented in Figure 7-4.

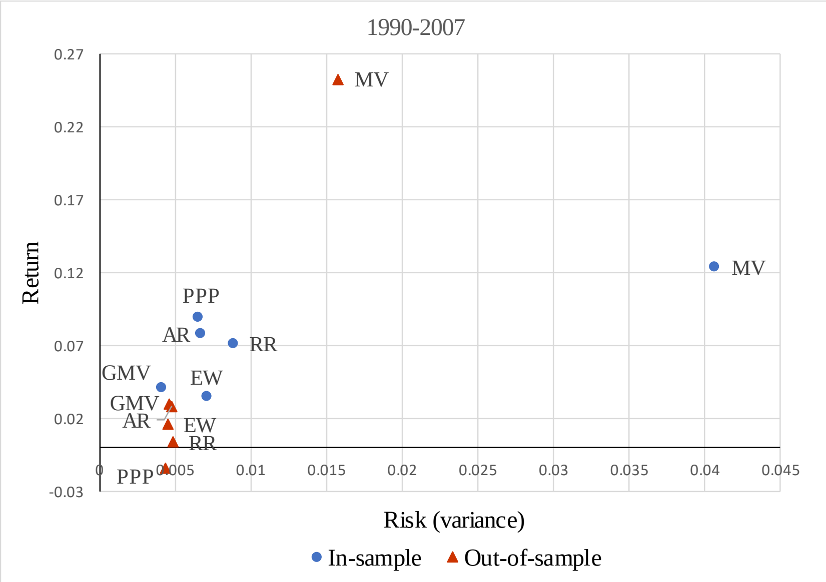
By analyzing the results for the non-robust portfolios, it can be confirmed that the classical mean-variance strategy reveals conflicting results when comparing in-sample and out-of-sample performances, since it stands out, simultaneously, as the solution with best in-sample performance and one of the portfolios with worst out-of-sample performances, in many of the time windows. These results support previous findings concerning the sensitivity of the MV portfolio to the estimation error and the effects of the input uncertainty in the optimization process (Best & Grauer, 1991a; Chopra & Ziemba, 1993; DeMiguel et al., 2009; Jagannathan & Ma, 2003). Regarding the EW portfolio, it is among the portfolios with lowest in-sample return, being the only portfolio with negative return in some windows (1996-2013; 1997-2104; 1998-2015; and 1999-2016). Out-of-sample, the EW solution is never the worst performing portfolio in terms of return, for all windows, and in terms of risk, for the majority of the windows.

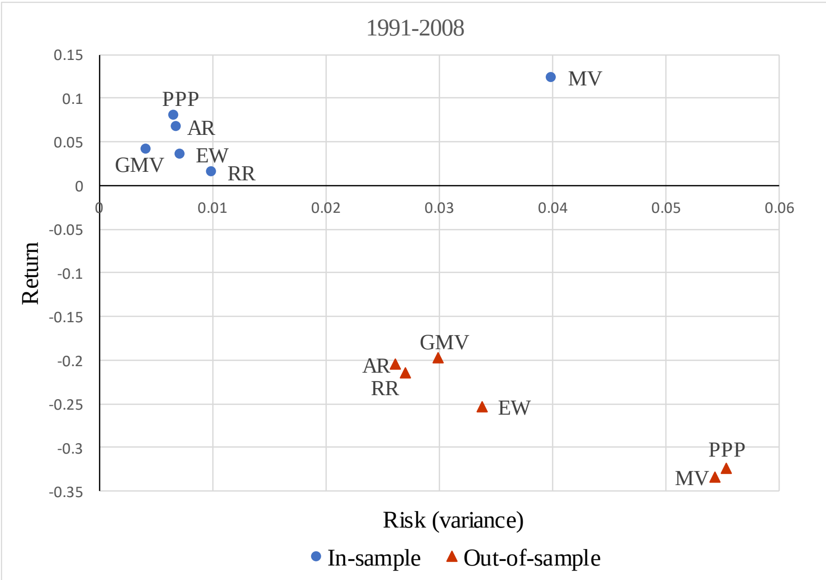
Concerning the GMV portfolio, although this portfolio shows worse in-sample return comparatively to the MV portfolio, a different trend can be observed when comparing out-of-sample performances. In fact, the GMV portfolio is a dominant solution comparatively to the MV and the EW portfolios in the majority of the time windows. Its better performance, relatively to the MV portfolio, can be explained by the fact that its computation requires only the estimates of variances and covariances of the asset returns, becoming less vulnerable to estimation errors. Furthermore, this portfolio is located at the left side of the scatter-plot in all windows, indicating low levels of risk, both in-sample and out-of-sample. Similarly to the MV portfolio, the PPP solution presents conflicting in-sample and out-of-sample results. In-sample, this portfolio is among the optimal solutions with higher return and lower risk, while out-of-sample, the PPP portfolio is among the optimal solutions with worst overall performance, in some of the windows under analysis (1990-2007; 1991-2008; 1998-2015; 1999-2016).

