INSERTION AND EXTENSION RESULTS FOR POINTFREE COMPLETE REGULARITY

JAVIER GUTIÉRREZ GARCÍA AND JORGE PICADO

Abstract. There are insertion-type characterizations in pointfree topology that extend well known insertion theorems in point-set topology for all relevant higher separation axioms with one notable exception: complete regularity. In this paper we fill this gap. The situation reveals to be an interesting and peculiar one: contrarily to what happens with all the other higher separation axioms, the extension to the pointfree setting of the classical insertion result for completely regular spaces characterizes a formally weaker class of frames introduced in this paper (called *completely c-regular frames*). The fact that any compact sublocale (quotient) of a completely regular frame is a C-sublocale (C-quotient) is obtained as a corollary.

Introduction

When moving from classical topology to pointfree topology the role of the category of topological spaces and continuous maps is taken by the category of locales and localic maps (and its dual category of frames and frame homomorphisms). Accordingly, the space of reals is taken by the locale of reals $\mathfrak{L}(\mathbb{R})$ ([2]) and the complete Boolean algebra of all subspaces of a space X is taken by the co-frame of all sublocales of a locale L (along this paper we shall make the latter a frame, that is, turned upside down, that we shall denote by $\mathcal{S}(L)$; see Section 1 below for the details).

Among the important examples of sublocales are, for each $a \in L$, the *closed sublocales* $\mathfrak{c}(a) = {\uparrow}a = \{b \in L \mid a \leq b\}$ and the *open sublocales* $\mathfrak{o}(a) = \{a \to b \mid b \in L\}$. The class of all closed sublocales is usually denoted by $\mathfrak{c}L$ and it is a subframe of $\mathcal{S}(L)$ isomorphic to the given frame L via the mapping $\mathfrak{c}\colon L \to \mathfrak{c}L$ given by the correspondence $a \mapsto \mathfrak{c}(a)$.

Now, the ℓ-ring $\mathsf{F}(L)$ of real functions on L ([7, 9]), that is, the ℓ-ring $\mathsf{C}(\mathcal{S}(L))$ of continuous real functions on $\mathcal{S}(L)$ (see [12] for more information), is formed by all frame homomorphisms

$$\mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L),$$

and partially ordered by

$$\begin{aligned} f \leq g \;&\equiv\; f(r,\text{—}) \leq g(r,\text{—}) \quad \text{for all } r \in \mathbb{Q} \\ &\Leftrightarrow\; g(\text{—},r) \leq f(\text{—},r) \quad \text{for all } r \in \mathbb{Q}. \end{aligned}$$

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An $f \in \mathsf{F}(L)$ is *lower* (resp. *upper*) *semicontinuous* if $f(r,—) \in \mathfrak{c}L$ (resp. $f(—,r) \in \mathfrak{c}L$) for every $r \in \mathbb{Q}$ and it is *continuous* if it is both lower and upper semicontinuous, that is, if $f(p,q) \in \mathfrak{c}L$ for every $p,q \in \mathbb{Q}$. We shall denote by $\mathsf{C}(L)$, $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$ the classes of continuous, lower semicontinuous, and upper semicontinuous members of $\mathsf{F}(L)$, respectively.

An insertion-type theorem in pointfree topology has the following structure. Let $\mathscr{F}, \mathscr{G}, \mathscr{H} \subseteq \mathsf{F}(L)$. Assume $f \in \mathscr{F}, g \in \mathscr{G}$ and $f \le g$. Then an insertion-type assertion states that

there exists an $h \in \mathscr{H}$ *such that* $f \le h \le g$.

The particular case $L = \mathcal{O}X$ for the topology $\mathcal{O}X$ of a space X gives the corresponding classical insertion theorem.

A fundamental example is the case $\mathscr{F} = \mathsf{USC}(L)$, $\mathscr{G} = \mathsf{LSC}(L)$, $\mathscr{H} = \mathsf{C}(L)$ that characterizes normal frames and extends the celebrated Katětov-Tong insertion theorem for normal spaces (see [11], [8] and [7]; recall that a frame L is normal if $a \vee b = 1$ implies the existence of $u, v \in L$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$):

Theorem. *The following are equivalent for a frame L:*

(i) *L is normal.*
(ii) *If $f \in \mathsf{USC}(L)$, $g \in \mathsf{LSC}(L)$ and $f \le g$, then there exists an $h \in \mathsf{C}(L)$ such that $f \le h \le g$.*

The corresponding extension theorem asserts that any closed sublocale (quotient) of a normal frame is a C-sublocale (C-quotient).

For more examples, characterizing monotonically normal, completely normal, perfectly normal, countably paracompact or extremally disconnected frames, consult [7] and [4].

Comparing this with the literature in classical topology there is one important case missing: complete regularity. (We thank Prof. Robert Lowen for pointing out this gap in a conversation with the second author.)

