FINITELY PRESENTABLE ALGEBRAS FOR FINITARY MONADS

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Abstract. For finitary regular monads $\mathbb{T}$ on locally finitely presentable categories we characterize the finitely presentable objects in the category of $\mathbb{T}$-algebras in the style known from general algebra: they are precisely the algebras presentable by finitely many generators and finitely many relations.

1. Introduction

If $\mathbb{T} = (T, \eta, \mu)$ is a finitary monad on Set, then the category $\mathsf{Set}^{\mathbb{T}}$ of its algebras is nothing else than the classical concept of a variety of algebras. An algebra A is called *finitely presentable* (in General Algebra) if it can be presented by a finite set of generators and a finite set of relations. This means that there exists a finite set X (of generators) such that A can be obtained as a quotient of the free algebra (TX, μ_X) modulo a finitely generated congruence E. A congruence E is *finitely generated* if there is a finite subrelation $R \subseteq E$ such that E is the smallest congruence on TX containing R.[1] For monads on Set, the above concept coincides with A being a finitely presentable object of $\mathsf{Set}^{\mathbb{T}}$ (see [3, Corollary 3.13]). In the present paper, we generalize this to finitary *regular monads* [9], i.e. those preserving regular epimorphisms, on locally finitely presentable categories $\mathscr{A}$ that have regular factorizations. We introduce the concept of a finitely generated congruence (see Definition 3.6) and prove that the finitely presentable objects of $\mathscr{A}^{\mathbb{T}}$ are precisely the quotients of free algebras (TX, μ_X) with X finitely presentable modulo finitely generated congruences. We also characterize finitely generated algebras for finitary monads; here no condition on the monad is required.

The presented results can be also formulated for locally λ-presentable categories and algebras for λ-accessible monads that are λ-presentable or λ-generated.

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[1] Note that this does not imply that E, regarded as a subalgebra of $(TX, \mu_X)^2$, is a finitely generated algebra.

2. Preliminaries

We present results about accessible monads in locally presentable categories. In the main text we concentrate on the finitary case, but all the proofs are easy to generalize. Recall from [3] that a category $\mathscr{A}$ is called locally λ-presentable if it is cocomplete and has a set of λ-presentable objects whose closure under λ-filtered colimits is all of $\mathscr{A}$. In case $\lambda = \aleph_0$ we speak about lfp (locally finitely presentable) categories. A monad is called λ-accessible if its endofunctor preserves λ-filtered colimits. In the case of $\aleph_0$ we speak about finitary monads.

2.1. Notation. (1) Given an lfp category $\mathscr{A}$, we choose a small set $\mathscr{A}_{\mathsf{fp}}$ of finitely presentable objects such that every finitely presentable object is isomorphic to one in $\mathscr{A}_{\mathsf{fp}}$. We consider this as a full subcategory of $\mathscr{A}$.

(2) Analogously for a set $\mathscr{A}_{\mathsf{fg}}$ representing all finitely generated objects. Recall that these are objects X such that the hom-functor $\mathscr{A}(X,-)$ preserves directed colimits of monomorphisms.

(3) Given an object X, by a finitely generated subobject is meant one represented by a monomorphism $m : M \rightarrowtail X$ with M finitely generated.

2.2. Remark. The following facts about lfp categories $\mathscr{A}$ can be found in [3]:

(1) $\mathscr{A}$ has a factorization system (E, M) with E = strong epimorphisms and M = monomorphisms [Proposition 1.61].

(2) For every object A form the canonical filtered diagram D_A from the coslice category $\mathscr{A}_{\mathsf{fp}}/A \to \mathscr{A}$ assigning to each morphism $f : X \to A$ the domain X. Then A is the canonical colimit of D_A [Proposition 1.57].

(3) An object A is finitely generated iff there exists B in $\mathscr{A}_{\mathsf{fp}}$ with a strong epimorphism from B to A [Proposition 1.69].

2.3. Remark. Colimits of filtered diagrams $D\colon \mathscr{D} \to \mathsf{Set}$ are precisely those cocones $c_i\colon D_i \to C$ ($i \in \mathsf{obj}\,\mathscr{D}$) of D that have the following properties:

(1) (c_i) is jointly surjective, i.e. $C = \bigcup c_i[D_i]$, and

(2) given i and elements $x, y \in D_i$ merged by c_i, then they are also merged by a connecting morphism $D_i \to D_j$ of D.

This is easy to see: for every cocone $c_i'\colon D_i \to C'$ of D define $f\colon C \to C'$ by choosing for every $x \in C$ some $y \in D_i$ with $x = c_i(y)$ and putting $f(x) = c_i'(y)$. By the two properties above, this is well defined and is unique with $f \cdot c_i = c_i'$ for all i.

Recall that an adjunction whose right adjoint is finitary is called a *finitary adjunction.*

2.4. Lemma. *Let $L \dashv R\colon \mathscr{B} \to \mathscr{A}$ be a finitary adjunction between lfp categories. Then we have:*

(1) *L preserves both finitely presentable objects and finitely generated ones;*

(2) *if L is fully faithful, then an object X is finitely presentable in $\mathscr{A}$ iff LX is finitely presentable in $\mathscr{B}$;*

(3) *if, moreover, L preserves monomorphisms, then X is finitely generated in $\mathscr{A}$ iff LX is finitely generated in $\mathscr{B}$.*

Proof.

(1) Let X be a finitely presentable object of $\mathscr{A}$ and let $D\colon \mathscr{D} \to \mathscr{B}$ be a filtered diagram. Then we have the following chain of natural isomorphisms

$$\begin{aligned}\mathscr{B}(LX, \operatorname{colim} D) &\cong \mathscr{A}(X, R(\operatorname{colim} D)) \\ &\cong \mathscr{A}(X, \operatorname{colim} RD) \\ &\cong \operatorname{colim}(\mathscr{A}(X, RD(-)) \\ &\cong \operatorname{colim}(\mathscr{B}(LX, D(-)).\end{aligned}$$

This shows that LX is finitely presentable in $\mathscr{B}$. Now if X is finitely generated in $\mathscr{A}$ and D is a directed diagram of monomorphisms, then RD is also a directed diagram of monomorphisms (since the right adjoint R preserves monomorphisms). Thus, the same reasoning proves LX to be finitely generated in $\mathscr{B}$.