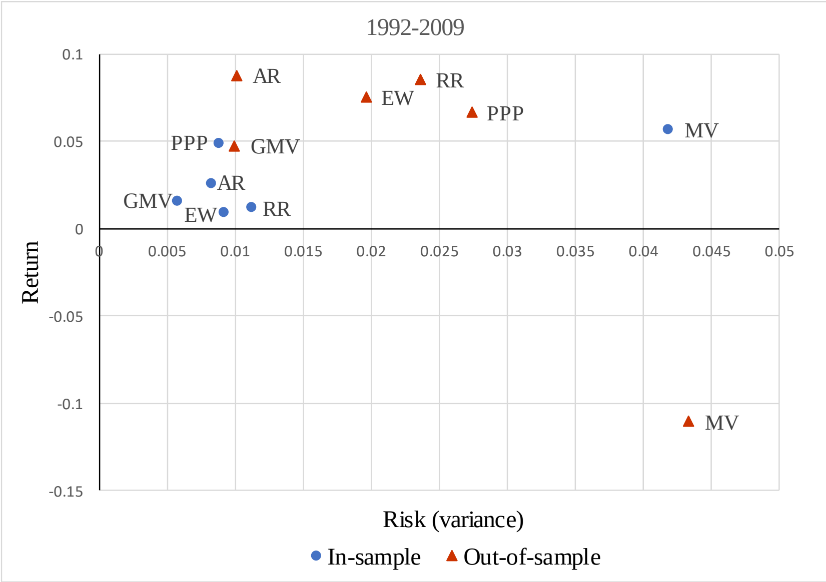
Regarding the ARD solution, this portfolio is generally located at the left side of the scatter-plot, among the optimal solutions with the lowest (in-sample and out-of-sample) risk and it even shows lower risk in out-of-sample data than in in-sample data, in some of the windows under analysis. Furthermore, the ARD portfolio stands out as a dominant solution comparatively to the PPP portfolio in 8 of the 10 windows, the MV portfolio in 7 of the 10 windows and the EW portfolio in 8 of the 10 windows. When compared to the GMV portfolio, the ARD presents higher return or lower risk, in the majority of the windows. Results presented in Table 7-4, regarding modified Sharpe ratio of all the investment strategies implemented in this study, also show that the ARD portfolio presents higher modified Sharpe ratio comparatively to the GMV portfolio in 6 of the 10 windows, comparatively to the MV portfolio in 7 of the 10 windows, and comparatively to the EW portfolio in 9 of the 10 windows. These results clearly reinforce the relevance of the proposed absolute robust methodology, since previous studies have confirmed the good performances of both the EW benchmark (DeMiguel et al., 2009) and the GMV portfolio (Chan et al., 1999; Jagannathan & Ma, 2003).

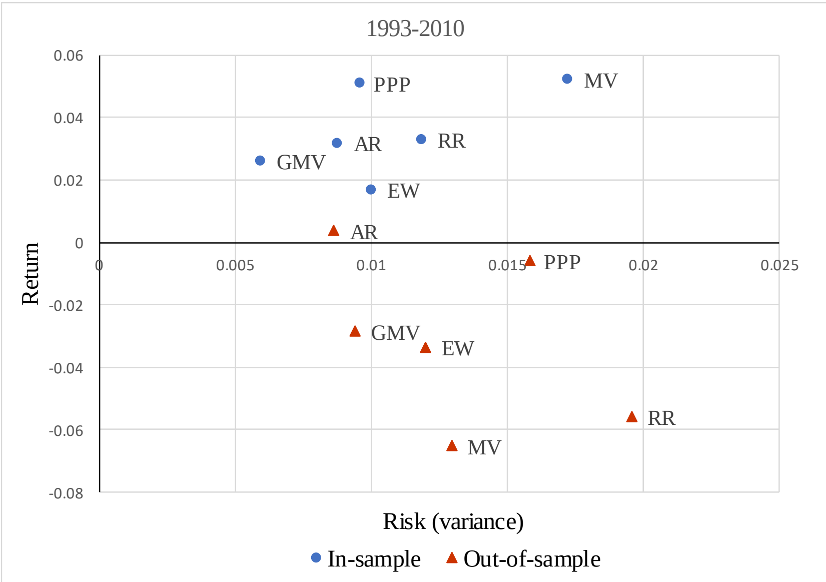
The RRD portfolio does not reveal results as good and consistent as the ARD portfolio, since it is among the portfolios with best performance in some of the windows under analysis (1991-2008; 1992-2009; 1995-2012; 1996-2013) and it is among the worst performing solutions in others (1993-2010; 1994-2011; 1997-2014). As for the modified Sharpe ratio, the RRD portfolio presents higher modified Sharpe ratio comparatively to the GMV portfolio in 3 of the 10 windows, and comparatively to the MV and EW portfolios in 5 of the 10 windows (Table 7-4).