Indeed, we know from [5] that

A space is completely regular if and only if for any $f, g\colon X \to [0,1]$, with f compact-like (i.e. $f^{-1}([t,1])$ is compact for all $t \in (0,1]$) and g lower semicontinuous satisfying $f \le g$, there exists a continuous $h\colon X \to [0,1]$ such that $f \le h \le g$.

This characterization of complete regularity holds for a very simple peculiar reason: every open U in X is the union of the compact subsets $\{x\}$, $x \in U$. Of course, when dealing with general frames one cannot imitate that: we may not have enough appropriate compact sublocales. The question naturally arises as to whether this insertion result continues to hold true in general frames. In this paper we address this question.

The case in frames reveals to be a peculiar one, with interesting ramifications: we show that the foregoing insertion result extends to completely regular frames but no longer characterizes complete regularity; among fit frames, it characterizes a formally wider class of frames that we introduce as *completely c-regular frames*. For that we need to revisit completely separated sublocales of a frame L (Section 2) and to introduce compact-like real functions on L (Section 4). The corresponding (Urysohn) separation-type lemma and (Tietze) extension-type theorem are also obtained (sections 4 and 5 respectively).

1. Preliminaries

Useful references for frames and locales are [10] and the recent [12]. Here we fix some notation and terminology and recall the relevant facts needed later on.

A *frame* L is a complete lattice with the distributive property

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\} \tag{1.1}$$

for all $a \in L$ and $S \subseteq L$; equivalently, it is a complete Heyting algebra with Heyting operation $\to$ satisfying the standard equivalence $a \wedge b \leq c$ if and only if $a \leq b \to c$. The *pseudocomplement* of an $a \in L$ is the element $a^* = a \to 0 = \bigvee\{b \in L \mid a \wedge b = 0\}$.

Frame homomorphisms are maps preserving all joins and all finite meets; the resulting category of frames and frame homomorphisms will be denoted by Frm and its opposite category is the category Loc of locales and localic maps. A typical frame is the lattice $\mathcal{O}X$ of all open sets of a topological space; if $f\colon X \to Y$ is a continuous map then $(U \mapsto f^{-1}[U])\colon \mathcal{O}Y \to \mathcal{O}X$ is obviously a frame homomorphism.

For any frame L, $k \in L$ is compact if $k \leq \bigvee \mathscr{X}$ implies $k \leq \mathscr{F}$ for some finite $\mathscr{F} \subseteq \mathscr{X}$, and L is called *compact* if its unit 1 is compact. L is *algebraic* if each $a \in L$ is a join of compact elements. Further, a frame L is called *completely regular* if $a = \bigvee\{b \in L \mid b \prec\!\!\prec a\}$ for each $a \in L$ where $b \prec\!\!\prec a$ (the "completely below" relation) means that there are $c_r \in L$ ($r \in \mathbb{Q} \cap [0,1]$) such that $c_0 = b$, $c_1 = a$ and $c_r \prec c_s$ (that is, ${c_r}^* \vee c_s = 1$) whenever $r < s$.

Sublocales. The lattice of subobjects of a locale L in Loc may be described in several equivalent ways (cf. [12]), most usually using congruences and nuclei. Here we use the *sublocale sets* (briefly, *sublocales*), that is, subsets $S \subseteq L$ such that

(S1) for every $A \subseteq S$, $\bigwedge A$ is in S, and
(S2) for every $s \in S$ and every $x \in L$, $x{\to}s$ is in S.

Each sublocale $S \subseteq L$ is also determined by the frame surjection (quotient map) $c_S\colon L{\to}S$ given by $c_S(x) = \bigwedge\{s \in S \mid s \geq x\}$ for all $x \in L$. E.g. the quotient maps $c_{\mathfrak{c}(a)}$ and $c_{\mathfrak{o}(a)}$ are given by $c_{\mathfrak{c}(a)}(x) = a \vee x$ and $c_{\mathfrak{o}(a)}(x) = a \to x$, respectively.

Moreover, each sublocale S of L is itself a frame with the same meets as in L, and since the Heyting operation $\to$ depends on the meet structure only, with the same Heyting operation. However the joins in S and L will not necessarily coincide:

$$\bigvee_{i\in I}^{S} a_i = c_S\Big(\bigvee_{i\in I} a_i\Big) \geq \bigvee_{i\in I} a_i.$$

It follows that $1_S = 1$ but in general $0_S \neq 0$. In particular

$$0_{\mathfrak{c}(a)} = a, \quad x \overset{\mathfrak{c}(a)}{\vee} y = x \vee y, \quad 0_{\mathfrak{o}(a)} = a^* \quad \text{and} \quad x \overset{\mathfrak{o}(a)}{\vee} y = a{\to}(x \vee y).$$

Intersections of sublocales are sublocales so we have a complete lattice which is indeed a *co-frame* [12], i.e. a complete lattice satisfying a distributive law dual to (1.1).