(2) Suppose that LX is finitely presentable in $\mathscr{B}$ and that $D\colon \mathscr{D} \to \mathscr{A}$ is a filtered diagram. Then we have the following chain of natural isomorphisms:

$$\begin{aligned}\mathscr{A}(X, \operatorname{colim} D) &\cong \mathscr{B}(LX, L(\operatorname{colim} D)) \\ &\cong \mathscr{B}(LX, \operatorname{colim} LD) \\ &\cong \operatorname{colim}(\mathscr{B}(LX, LD(-)) \\ &\cong \operatorname{colim}(\mathscr{A}(X, D(-))\end{aligned}$$

Indeed, the first and last step use that L is fully faithful, the second step that L is finitary and the third one that LX is finitely presentable in $\mathscr{B}$.

(3) If LX is finitely generated in $\mathscr{B}$ and $D\colon \mathscr{D} \to \mathscr{A}$ a directed diagram of monomorphisms, then so is LD since L preserves monomorphisms. Thus the same reasoning as in (2) shows that X is finitely generated in $\mathscr{A}$. ∎

2.5. Lemma. *Let $\mathscr{A}$ be an lfp category and I a set. An object in the power category $\mathscr{A}^I$ is finitely presentable iff its components*

(1) *are finitely presentable in $\mathscr{A}$, and*

(2) *all but finitely many are initial objects.*

Proof. Denote by 0 and 1 the initial and terminal objects, respectively. Note that for every $i \in I$ there are two fully faithful functors $L_i, R_i\colon \mathscr{A} \hookrightarrow \mathscr{A}^I$ defined by:

$$\big(L_i(X)\big)_j = \begin{cases} X & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad \big(R_i(X)\big)_j = \begin{cases} X & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

For every $i \in I$ there is also a canonical projection $\pi_i\colon \mathscr{A}^I \to \mathscr{A}$, $\pi_i\big((X_j)_{j\in I}\big) = X_i$. We have the following adjunctions:

$$L_i \dashv \pi_i \dashv R_i.$$

Sufficiency. Let $A = (A_i)_{i \in I}$ satisfy (1) and (2), then $L_i(A_i)$ is finitely presentable in $\mathscr{A}^I$ by Lemma 2.4(1). Thus, so is A, since it is the finite coproduct of those $L_i(A_i)$, with A_i not initial.

Necessity. Let $A = (A_i)_{i \in I}$ be finitely presentable in $\mathscr{A}^I$. Then for every $i \in I$, $\pi_i(A)$ is finitely presentable in $\mathscr{A}$ by Lemma 2.4(1), proving item (1). To verify (2), for every finite set $J \subseteq I$, let A_J have the components A_j for every $j \in J$ and 0 otherwise. These objects A_J form an obvious directed diagram with a colimit cocone $a_J \colon A_J \to A$. Since A is finitely presentable, there exists J_0 such that id_A factorizes through a_{J_0}, i.e. a_{J_0} is a split epimorphism. Since a split quotient of an initial object is initial, we conclude that (2) holds. ∎

3. Finitely Presentable Algebras

In the introduction we have recalled the definition of a finitely presentable algebra from General Algebra and the fact that for a finitary monad $\mathbb{T}$ on Set, this is equivalent to A being a finitely presentable object of $\mathsf{Set}^{\mathbb{T}}$. We now generalize this to finitary *regular monads* [9], i.e. those preserving regular epimorphisms, on lfp categories that have regular factorizations.

First, we turn to characterizing finitely generated algebras for *arbitrary* finitary monads.

3.1. Remark. Let $\mathbb{T}$ be a finitary monad on an lfp category $\mathscr{A}$. Then the Eilenberg-Moore category $\mathscr{A}^{\mathbb{T}}$ is also lfp [3, Remark 2.78]. Thus, it has (strong epi, mono)-factorizations. The monomorphisms in $\mathscr{A}^{\mathbb{T}}$, representing *subalgebras*, are precisely the $\mathbb{T}$-algebra morphisms carried by a monomorphism of $\mathscr{A}$ (since the forgetful functor $\mathscr{A}^{\mathbb{T}} \to \mathscr{A}$ creates limits). The strong epimorphisms of $\mathscr{A}^{\mathbb{T}}$, representing *strong quotient algebras*, need not coincide with those carried by strong epimorphisms of $\mathscr{A}$ – we do not assume that $\mathbb{T}$ preserves strong epimorphisms.

Recall our terminology that a finitely generated subobject of an object A is one represented by a monomorphism $m \colon M \rightarrowtail A$ with M a finitely generated object.

3.2. Notation. Throughout this section given a $\mathbb{T}$-algebra morphism $f \colon X \to Y$ we denote by $\mathsf{Im}\, f$ its image in $\mathscr{A}^{\mathbb{T}}$. That is, we have a strong epimorphism $e \colon X \twoheadrightarrow \mathsf{Im}\, f$ and a monomorphism $m \colon \mathsf{Im}\, f \rightarrowtail B$ in $\mathscr{A}^{\mathbb{T}}$ with $f = m \cdot e$.

3.3. Definition. *An algebra (A, a) for $\mathbb{T}$ is said to be* generated by a subobject $m \colon M \rightarrowtail A$ *of the base category $\mathscr{A}$ if no proper subalgebra of (A, a) contains m.*

The phrase "(A, a) is generated by a finitely generated subobject" may sound strange, but its meaning is clear: there exists a subobject $m \colon M \rightarrowtail A$ with M in $\mathscr{A}_{\mathsf{fg}}$ such that m does not factorize through any proper subalgebra of (A, a).

3.4. Example. The free algebras on finitely presentable objects are shortly called *ffp algebras* below: they are the algebras (TX, μ_X) with X finitely presentable.

(1) Every ffp algebra is generated by a finitely generated object: factorize the unit $\eta_X \colon X \to TX$ in $\mathscr{A}$ as a strong epimorphism $e \colon X \twoheadrightarrow M$ (thus, M is finitely generated by Remark 2.2(5)) followed by a monomorphism $m \colon M \rightarrowtail TX$. Using the universal property, it is easy to see that m generates (TX, μ_X); indeed, suppose we had a subalgebra $s \colon (A, a) \rightarrowtail (TX, \mu_X)$ containing m, via $n \colon M \rightarrowtail A$, say. Then the unique extension of $n \cdot e \colon X \to A$ to a $\mathbb{T}$-algebra morphism $h \colon (TX, \mu_X) \to (A, a)$ satisfies $s \cdot h = \mathsf{id}_{(TX,\mu_X)}$. Thus, s is an isomorphism.

(2) Every ffp algebra is finitely presentable in $\mathscr{A}^{\mathbb{T}}$: apply Lemma 2.4 to the forgetful functor $R \colon \mathscr{A}^{\mathbb{T}} \to \mathscr{A}$ and its left adjoint $LX = (TX, \mu_X)$.

