Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 17–24

STABILITY PROPERTIES CHARACTERISING n-PERMUTABLE CATEGORIES

PIERRE-ALAIN JACQMIN AND DIANA RODELO

ABSTRACT: The purpose of this paper is two-fold. A first and more concrete aim is to characterise n-permutable categories through certain stability properties of regular epimorphisms. These characterisations allow us to recover the ternary terms and the $(n+1)$-ary terms describing n-permutable varieties of universal algebras.

A second and more abstract aim is to explain two proof techniques, by using the above characterisation as an opportunity to provide explicit new examples of their use:

- an *embedding theorem* for n-permutable categories which allows us to follow the varietal proof to show that an n-permutable category has certain properties;
- the theory of *unconditional exactness properties* which allows us to remove the assumption of the existence of colimits, in particular when we use the *approximate co-operations* approach to show that a regular category is n-permutable.

KEYWORDS: Mal'tsev category, Goursat category, n-permutable category, embedding theorem, unconditional exactness property.
AMS SUBJECT CLASSIFICATION (2000): 08A30, 08B05, 08C05, 18B15, 18B99, 18C99, 18A32.

Introduction

A variety of universal algebras is called *n-permutable*, $n \geqslant 2$, when any pair of congruences R and S on a same algebra n-permutes: $(R,S)_n = (S,R)_n$, where $(R,S)_n$ denotes the composite $RSRS\cdots$ of R and S, n times. This notion determines a sequence of families of varieties, whose first two instances are well known: for $n=2$, we regain 2-permutable varieties [23], better known as *Mal'tsev varieties*; for $n=3$, these are the 3-permutable varieties, sometimes referred to as *Goursat varieties*. The property of n-permutability makes sense in any regular category and was generalised to this categorical context in [7], where *n-permutable categories* were first studied.

Received 15th May 2017.

Financial support from FNRS grant 1.A741.14 is gratefully acknowledged by the first author.

The second author acknowledges partial financial assistance by Centro de Matemática da Universidade de Coimbra—UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

A variety is a Mal'tsev variety precisely when its theory contains a ternary operation p such that the identities $p(x,y,y)=x$ and $p(x,x,y)=y$ hold [23]. 3-permutable varieties are characterised by the existence two ternary operations r and s satisfying the identities $r(x,y,y) = x$, $r(x,x,y) = s(x,y,y)$ and $s(x,x,y) = y$ [13]. Equivalently, they are characterised by the existence of quaternary operations p and q such that the identities $p(x,y,y,z) = x$, $p(x,x,y,y) = q(x,x,y,y)$ and $q(x,y,y,z) = z$ hold [13]. For an n-permutable variety, its theory contains $n-1$ ternary operations or, equivalently, $n-1$ $(n+1)$-ary operations where similar generalised identities hold ([13], see Theorems 1.1 and 1.2 below).

It was shown in [10, 12] that regular Goursat categories can be characterised by certain stability properties for regular epimorphisms. The stability property from [12] (which is recalled here as Proposition 2.2), visualised by means of a cube, allows us to recover the ternary terms r and s which describe 3-permutable varieties. The characterisation from [10], stated through a *Goursat pushout* diagram, allows us to obtain the quaternary varietal terms p and q.

Purpose of the paper. Our first aim with this paper is to extend the above characterisations of Goursat categories from [10, 12] to the general case of n-permutable categories. The idea is to obtain a property involving a diagram which allows us to recover the varietal terms describing n-permutable varieties of universal algebras. We first prove that a certain stability property of regular epimorphisms characterises n-permutable categories for $n \geqslant 3$ (Theorem 3.3); and this allows us to recover the ternary terms (Section 4). Then, we prove a similar characterisation from which we can recover the $(n+1)$-ary terms. For this second result, we have to distinguish four cases, depending on the congruence class modulo 4 of n (Theorems 5.1, 5.3, 5.4 and 5.5).

The proofs we give of these characterisations require two special techniques which were developed in [17, 16, 15]. A second main aim of this paper is to use the above characterisations as an opportunity to give concrete and new examples of where and how such techniques may be applied.

First proof technique: translating varietal proofs. The first technique arose from a wider historical context. Since the 1990's, with the introduction of Mal'tsev [9], Goursat [7] and protomodular [4] categories, many non-abelian categorical-algebraic properties have been studied. Several of those, when considered in a varietal context, led to characterisations given in terms of operations and identities. In some cases, the terms and axioms are quite simple to express: for instance, as recalled above, this happens with 2-permutability of congruences. Eventually, such a translation process leads to varietal proofs which are often easier than the corresponding categorical ones. A question thus arises:

How can one turn a (simple) varietal proof using operations and identities into a categorical proof?

In [19] Z. Janelidze proposed a matrix presentation of some of these categorical conditions in order to study them in a unified way. Such a presentation applies, for instance, to the collections of Mal'tsev, unital [5], strongly unital [5] and subtractive [18] categories. With the more general type of matrices from [20], one can also include n-permutable categories as examples. Note that these are collections of categories whose varietal counterpart has a simple characterisation in terms of operations and identities; their "categorical version" may be described in terms of properties of (internal) relations. For each categorical condition, the matrix is built according to the corresponding varietal operations and identities. At the same time, the varietal information put into the matrix allows us to deduce properties concerning relations, which translates into the categorical condition in question. Thus, the matrix gives a link between the varietal and categorical worlds. In particular, the matrix presentation makes it possible to turn varietal proofs into categorical ones: we write the varietal proof into the appropriate matrix system to get the right (but sometimes unintuitive) relation to which we apply the assumptions on the category. This technique has been explained and used in [22].

In parallel with the development of the matrix approach, D. Bourn and Z. Janelidze showed in [6] that in regular Mal'tsev categories with binary coproducts, the objects can be equipped with an approximate Mal'tsev co-operation. Using this co-operation, many varietal proofs for Mal'tsev varieties can be extended to the context of regular Mal'tsev categories with binary coproducts. This result has been generalised for Janelidze's simple matrix conditions in [21], for n-permutable categories in [24] and for Janelidze's generalised matrix conditions in [17, 15].

On the one hand, the matrix technique is sometimes unintuitive and may give rise to rather long proofs (see Remark 2.4). Moreover, the link between the initial varietal proof and the deduced categorical proof is often hard to see. On the other hand, with the technique of approximate co-operations, the categorical proof obtained is a simple adjustment of the original varietal proof. However, the additional assumption on the existence of binary coproducts is unwelcome. To overcome these flaws, in [14, 17, 15] new embedding theorems have been developed for Janelidze's generalised matrix conditions.