For notational reasons, we make this co-frame into a frame $\mathcal{S}(L)$ by considering the opposite ordering

$$S_1 \le S_2 \quad \Leftrightarrow \quad S_2 \subseteq S_1.$$

Thus, given $\{S_i \in \mathcal{S}(L) \mid i \in I\}$, we have

$$\bigvee_{i\in I} S_i = \bigcap_{i\in I} S_i \quad \text{and} \quad \bigwedge_{i\in I} S_i = \Big\{ \bigwedge A : A \subseteq \bigcup_{i\in I} S_i \Big\}.$$

Further, $\{1\}$ is the top and L is the bottom in $\mathcal{S}(L)$ that we simply denote by 1 and 0.

The *closure* $\overline{S}$ of a sublocale $S \in \mathcal{S}(L)$ is the largest closed sublocale smaller than S, and is given by the formula $\overline{S} = \uparrow(\bigwedge S)$. We shall denote the closed sublocales of a sublocale S of L by $\mathfrak{c}^S(a)$.

Facts 1.1. *For every $a, b \in L$, $A \subseteq L$ and $S, T \in \mathcal{S}(L)$, we have:*

(1) $\mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$ *and* $\mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$.
(2) $\bigvee_{a\in A} \mathfrak{c}(a) = \mathfrak{c}(\bigvee A)$ *and* $\bigwedge_{a\in A} \mathfrak{o}(a) = \mathfrak{o}(\bigvee A)$.
(3) $\mathfrak{c}(a) \vee \mathfrak{o}(a) = 1$ *and* $\mathfrak{c}(a) \wedge \mathfrak{o}(a) = 0$.
(4) $\mathfrak{o}(a) \ge \mathfrak{c}(b)$ *if and only if* $a \wedge b = 0$.
(5) $\mathfrak{o}(a) \le \mathfrak{c}(b)$ *if and only if* $a \vee b = 1$.
(6) $\overline{1} = 1$, $\overline{S} \le S$, $\overline{\overline{S}} = \overline{S}$, *and* $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$.
(7) $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$.
(8) $\mathfrak{c}(a) \vee S$ *is the closed sublocale* $\mathfrak{c}^S(c_S(a))$ *of* S.
(9) *If T is a closed sublocale of S then $T = \mathfrak{c}(x) \vee S$ for some $x \in S$.*

A sublocale S is said to be *compact* if it is compact as a frame. Equivalently:

Fact 1.2. *A sublocale S of L is compact if and only if for each $\{a_i\}_{i\in I} \subseteq L$ such that $\bigwedge_{i\in I} \mathfrak{o}(a_i) \le S$, there exists a finite $J \subseteq I$ such that $\bigwedge_{i\in J} \mathfrak{o}(a_i) \le S$.*

Proof. Just notice that

$$\bigwedge_{i\in I} \mathfrak{o}(a_i) = \mathfrak{o}\Big(\bigvee_{i\in I} a_i\Big) \le S \iff 1 = \mathfrak{c}\Big(\bigvee_{i\in I} a_i\Big) \vee S = \mathfrak{c}^S\Big(c_S\Big(\bigvee_{i\in I} a_i\Big)\Big)$$

$$\iff 1 = c_S\Big(\bigvee_{i\in I} a_i\Big) = \bigvee^{S}_{i\in I} c_S(a_i). \quad \square$$

Note that in the co-frame of sublocales this just says that *a sublocale is compact iff every open cover has a finite subcover.*

Corollary 1.3. *An element $k \in L$ is compact iff the sublocale $\mathfrak{o}(k)$ is compact.* $\square$

Fact 1.4. *If S is a compact sublocale of a frame L and T is a closed sublocale of S then T is a compact sublocale of L.*

Real functions. The frame $\mathfrak{L}(\mathbb{R})$ of reals is the frame specified by generators $(p, —)$ and $(—, q)$ for $p, q \in \mathbb{Q}$, and defining relations

(R1) $(p, —) \wedge (—, q) = 0$ whenever $p \ge q$,
(R2) $(p, —) \vee (—, q) = 1$ whenever $p < q$,
(R3) $(p, —) = \bigvee_{r>p}(r, —)$ and $(—, q) = \bigvee_{s<q}(—, s)$, for every $p, q \in \mathbb{Q}$,
(R4) $\bigvee_{p\in\mathbb{Q}}(p, —) = 1 = \bigvee_{q\in\mathbb{Q}}(—, q)$.

For the alternative definition of $\mathfrak{L}(\mathbb{R})$ by the generators $(p,q) \in \mathbb{Q}\times\mathbb{Q}$ and appropriate relations, see [2] or [12, Chapter XIV]. Continuous real functions are usually defined as frame homomorphisms $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ (see [2]). As proved in [7], after the isomorphism $\mathfrak{c}\colon L \to \mathfrak{c}L$, they can be identified with the elements of $\mathsf{C}(L) = \mathsf{LSC}(L)\cap\mathsf{USC}(L)$. In what follows, we will freely refer to continuous real function as both the real function $f \in \mathsf{C}(L)$ and the unique frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathfrak{c}\cdot\varphi = f$.