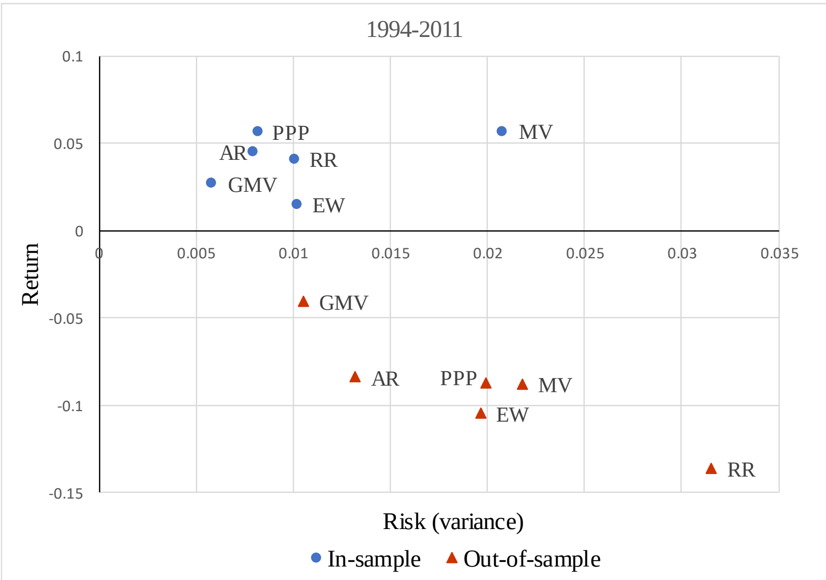
Figure 7-4: In-sample and out-of-sample risks and returns of the RRD, ARD, PPP, MV, EW and GMV portfolios computed for each time window.