As a consequence of one of these embedding theorems, in an n-permutable context we now only have to produce a proof of theorems about finite limits and regular epimorphisms in a particular essentially algebraic (i.e., locally presentable) category in order to obtain the same result in a categorical context. Moreover, as we shall see, it seems that transforming a varietal proof into a proof for this particular category is always straightforward. In each of our characterising theorems, we will use this embedding theorem to prove one implication of our characterisations.

Second proof technique: removing colimit assumptions. The second technique, from [16, 15], is used in this paper to prove the converse implications. We first show these implications under the additional condition that the category has binary coproducts, so that we can apply the characterisation of n-permutable categories with binary coproducts via approximate Hagemann-Mitschke co-operations [24]. Then, in order to remove this assumption on coproducts, we are going to use the theory of unconditional exactness properties [16, 15]. This is a class of exactness properties, stable under the cofiltered limit completion $\mathcal{C} \hookrightarrow \mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$, where $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})$ is the category of left exact functors from $\mathcal{C}$ to the category Set of sets. We thus have to check that, in a regular context, the characterising properties in our theorems are unconditional exactness properties. To achieve this, we need to check, among other things, that certain categories generated by some finite conditional graphs are finite. We develop an algorithm to do this easily. Once this check is done, we know that if a regular category satisfies the property under consideration, then so does the cocomplete category $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$. Using the first part of the proof, we will be able to conclude that it is n-permutable and, hence, that so is $\mathcal{C}$, since the full embedding $\mathcal{C} \hookrightarrow \mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ preserves finite limits and colimits.

Structure of the paper. The paper is organised as follows. In the first section, we recall the characterisations of n-permutable varieties, the corresponding embedding theorems and prove (recall) some characterisations of n-permutable categories using the calculus of relations (Theorem 1.3). In the second section, we characterise those categories via a stability property for regular epimorphisms, using binary coproducts (Theorem 2.3). In Section 3, we prove the assumption on coproducts may be removed from this theorem using unconditional exactness properties (Theorem 3.3). Section 4 uses this result to extract the ternary operations for n-permutable varieties. Finally, the last section is devoted to the other characterisation of n-permutable categories (Theorems 5.1, 5.3, 5.4 and 5.5) which gives the $(n+1)$-ary operations in the varietal case.

Acknowledgements

The authors would like to warmly thank Tim Van der Linden for his useful comments to improve the introduction of this paper and make the body of the text easier to read.

1. n-permutable categories

A variety of universal algebras has *n-permutable* congruences when any pair of congruences R and S on a same algebra is such that

$$(R,S)_n = (S,R)_n,$$

where $(R,S)_n$ denotes the composite $RSRS\cdots$, of n alternating factors. This property was studied in a categorical context in [7] and led to the notion of n-permutable category, defined as a regular category [1] whose equivalence relations are n-permutable. We recall that 2-permutable categories are usually called regular Mal'tsev categories [8] and 3-permutable categories are also known as Goursat categories [8, 7].

It is well known that a Mal'tsev variety $\mathcal{V}$ of universal algebras is such that its theory admits a ternary operation p satisfying $p(x,y,y) = x$ and $p(x,x,y) = y$ [23]. For the strictly weaker notion of a 3-permutable variety, the corresponding theory admits ternary operations r and s satisfying $r(x,y,y) = x$, $r(x,x,y) = s(x,y,y)$ and $s(x,x,y) = y$ [13]. In general, n-permutable varieties can be characterised by the existence of ternary operations satisfying suitable identities:

Theorem 1.1 (Theorem 2 of [13]). *Let $n \geqslant 2$ and let $\mathcal{V}$ be a variety of universal algebras. The following statements are equivalent:*

(i) *the congruence relations of every algebra of $\mathcal{V}$ are n-permutable;*
(ii) *there exist ternary algebraic operations $w_1, \ldots, w_{n-1}$ of $\mathcal{V}$ for which the identities*

$$\begin{cases} w_1(x,y,y) = x, \\ w_i(x,x,y) = w_{i+1}(x,y,y), \quad \textit{for } i \in \{1,\ldots,n-2\}, \\ w_{n-1}(x,x,y) = y \end{cases}$$

hold.

In order to prove the stability properties characterising n-permutable categories, we will need the embedding theorem from [17, 15]. The essentially algebraic n-permutable category $\mathrm{Mod}(\Gamma_n)$ has as objects S_n-sorted sets $(A_s)_{s\in S_n}$ (for a fixed set S_n of sorts), equipped with, among others, for each $s \in S_n$, an injective operation $\alpha^s \colon s \to (s,0)$, $n-1$ ternary operations $w_1^s, \ldots, w_{n-1}^s \colon s^3 \to (s,0)$ and a partial operation $\pi^s \colon (s,0) \to s$ defined exactly on the image of α^s such that the identities

$$\begin{cases} w_1^s(x,y,y) = \alpha^s(x), \\ w_i^s(x,x,y) = w_{i+1}^s(x,y,y), \quad \text{for } i \in \{1,\ldots,n-2\}, \\ w_{n-1}^s(x,x,y) = \alpha^s(y), \\ \pi^s(\alpha^s(x)) = x \end{cases} \tag{1}$$

hold for each $x, y \in A_s$. Roughly speaking, the embedding theorem then says that to prove a theorem about finite limits and regular epimorphisms in an n-permutable context, it is enough to prove it in $\mathrm{Mod}(\Gamma_n)$, supposing that the given regular epimorphisms are actually S_n-sorted functions which are surjective in each sort.

It is also well-known that n-permutable varieties may also be characterised by the existence of $(n+1)$-ary operations:

Theorem 1.2 (Theorem 1 of [13]). *Let $n \geqslant 2$ and let $\mathcal{V}$ be a variety of universal algebras. The following statements are equivalent:*

(i) *the congruence relations of every algebra of $\mathcal{V}$ are n-permutable;*

(ii) *there exist $(n+1)$-ary algebraic operations $v_0, v_1, \ldots, v_{n-1}, v_n$ of $\mathcal{V}$ for which the identities*

$$\begin{cases} v_0(x_0,\ldots,x_n) = x_0, & \\ v_{i-1}(x_0,x_0,x_2,x_2,\ldots) = v_i(x_0,x_0,x_2,x_2,\ldots), & i \text{ even}, \\ v_{i-1}(x_0,x_1,x_1,x_3,x_3,\ldots) = v_i(x_0,x_1,x_1,x_3,x_3,\ldots), & i \text{ odd}, \\ v_n(x_0,\ldots,x_n) = x_n & \end{cases}$$

hold.

Since v_0 and v_n are just projections, one usually simply refers to the main $n-1$ terms $v_1, \ldots, v_{n-1}$, where $v_1(x_0,x_1,x_1,x_2,x_2,\ldots) = x_0$ replaces the first and third identities, when $i=1$, and $v_{n-1}(\ldots,x_{j-1},x_{j-1},x_j) = x_j$ replaces the last and the second or third identities, when $i=n$ even or odd, respectively.