In order to define a continuous real function $f \in \mathsf{C}(L)$ it suffices to consider two maps from $\mathbb{Q}$ to L that turn the defining relations (R1)–(R4) of $\mathfrak{L}(\mathbb{R})$ into identities in L. This can be easily stated via scales: a *scale* in L (see [7]) is a family $(s_p)_{p\in\mathbb{Q}}$ of elements of L satisfying

(1) $s_q \prec s_p$ whenever $p < q$, and
(2) $\bigvee_{p\in\mathbb{Q}} s_p = 1 = \bigvee_{p\in\mathbb{Q}} {s_p}^*$.

In fact, for each scale $(s_r)_{r\in\mathbb{Q}}$ the formulas $f(p,\text{—}) = \bigvee_{r>p} s_r$ and $f(\text{—},q) = \bigvee_{r<q} {s_r}^*$ $(p,q \in \mathbb{Q})$ determine an $f \in \mathsf{C}(L)$. In the particular case of $L = \mathcal{S}(M)$ for some frame M, if every s_r is a closed (resp. open, resp. clopen) sublocale then $f \in \mathsf{LSC}(M)$ (resp. $f \in \mathsf{USC}(M)$, resp. $f \in \mathsf{C}(M)$).

Example 1.5 (Constant functions)**.** Let r be a rational number. We denote by $\boldsymbol{r}$ the real function defined for each $p,q \in \mathbb{Q}$ by

$$\boldsymbol{r}(p,\text{—}) = \begin{cases} 1 & \text{if } p < r, \\ 0 & \text{if } p \geq r, \end{cases} \quad \text{and} \quad \boldsymbol{r}(\text{—},q) = \begin{cases} 0 & \text{if } q \leq r, \\ 1 & \text{if } q > r. \end{cases}$$

Example 1.6 (Characteristic functions)**.** Let c be a complemented element of L. We denote by χ_c the real function defined for each $p,q \in \mathbb{Q}$ by

$$\chi_c(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0, \\ c^* & \text{if } 0 \leq p < 1, \\ 0 & \text{if } p \geq 1, \end{cases} \quad \text{and} \quad \chi_c(\text{—},q) = \begin{cases} 0 & \text{if } q \leq 0, \\ c & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1. \end{cases}$$

2. Complete regularity and completely separated sublocales

The notion of complete separation in pointfree topology was first introduced in [1] in terms of quotient maps and cozero elements and equivalently reformulated in [6] in terms of sublocales and continuous real functions.

Let S and T be sublocales of L. They are said to be *completely separated* if there exists an $f \in \mathsf{C}(L)$ such that

$$f(0,\text{—}) \leq S \quad \text{and} \quad f(\text{—},1) \leq T.$$

Equivalently, this means that the corresponding quotient maps c_S and c_T are completely separated, i.e. there exists a frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $c_S(\varphi(0,\text{—})) = 0_S$ and $c_T(\varphi(\text{—},1)) = 0_T$.

Remarks 2.1. (i) 0 and 1 are completely separated by χ_0.
(ii) If S and T are completely separated, then any sublocales $U \geq S$ and $V \geq T$ are completely separated as well.
(iii) Sublocales S and T are completely separated iff $\overline{S}$ and $\overline{T}$ are completely separated.

(iv) If S and T_i $(i = 1, 2)$ are completely separated, then S and $T_1 \wedge T_2$ are also completely separated.

The following theorem from [6, Thm. 4.2] will be crucial in our approach.

Theorem 2.2. *Let L be a frame and let $f, g \in \mathsf{F}(L)$. Then the following are equivalent:*

(i) *There exists $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.*

(ii) *The sublocales $f(—, q)$ and $g(p, —)$ are completely separated for every $p < q$ in $\mathbb{Q}$.* □

Proposition 2.3. *Let S and T be sublocales of L. Then S and T are completely separated if and only if there exists an $f \in \mathsf{C}(L)$ such that*

$$\chi_{\overline{S}} \leq f \leq \chi_{\overline{T}^*}$$

Proof. Let S and T be sublocales. Then

$$\chi_{\overline{S}}(—, q) = \begin{cases} 0 & \text{if } q \leq 0, \\ \overline{S} & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1, \end{cases} \quad \text{and} \quad \chi_{\overline{T}^*}(p, —) = \begin{cases} 1 & \text{if } p < 0, \\ \overline{T} & \text{if } 0 \leq p < 1, \\ 0 & \text{if } p \geq 1, \end{cases}$$

and the result follows immediately by Remarks 2.1 (i) and (iii) and Theorem 2.2. □

The following is also included in [6] (Remark 3.5).

Remark 2.4. Let $a, b \in L$. Then $b \prec\!\prec a$ if and only if $\mathfrak{o}(b)$ and $\mathfrak{c}(a)$ are completely separated.