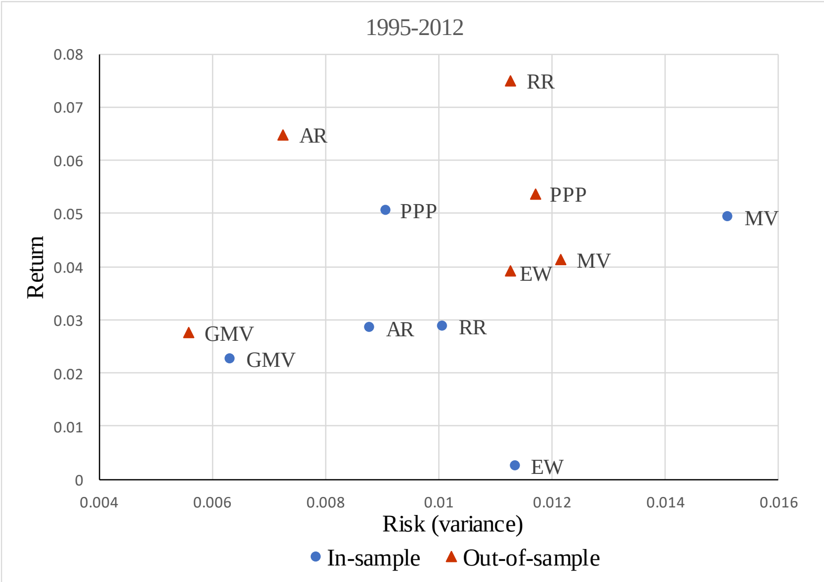


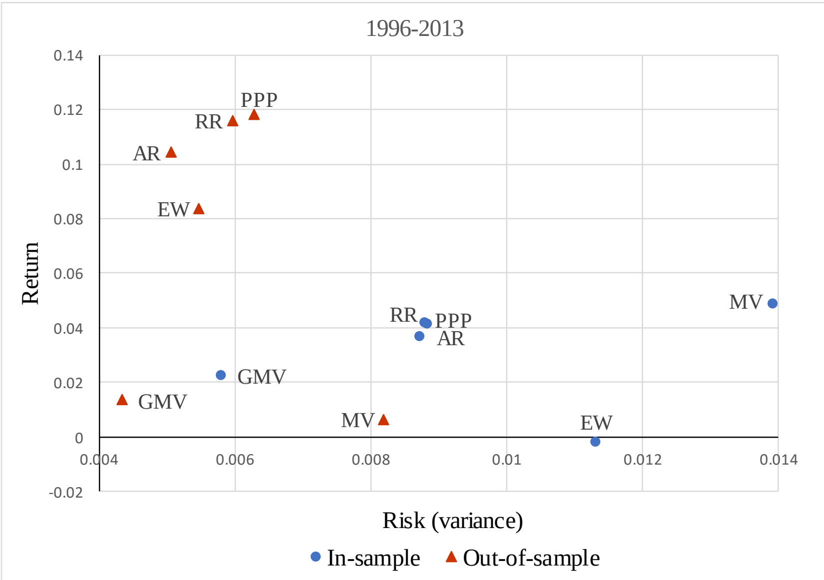


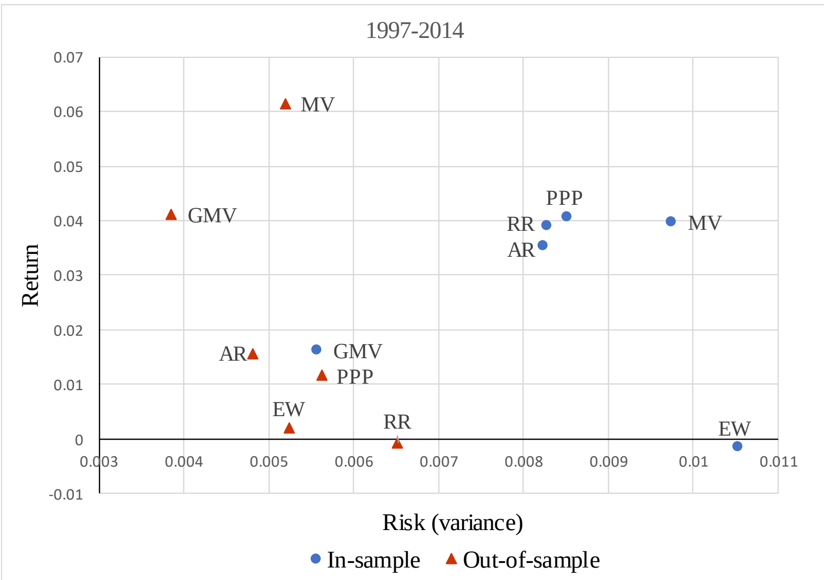

1992-2009
Return
Risk (variance)
In-sample
Out-of-sample
AR
RR
EW
PPP
MV
PPP
GMV
AR
GMV
EW
RR
MV

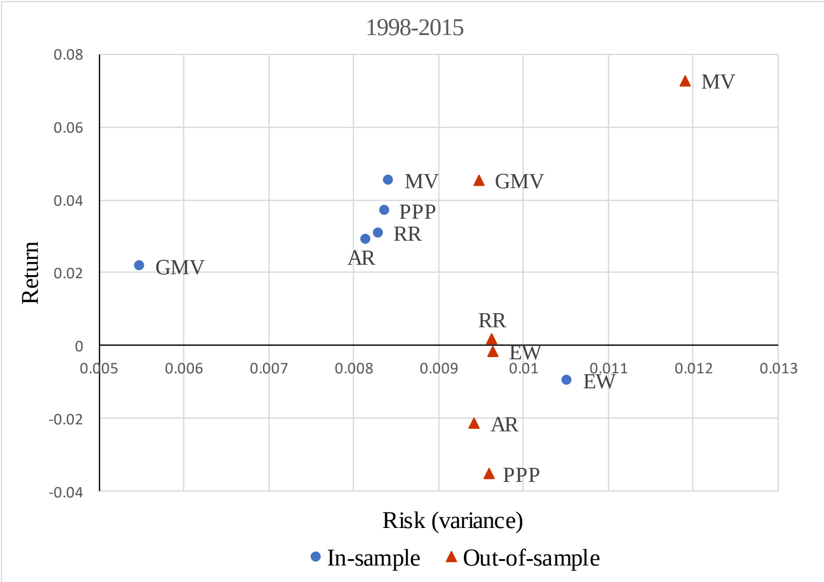

1993-2010
Return
Risk (variance)
In-sample
Out-of-sample
PPP
MV
AR
RR
GMV
EW
AR
PPP
GMV
EW
RR
MV