Analogously to the ternary operation case, we also have an embedding theorem for n-permutable categories using the $(n+1)$-ary operations [17, 15]. This time, we use the essentially algebraic n-permutable category $\mathrm{Mod}(\Gamma'_n)$. It has as objects S'_n-sorted sets $(A_s)_{s\in S'_n}$ (for a fixed set S'_n of sorts), equipped with, among others, for each $s \in S'_n$, an injective operation $\alpha^s\colon s \to (s,0)$, $(n+1)$-ary operations $v^s_0,\ldots,v^s_n\colon s^{n+1} \to (s,0)$ and a partial operation $\pi^s\colon (s,0) \to s$ defined exactly on the image of α^s such that the identities

$$\begin{cases} v^s_0(x_0,\ldots,x_n) = \alpha^s(x_0), & \\ v^s_{i-1}(x_0,x_0,x_2,x_2,\ldots) = v^s_i(x_0,x_0,x_2,x_2,\ldots), & i \text{ even}, \\ v^s_{i-1}(x_0,x_1,x_1,x_3,x_3,\ldots) = v^s_i(x_0,x_1,x_1,x_3,x_3,\ldots), & i \text{ odd}, \\ v^s_n(x_0,\ldots,x_n) = \alpha^s(x_n), & \\ \pi^s(\alpha^s(x)) = x & \end{cases} \tag{2}$$

hold for $x, x_0, x_1, \ldots, x_n \in A_s$. Roughly speaking, the embedding theorem then says that to prove a theorem about finite limits and regular epimorphisms in an n-permutable context, it is enough to prove it in $\mathrm{Mod}(\Gamma'_n)$, supposing that the given regular epimorphisms are actually S'_n-sorted functions which are surjective in each sort.

We conclude this section with a result that gathers some known characterisations of n-permutable categories as well as a new one, given by condition (iv), which will be useful in the next section.

Theorem 1.3. *Let $n \geqslant 2$ and let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

(i) $\mathcal{C}$ *is an n-permutable category;*
(ii) $(P, P^\circ)_{n+1} \leqslant (P, P^\circ)_{n-1}$, *for any relation* P*;*
(iii) $E^\circ \leqslant E^{n-1}$, *for any reflexive relation* E*;*
(iv) $(1_X \wedge T)T^\circ(1_X \wedge T) \leqslant T^{n-1}$, *for any relation* T *on an object* X*.*

Proof: (i) $\Leftrightarrow$ (ii). By Theorem 3.5 of [7].
(i) $\Leftrightarrow$ (iii). By Theorem 1 in [22].
(i) $\Leftrightarrow$ (iv). This type of equivalence was mentioned at the end of [22]. Note that condition (iv) is a stronger version of condition (iii), so we only have to prove that (i) $\Rightarrow$ (iv); in fact, we shall prove that (ii) $\Rightarrow$ (iv). In a regular context, it suffices to give a proof in set-theoretical terms due to Barr's embedding theorem [2]. Suppose that xTx, yTx and yTy. We want to prove that $xT^{n-1}y$, i.e., that $xT\mu_1T\mu_2\cdots\mu_{n-3}T\mu_{n-2}Ty$, for some $\mu_1, \ldots, \mu_{n-2} \in X$.
Case $n = 2k-1$ is odd: We define a relation P from X^k to X^k by: $(a_1, a_2, \ldots, a_{k-1}, a_k)P(b_1, b_2, \ldots, b_{k-1}, b_k)$ if and only if

$$\begin{cases} a_iTb_i, & 1 \leqslant i \leqslant k \\ b_jTa_{j+1}, & 1 \leqslant j \leqslant k-1. \end{cases}$$

From

$$\begin{array}{rcl} (x, \ldots, x) & P & (x, \ldots, x) \\ (y, x, \ldots, x) & P & (x, \ldots, x) \\ (y, x, \ldots, x) & P & (y, x, \ldots, x) \\ (y, y, x, \ldots, x) & P & (y, x, \ldots, x) \\ & \vdots & \\ (y, \ldots y, x) & P & (y, \ldots, y, x) \\ (y, \ldots, y) & P & (y, \ldots, y, x), \end{array}$$

we get $(x, \ldots, x)(P, P^\circ)_{2k=n+1}(y, \ldots, y)$. By assumption, we conclude that $(x, \ldots, x)(P, P^\circ)_{n-1}(y, \ldots, y)$, i.e.,

$$\begin{array}{rcl} (x, \ldots, x) & P & (\nu_{11}, \ldots, \nu_{1k}) \\ (\nu_{21}, \ldots, \nu_{2k}) & P & (\nu_{11}, \ldots, \nu_{1k}) \\ (\nu_{21}, \ldots, \nu_{2k}) & P & (\nu_{31}, \ldots, \nu_{3k}) \\ (\nu_{41}, \ldots, \nu_{4k}) & P & (\nu_{31}, \ldots, \nu_{3k}) \\ & \vdots & \\ (\nu_{n-3,1}, \ldots, \nu_{n-3,k}) & P & (\nu_{n-2,1}, \ldots, \nu_{n-2,k}) \\ (y, \ldots, y) & P & (\nu_{n-2,1}, \ldots, \nu_{n-2,k}), \end{array}$$

for some $(\nu_{i1}, \ldots, \nu_{ik}) \in X^k$, $1 \leqslant i \leqslant n-2$. From these relations, we get

$$xT\nu_{11},\ \nu_{11}T\nu_{22},\ \nu_{22}T\nu_{32},\ \nu_{32}T\nu_{43},\ \ldots,\ \nu_{n-3,k-1}T\nu_{n-2,k-1} \quad \text{and} \quad \nu_{n-2,k-1}Ty,$$

which implies that $xT^{n-1}y$.