From Proposition 2.3 and Remark 2.4 it follows immediately that:

Corollary 2.5. *Let $a, b \in L$. Then $b \prec\!\prec a$ if and only there exists an $f \in \mathsf{C}(L)$ such that $\chi_{\overline{\mathfrak{o}(b)}} \leq f \leq \chi_{\mathfrak{o}(a)}$.* □

3. Variants of [complete] regularity in frames

Recall that a topological space $(X, \mathcal{O}X)$ is *regular* if for any $U \in \mathcal{O}X$ and each $x \in U$ there exists $V \in \mathcal{O}X$ such that $x \in V \subseteq \overline{V} \subseteq U$. The following characterizations are easy to get:

$$\begin{aligned} (X, \mathcal{O}X) \text{ is regular} \quad &\Longleftrightarrow \quad U = \bigcup \{V \in \mathcal{O}X \mid \overline{V} \subseteq U\} \quad \text{for every } U \in \mathcal{O}X \quad (*) \\ &\Longleftrightarrow \quad \text{For every compact } K \subseteq X \text{ and } U \in \mathcal{O}X \text{ such that } K \subseteq U, \text{ there exists } V \in \mathcal{O}X \text{ such that } K \subseteq V \text{ and } \overline{V} \subseteq U. \quad (**) \end{aligned}$$

Note that it is the direct translation of condition $(*)$ from $\mathcal{O}X$ to an arbitrary frame that is taken as the usual definition of a regular frame: a frame L is *regular* if $a = \bigvee\{b \in L \mid b \prec a\}$ for each $a \in L$, or, equivalently, if

$$\mathfrak{o}(a) = \bigwedge \{\mathfrak{o}(b) \mid b \prec a\} \quad \text{for each } a \in L.$$

What does the translation of condition $(**)$ to general frames produce? We first note the following:

Proposition 3.1. *Let $(X, \mathcal{O}X)$ be a topological space. Then X is regular if and only if for every compact sublocale S and every $U \in \mathcal{O}X$ satisfying $\mathfrak{o}(U) \leq S$, there exists $V_S \in \mathcal{O}X$ such that $\mathfrak{o}(V_S) \leq S$ and $V_S \prec U$.*

Proof. Let S be a compact sublocale and $U \in \mathcal{O}X$ such that $\mathfrak{o}(U) \leq S$. Then

$$\bigwedge\{\mathfrak{o}(V) \mid V \prec U\} = \mathfrak{o}(U) \leq S$$

and a use of Fact 1.2 gives $\{V_i\}_{i=1}^n \subseteq L$ such that $V_i \prec U$ for every $i \in \{1, \ldots, n\}$ and $\bigwedge_{i=1}^n \mathfrak{o}(V_i) \leq S$. Take $V_S = \bigcup_{i=1}^n V_i$. Then $\mathfrak{o}(V_S) = \bigwedge_{i=1}^n \mathfrak{o}(V_i) \leq S$ and $\overline{V_S} = \bigcup_{i=1}^n \overline{V_i} \subseteq U$.

For the converse: First note that since $X \setminus \overline{\{x\}}$ is prime for each $x \in X$, it follows from [12, III.10.1] that $S_x = \{X \setminus \overline{\{x\}}, X\}$ (the *one-point* sublocale corresponding to the familiar homeomorphism $\mathcal{O}X \to \mathbf{2}$ determined by $x \in X$, given by $U \mapsto 1$ iff $x \in U$) is a compact sublocale of $\mathcal{O}X$. Moreover, given $U \in \mathcal{O}X$, we have that

$$\begin{aligned} \mathfrak{o}(U) \leq S_x \quad &\iff \quad X \setminus \overline{\{x\}} \in \mathfrak{o}(U) \quad \iff \quad U \to X \setminus \overline{\{x\}} = X \setminus \overline{\{x\}} \\ &\iff \quad U \not\leq X \setminus \overline{\{x\}} \quad \iff \quad U \cap \overline{\{x\}} \neq \varnothing \quad \iff \quad x \in U. \end{aligned}$$

Let $U \in \mathcal{O}X$ and $x \in U$. Then S_x is a compact sublocale such that $\mathfrak{o}(U) \leq S_x$ and hence there exists $V_x \in \mathcal{O}X$ such that $V_x \prec U$ and $\mathfrak{o}(V_x) \leq S_x$. It follows that $x \in V_x$ and $\overline{V_x} \subseteq U$. □

In the same vein (just replacing $\prec$ by $\prec\!\!\prec$), we have also the following:

Proposition 3.2. *Let $(X, \mathcal{O}X)$ be a topological space. Then X is completely regular if and only if for every compact sublocale S and every $U \in \mathcal{O}X$ satisfying $\mathfrak{o}(U) \leq S$, there exists $V_S \in \mathcal{O}X$ such that $\mathfrak{o}(V_S) \leq S$ and $V_S \prec\!\!\prec U$.* □

This suggests the following variant of [complete] regularity in frames:

Definition 3.3. Let L be a frame. L is said to be *c-regular* (resp. *completely c-regular*) if for each compact sublocale S of L and each $a \in L$ such that $\mathfrak{o}(a) \leq S$, there exists $b_S \in L$ such that $\mathfrak{o}(b_S) \leq S$ and $b_S \prec a$ (resp. $b_S \prec\!\!\prec a$).