3.5. Theorem. *For every finitary monad $\mathbb{T}$ on an lfp category $\mathscr{A}$ the following conditions on an algebra (A, a) are equivalent:*

(1) *(A, a) is generated by a finitely generated subobject.*

(2) *(A, a) is a strong quotient algebra of an ffp algebra.*

(3) *(A, a) is a finitely generated object of $\mathscr{A}^{\mathbb{T}}$.*

Proof. (3) $\Longrightarrow$ (2) For every $f \colon X \to A$ with $X \in \mathscr{A}_{\mathsf{fp}}$ consider its unique extension to a $\mathbb{T}$-algebra morphism $a \cdot Tf \colon (TX, \mu_X) \to (A, a)$ and form its factorization in $\mathscr{A}^{\mathbb{T}}$:

$$\begin{array}{ccc} TX & \xrightarrow{e_f} & \mathsf{Im}(a \cdot Tf) \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle m_f} \\ TA & \xrightarrow[a]{} & A \end{array}$$

Now observe that $a \colon (TA, \mu_A) \to (A, a)$ is a strong epimorphism in $\mathscr{A}^{\mathbb{T}}$; in fact, the laws of Eilenberg-Moore algebras for $\mathbb{T}$ imply that a is the coequalizer of

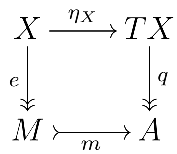
$$(TTA, \mu_{TA}) \underset{\mu_A}{\overset{Ta}{\rightrightarrows}} (TA, \mu_A).$$

From Remark 2.2(2) and the finitarity of the functor T we deduce that $Tf \colon (TX, \mu_X) \to (TA, \mu_A)$, $f \in \mathscr{A}_{\mathsf{fp}}/A$, is a filtered colimit in $\mathscr{A}^{\mathbb{T}}$. It follows from [2, Lemma 2.9] that (A, a) is a directed union of the images of $a \cdot TF$ for f in $\mathscr{A}_{\mathsf{fp}}/A$, that is, the directed union of the subobjects m_f for f in $\mathscr{A}_{\mathsf{fp}}/A$.

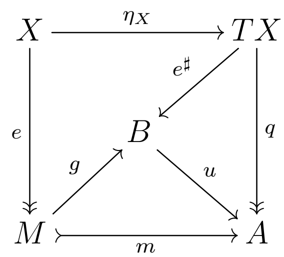
Now since (A, a) is finitely generated, id_A factorizes through one of the corresponding colimit injections $m_f \colon \mathsf{Im}(a \cdot Tf) \rightarrowtail A$ for some $f \colon X \to A$ in $\mathscr{A}_{\mathsf{fp}}/A$. Therefore m_f is split epic, whence an isomorphism, and A is a strong quotient of (TX, μ_X) via e_f, as desired.

(2) $\Longrightarrow$ (1) Let $q \colon (TX, \mu_X) \twoheadrightarrow (A, a)$ be a strong epimorphism in $\mathscr{A}^{\mathbb{T}}$ with X finitely presentable in $\mathscr{A}$. Factorize $q \cdot \eta_X$ as a strong epimorphism followed by a monomorphism

in $\mathscr{A}$:

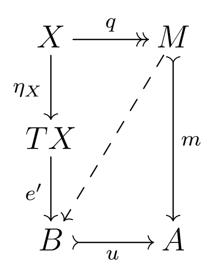


Then M is finitely generated in $\mathscr{A}$ by Remark 2.2(5). We shall prove that for every subalgebra $u\colon (B,b) \rightarrowtail (A,a)$ containing m (i.e. such that there is a morphism $g\colon M \to B$ in $\mathscr{A}$ with $u \cdot g = m$) u is invertible. Let $e^\sharp\colon (TX, \mu_X) \to (B,b)$ be the unique extension of $g \cdot e$ to a $\mathbb{T}$-algebra morphism:



Then we see that $u \cdot e^\sharp = q$ because this triangle of $\mathbb{T}$-algebra morphisms commutes when precomposed by the universal morphism η_X. Since q is strongly epic, so is u, and therefore u is an isomorphism, as desired.

(1) $\Longrightarrow$ (2) Let $m\colon M \rightarrowtail A$ be a finitely generated subobject of A that generates (A,a). By Remark 2.2(5), there exists a strong epimorphism $q\colon X \twoheadrightarrow M$ in $\mathscr{A}$ with X finitely presentable. The unique extension $e = (m \cdot q)^\sharp\colon (TX, \mu_X) \to (A,a)$ to a $\mathbb{T}$-algebra morphism is an *extremal epimorphism*, i.e. if e factorizes through a subalgebra $u\colon (B,b) \rightarrowtail (A,a)$, then u is an isomorphism. To prove this, recall that u is also monic in $\mathscr{A}$. Given $e = u \cdot e'$ we use the diagonal fill-in property in $\mathscr{A}$:



Since m generates (A,a), this proves that u is an isomorphism. In a complete category every extremal epimorphism is strong, thus we have proven (2).

By Remark 2.2(5) and the fact that ffp algebras are finitely presentable in $\mathscr{A}^\mathbb{T}$ (see Example 3.4(b)) we have (2) $\Longrightarrow$ (3). ■

As usual, by a *congruence* on a $\mathbb{T}$-algebra (A,a) is meant a subalgebra $(K,k) \rightarrowtail (A,a) \times (A,a)$ whose restricted projections form a kernel pair $\ell, r\colon (K,k) \rightrightarrows (A,a)$ of some $\mathbb{T}$-algebra morphism. Given a coequalizer $q\colon (A,a) \twoheadrightarrow (B,b)$ of ℓ, r in $\mathscr{A}^\mathbb{T}$, then (B,b) is called the *quotient algebra of* (A,a) *modulo* (K,k).

3.6. Definition. *By a* finitely generated congruence *is meant a congruence* $\ell, r\colon (K,k) \rightrightarrows (A,a)$ *such that there exists a finitely generated subalgebra* $m\colon (K',k') \rightarrowtail (K,k)$ *in* $\mathscr{A}^{\mathbb{T}}$ *for which the quotient of* (A,a) *modulo* (K,k) *is also a coequalizer of* $\ell \cdot m$ *and* $r \cdot m$*:*

$$(K',k') \overset{m}{\rightarrowtail} (K,k) \underset{r}{\overset{\ell}{\rightrightarrows}} (A,a) \overset{q}{\twoheadrightarrow} (B,b).$$

In the next theorem we assume that our base category has regular factorizations, i.e. (regular epi,mono)-factorizations.