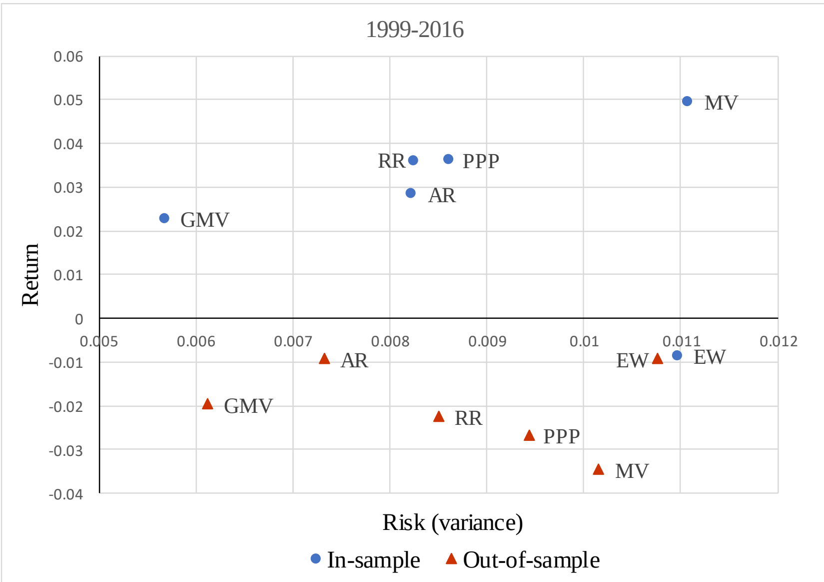

1994-2011
Return
Risk (variance)
In-sample
Out-of-sample
PPP
MV
AR
RR
GMV
EW
GMV
AR
PPP
MV
EW
RR


1995-2012
Return
Risk (variance)
In-sample
Out-of-sample
RR
AR
PPP
MV
EW
GMV
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.004
0.006
0.008
0.01
0.012
0.014
0.016


1996-2013
Return
Risk (variance)
In-sample
Out-of-sample
PPP
RR
AR
EW
MV
GMV
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0.004
0.006
0.008
0.01
0.012
0.014


1997-2014
Return
Risk (variance)
In-sample
Out-of-sample
MV
GMV
PPP
RR
AR
EW
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.01
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01
0.011





The optimal portfolios were computed considering the in-sample period length of 17 years and the parameters combination $\gamma = 0.5$ *and* $S = 500$.

From the analysis of Figure 7-4 it is also possible to observe how much each portfolio deviates from what would be expected looking at the corresponding in-sample performances. No portfolio has an out-of-sample performance that is systematically close to the in-sample performance in all of the windows under analysis. Actually, it can be observed that, in many of these windows, the MV approach stands out as generating the solutions that exhibit the highest deviations from the in-sample performances and, from that point of view, are the less consistent solutions.

The outperformance of the ARD portfolio comparatively to the RRD and PPP portfolios is confirmed when out-of-sample regret (Table 7-3) and modified Sharpe ratio (Table 7-4) are compared. The ARD portfolio presents higher modified Sharpe ratio comparatively to the PPP portfolio in 9 of the 10 windows (except in 1996-2013), and comparatively to the RRD portfolio in 8 of the 10 windows. Analyzing the robustness in terms of the utility loss for the investor (regret), it can be observed that the ARD portfolio is systematically more robust than the PPP and the RRD portfolios, revealing lower regrets in most of the windows. In fact, the ARD portfolio presents lower regret comparatively to the PPP portfolio in 9 of the 10 windows, and comparatively to the RRD portfolio in 6 of the 10 windows.

Table 7-3: Out-of-sample regret of the RRD, ARD and PPP portfolios

	PPP	**RRD**	**ARD**
2007	2.496E-02	2.489E-02	2.480E-02
2008	7.679E-03	7.120E-03	7.070E-03
2009	3.565E-04	2.908E-04	2.949E-04
2010	9.170E-03	9.367E-03	9.140E-03
2011	8.052E-03	8.253E-03	8.043E-03
2012	4.007E-04	3.224E-04	3.639E-04
2013	2.055E-02	2.056E-02	2.060E-02
2014	2.097E-03	2.145E-03	2.082E-03
2015	1.178E-02	1.163E-02	1.172E-02
2016	1.148E-02	1.146E-02	1.141E-02

This table shows the out-of-sample regret of the optimal portfolios by out-of-sample year. Results are presented for the in-sample period length of 17 years and the parameters combination $\gamma = 0.5$ *and* $S = 500$.

The analysis of the characteristics' coefficients of the ARD, RRD and PPP portfolios by out-of-sample year revealed no pattern between the value of the coefficient and the performance of the optimal solutions.

Table 7-4: Out-of-sample modified Sharpe ratio (S_I) of the RRD, ARD, PPP, MV, GMV and EW portfolios

	PPP	RRD	ARD	MV	GMV	EW
2007	-3.412E-03	-2.340E-03	-6.219E-04	1.715E+00	-5.030E-04	-1.436E-03
2008	-8.538E-02	-4.186E-02	-3.942E-02	-8.704E-02	-4.095E-02	-5.380E-02
2009	2.923E-01	4.385E-01	6.950E-01	-2.669E-02	2.968E-01	4.090E-01
2010	-1.801E-03	-8.959E-03	-4.326E-04	-8.371E-03	-3.577E-03	-4.588E-03
2011	-1.318E-02	-2.518E-02	-1.031E-02	-1.381E-02	-4.757E-03	-1.549E-02
2012	4.840E-01	6.933E-01	7.447E-01	3.629E-01	3.515E-01	3.566E-01
2013	1.496E+00	1.502E+00	1.471E+00	6.988E-02	2.085E-01	1.136E+00
2014	1.414E-01	-1.559E-04	2.085E-01	8.388E-01	6.459E-01	1.299E-02
2015	-3.351E-03	2.664E-02	-1.974E-03	6.749E-01	4.755E-01	-6.220E-05
2016	-2.189E-03	-1.685E-03	-4.429E-04	-3.072E-03	-1.220E-03	-5.366E-04