The following is almost obvious:

Proposition 3.4. *Let L be a frame. If L is [completely] regular, then it is [completely] c-regular.*

Proof. The proof follows the lines of the first implication in Proposition 3.1. □

What about the converse implication? As it follows from Propositions 3.1 and 3.2, it clearly holds for spatial frames. More generally, [complete] regularity coincides with [complete] c-regularity whenever any $a \in L$ satisfies

$$\mathfrak{o}(a) = \bigwedge\{S \in \mathcal{S}(L) \mid \mathfrak{o}(a) \leq S \text{ and } (S \text{ clopen or compact})\}. \tag{3.4.1}$$

Indeed, let L be c-regular and $a \in L$. If $\mathfrak{o}(a) \leq S$ and S is clopen (i.e. $S = \mathfrak{c}(x) = \mathfrak{o}(x^*)$ for some complemented $x \in L$), then $x^* \prec a$ and thus by (3.4.1), $S \geq \bigwedge\{\mathfrak{o}(b) \mid b \prec a\}$. On the other hand, for each compact sublocale S satisfying $\mathfrak{o}(a) \leq S$, by c-regularity there exists $b_S \in L$ such that $\mathfrak{o}(b_S) \leq S$ and $b_S \prec a$. It follows that $S \geq \mathfrak{o}(b_S) \geq \bigwedge\{\mathfrak{o}(b) \mid b \prec a\}$. Hence

$$\mathfrak{o}(a) = \bigwedge\{S \in \mathcal{S}(L) \mid \mathfrak{o}(a) \leq S \text{ and } (S \text{ clopen or compact})\} \geq \bigwedge\{\mathfrak{o}(b) \mid \mathfrak{o}(a) \leq b \prec a\}$$

and since the converse inequality is always true we conclude that a is regular. A similar argument applies in the case of complete regularity.

Note that any complemented $c \in L$ satisfies (3.4.1) since in that case $\mathfrak{o}(c)$ is clopen, and the join of any set of elements satisfying (3.4.1) also satisfies (3.4.1). Therefore, each zero-dimensional frame (in particular, each Boolean frame) satisfies (3.4.1). On the other hand, by Corollary 1.3, any compact element satisfies (3.4.1). Consequently algebraic frames also satisfy (3.4.1).

Question 3.5. The question is left open whether every [completely] c-regular frame is [completely] regular. As shown above, the two notions coincide for a wide class of frames, namely the ones satisfying condition (3.4.1) for any $a \in L$ — including e.g. all spatial frames and algebraic frames—, but we believe this not to be the case in general. However a proof of this has eluded us so far.

4. Insertion theorem: not quite like the classical case

Given a frame L, we say that an $f \in \mathsf{F}(L)$ is *upper compact-like* (resp. *lower compact-like*) if $f(\text{—}, p)$ (resp. $f(p, \text{—})$) is a compact sublocale of L for every $p \in \mathbb{Q}$. As a first example we note the following obvious proposition:

Proposition 4.1. *Let S be a complemented sublocale of a frame L. Then:*

(i) *χ_S is upper compact-like if and only if S is compact.*
(ii) *χ_S is lower compact-like if and only if S^* is compact.* □

Proposition 4.2. (i) *If L is compact, then all upper (resp. lower) semicontinuous functions on L are upper (resp. lower) compact-like.*

(ii) *If L is Hausdorff or fit, then all upper (resp. lower) compact-like functions on L are upper (resp. lower) semicontinuous.*

Proof. (i) is a consequence of Fact 1.4 and (ii) follows immediately from the fact that in any Hausdorff (or fit) frame, compact sublocales are closed [13]. □

In order to obtain our insertion result we need first the following Urysohn-type separation result.

Lemma 4.3. *The following statements are equivalent for any frame L:*

(i) *L is completely c-regular.*
(ii) *Every two sublocales S and T of L such that $S \vee T = 1$, one of which is compact and the other closed, are completely separated.*

Proof. (i)⇒(ii): Let S and T be sublocales such that $S \vee T = 1$, with S being compact and $T = \mathfrak{c}(a)$. Then we have $S \vee \mathfrak{c}(a) = 1$ iff $\mathfrak{o}(a) \leq S$ and therefore, by hypothesis, there exists $b_S \in L$ such that $\mathfrak{o}(b_S) \leq S$ and $b_S \prec\!\!\prec a$. By Proposition 2.4, $\mathfrak{o}(b_S)$ and $T = \mathfrak{c}(a)$ are completely separated and, finally, by Remark 2.1 (ii) so are S and T.