3.7. Theorem. *Let* $\mathbb{T}$ *be a regular, finitary monad on an lfp category* $\mathscr{A}$ *with regular factorizations. For every* $\mathbb{T}$*-algebra* (A,a) *the following conditions are equivalent:*

(1) (A,a) *is a quotient of an ffp algebra modulo a finitely generated congruence.*

(2) (A,a) *is a coequalizer of a parallel pair of* $\mathbb{T}$*-algebra morphisms between ffp algebras:*

$$(TY,\mu_Y) \underset{g}{\overset{f}{\rightrightarrows}} (TX,\mu_X) \overset{e}{\twoheadrightarrow} (A,a) \qquad (X, Y \text{ in } \mathscr{A}_{\mathsf{fp}}).$$

(3) (A,a) *is a finitely presentable object of* $\mathscr{A}^{\mathbb{T}}$.

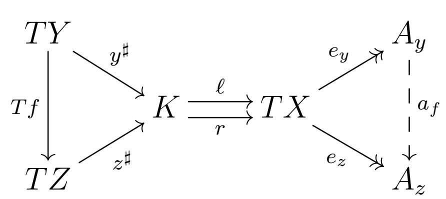
Proof. (2) $\Longrightarrow$ (3) Since finitely presentable objects are closed under finite colimits, this follows from Example 3.4(2).

(3) $\Longrightarrow$ (1) First note that the classes of regular and strong epimorphisms in $\mathscr{A}^{\mathbb{T}}$ coincide; indeed, since $\mathbb{T}$ preserves regular epimorphisms, the regular factorizations of $\mathscr{A}$ lift to $\mathscr{A}^{\mathbb{T}}$ (see [9, Proposition 4.17]). Then, by Theorem 3.5, (A,a) is a regular quotient of an ffp algebra via $q\colon (TX,\mu_X) \twoheadrightarrow (A,a)$, say.

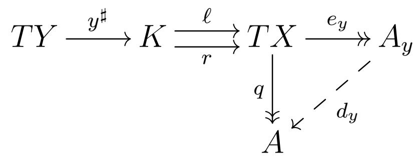
Now take the kernel pair $\ell, r\colon (K,k) \rightrightarrows (TX,\mu_X)$ of q in $\mathscr{A}^{\mathbb{T}}$ and note that q is its coequalizer. We are going to prove that the congruence (ℓ, r) is finitely generated. We first verify some of its properties. Write K in $\mathscr{A}$ as the filtered colimit of its canonical filtered diagram $D_K\colon \mathscr{A}_{\mathsf{fp}}/K \to \mathscr{A}$ (see Remark 2.2(2)) with the colimit injections $y\colon Y \to K$. Take the unique extension $y^\sharp\colon (TY,\mu_Y) \to (K,k)$ to a $\mathbb{T}$-algebra morphism and form the following coequalizer in $\mathscr{A}^{\mathbb{T}}$:

$$(TY,\mu_Y) \overset{y^\sharp}{\longrightarrow} (K,k) \underset{r}{\overset{\ell}{\rightrightarrows}} (TX,\mu_X) \overset{e_y}{\twoheadrightarrow} (A_y,a_y)\,.$$

(a) This defines a filtered diagram $\overline{D}\colon \mathscr{A}_{\mathsf{fp}}/K \to \mathscr{A}^{\mathbb{T}}$ taking y to (A_y,a_y). In fact, for every morphism $f\colon (Y,y) \to (Z,z)$ in $\mathscr{A}_{\mathsf{fp}}/K$ we obtain a $\mathbb{T}$-algebra morphism $a_f\colon (A_y,a_y) \to (A_z,a_z)$ using the following diagram in $\mathscr{A}^{\mathbb{T}}$ (where we drop the algebra structures):



Note that a_f is a regular epimorphism in $\mathscr{A}^{\mathbb{T}}$. Furthermore, for every $y\colon Y \to K$ in $\mathscr{A}_{\mathsf{fp}}/K$ we also obtain a morphism $d_y\colon A_y \to A$ such that $d_y \cdot e_y = q$:

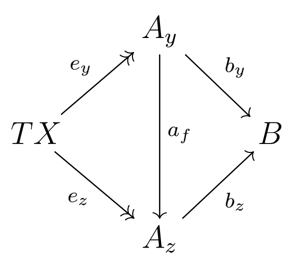


These morphisms d_y form a cocone on the diagram $\overline{D}$; indeed, we have for every morphism $f\colon (Y, y) \to (Z, z)$ of $\mathscr{A}_{\mathsf{fp}}/K$ that

$$d_z \cdot a_f \cdot e_y = d_z \cdot e_z = q = d_y \cdot e_y,$$

and we conclude that $d_z \cdot a_f = d_y$ since e_y is epic.

(b) We now show that $(A, a) = \operatorname{colim} \overline{D}$ with colimit injections $d_y : (A_y, a_y) \to (A, a)$. Given a cocone $b_y\colon (A_y, a_y) \to (B, b)$ of $\overline{D}$, we prove that it factorizes uniquely through (d_y). We first note that all the morphisms $b_y \cdot e_y$ are equal because the diagram D_K is filtered and for every morphism $f\colon (Y, y) \to (Z, z)$ in $\mathscr{A}_{\mathsf{fp}}/K$ we have the commutative diagram below:



Let us call the above morphism $q'\colon TX \to B$, and observe that for every $y\colon Y \to K$ in $\mathscr{A}_{\mathsf{fp}}/K$ we have

$$q' \cdot \ell \cdot y^\sharp = b_y \cdot e_y \cdot \ell \cdot y^\sharp = b_y \cdot e_y \cdot r \cdot y^\sharp = q' \cdot r \cdot y^\sharp.$$

The cocone of morphisms $y^\sharp\colon TY \to K$ is collectively epic since so is the colimit cocone $y\colon Y \to K$, and therefore $q' \cdot \ell = q' \cdot r$. Thus, there exists a unique factorization $h\colon A \to B$ of q' through $q = \mathsf{coeq}(\ell, r)$, i.e. $h \cdot q = q'$. We now have, for every $y\colon Y \to K$ in $\mathscr{A}_{\mathsf{fp}}/K$,

$$h \cdot d_y \cdot e_y = h \cdot q = q' = b_y \cdot e_y,$$

which implies $h \cdot d_y = b_y$ using that e_y is epic.