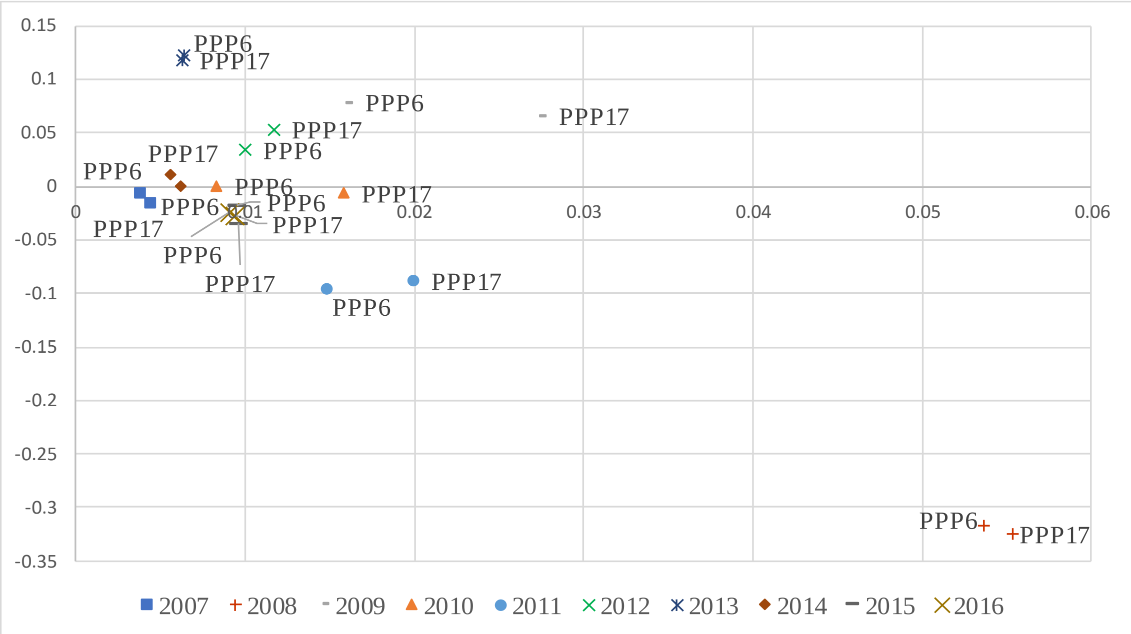
This table shows the out-of-sample modified Sharpe ratio (S_I) of the optimal portfolios by out-of-sample year. Results are presented for the in-sample period length of 17 years and the parameters combination $\gamma = 0.5$ and $S = 500$.

7.3.3.3 Effect of the in-sample period variation over the PPP portfolio

The out-of-sample risks and returns of the PPP portfolios proposed by Brandt et al. (2009) were also compared in each of the out-of-sample years, according to the in-sample period length used in their computation. Results are presented for a value of 0.5 for the risk aversion parameter in Figure 7-5, but general conclusions prevail for all the risk aversion parameter's values under study.

Although, the PPP17 (PPP portfolio computed using an in-sample period length of 17 years) and the PPP6 (PPP portfolio computed using an in-sample period length of 6 years) are very similar in some of the out-of-sample years, in terms of the location in the risk/return space, an important outcome can be observed. The PPP6 portfolio is a dominant solution comparatively to the PPP17 portfolio in 6 of the 10 out-of-sample years, while it presents higher out-of-sample return in 7 of the 10 out-of-sample periods and lower risk in 8 of the 10 out-of-sample periods under analysis. In fact, the PPP17 portfolio is a dominant solution comparatively to the PPP6 portfolio in only 1 of the 10 out-of-sample periods. This important observation suggests that the parametric portfolio policy is sensitive to the length of the in-sample period. As analysed in Table 7-2, when the in-sample period length is reduced, the parametric portfolio policy tends to increase the representation of value firms (by increasing the value of the BTM coefficient), small firms (by decreasing the value of the ME coefficient) and past winners (by increasing the value of the MOM coefficient).

Figure 7-5: Out-of-sample risks and returns of the PPP portfolio.



Portfolios' risk and return were computed by out-of-sample year and considering an in-sample period length of 17 years and a value of 0.5 for the risk aversion parameter.

