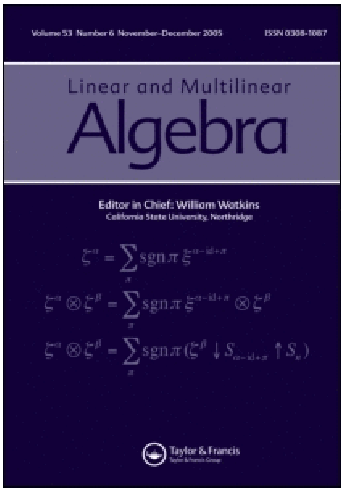

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A note on the multiplicities of the eigenvalues of a graph

C. M. DA Fonseca [ab]
[a] Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal [b] Communicated by D.Cvetkovi,

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A note on the multiplicities of the eigenvalues of a graph

C. M. DA FONSECA*

Departamento de Matemática, Universidade de Coimbra,
3001-454 Coimbra, Portugal

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Let $A(G)$ be a Hermitian matrix whose graph is a given graph G. From the interlacing theorem, it is known that $m_{A(G\setminus i)}(\theta) \geq m_{A(G)}(\theta) - 1$, where $m_{A(G)}(\theta)$ is the multiplicity of the eigenvalue θ of $A(G)$. In this note we improve this inequality for some paths with more than one vertex.

Keywords: Multiplicity; Eigenvalues; Graph; Tree; Hermitian matrices

Mathematics Subject Classifications: 15A18; 15A57; 05C50; 05C05

1. Introduction and preliminaries

Spectra of ordinary adjacency matrices of graphs are relatively well known, but for more general adjacency matrices this cannot be said. In the last few years, motivated essentially by the works of Genin and Maybee [1] as well as Parter [2], Johnson, Leal Duarte and others (cf [3–6,15]) developed the study of the multiplicities of eigenvalues of real acyclic matrices.

Given an undirected finite graph G, possibly with loops, we write $i \sim j$, if the vertices i and j are adjacent. If S is a subset of the vertex set of G, then $G\backslash S$ is the subgraph of G induced by the vertices not in S. In particular, if $i \in V(G)$, then $G \setminus i$ is the graph obtained by removing i and all of its incident edges. For more details on graph theory, the reader is referred to [7,8].

Let $A = (a_{ij})$ be a Hermitian matrix. The (weighted) graph of A, $G(A)$, is determined entirely by the off-diagonal entries of A: the vertex set is $\{1, \ldots, n\}$ and i and j are adjacent if and only if $a_{ij} \neq 0$. Given a graph G, a matrix whose graph is G is denoted by $A(G)$. In particular, if A is a 01-matrix, with main diagonal equal to zero, then A

*Email: cmf@mat.uc.pt

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is the adjacency matrix of $G(A)$. Further, $\mathcal{H}(G)$ denotes the set of all $n \times n$ Hermitian matrices which share a common graph G, i.e.,

$$\mathcal{H}(G) = \{A \mid A = A^*,\ G(A) = G\}.$$

We denote by $\varphi(G, \lambda)$, or simply $\varphi(G)$, the characteristic polynomial of $A(G)$, i.e., $\varphi(G, \lambda) = \det(\lambda I - A(G))$, sometimes referred to as the characteristic polynomial of G.

The general interlacing theorem between the eigenvalues of a Hermitian matrix and any principal submatrix is well known in the literature (see e.g. [9]).

Theorem 1.1 *Let G be a graph on n vertices and $A(G) \in \mathcal{H}(G)$. Then all eigenvalues of $A(G)$ are real, say $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Furthermore, if i is a vertex in G and $\mu_1 \le \mu_2 \le \cdots \le \mu_{n-1}$ are the eigenvalues of $A(G \setminus i)$, then*

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \mu_{n-1} \le \lambda_n,$$

i.e., the eigenvalues of $A(G)$ interlace those of $A(G \setminus i)$.