Uniqueness of h with the latter property follows immediately from q being epic: if $k\colon A \to B$ fulfils $k \cdot d_y = b_y$ for every y in $\mathscr{A}_{\mathsf{fp}}/K$, we have

$$k \cdot q = k \cdot d_y \cdot e_y = b_y \cdot e_y = q' = h \cdot q.$$

(c) Now use that (A, a) is finitely presentable in $\mathscr{A}^{\mathbb{T}}$ to see that there exists some $w\colon W \to K$ in $\mathscr{A}_{\mathsf{fp}}/K$ and a $\mathbb{T}$-algebra morphism $s\colon (A, a) \to (A_w, a_w)$ such that $d_w \cdot s =$

id_A. Then $s \cdot d_w$ is an endomorphism of the $\mathbb{T}$-algebra (A_w, a_w) satisfying $d_w \cdot (s \cdot d_w) = d_w$. Since e_w is a coequalizer of a parallel pair of $\mathbb{T}$-algebra morphisms between ffp algebras, (A_w, a_w) is finitely presentable by Example 3.4(2). The colimit injection d_w merges $s \cdot d_w$ and id_{A_w}, hence there exists a morphism $f\colon (W, w) \to (Y, y)$ of $\mathscr{A}_{\mathsf{fp}}/K$ with a_f merging them too, i.e. such that $a_f \cdot (s \cdot d_w) = a_f$. This implies that $d_y\colon (A_y, a_y) \to (A, a)$ is an isomorphism with inverse $a_f \cdot s$. Indeed, we have

$$d_y \cdot (a_f \cdot s) = d_w \cdot s = \mathsf{id}_A,$$

and for $(a_f \cdot s) \cdot d_y = \mathsf{id}_{A_y}$ we use that a_f is epic:

$$(a_f \cdot s) \cdot d_y \cdot a_f = a_f \cdot s \cdot d_w = a_f.$$

Thus (A, a) is a quotient of the ffp algebra (TX, μ_X) modulo the congruence (ℓ, r).

(d) We are ready to prove that $\ell, r\colon (K, k) \rightrightarrows (TX, \mu_X)$ is a finitely generated congruence. Take the regular factorization of $y^\sharp\colon (TY, \mu_Y) \to (K, k)$ for y in (c):

$$y^\sharp = \big((TY, \mu_Y) \xrightarrow{e} \mathsf{Im}(y^\sharp) \xrightarrow{m} (K, k) \big).$$

Then e_y is also the coequalizer of $\ell \cdot m$ and $r \cdot m$, and $\mathsf{Im}(y^\sharp)$ is a finitely generated $\mathbb{T}$-algebra by Theorem 3.5, as desired.

(1) $\Longrightarrow$ (2) We are given a regular epimorphism $e\colon (TX, \mu_X) \twoheadrightarrow (A, a)$ with X finitely presentable in $\mathscr{A}$ and a pair $\ell', r'\colon (K', k') \rightrightarrows (TX, \mu_X)$ with (K', k') finitely generated, whose coequalizer is e. By Theorem 3.5, there exists a regular quotient $q\colon (TY, \mu_Y) \twoheadrightarrow (K', k')$ with Y finitely presentable in $\mathscr{A}$. Since e is a coequalizer of ℓ', r', it is also a coequalizer of the pair $\ell' \cdot q, r' \cdot q$. ∎

3.8. Open Problem. Are (1)–(3) above equivalent for all finitary monads (not necessarily regular ones)?

4. Finitary Monads on Sets

For every lfp category $\mathscr{A}$ we are now going to characterize the finitely presentable and finitely generated objects of the category $\mathsf{Mnd_f}(\mathscr{A})$ of finitary monads on $\mathscr{A}$. They do not coincide in case $\mathscr{A} = \mathsf{Set}$, in contrast with the result of [2] that in the category of all finitary set functors every finitely generated object is finitely presentable.

4.1. Remark. The finitely generated objects of the category $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ of all finitary set functors were characterized in [2] as precisely the *super-finitary* set functors. These are the quotient functors of the polynomial functors $H_\Sigma X = \coprod_{n \in \mathbb{N}} \Sigma_n \times X^n$ for signatures Σ of finitely many finitary operations.

4.2. Example. As an application of Theorem 3.5, we generalize from [2, Corollary 3.31] the fact that $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ has as finitely generated objects precisely the super-finitary functors, to all lfp categories $\mathscr{A}$. Denote by

$$[\mathscr{A}, \mathscr{A}]_{\mathsf{fin}}$$

the category of all finitary endofunctors of $\mathscr{A}$. An example is the polynomial functor H_Σ for every signature Σ in the sense of Kelly and Power [6] that we now recall. A signature Σ is a collection of objects Σ_n of $\mathscr{A}$ indexed by objects $n \in \mathscr{A}_{\mathsf{fp}}$. Let $|\mathscr{A}_{\mathsf{fp}}|$ be the discrete category of objects of $\mathscr{A}_{\mathsf{fp}}$, then the functor category

$$\mathsf{Sig}(\mathscr{A}) = \mathscr{A}^{|\mathscr{A}_{\mathsf{fp}}|}$$

is called the *category of signatures*. Its morphisms from $\Sigma \to \Sigma'$ are collections of morphisms $e_n \colon \Sigma_n \to \Sigma'_n$ for $n \in |\mathscr{A}_{\mathsf{fp}}|$. The *polynomial functor* H_Σ is the coproduct of the endofunctors $\mathscr{A}(n, -) \bullet \Sigma_n$, where $\bullet$ denotes copowers of Σ_n, that is, for an object X we have:

$$H_\Sigma X = \coprod_{n \in \mathscr{A}_{\mathsf{fp}}} \mathscr{A}(n, X) \bullet \Sigma_n.$$

We obtain an adjoint situation

$$[\mathscr{A}, \mathscr{A}]_{\mathsf{fin}} \underset{\Phi}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{Sig}(\mathscr{A})$$

where the forgetful functor U takes a finitary endofunctor F to the signature

$$UF = \Sigma \quad \text{with} \quad \Sigma_n = Fn \qquad (n \in \mathscr{A}_{\mathsf{fp}})$$

and Φ takes a signature Σ to the polynomial endofunctor $\Phi\Sigma = H_\Sigma$. The resulting monad $\mathbb{T}$ is given by

$$(T\Sigma)_n = \coprod_{m \in \mathscr{A}_{\mathsf{fp}}} \mathscr{A}(m, n) \bullet \Sigma_m.$$

4.3. Remark. It is easy to see that the forgetful functor U is monadic. Indeed, this follows from Beck's theorem since U has a left-adjoint and creates all colimits. The latter is clear since colimits of functors are formed object-wise, and for finitary functors they are determined on the finitely presentable objects.