Case $n = 2k$ is even: We define a relation P similarly as in the odd case. We have the same relations on P as above, with an extra relation

$$(y, \ldots, y) \quad P \quad (y, \ldots, y),$$

so that $(x, \ldots, x)(P, P^{\circ})_{2k+1=n+1}(y, \ldots, y)$. By assumption, we conclude that $(x, \ldots, x)(P, P^{\circ})_{n-1}(y, \ldots, y)$. We then get similar relations for P as in the odd case, but with $(y, \ldots, y)$ ending on the right in the last relation

$$\begin{array}{rcl}
(x, \ldots, x) & P & (\nu_{11}, \ldots, \nu_{1k}) \\
(\nu_{21}, \ldots, \nu_{2k}) & P & (\nu_{11}, \ldots, \nu_{1k}) \\
(\nu_{21}, \ldots, \nu_{2k}) & P & (\nu_{31}, \ldots, \nu_{3k}) \\
(\nu_{41}, \ldots, \nu_{4k}) & P & (\nu_{31}, \ldots, \nu_{3k}) \\
 & \vdots & \\
(\nu_{n-2,1}, \ldots, \nu_{n-2,k}) & P & (\nu_{n-3,1}, \ldots, \nu_{n-3,k}) \\
(\nu_{n-2,1}, \ldots, \nu_{n-2,k}) & P & (y, \ldots, y).
\end{array}$$

From these relations, we get

$$xT\nu_{11},\ \nu_{11}T\nu_{22},\ \nu_{22}T\nu_{32},\ \nu_{32}T\nu_{43},\ \ldots,\ \nu_{n-3,k-1}T\nu_{n-2,k} \quad \text{and} \quad \nu_{n-2,k}Ty,$$

which implies that $xT^{n-1}y$. ∎

Remark 1.4. The proof of Theorem 1.3 (ii) $\Rightarrow$ (iv) followed the matrix technique mentioned in the introduction. A varietal proof of this property was put into an appropriate matrix, from which we deduced the relation P. Although P caries the precise information we need, the varietal proof behind its construction is not visible at first glance. In the next sections, we rather give the main proofs using embedding theorems for n-permutable categories, which are mere adjustments of the varietal proofs.

2. Stability property for regular epimorphisms

In this section we are going to study a certain type of stability property for regular epimorphisms (in a regular category) which characterises n-permutable categories. The idea is to obtain, for arbitrary $n \geqslant 3$, similar properties as those known for the Mal'tsev and the Goursat cases.

For Mal'tsev categories, the stability property involves a commutative (upward and downward) cube where the left and right faces are pullbacks of split epimorphisms along arbitrary morphisms

$$\begin{tikzcd}
X \times_Y Z \arrow[rr, "\lambda"] \arrow[dd, shift right] \arrow[dr] & & U \times_V W \arrow[dd, dotted, shift right] \arrow[dr] & \\
& Z \arrow[rr, "\delta"] \arrow[dd, shift right, "g"'] & & W \arrow[dd, shift right, "k"'] \\
X \arrow[rr, dotted, "\gamma"] \arrow[dr, "f"'] \arrow[uu, shift right] & & U \arrow[dr, dotted, "h"] \arrow[uu, dotted, shift right] & \\
& Y \arrow[rr, "\beta"'] \arrow[uu, shift right] & & V. \arrow[uu, shift right]
\end{tikzcd} \tag{3}$$

Proposition 2.1. [11] *Let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

(i) *$\mathcal{C}$ is a Mal'tsev category;*
(ii) *for any diagram (3), if γ, β and δ are regular epimorphisms, then the comparison morphism λ is also a regular epimorphism.*

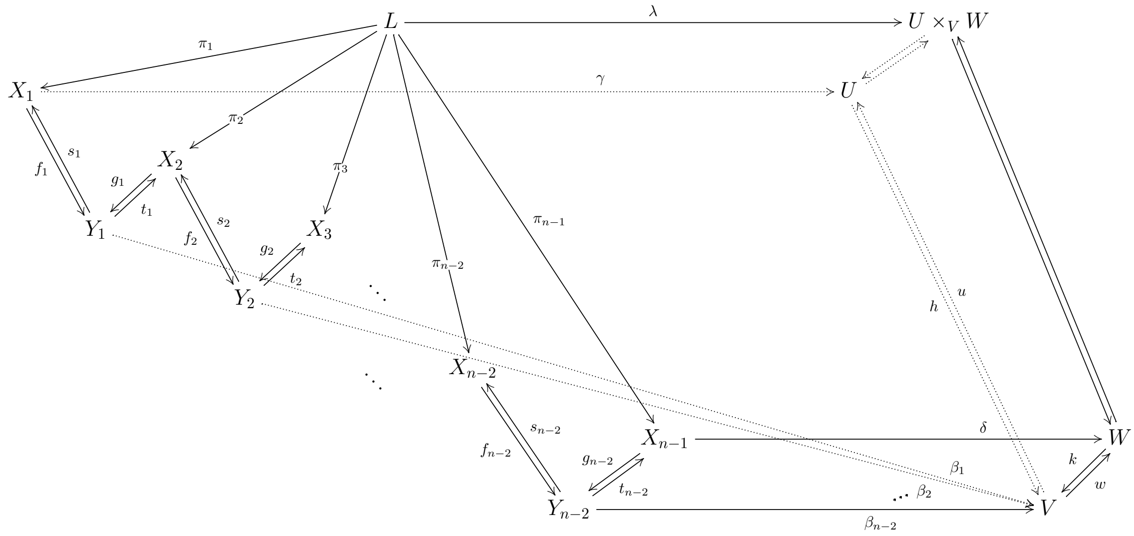
For the weaker notion of a Goursat category, the stability property now involves a cube where the left and right faces are pullbacks of split epimorphisms and where β, γ, δ and λ commute with the split epimorphisms and with their sections

$$\begin{tikzcd}
X \times_Y Z \arrow[rr, "\lambda"] \arrow[dd, shift right] \arrow[dr, shift left] & & U \times_V W \arrow[dd, dotted, shift right] \arrow[dr, shift left] & \\
& Z \arrow[rr, "\delta"] \arrow[dd, shift right, "g"'] \arrow[ul, shift left] & & W \arrow[dd, shift right, "k"'] \arrow[ul, shift left] \\
X \arrow[rr, dotted, "\gamma"] \arrow[dr, shift left, "f"'] \arrow[uu, shift right] & & U \arrow[dr, dotted, shift left, "h"] \arrow[uu, dotted, shift right] & \\
& Y \arrow[rr, "\beta"'] \arrow[uu, shift right] \arrow[ul, shift left] & & V. \arrow[uu, shift right] \arrow[ul, dotted, shift left]
\end{tikzcd} \tag{4}$$

Proposition 2.2. [12] *Let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

(i) *$\mathcal{C}$ is a Goursat category;*
(ii) *for any diagram (4), if γ, β and δ are regular epimorphisms, then the comparison morphism λ is also a regular epimorphism.*