This theorem has a well-known corollary for tridiagonal matrices already proved elsewhere.

Corollary 1.2 *Let P be a path on n vertices and $A \in \mathcal{H}(P)$. Then A has n distinct real eigenvalues.*

In this note, we prove some relations between the multiplicities of an eigenvalue whenever a path is taken away from the graph. In particular, if the graph is a tree, a connected graph without cycles, then the multiplicity of an eigenvalue cannot go down by more than 1. This result will be a natural generalization of a consequence of Theorem 1.1 for trees. An example will be given.

2. Some properties of a characteristic polynomial of a graph

There are important similarities between orthogonal polynomials (for more details see e.g. [10]) and the characteristic polynomial of a tree. Heilmann and Lieb [11] have already connected orthogonal and matchings polynomials (cf, e.g. [12,13]).

For any vertices i and j of the graph of A, say G, define $w_{ij}(A) = a_{ij}$. Given a (weighted) path P in G with more than one vertex, let us define $w(P) = \prod_{(k,\ell)} w_{k,\ell}(P)$, where the product is taken over the weights of the edges (k, ℓ) of P. We denote by $\mathcal{P}_{ij}$, the set of all paths connecting the vertex i to the vertex j. The polynomial

$$\varphi_{ij}(G, \lambda) = \sum_{P \in \mathcal{P}_{ij}} w(P)\varphi(G \setminus P, \lambda)$$

can be regarded as the ij-entry of the adjoint of $\lambda I - A(G)$ (cf [13]). Notice that $\varphi_{ji}(G, \lambda) = \overline{\varphi_{ij}(G, \lambda)}$. Therefore

$$\varphi(G, \lambda)\varphi(G \backslash i, \mu) - \varphi(G, \mu)\varphi(G \backslash i, \lambda) = (\lambda - \mu) \sum_{j=1}^{n} \varphi_{ij}(G, \lambda)\varphi_{ij}(G, \mu)\,, \tag{2.1}$$

for any $i \in \{1, \ldots, n\}$.

When A is acyclic, i.e., the graph of A is a tree T, since $\mathcal{P}_{ij}$ has only one element, say P_{ij}, we get, for any vertex i,

$$\varphi(T,\lambda)\varphi(T\backslash i,\mu) - \varphi(T,\mu)\varphi(T\backslash i,\lambda) = (\lambda-\mu)\sum_{j=1}^{n}\left|w(P_{ij})\right|^2\varphi(T\backslash P_{ij},\lambda)\varphi(T\setminus P_{ij},\mu).$$

In analogy to the orthogonal polynomials, this equality is the so-called Christoffel–Darboux Identity (cf [10,14]).

From (2.1) several identities can be derived:

Lemma 2.1 *Let G be a (weighted) graph on n vertices. For every pair of vertices i, j of G,*

$$\varphi'(G,\lambda)\varphi(G\backslash i,\lambda) - \varphi(G,\lambda)\varphi'(G\backslash i,\lambda) = \sum_{k=1}^{n}\varphi_{ik}(G,\lambda)^2$$

$$\varphi'(G,\lambda)^2 - \varphi''(G,\lambda)\varphi(G,\lambda) = \sum_{k,\ell=1}^{n}\varphi_{k\ell}(G,\lambda)^2 \tag{2.2}$$

$$\varphi(G\backslash i,\lambda)\varphi(G\backslash j,\lambda) - \varphi(G\backslash ij,\lambda)\varphi(G,\lambda) = \varphi_{ij}(G,\lambda)^2. \tag{2.3}$$