Thus, we see that the category $[\mathscr{A}, \mathscr{A}]_{\mathsf{fin}}$ is equivalent to the Eilenberg-Moore category of the monad $\mathbb{T}$. By Theorem 3.5, finitely generated objects of $[\mathscr{A}, \mathscr{A}]_{\mathsf{fin}}$ are precisely the strong quotients of ffp algebras for $\mathbb{T}$. Now by Lemma 2.5, a signature Σ is finitely presentable in $\mathscr{A}^{|\mathscr{A}_{\mathsf{fp}}|}$ iff for the initial object 0 of $\mathscr{A}$ we have

$$\Sigma_n = 0 \text{ for all but finitely many } n \in \mathscr{A}_{\mathsf{fp}}$$

and

$$\Sigma_n \text{ is finitely presentable for every } n \in \mathscr{A}_{\mathsf{fp}}.$$

Let us call such signatures *super-finitary*. We thus obtain the following result.

4.4. Corollary. *For an lfp category $\mathscr{A}$, a finitary endofunctor is finitely generated in $[\mathscr{A},\mathscr{A}]_{\mathsf{fin}}$ iff it is a strong quotient of a polynomial functor H_Σ with Σ super-finitary.*

4.5. Example. Another application of results of Section 3 is to the category

$$\mathsf{Mnd_f}(\mathscr{A})$$

of all finitary monads on an lfp category $\mathscr{A}$. Lack proved in [8] that this category is also monadic over the category of signatures. More precisely, for the forgetful functor $V\colon \mathsf{Mnd_f}(\mathscr{A}) \to [\mathscr{A},\mathscr{A}]_{\mathsf{fin}}$ the composite

$$UV\colon \mathsf{Mnd_f}(\mathscr{A}) \to \mathsf{Sig}(\mathscr{A})$$

is a monadic functor. Recall from Barr [4] that every finitary endofunctor H generates a free monad; let us denote it by H^*. The corresponding free monad $\mathbb{T}$ for UV assigns to every signature Σ the signature derived from the free monad on Σ (w.r.t. UV), or, equivalently, from the free monad H_Σ^* on the polynomial endofunctor H_Σ. Thus the monad $\mathbb{T}$ is given by the following rule for Σ in $\mathsf{Sig}(\mathscr{A})$:

$$(T\Sigma)_n = H_\Sigma^* n \qquad \text{for all } n \in \mathscr{A}_{\mathsf{fp}}.$$

(Example: if $\mathscr{A} = \mathsf{Set}$ then H_Σ^* assigns to every set X the set $H_\Sigma^* X$ of all Σ-terms with variables in X.) In general, it follows from [1] that the underlying functor of H_Σ^* is the colimit of the following ω-chain in $[\mathscr{A},\mathscr{A}]_{\mathsf{fin}}$:

$$\mathsf{Id} \xrightarrow{w_0} H_\Sigma + \mathsf{Id} \xrightarrow{H_\Sigma w_0 + \mathsf{id}} H_\Sigma(H_\Sigma + \mathsf{Id}) + \mathsf{Id} \longrightarrow \cdots W_n \xrightarrow{w_n} W_{n+1} \longrightarrow \cdots$$

Here, $W_0 = \mathsf{Id}$ and $W_{n+1} = H_\Sigma W_n + \mathsf{Id}$. Moreover, $w_0\colon \mathsf{Id} \to H_\Sigma + \mathsf{Id}$ is the coproduct injection, while $w_{n+1} = H_\Sigma w_n + \mathsf{id}$. The monad H_Σ^* is thus the free $\mathbb{T}$-algebra on Σ and the ffp algebras are precisely H_Σ^* for Σ super-finitary.

4.6. Definition. *Let Σ be a signature in an lfp category $\mathscr{A}$.*

(1) *By a* Σ-equation *is meant a parallel pair*

$$f, f'\colon k \longrightarrow H_\Sigma^* n \qquad \textit{with } k, n \in \mathscr{A}_{\mathsf{fp}}$$

of morphisms in $\mathscr{A}$.

(2) *A quotient $c\colon H_\Sigma^* \to \mathbb{M}$ in $\mathsf{Mnd_f}(\mathscr{A})$ is said to* satisfy *the equation if its n-component merges f and f' (i.e. $c_n \cdot f = c_n \cdot f'$).*

(3) *By a* presentation *of a monad $\mathbb{M}$ in $\mathsf{Mnd_f}(\mathscr{A})$ is meant a signature Σ and a collection of Σ-equations such that the least quotient of H_Σ^* satisfying all of the given equations has the form $c\colon H_\Sigma^* \twoheadrightarrow \mathbb{M}$.*

(4) *The presentation is called* super-finitary *if Σ is super-finitary and we have a finite set of equations.*

If $\mathscr{A} = \mathsf{Set}$, this is the classical concept of a presentation of a variety by a signature and equations. Indeed, given a pair $f, f' \colon 1 \to H_\Sigma^* n$, which is a pair of Σ-terms in n variables, satisfaction of the equation $f = f'$ in the sense of General Algebra means precisely $c_n \cdot f = c_n \cdot f'$. And a general pair $f, f' \colon k \to H_\Sigma^* n$ can be substituted by k pairs of terms in n variables.

4.7. Remark.

(1) Every finitary monad $\mathbb{M}$ has a presentation. Indeed, since this is an algebra for the monad $\mathbb{T}$ of Example 4.2, it is a coequalizer of a parallel pair of monad morphisms between free algebras for $\mathbb{T}$:

$$H_\Gamma^* \underset{r}{\overset{\ell}{\rightrightarrows}} H_\Sigma^* \xrightarrow{\;c\;} \mathbb{M}$$

To give a monad morphism ℓ is equivalent to giving a signature morphism

$$\ell_n \colon \Gamma_n \longrightarrow H_\Sigma^* n \qquad (n \in |\mathscr{A}_{\mathsf{fp}}|).$$

Analogously for $r \mapsto (r_n)$. Thus, to say that c merges ℓ and r is the same as to say that $\mathbb{M}$ satisfies the equations $\ell_n, r_n \colon \Gamma_n \to H_\Sigma^* n$ for all n. And the above coequalizer c is the least such quotient.

(2) Every equation $f, f' \colon k \to H_\Sigma^* n = \operatorname{colim}_{r \in \mathbb{N}} W_r n$ can be substituted, for some number r (the "depth" of the terms), by an equation $g, g' \colon k \to W_r n$. This follows from k being finitely presentable.