We now give a stability property for regular epimorphisms which shall characterise n-permutable categories for $n \geqslant 3$. Consider the following diagram



$$(5)$$

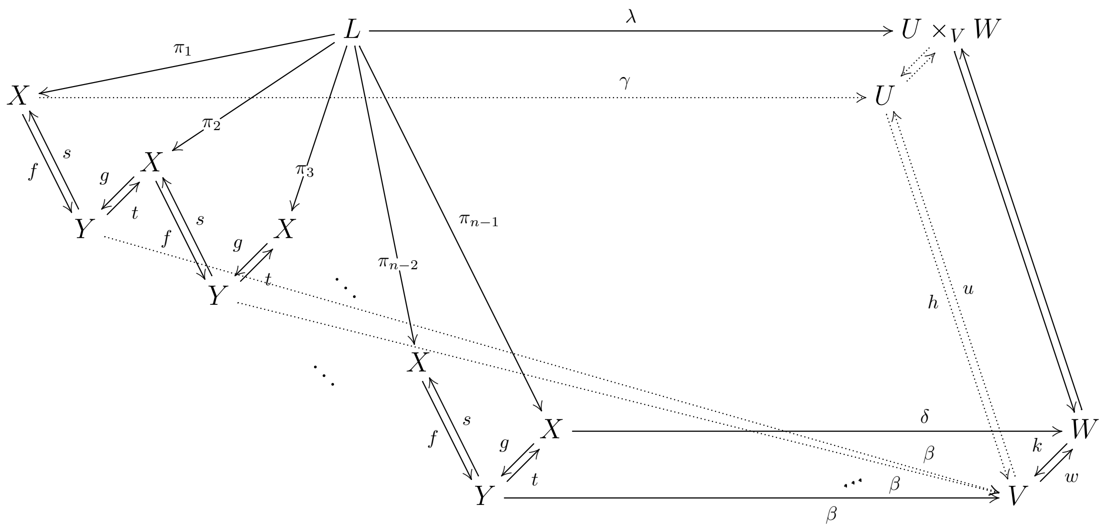
where the left side is a limit of split epimorphisms and the right quadrangle is a pullback of split epimorphisms. We assume that the arrows γ, β_1 and δ, β_{n-2} commute with the corresponding split epimorphisms as well as with their splittings. We also assume that β_j is a split epimorphism, split by i_j, for $2 \leqslant j \leqslant n-2$, and are such that the following squares commute (also with the vertical splittings),

$$\begin{tikzcd}
X_j \arrow[r, "f_j"] \arrow[d, shift right, "g_{j-1}"'] & Y_j \arrow[d, shift right, "\beta_j"'] \\
Y_{j-1} \arrow[r, "\beta_{j-1}"'] \arrow[u, shift right, "t_{j-1}"'] & V \arrow[u, shift right, "i_j"']
\end{tikzcd}$$

i.e.,

$$\begin{cases} \beta_j f_j = \beta_{j-1} g_{j-1} \\ f_j t_{j-1} = i_j \beta_{j-1} \end{cases}, \qquad 2 \leqslant j \leqslant n-2. \tag{6}$$

We will also consider the above diagram in the case when $f_1 = \cdots = f_{n-2} = f$, $g_1 = \cdots = g_{n-2} = g$, $\beta_1 = \cdots = \beta_{n-2} = \beta$



(7)

and β is endowed with two splittings i and i' so that now

$$\begin{cases} \beta f = \beta g \\ ft = i\beta \\ gs = i'\beta. \end{cases} \tag{8}$$

Theorem 2.3. *Let $n \geqslant 3$ and let $\mathcal{C}$ be a regular category with binary coproducts. The following statements are equivalent:*

(i) *$\mathcal{C}$ is an n-permutable category;*
(ii) *if γ and δ in diagram* (5) *are regular epimorphisms, then the comparison morphism λ is also a regular epimorphism;*
(iii) *if γ and δ in diagram* (7) *are regular epimorphisms, then the comparison morphism λ is also a regular epimorphism.*

Proof: (i) $\Rightarrow$ (ii) Using the embedding theorem for n-permutable categories, it is enough to prove this implication in $\mathrm{Mod}(\Gamma_n)$. So let $s \in S_n$ and $(a, b) \in (U \times_V W)_s$. Since regular epimorphisms are supposed to be surjective in each sort, there exist $x_1 \in (X_1)_s$ and $x_{n-1} \in (X_{n-1})_s$ such that $\gamma(x_1) = a$ and $\delta(x_{n-1}) = b$. Since $h(a) = k(b)$, we get

$$\beta_1 f_1(x_1) = h\gamma(x_1) = k\delta(x_{n-1}) = \beta_{n-2}g_{n-2}(x_{n-1}). \tag{9}$$

Using the identities of (6) repeatedly, for $1 \leqslant j \leqslant n-2$ we see that

$$\beta_j g_j s_{j+1} g_{j+1} \cdots s_{n-2} g_{n-2}(x_{n-1}) = \beta_{n-2} g_{n-2}(x_{n-1}) \tag{10}$$

(for $j = n-2$, just write $\beta_{n-2}g_{n-2}(x_{n-1})$ on the left) and

$$\beta_j f_j t_{j-1} f_{j-1} \cdots t_1 f_1(x_1) = \beta_1 f_1(x_1) \tag{11}$$

(for $j = 1$, just write $\beta_1 f_1(x_1)$ on the left). We then deduce

$$\gamma s_1 g_1 \cdots s_{n-2} g_{n-2}(x_{n-1}) \overset{(10)(9)}{=} \gamma s_1 f_1(x_1) \tag{12}$$

and

$$\delta t_{n-2} f_{n-2} \cdots t_1 f_1(x_1) \overset{(11)(9)}{=} \delta t_{n-2} g_{n-2}(x_{n-1}). \tag{13}$$

For $2 \leqslant j \leqslant n-2$, we also find

$$\begin{aligned} f_j t_{j-1} g_{j-1} s_j g_j \cdots s_{n-2} g_{n-2}(x_{n-1}) &\overset{(6)}{=} i_j \beta_{j-1} g_{j-1} s_j g_j \cdots s_{n-2} g_{n-2}(x_{n-1}) \\ &\overset{(10)}{=} i_j \beta_{n-2} g_{n-2}(x_{n-1}) \\ &\overset{(9)}{=} i_j \beta_1 f_1(x_1) \\ &\overset{(11)}{=} i_j \beta_{j-1} f_{j-1} t_{j-2} f_{j-2} \cdots t_1 f_1(x_1) \\ &\overset{(6)}{=} f_j t_{j-1} f_{j-1} t_{j-2} f_{j-2} \cdots t_1 f_1(x_1). \end{aligned} \tag{14}$$

Now, we define

$$\overline{x_1} = w_1^s(x_1, s_1 f_1(x_1), s_1 g_1 \cdots s_{n-2} g_{n-2}(x_{n-1})) \in (X_1)_{(s,0)}$$

and for $2 \leqslant j \leqslant n-1$,

$$\begin{aligned} \overline{x_j} = w_j^s(t_{j-1} f_{j-1} \cdots t_1 f_1(x_1), t_{j-1} g_{j-1}(s_j g_j \cdots s_{n-2} g_{n-2})(x_{n-1}), \\ s_j g_j \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ \in (X_j)_{(s,0)}. \end{aligned}$$