7.4 Conclusions

In this chapter we propose a relative robust and absolute robust parametric portfolio policies. For the relative robust parametric portfolio approach, where the maximum regret is minimized, regret is defined as the utility loss for the investor resulting from choosing a given portfolio instead of choosing the optimal portfolio for the realized scenario. In the absolute robust approach, the minimum investor's expected utility in the worst-case scenario is maximized. Results are analyzed for different in-sample period lengths and values of the risk aversion parameter, and relevant conclusions are drawn regarding the real benefits of the proposed methodology comparatively to the non-robust parametric portfolio policies presented by Brandt et al. (2009) and other benchmarks established in the portfolio selection theory.

The results suggest that both absolute robust and relative robust parametric portfolio policies seem to marginally benefit from the use of long-term past data in order to estimate the input parameters. Concerning the uncertainty set cardinality, increasing the number of scenarios generally improves the overall performance of the ARD portfolio, while it seems to have no direct effect over the RRD portfolio. Regarding the PPP portfolio, its overall in-sample performance deteriorates when reducing the length of the in-sample period. Looking at out-of-sample results, an important outcome concerning the effect of the in-sample period length stands out. Results suggest that the parametric portfolio policy is sensitive to the length of the in-sample period, benefiting from its reduction.

The overall results suggest that the proposed absolute robust parametric portfolio generally outperforms the remaining investment strategies applied in this study, in at least one of the performance measures considered - abnormal return, return, risk, modified Sharpe ratio and/or regret. These results clearly reinforce the relevance of the proposed absolute robust parametric portfolio policy, since previous studies have confirmed the good performances of both the EW benchmark (DeMiguel et al., 2009) and the GMV portfolio (Chan et al., 1999; Jagannathan & Ma, 2003). Although outperforming the PPP portfolio in many of the windows under analysis, the RRD portfolios do not show results as good and consistent as the ARD portfolio, since they are among the portfolios with the best performance in some of the windows under analysis and they are among the worst performing solutions in others. This was an unexpected finding comparatively to the results presented by Caçador et al. (2019) who confirmed the dominance of the relative robust approach over the absolute robust approach. These dissimilar outcomes suggest that the performance of the different robust optimization approaches is model-dependent.

Overall, our findings confirm that the robust methodology leads to the performance enhancement of known models described in the literature, specifically of the parametric portfolio policies presented by Brandt et al. (2009), reinforcing the relevance of robust optimization within the field of portfolio selection under uncertainty.

Chapter 8

Final considerations

The work developed and the main results achieved have been described in the preceding sections. We believe that our study constitutes a valuable contribution for the assertion of robust optimization, in particular of relative robust models, within the field of portfolio selection under uncertainty. Nonetheless, we are aware of the numerous alternative directions that could have been taken in order to deeply explore the benefits of applying the RO methodology to the portfolio selection theory. It is now time to address the main limitations of our study and to suggest directions for future research.

We developed new robust portfolio optimization models and new robustness measures by extending and combining established methodologies in this field of research. By presenting different relative robust models under different definitions of regret, based on different objective functions, we examined the main contribution of the RO methodology when applied to portfolio optimization models subject to estimation errors of different magnitudes. In chapter 5, we developed a regret measure corresponding to the investor's utility loss (resulting from choosing a given portfolio instead of choosing the optimal portfolio of the realized scenario for the uncertain parameters) using a CRRA utility function and its approximation given by the second order Taylor expansion of the expected utility function around the expected return of the portfolio. Both models proposed in this chapter (models A and B) used this regret measure and considered uncertainty in the first and second order moments of asset returns and the portfolio's weights as decision variables. In chapter 7, a measure of regret based on the investor's utility loss and the same CRRA utility function was also presented but, in this case, we considered the average utility the investor would have obtained by implementing the methodology over the historical sample period instead of its approximation given by the second order Taylor expansion. Also, in this chapter (model D), the portfolio's weights were defined as a function of the firm's characteristics (hence, the model's decision variables corresponded to the coefficients of the firm's characteristics) and uncertainty was considered in the asset returns, the asset weights in the benchmark portfolio used and in the asset characteristics. An essentially different measure of regret was presented in chapter 6, where regret was defined as the increase in the investment risk resulting from choosing a given portfolio instead of choosing the optimal portfolio of the realized scenario for the uncertain parameters. In this model (model C), the decision variables corresponded to the portfolio's weights and uncertainty was considered in the sample covariance matrix only.