3. Relations between the multiplicities

Throughout, $m_A(\theta)$ denotes the (algebraic) multiplicity of the eigenvalue θ of a Hermitian matrix A. From Theorem 1.1 we have

$$m_{A(T\backslash i)}(\theta) = m_{A(G)}(\theta)+1, \quad m_{A(G\backslash i)}(\theta) = m_{A(G)}(\theta), \quad \text{or} \quad m_{A(G\backslash i)}(\theta) = m_{A(G)}(\theta)-1,$$

for any matrix $A(G)$ in $\mathcal{H}(G)$, and for any vertex i in G. Notice that G has at least one vertex such that $m_{A(G\backslash i)}(\theta) = m_{A(G)}(\theta)-1$. Indeed, the multiplicity of θ as zero of $\varphi'(G,\lambda)$ is $m_{A(G)}(\theta)-1$. If $m_{A(T\backslash i)}(\theta) \geq m_{A(G)}(\theta)$ for all vertices i in G, then the multiplicity of θ as zero of $\varphi'(T,\lambda)$ is at least $m_{A(T)}(\theta)$, since $\varphi'(G,\lambda) = \sum_{k=1}^{n}\varphi(G\backslash k,\lambda)$.

We now state the main result of this note.

Theorem 3.1 *Let P be a path in the graph G and $A(G)$ in $\mathcal{H}(G)$. If θ is an eigenvalue of $A(G)$, then the multiplicity of θ as zero of each $\varphi_{ij}(G,\lambda)$ is at least $m_{A(G)}(\theta)-1$. In particular, if P is a path which does not intersect any cycle in G, then $m_{A(G\backslash P)}(\theta) \geq m_{A(G)}(\theta)-1$.* ■

Proof Suppose that θ is an eigenvalue of $A(G)$ with $m_{A(G)}(\theta) > 1$. Then θ is a zero of $\varphi'(G,\lambda)^2 - \varphi''(G,\lambda)\varphi(G,\lambda)$ with multiplicity at least $2m_A(\theta)-2$. From (2.2), θ is a zero of the nonnegative sum $\sum_{i,j=1}^{n}\varphi_{ij}(G,\lambda)^2$, and therefore θ as a zero of each $\varphi_{ij}(G,\lambda)$ has multiplicity at least $m_{A(G)}(\theta)-1$.

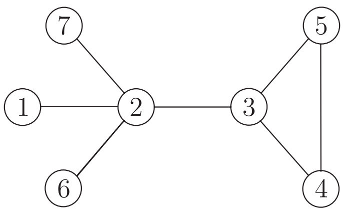
Corollary 3.2 *Let P be a path in the tree T and $A(T) \in \mathcal{H}(T)$. If θ is an eigenvalue of $A(T)$, then $m_{A(T\backslash P)}(\theta) \geq m_{A(T)}(\theta)-1$.*

We finally point out that if $m_{A(G\backslash P_{ij})}(\theta) = m_{A(G)}(\theta) - 1$ for some vertex j such that P_{ij} does not intersect any cycle of G, then $m_{A(G\backslash i)}(\theta) = m_{A(G)}(\theta) - 1$. In fact, suppose that $m_{A(G\backslash i)}(\theta) \geq m_{A(G)}(\theta)$. For any $j(\neq i)$ the multiplicity of θ as a zero of $\varphi(G\backslash i, \lambda)\varphi(G\backslash j, \lambda) - \varphi(G\backslash ij, \lambda)\varphi(G, \lambda)$ is at least $2m_{A(G)}(\theta) - 1$. Indeed, by (2.3), it is at least $2m_{A(G)}(\theta)$, hence $m_{A(G\backslash P_{ij})}(\theta) \geq m_{A(G)}(\theta)$.

We end this note with an example. Set

$$A = \begin{pmatrix} 2 & -i & 0 & 0 & 0 & 0 & 0 \\ i & -1 & 1/2 & 0 & 0 & 1 & 1-i \\ 0 & 1/2 & -3 & 2 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1+i & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The graph of A, say G, is



For the Hermitian matrix A, 1 is an eigenvalue of multiplicity two. Consider the path 627. Then 1 is also an eigenvalue of $A(G\backslash 627)$ with multiplicity one. Hence, we may conclude $m_{A(G\backslash 6)}(1) = m_{A(G)}(1) - 1 = 1$.

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