We can compute the following identities:

$$\begin{aligned} f_1(\overline{x_1}) &= w_1^s(f_1(x_1), f_1 s_1 f_1(x_1), f_1 s_1 g_1 \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ &= w_1^s(f_1(x_1), f_1(x_1), g_1 s_2 \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ &\overset{(1)}{=} w_2^s(f_1(x_1), g_1 s_2 \cdots s_{n-2} g_{n-2}(x_{n-1}), g_1 s_2 \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ &= w_2^s(g_1 t_1 f_1(x_1), g_1 t_1 g_1 s_2 \cdots s_{n-2} g_{n-2}(x_{n-1}), g_1 s_2 \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ &= g_1(\overline{x_2}) \end{aligned}$$

and for each $2 \leqslant j \leqslant n-2$,

$$\begin{array}{rcl} f_j(\overline{x_j}) & = & w^s_j(f_j t_{j-1} f_{j-1} \cdots t_1 f_1(x_1), f_j t_{j-1} g_{j-1} s_j g_j \cdots s_{n-2} g_{n-2}(x_{n-1}), \\ & & \hfill f_j s_j g_j \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ & \overset{(14)}{=} & w^s_j(f_j t_{j-1} f_{j-1} \cdots t_1 f_1(x_1), f_j t_{j-1} f_{j-1} \cdots t_1 f_1(x_1), \\ & & \hfill g_j \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ & \overset{(1)}{=} & w^s_{j+1}(f_j t_{j-1} f_{j-1} \cdots t_1 f_1(x_1), g_j \cdots s_{n-2} g_{n-2}(x_{n-1}), \\ & & \hfill g_j \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ & = & w^s_{j+1}(g_j t_j f_j \cdots t_1 f_1(x_1), g_j s_{j+1} \cdots s_{n-2} g_{n-2}(x_{n-1}), \\ & & \hfill g_j s_{j+1} \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ & = & g_j(\overline{x_{j+1}}). \end{array}$$

This exactly means that $(\overline{x_1}, \ldots, \overline{x_{n-1}}) \in L_{(s,0)}$. Moreover, the equalities

$$\begin{array}{rcl} \gamma(\overline{x_1}) & = & w^s_1(\gamma(x_1), \gamma s_1 f_1(x_1), \gamma s_1 g_1 \cdots s_{n-2} g_{n-2}(x_{n-1})) \\ & \overset{(12)}{=} & w^s_1(\gamma(x_1), \gamma s_1 f_1(x_1), \gamma s_1 f_1(x_1)) \\ & \overset{(1)}{=} & \alpha^s(\gamma(x_1)) \\ & = & \alpha^s(a) \end{array}$$

and

$$\begin{array}{rcl} \delta(\overline{x_{n-1}}) & = & w^s_{n-1}(\delta t_{n-2} f_{n-2} \cdots t_1 f_1(x_1), \delta t_{n-2} g_{n-2}(x_{n-1}), \delta(x_{n-1})) \\ & \overset{(13)}{=} & w^s_{n-1}(\delta t_{n-2} g_{n-2}(x_{n-1}), \delta t_{n-2} g_{n-2}(x_{n-1}), \delta(x_{n-1})) \\ & \overset{(1)}{=} & \alpha^s(\delta(x_{n-1})) \\ & = & \alpha^s(b) \end{array}$$

hold. This implies that

$$\lambda(\overline{x_1}, \ldots, \overline{x_{n-1}}) = (\alpha^s(a), \alpha^s(b)) = \alpha^s(a, b)$$

and

$$(a, b) \overset{(1)}{=} \pi^s(\alpha^s(a, b)) = \pi^s(\lambda(\overline{x_1}, \ldots, \overline{x_{n-1}})) \in \mathrm{Im}(\lambda)_s$$

which proves that λ is a regular epimorphism.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) By Theorem 4 from [24], we must show that, for any object X in $\mathcal{C}$, the morphism ξ_X in the dotted limit of the outer diagram of solid arrows

$$\begin{tikzcd}
 & & & X \arrow[llld, "\iota_1"'] \arrow[rrrd, "\iota_2"] & & \\
2X & & & & & 2X \\
 & & & A(X) \arrow[uu, dotted, "\xi_X"'] \arrow[dlll, dotted, "(w_1)_X"'] \arrow[ddll, dotted, "(w_2)_X"] \arrow[dddl, dotted, "(w_3)_X"] \arrow[ddr, dotted, "(w_{n-2})_X"'] \arrow[drr, dotted, "(w_{n-1})_X"] & & \\
3X \arrow[uu, "1_X+\nabla_X"] \arrow[d, "\nabla_X+1_X"'] & & & & & 3X \arrow[uu, "\nabla_X+1_X"'] \arrow[d, "1_X+\nabla_X"] \\
2X & 3X \arrow[l, "1_X+\nabla_X"] \arrow[d] & & \cdots & 3X \arrow[r, "\nabla_X+1_X"'] & 2X \\
 & 2X & 3X \arrow[l] & & &
\end{tikzcd} \tag{15}$$

is a regular epimorphism.

We consider diagram (7), where $f = \delta = \nabla_X + 1_X$, $s = (\iota_2\ \iota_3)$, $g = \gamma = 1_X + \nabla_X$, $t = (\iota_1\ \iota_2)$, $\beta = k = h = \nabla_X$, $w = i = \iota_1$ and $u = i' = \iota_2$. By assumption, λ is a regular epimorphism.

We now consider the induced morphism j

$$\begin{tikzcd}
X \arrow[dr, dotted, "j"] \arrow[drr, bend left, "\iota_2"] \arrow[ddr, bend right, "\iota_1"'] & & \\
 & \mathrm{Eq}(\nabla_X) \arrow[r, "\rho_2"] \arrow[d, "\rho_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & 2X \arrow[d, "\nabla_X"] \\
 & 2X \arrow[r, "\nabla_X"'] & X
\end{tikzcd}$$

and its pullback along the regular epimorphism λ

$$\begin{tikzcd}
Q \arrow[r, two heads, "\lambda'"] \arrow[d, "j'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "j"] \\
L \arrow[r, two heads, "\lambda"'] & \mathrm{Eq}(\nabla_X).
\end{tikzcd}$$