4.8. Theorem. *Let $\mathscr{A}$ be an lfp category with regular factorizations. A finitary monad is, as an object of $\mathsf{Mnd_f}(\mathscr{A})$,*

(1) *finitely generated iff it has a presentation by Σ-equations for a super-finitary signature Σ, and*

(2) *finitely presentable iff it has a presentation by finitely many Σ-equations for a super-finitary signature Σ .*

Proof. (1) Let $\mathbb{M}$ have a presentation with Σ super-finitary. Then $\mathbb{M}$ is a (regular) quotient of an ffp-algebra H_Σ^* for $\mathbb{T}$, thus, it is finitely generated by Theorem 3.5.

Conversely, if $\mathbb{M}$ is finitely generated, it is a (strong) quotient $c \colon H_\Sigma^* \twoheadrightarrow \mathbb{M}$ for Σ super-finitary. It is sufficient to show that c is a regular epimorphism in $\mathsf{Mnd_f}(\mathscr{A})$, then the argument that $\mathbb{M}$ has a presentation using Σ is as in Remark 4.7.

Since $\mathscr{A}$ has regular factorizations, so does $\mathsf{Sig}(\mathscr{A}) = \mathscr{A}^{|\mathscr{A}_{\mathsf{fp}}|}$. And the monad $\mathbb{T}$ on $\mathsf{Sig}(\mathscr{A})$ given by

$$(T\Sigma)_n = H_\Sigma^* n \qquad (n \in \mathscr{A}_{\mathsf{fp}})$$

is regular. Indeed, for every regular epimorphism $e \colon \Sigma \twoheadrightarrow \Gamma$ in $\mathsf{Sig}(\mathscr{A})$ we have regular epimorphisms $e_n \colon \Sigma_n \twoheadrightarrow \Gamma_n$ in $\mathscr{A}$ ($n \in \mathscr{A}_{\mathsf{fp}}$), and the components of Te are the morphisms

$$(Te)_m = \coprod_{n \in |\mathscr{A}_{\mathsf{fp}}|} \mathscr{A}(n, m) \bullet e_n \qquad (m \in \mathscr{A}_{\mathsf{fp}}).$$

Since coproducts of regular epimorphisms in $\mathscr{A}$ are regular epimorphisms, we conclude that each $(Te)_m$ is regularly epic in $\mathscr{A}$. Thus, Te is regularly epic in $\mathsf{Sig}(\mathscr{A})$.

Consequently, the category of $\mathbb{T}$-algebras has regular factorizations. Since c is a strong epimorphism, it is regular.

(2) We can apply Theorem 3.7: an algebra $\mathbb{M}$ for $\mathbb{T}$ is finitely presentable iff it is a coequalizer in $\mathsf{Mnd_f}(\mathscr{A})$ as follows:

$$H_\Gamma^* \underset{r}{\overset{\ell}{\rightrightarrows}} H_\Sigma^* \xrightarrow{c} \mathbb{M}$$

for some super-finitary signatures Γ and Σ. By the preceding remark, we can substitute ℓ and r by a collection of equations $\Gamma_n \rightrightarrows H_\Sigma^* n$, and since Γ is super-finitary, this collection is finite. Therefore, every finitely presentable monad in $\mathsf{Mnd_f}(\mathscr{A})$ has a super-finitary presentation.

Conversely, let $\mathbb{M}$ be presented by a super-finitary signature Σ and equations

$$f_i, f_i' \colon k_i \longrightarrow H_\Sigma^* n_i \qquad (i = 1, \dots, r).$$

Let Γ be the super-finitary signature with

$$\Gamma_k = \coprod_{\substack{i \in I \\ k_i = k}} k_i \qquad (\text{for all } k \in \mathscr{A}_{\mathsf{fp}})$$

Then we have signature morphisms

$$f, f' \colon \Gamma \longrightarrow T(\Sigma)$$

derived from the given pairs in an obvious way. For the corresponding monad morphisms

$$\bar{f}, \bar{f}' \colon H_\Gamma^* \longrightarrow H_\Sigma^*$$

we see that the coequalizer of this pair is the smallest quotient $c \colon H_\Sigma^* \twoheadrightarrow \mathbb{M}$ with $c_{n_i} \cdot f_i = c_{n_i} \cdot f_i'$ for all $i = 1, \dots, n$. This follows immediately from the fact that c is a regular epimorphism in $\mathsf{Mnd_f}(\mathscr{A})$. Indeed, since $\mathscr{A}$ has regular factorizations, so does $\mathsf{Sig}(\mathscr{A})$, a power of $\mathscr{A}$. Moreover, $\mathbb{T}$ is a regular monad, thus the category $\mathsf{Mnd_f}(\mathscr{A})$ of its algebras has regular factorizations, thus, every strong epimorphism is regular. ∎

4.9. COROLLARY. *A finitary monad on* Set *is a finitely presentable object of* $\mathsf{Mnd_f}(\mathsf{Set})$ *iff the corresponding variety of algebras has a presentation (in the classical sense) by finitely many operations and finitely many equations.*

Most of "everyday" varieties (groups, lattices, boolean algebras, etc.) yield finitely presentable monads. Vector spaces over a field K yield a finitely presentable monad iff K is finite – equivalently, that monad is finitely generated. However, there are finitely generated monads in $\mathsf{Mnd_f}(\mathsf{Set})$ that fail to be finitely presentable. We prove that the classes of finitely presentable and finitely generated objects differ in $\mathsf{Mnd_f}(\mathsf{Set})$ by relating monads to monoids via an adjunction.

4.10. Remark.

(1) Recall that every set functor has a unique strength. This follows from the result by Kock [7] that a strength of an endofunctor on a closed monoidal category bijectively corresponds to a way of making that functor enriched (see also Moggi [10, Proposition 3.4]).