(ii)⇒(i): Let $a \in L$ and let S be a compact sublocale such that $\mathfrak{o}(a) \leq S$. Since $\mathfrak{c}(a) \vee S = 1$, it follows by hypothesis that S and $\mathfrak{c}(a)$ are completely separated. Hence there exists $h \in \mathsf{C}(L)$ such that $h(0, \text{—}) \leq \mathfrak{c}(a)$ and $h(\text{—}, 1) \leq S$ and thus $h\left(\text{—}, \frac{1}{2}\right) \leq h\left(\frac{1}{2}, \text{—}\right)^* \leq$

$h(—,1) \le S$. Let $b_S \in L$ such that $h\left(\frac{1}{2},—\right)^* = \mathfrak{o}(b_S)$. Then $\mathfrak{o}(b_S)$ are $\mathfrak{c}(a)$ are completely separated, i.e. $b_S \prec\!\prec a$. □

Remark 4.4. We point out that a proof that complete regularity implies the statement (ii) above appears in [3, Lemma 2.1].

We can now prove our insertion-type result for completely c-regular frames:

Theorem 4.5. *Let L be a completely c-regular frame. If $f, g \in \mathsf{F}(L)$, f is upper compact-like, g is lower semicontinuous and $f \le g$, then there exists $h \in \mathsf{C}(L)$ such that $f \le h \le g$.*

The converse holds for frames in which every compact sublocale is complemented (in particular, in any Hausdorff or fit frame).

Proof. Let $f, g \in \mathsf{F}(L)$ such that f is upper compact-like, g is lower semicontinuous and $f \le g$. Then $f(—,q)$ is compact, $g(p,—)$ is closed and $f(—,q) \vee g(p,—) = 1$ for each $p < q$ in $\mathbb{Q}$. It follows from Lemma 4.3 that $f(—,q)$ and $g(p,—)$ are completely separated. We conclude from Theorem 2.2 that there exists $h \in \mathsf{C}(L)$ such that $f \le h \le g$.

Conversely, let L be a frame in which every compact sublocale is complemented. By Lemma 4.3 it suffices to check that any sublocales S and T such that $S \vee T = 1$, with S compact (hence complemented) and $T = \mathfrak{c}(a)$ closed, are completely separated. But $S \ge \mathfrak{o}(a)$ and so $\chi_S \le \chi_{\mathfrak{o}(a)}$ with χ_S being upper compact-like and $\chi_{\mathfrak{o}(a)}$ lower semicontinuous. It then follows that there exists $h \in \mathsf{C}(L)$ such that $\chi_S \le h \le \chi_{\mathfrak{o}(a)}$, that is $h(—,1) \le S$ and $h(0,—) \le \mathfrak{c}(a)$. □

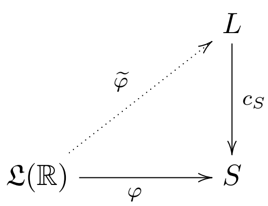
Remark 4.6. Recall from [9] that there is a dual order-automorphism $-(\cdot)\colon \mathsf{F}(L) \to \mathsf{F}(L)$ defined by $(-f)(—,r) = f(-r,—)$ and $(-f)(r,—) = f(—,-r)$ for every $r \in \mathbb{Q}$, and that $f \in \mathsf{F}(L)$ is upper semicontinuous (resp. upper compact-like) if and only if $-f$ is lower semicontinuous (resp. lower compact-like). Using this, one readily gets the following dual result:

> *If L is a completely c-regular frame and $f, g \in \mathsf{F}(L)$ are such that f is upper semicontinuous, g is lower compact-like and $f \le g$, then there exists $h \in \mathsf{C}(L)$ such that $f \le h \le g$.*

Remark 4.7. It should be noted that the statement (ii) of Urysohn-type Lemma 4.3 is just the particularization to characteristic functions of the foregoing insertion statement.

5. Extension theorem

Let S be a sublocale of L. Recall from [1] that a frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to S$ is said to have an *extension* to L if there exists a frame homomorphism $\widetilde{\varphi}\colon \mathfrak{L}(\mathbb{R}) \to L$ such that the diagram



commutes.

An $f \in \mathsf{C}(S)$ has a *continuous extension* to L if the associated frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to S$ (such that $f = \mathfrak{c}_S \cdot \varphi$) has an extension to L. The sublocale S is then said to be *C-embedded* if every $f \in \mathsf{C}(S)$ has a continuous extension to L.

Theorem 5.1. *Every compact closed sublocale of a completely c-regular frame L is C-embedded in L.*

Proof. Let S be a compact closed sublocale and let $\varphi\colon \mathfrak{L}(\mathbb{R}) \to S$ be a frame homomorphism. Clearly, there exist $p, q \in \mathbb{Q}$ such that $\varphi(p,—) = 1$ and $\varphi(—,q) = 1$ because S is compact and $\bigvee\{(p,—) \mid p \in \mathbb{Q}\} = 1 = \bigvee\{(—,q) \mid q \in \mathbb{Q}\}$ in $\mathfrak{L}(\mathbb{R})$, and thus $-\boldsymbol{n} \le \varphi \le \boldsymbol{n}$ for some natural n. Now, define an antitone $\mathcal{S} = (S_r \mid r \in \mathbb{Q}) \subseteq \mathcal{S}(L)$ as follows:

$$S_r = \begin{cases} 0 & \text{if } r \ge n, \\ \mathfrak{c}(\varphi(—,-r)) & \text{if } -n \le r < n, \\ 1 & \text{if } r < -n. \end{cases}$$