We get a cone for the outer diagram in (15):

$$\begin{tikzcd}
 & & & X \arrow[dlll, "\iota_1"'] \arrow[drrr, "\iota_2"] & & & \\
2X & & & & & & 2X \\
 & & & Q \arrow[uu, two heads, "\lambda'"'] \arrow[ddlll, "\pi_1 j'"'] \arrow[dddll, "\pi_2 j'"] \arrow[ddddl, "\pi_3 j'"] \arrow[ddrrr, "\pi_{n-1} j'"] \arrow[dddrr, "\pi_{n-2} j'"'] & & & \\
 & & & & & & \\
3X \arrow[uuu, "1_X+\nabla_X"] \arrow[d, "\nabla_X+1_X"'] & & & & & & 3X \arrow[uuu, "\nabla_X+1_X"'] \arrow[d, "1_X+\nabla_X"] \\
2X & 3X \arrow[l, "1_X+\nabla_X"] \arrow[d] & & & & 3X \arrow[r, "\nabla_X+1_X"'] & 2X \\
 & 2X & 3X \arrow[l] & & \cdots & &
\end{tikzcd}$$

As a consequence, there exists a unique $\tau\colon Q \to A(X)$ such that, in particular, $\xi_X \tau = \lambda'$. Finally, ξ_X is a regular epimorphism since so is λ'. ∎

Remark 2.4. Instead of using the embedding theorem for n-permutable categories to prove (i) ⇒ (ii) of Theorem 2.3, one could have used the matrix presentation from [20] and Barr's embedding theorem for regular categories [2]. Then, instead of defining the elements $\overline{x_1}, \dots, \overline{x_{n-1}}$, we could have considered a relation T defined on $Y_1 \times \cdots \times Y_{n-2} \times U \times W$ by

$$(\varphi_1, \dots, \varphi_{n-2}, a_1, b_1) T (\psi_1, \dots, \psi_{n-2}, a_2, b_2)$$

if and only if there exists $\chi_i \in X_i$, for each $1 \leqslant i \leqslant n-1$, such that

$$\begin{array}{ll} g_j(\chi_{j+1}) = \varphi_j, & \gamma(\chi_1) = a_1, \\ f_j(\chi_j) = \psi_j, & \delta(\chi_{n-1}) = b_2, \end{array}$$

for all $1 \leqslant j \leqslant n-2$.

For

$$c = (f_1(x_1), f_2 t_1 f_1(x_1), \dots, f_{n-2} t_{n-3} f_{n-3} \cdots t_1 f_1(x_1), \gamma(x_1), \delta t_{n-2} g_{n-2}(x_{n-1}))$$

and

$$d = (g_1 s_2 g_2 \cdots s_{n-2} g_{n-2}(x_{n-1}), \dots, g_{n-3} s_{n-2} g_{n-2}(x_{n-1}), g_{n-2}(x_{n-1}), \\ \gamma s_1 f_1(x_1), \delta(x_{n-1}))$$

we then have:

cTc, by setting $\chi_1 = x_1$ and $\chi_j = t_{j-1}f_{j-1}\cdots t_1 f_1(x_1)$, $2 \leqslant j \leqslant n-1$. Note that $\delta(\chi_{n-1}) \overset{(13)}{=} \delta t_{n-2}g_{n-2}(x_{n-1})$.

dTc, by setting $\chi_1 = s_1 f_1(x_1)$ and $\chi_j = t_{j-1}g_{j-1}s_j g_j \cdots s_{n-2}g_{n-2}(x_{n-1})$, $2 \leqslant j \leqslant n-1$ (here $\chi_{n-1} = t_{n-2}g_{n-2}(x_{n-1})$). We have $f_j(\chi_j) \overset{(14)}{=} f_j t_{j-1} f_{j-1}\cdots t_1 f_1(x_1)$ for each $2 \leqslant j \leqslant n-2$.

dTd, by setting $\chi_j = s_j g_j \cdots s_{n-2}g_{n-2}(x_{n-1})$, $1 \leqslant j \leqslant n-2$, and $\chi_{n-1} = x_{n-1}$. Note that $\gamma(\chi_1) \overset{(12)}{=} \gamma s_1 f_1(x_1)$.

From

$$c(1_{Y_1\times\cdots\times Y_{n-2}\times U\times W} \wedge T)T^{\circ}(1_{Y_1\times\cdots\times Y_{n-2}\times U\times W} \wedge T)d$$

and Theorem 1.3, we obtain $cT^{n-1}d$, i.e.,

$$\begin{aligned}
&cT(\xi_{11},\ldots,\xi_{1,n-2},a_1,b_1)\,T\,(\xi_{21},\ldots,\xi_{2,n-2},a_2,b_2)\\
&\cdots\\
&(\xi_{n-3,1},\ldots,\xi_{n-3,n-2},a_{n-3},b_{n-3})\,T\,(\xi_{n-2,1},\ldots,\xi_{n-2,n-2},a_{n-2},b_{n-2})Td,
\end{aligned}$$

for some $\xi_{i,j} \in Y_i$, $1 \leqslant i,j \leqslant n-2$, $a_1,\ldots,a_{n-2} \in U$ and $b_1,\ldots,b_{n-2} \in W$. We then get

$\exists \chi_{11} \in X_1,\ldots,\chi_{1,n-1} \in X_{n-1}$ such that

$$\begin{cases} \cdots \\ f_1(\chi_{11}) = \xi_{11},\ldots, \quad \gamma(\chi_{11}) = a = \gamma(x_1), \end{cases}$$

$\exists \chi_{21} \in X_1, \chi_{22} \in X_2,\ldots,\chi_{2,n-1} \in X_{n-1}$ such that

$$\begin{cases} g_1(\chi_{22}) = \xi_{11},\ldots \\ \ldots, f_2(\chi_{22}) = \xi_{22},\ldots \end{cases}$$

$\cdots$

$\exists \chi_{n-2,1} \in X_1,\ldots,\chi_{n-2,n-2} \in X_{n-2}, \chi_{n-2,n-1} \in X_{n-1}$ such that

$$\begin{cases} \ldots, g_{n-3}(\chi_{n-2,n-2}) = \xi_{n-3,n-3},\ldots \\ \ldots, f_{n-2}(\chi_{n-2,n-2}) = \xi_{n-2,n-2},\ldots \end{cases}$$

$\exists \chi_{n-1,1} \in X_1,\ldots,\chi_{n-1,n-1} \in X_{n-1}$ such that

$$\begin{cases} \ldots, g_{n-2}(\chi_{n-1,n-1}) = \xi_{n-2,n-2}, \quad \delta(\chi_{n-1,n-1}) = b = \delta(x_{n-1}). \\ \cdots \end{cases}$$

It now easily follows that $(\chi_{11},\ldots,\chi_{n-1,n-1}) \in L$ is such that

$$\lambda(\chi_{11},\ldots,\chi_{n-1,n-1}) = (a,b),$$

thus λ is a regular epimorphism.