The real benefits of the robust portfolios from the investor perspective were assessed by examining whether the RO methodology adds value to the investment decision problem and mitigates the impact of the estimation errors on the computation of the optimal solution. All the proposed models were tested through the implementation of empirical applications that compared in-sample and out-of-sample performances of robust and non-robust portfolios, applying a large set of different performance measures. Furthermore, we located the computed portfolios in the risk-return space, highlighting the dominant strategies from the set of the implemented strategies in each chapter. For each of the proposed robust portfolios, we used its non-robust counterpart, as well as other non-robust and robust portfolios, as benchmarks in the performance assessment in order to analyze whether the robust methodology allowed the improvement of the performance of current models available in the literature. We confirmed that the proposed robust portfolios generally outperform the non-robust benchmarks implemented in our study, with the exception of the GMV portfolio. Furthermore, the proposed robust portfolios are generally more robust (concerning utility loss for the investor) and provide more consistent results (between in-sample and the out-of-sample performances) comparatively to the non-robust benchmarks. These results were also verified when the proposed robust portfolios were compared with other robust solutions already described in the literature. Overall, the empirical evidences found in our study support the potential of the RO methodology in the mitigation of the estimation errors on the computation of the optimal solution. We also examined the relevance of the proposed robust models for different levels of the investor's risk preference. This analysis confirmed the main strengths of the new methodologies proposed that can be seen as valid alternatives for those investors who can be more affected by the methodological weakness of the classical MV strategy. Finally, we investigated the effect of considering uncertainty sets with different number of scenarios as well as long-term historical data over short-term historical data in the definition of the uncertainty set by estimating samples and in-sample sets of different lengths. The empirical evidences showed that reducing the in-sample period length seems to have no substantial effect either in the exposure of the proposed robust portfolios to individual assets or in the consistency of their out-of-sample results. These outcomes highlight the utility of the proposed robust models in the presence of limited data. The use of the GA played an essential role in this latter analysis. The evolutionary algorithm was fundamental to overcome the difficulties raised by computational complexity of the robust portfolio optimization problems and to explore more parameter combinations without sacrificing the computational times and the research objectives.

The relative robust and the absolute robust approaches were compared by analyzing the performance of relative robust and absolute robust portfolios, emphasizing their main advantages and limitations. Generally, no substantial differences concerning the performances of the relative robust portfolio and the corresponding absolute robust portfolio were found. Both relative robust and absolute robust strategies generated robust solutions that were highly diversified and showed similar (low) risk, similar consistency between in-sample and out-of-sample results and similar robustness in terms of utility loss for the investor. Nonetheless, in some of the developed works, the general dominance of the relative robust approach over the absolute robust approach was confirmed, while in others the opposite was found. These dissimilar outcomes observed among the relative robust model and the corresponding absolute robust model, defined for different objective functions and subject to estimations errors of different magnitudes, suggest that the overperformance of one of the robust optimization approaches comparatively to the other is model-dependent.

It is important to notice that the results that were presented and previously outlined were obtained for a particular asset class. We used the stocks in the constituent list of two of the most important indices of Europe: the Dax and the EURO STOXX 50 indexes. Hence, it is not possible to generalize our results to other asset classes due to the particular characteristics of these assets. We consider this the major limitation of our study. But that are other limitations that deserve our attention. The datasets used in the empirical applications had a small number of assets: between 19 and 28 for the dataset constructed with the assets in the constituent list of the DAX index and between 28 and 48 for the datasets constructed with the assets in the constituent list of the EURO STOXX 50 index. While considered a limitation, the selection of stocks in the constituent list of market indices is a common practice in the portfolio selection literature. In particular, the selection of indices with a small number of stocks reduces the dimensionality of the optimization problem and, thus, the computational time necessary to compute the optimal solutions. We highlight one other limitation related to the confirmation of our results. We did not perform the analysis of the statistical significance of the differences found between the performance of the proposed robust portfolios and the performance of the benchmarks used in this study. Two main reasons explain this decision. The first reason lies in the characteristics of the OS periods. We used annual OS periods and computed annualized performance measures, resulting in an insufficient number of observations to that end. Increasing the number of time windows was also not possible, since we used a risk-free asset that has data only available from September 2004 onwards for computing the modified Sharpe ratios. The second reason is related to the main objectives of our study. We were not overly concerned with the statistical significance of the differences that were found; instead we were more concerned in analyzing the behavior of the proposed robust methodologies, in comparison with alternative robust and non-robust methodologies, and we feel that we were already doing it by using a large set of different performance measures.

Future research will focus on overcoming some of the limitations previously described. Namely, in analysing the performance of the proposed robust methodologies in different datasets: with a larger number of assets, with different asset classes. Furthermore, we intend to deeply analyze the behavior of the proposed robust methodologies with lengthier series of asset prices, allowing us to calculate the statistical differences between the performance measures. We are also interest in deeply analyzing the characteristics of the assets that are selected by the non-robust and the proposed robust models in order to understand their similar, yet different, performances.

While the RO methodology is gaining significant relevance within the theory of portfolio selection under uncertainty, we believe that it has not yet been used to its full potential in this field of research. We hope that this study will contribute to enhance its dissemination among quantitative portfolio managers and its use by general decision makers.

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