Since each S_r is complemented and $\bigvee_{r\in\mathbb{Q}} S_r = 1 = \bigvee_{r\in\mathbb{Q}} {S_r}^*$, it follows that $\mathcal{S}$ is a scale that generates an $f_1 \in \mathsf{F}(L)$. Let $f = -f_1$. Then

$$f(p,—) = f_1(—,-p) = \bigvee_{s<-p} {S_s}^* = \begin{cases} 0 & \text{if } p \ge n, \\ \bigvee_{r>p} \mathfrak{o}(\varphi(—,r)) & \text{if } -n \le p < n, \\ 1 & \text{if } p < -n, \end{cases} \quad \text{and}$$

$$f(—,q) = f_1(-q,—) = \bigvee_{s>-q} S_s = \begin{cases} 1 & \text{if } q > n \\ \mathfrak{c}(\varphi(—,q)) & \text{if } -n < q \le n, \\ 0 & \text{if } q \le -n. \end{cases}$$

Since S is closed then, for each $-n < q \le n$, $f(—,q) = \mathfrak{c}(\varphi(—,q)) = \mathfrak{c}(\varphi(—,q)) \vee S$ is a closed sublocale of S. Further, since S is compact, by Fact 1.4 we may conclude that $f(—,q)$ is compact. Hence f is upper compact-like.

Moreover, $\mathcal{T} = (T_r \mid r \in \mathbb{Q}) \subseteq \mathcal{S}(L)$ defined by

$$T_r = \begin{cases} 0 & \text{if } r \ge n, \\ \mathfrak{c}(\varphi(r,—)) & \text{if } -n \le r < n, \\ 1 & \text{if } r < -n, \end{cases}$$

is also a scale. The corresponding $g \in \mathsf{F}(L)$ is now given by

$$g(p,—) = T_p \quad \text{and} \quad g(—,q) = \begin{cases} 1 & \text{if } q > n, \\ \bigvee_{r<q} \mathfrak{o}(\varphi(r,—)) & \text{if } -n < q \le n, \\ 0 & \text{if } q \le -n, \end{cases}$$

and it is lower semicontinuous since $g(p,—)$ is closed for every p.

Finally, $\varphi(—,r) \vee \varphi(p,—) = 1$ for each $-n \le p < r < n$ and thus $\mathfrak{o}(\varphi(—,r)) \le \mathfrak{c}(\varphi(—,p))$. Hence

$$f(p,—) = \bigvee_{r>p} \mathfrak{o}(\varphi(—,r)) \le \mathfrak{c}(\varphi(—,p)) = g(p,—),$$

that is, $f \le g$.

It then follows from Theorem 4.5 that there exists $h \in \mathsf{C}(L)$ such that $f \le h \le g$. Consequently, $h(p,\text{—}) = 0 = \mathfrak{c}(\varphi(p,\text{—}))$ for every $p \ge n$, $h(p,\text{—}) = 1 = \mathfrak{c}(\varphi(p,\text{—}))$ for every $p < -n$ and, for each $-n \le p < n$, we have

$$\mathfrak{c}(\varphi(p,\text{—})) = \bigvee_{r>p} \mathfrak{c}(\varphi(r,\text{—})) \le \bigvee_{r>p} \mathfrak{o}(\varphi(\text{—},r)) = f(p,\text{—}) \le h(p,\text{—}) \le g(p,\text{—}) = \mathfrak{c}(\varphi(p,\text{—})).$$

Similarly, $h(\text{—},q) = 1 = \mathfrak{c}(\varphi(\text{—},q))$ for every $q > n$, $h(\text{—},q) = 0 = \mathfrak{c}(\varphi(\text{—},q))$ for every $q \le -n$ and, for each $-n < q \le n$,

$$\mathfrak{c}(\varphi(\text{—},q)) = \bigvee_{r<q} \mathfrak{c}(\varphi(\text{—},r)) \le \bigvee_{r<q} \mathfrak{o}(\varphi(r,\text{—})) = g(\text{—},q) \le h(\text{—},q) \le f(\text{—},q) = \mathfrak{c}(\varphi(\text{—},q)).$$

We then conclude that $h = \mathfrak{c} \cdot \varphi$ and hence $\widetilde{\varphi} = \mathfrak{c}^{-1} \cdot h\colon \mathfrak{L}(\mathbb{R}) \to L$ is the desired extension of φ. □

Of course, when L is fit (or Hausdorff) and completely c-regular, Theorem 5.1 asserts that any compact sublocale of L is C-embedded in L. In particular:

Corollary 5.2. *Every compact sublocale of a completely regular frame L is C-embedded in L.* □

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Department of Mathematics, University of the Basque Country UPV/EHU, Apdo. 644, 48080 Bilbao, Spain
E-mail address: javier.gutierrezgarcia@ehu.es
URL: www.ehu.es/javiergutierrezgarcia/

CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: picado@mat.uc.pt
URL: www.mat.uc.pt/~picado/