(2) For every monad (T, η, μ) on Set we have a canonical strength, i.e. a family of morphisms

$$s_{X,Y} \colon TX \times Y \to T(X \times Y)$$

natural in X and Y and such that the following diagrams commute for all sets X, Y, Z:

$$\begin{array}{ccc} TX \times 1 & \xrightarrow{s_{1,X}} & T(X \times 1) \\ & \cong \searrow \quad \swarrow \cong & \\ & TX & \end{array} \qquad \begin{array}{ccc} TX \times Y \times Z & \xrightarrow{s_{X,Y} \times Z} & T(X \times Y) \times Z \\ & \searrow^{s_{X,Y \times Z}} & \downarrow s_{X \times Y, Z} \\ & & T(X \times Y \times Z) \end{array}$$

$$\begin{array}{ccc} TX \times Y & \xrightarrow{s_{X,Y}} & T(X \times Y) \\ \eta_X \times Y \uparrow & \nearrow_{\eta_{X \times Y}} & \\ X \times Y & & \end{array} \qquad \begin{array}{ccccc} TTX \times Y & \xrightarrow{s_{TX,Y}} & T(TX \times Y) & \xrightarrow{Ts_{X,Y}} & TT(X \times Y) \\ \mu_X \times Y \downarrow & & & & \downarrow \mu_{X \times Y} \\ TX \times Y & & \xrightarrow{s_{X,Y}} & & T(X \times Y) \end{array}$$

In fact, one defines the canonical strength by the commutativity of the following diagrams for every element $y \colon 1 \to Y$:

$$\begin{array}{ccc} TX \times Y & \xrightarrow{s_{X,Y}} & T(X \times Y) \\ \uparrow TX \times y & & \uparrow T(X \times y) \\ TX \times 1 & \cong TX \cong & T(X \times 1) \end{array}$$

4.11. Notation.

(1) We denote by $R \colon \mathsf{Mnd_f}(\mathsf{Set}) \to \mathsf{Mon}$ the functor sending a monad (T, η, μ) with strength s to the monoid $T1$ with unit $\eta_1 \colon 1 \to T1$ and the following multiplication:

$$m \colon T1 \times T1 \xrightarrow{s_{1,T1}} T(1 \times T1) \xrightarrow{\cong} TT1 \xrightarrow{\mu_1} T1.$$

For example, the finite power-set monad $\mathcal{P}_\mathsf{f}$ induces the monoid $(\{0,1\}, \wedge, 1)$ of boolean values with conjunction as multiplication.

(2) We define a functor $L \colon \mathsf{Mon} \to \mathsf{Mnd_f}(\mathsf{Set})$ as follows. For every monoid $(M, *, 1_M)$ we have the monad LM of free M-sets with the following object assignment, unit and multiplication:

$$LM(X) = M \times X, \quad \eta_X \colon x \mapsto (1_M, x), \quad \mu_X \colon (n, (m, x)) \mapsto (n * m, x).$$

This extends to a functor $L \colon \mathsf{Mon} \to \mathsf{Mnd_f}(\mathsf{Set})$ by $(Lf)_X = f \times \mathsf{id}_X$ for monoid homomorphisms f.

4.12. Proposition. *We have an adjoint situation $L \dashv R$ with the following unit ν and counit ϵ:*

$$\nu_M \colon M \xrightarrow{\cong} M \times 1 = RLM$$
$$\epsilon_T \colon LRT = T1 \times (-) \xrightarrow{s_{1,-}} T(1 \times (-)) \xrightarrow{T\cong} T,$$

for a monad T with strength s.

Proof. It is not hard to see that ν_M is a monoid morphism because the monoid structure in $M \times 1 = RLM$ boils down to the monoid structure of M. Furthermore, ν_M is clearly natural in M.

For every monad T, ϵ_T is a natural transformation $T1 \times (-) \to T$ because of the naturality of the strength s. The axioms for the strength imply that $s_{1,-} \colon T1 \times (-) \to T(1 \times (-))$ is a monad morphism by straightforward diagram chasing. To see ν and ϵ establish an adjunction, it remains to check the triangle identities:

- The identity $\epsilon_{LM} \cdot L\nu_M = \mathsf{id}_{LM}$ is just the associativity of the product:

$$LM \xrightarrow{L\nu_M} LRLM \xrightarrow{\qquad\qquad \epsilon_{LM} \qquad\qquad} LM$$
$$M \times (-) \xrightarrow{\nu_M \times (-)} (M \times 1) \times (-) \xrightarrow{\cong} M \times (1 \times (-)) \xrightarrow{M \times \cong} M \times (-)$$

  The composite is obviously the identity on $M \times (-)$.

- The identity $R\epsilon_T \cdot \nu_{RT} = \mathsf{id}_{RT}$ follows directly from the first axiom of strength:

$$RT \xrightarrow{\nu_{RT}} RLRT \xrightarrow{\qquad R\epsilon_T \qquad} RT$$
$$T1 \xrightarrow{\nu_{T1}} T1 \times 1 \xrightarrow{s_{1,1}} T(1 \times 1) \xrightarrow{\cong} T1, \qquad s_{1,1} \cdot \nu_{T1} = T\nu_1$$

■

4.13. Corollary. *In the category of finitary monads on* Set *the classes of finitely presentable and finitely generated objects do not coincide.*

Proof. Note that from the fact that the unit of the adjunction $L \dashv R$ is an isomorphism we see that L is fully faithful. Thus, we may regard Mon as a full coreflective subcategory of $\mathsf{Mnd_f}(\mathsf{Set})$. Furthermore, the right adjoint R preserves filtered colimits; this follows from the fact that filtered colimits in $\mathsf{Mnd_f}(\mathsf{Set})$ are created by the forgetful functor into $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ where they are formed object-wise. In addition, L preserves monomorphisms; in fact, for an injective monoid morphism $m \colon M \rightarrowtail M'$ the monad morphism $Lm \colon LM \to LM'$ is monic since all its components $m \times \mathsf{id}_X \colon M \times X \to M' \times X$ are. By Lemma 2.4, we therefore have that a monoid M is finitely presentable (resp. finitely generated) if and only if the monad LM is finitely presentable (resp. finitely generated).

In the category Mon of monoids finitely presentable and finitely generated objects do not coincide; see Campbell et al. [5, Example 4.5]. Thus the same holds for $\mathsf{Mnd_f}(\mathsf{Set})$. ■

4.14. Remark. For λ-accessible monads $\mathbb{T}$ on locally λ-presentable categories $\mathscr{A}$ all the above results have an appropriate statement and completely analogous proofs. Let $\mathscr{A}_\lambda$ be a set representing all λ-presentable objects of $\mathscr{A}$. Free algebras (TX, μ_X) with X λ-presentable are called λ-fp algebras. Then we have in particular the following:

(1) An algebra (A, a) is λ-generated in $\mathscr{A}^{\mathbb{T}}$ iff it is generated by a λ-generated subobject of A. These are precisely the strong quotients of λ-fp algebras.

(2) Suppose $\mathscr{A}$ has regular factorizations and T preserves regular epimorphisms. An algebra is λ-presentable in $\mathscr{A}^{\mathbb{T}}$ iff it is a quotient of a λ-fp algebra modulo a λ-generated congruence. These are precisely the coequalizers of parallel pairs between λ-fp algebras.

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