3. Avoiding coproducts

The statement of Theorem 2.3 does not use the hypothesis on the existence of coproducts, while its proof does. The natural question is: Is this assumption really needed? To give a negative answer to this question, we will use the following result.

Theorem 3.1. [16, 15] *Let P be an unconditional exactness property. If a small finitely complete category $\mathcal{C}$ satisfies P, then* $\mathrm{Lex}(\mathcal{C},\mathrm{Set})^{\mathrm{op}}$ *also satisfies P.*

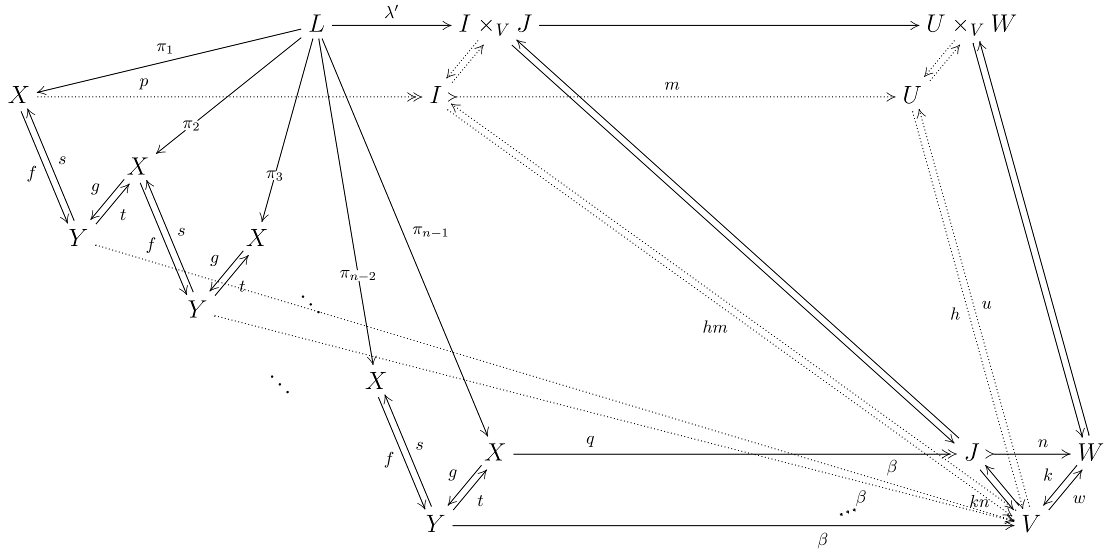
Here, $\mathrm{Lex}(\mathcal{C},\mathrm{Set})$ denotes the complete and cocomplete category of finite limit preserving functors from $\mathcal{C}$ to Set. We recall that the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Lex}(\mathcal{C},\mathrm{Set})^{\mathrm{op}}$ is full and faithful and preserves finite limits and colimits. Roughly speaking, an unconditional exactness property is a property of the form: for any diagram of a fixed finite shape, if we build some specified finite (co)limits, then some specified morphism is an isomorphism. For more details, see [16, 15]. Regularity is such an example. Let us prove that the property

$$\text{regular} + 2.3.\text{(iii)}$$

is also of this form.

Firstly, let us remove the assumption 'if γ and δ are regular epimorphisms' in 2.3.(iii) (since the property has to be 'unconditional'). This is easy since in a regular context, 2.3.(iii) is equivalent to the following:

(iii') For any diagram (7) in $\mathcal{C}$, if mp and nq are the (regular epimorphism, monomorphism) factorisations of γ and δ respectively, then the factorisation λ' of $p\pi_1$ and $q\pi_{n-1}$ through the pullback of hm and kn is a regular epimorphism.



(16)

Therefore, it remains to prove that the shape of the diagram we started with to build (7) is a finite category. In order to do it, we consider the adjunction

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{CondGraph} \arrow[l, bend right, "\mathrm{Path}"'] \end{tikzcd}$$

where CondGraph is the category of small conditional graphs (see Section 5.1 in [3]). If $\mathcal{G}$ is a conditional graph and $X \in \mathrm{ob}\,\mathcal{G}$, we denote by $\mathcal{G}\backslash X$ the conditional graph constructed from $\mathcal{G}$ by removing X and all arrows from or to X. The conditions of $\mathcal{G}\backslash X$ are all pairs of parallel paths giving rise to the same arrow in $\mathrm{Path}(\mathcal{G})$. Moreover, if $X_1, \dots, X_n$ are objects of $\mathcal{G}$, $\mathcal{G}\backslash\{X_1, \dots, X_n\}$ is recursively defined as $(\mathcal{G}\backslash\{X_1, \dots, X_{n-1}\})\backslash X_n$, with $\mathcal{G}\backslash\{X_1\} = \mathcal{G}\backslash X_1$.

Lemma 3.2. *Let $\mathcal{G}$ be a finite conditional graph and X one of its objects. Suppose that for any pair of arrows $f\colon Y \to X$ and $g\colon X \to Z$ in $\mathcal{G}$, there exists a path $(h_1, \dots, h_n)\colon Y \to Z$ in $\mathcal{G}\backslash X$ such that the composites gf and $h_n \cdots h_1$ are equal in* $\mathrm{Path}(\mathcal{G})$. *Then* $\mathrm{Path}(\mathcal{G})$ *is finite if and only if* $\mathrm{Path}(\mathcal{G}\backslash X)$ *is finite.*

Proof: Firstly, we can suppose without loss of generality that all conditions of $\mathcal{G}\backslash X$ are in $\mathcal{G}$. Indeed, we can add them to the ones of $\mathcal{G}$ without changing $\mathrm{Path}(\mathcal{G})$.

The inclusion morphism $\mathcal{G}\backslash X \hookrightarrow \mathcal{G}$ gives rise to a faithful functor

$$\mathrm{Path}(\mathcal{G}\backslash X) \to \mathrm{Path}(\mathcal{G})$$

which shows the ‘only if’ part.

Let us now prove the 'if part'. If $\mathrm{Path}(\mathcal{G}\backslash X)$ is finite, there exists an $N \in \mathbb{N}$ such that any morphism of $\mathrm{Path}(\mathcal{G}\backslash X)$ can be represented by a path of at most N arrows of $\mathcal{G}\backslash X$.

Consider a path $(f_1, \ldots, f_n)$ in $\mathcal{G}$. By the assumption on X, we know that this path is equal in $\mathrm{Path}(\mathcal{G})$ to a path $(h_1, \ldots, h_m)$ where $h_2, \ldots, h_{m-1}$ are in $\mathcal{G}\backslash X$. Thus we can suppose $m - 2 \leqslant N$ which proves that $\mathrm{Path}(\mathcal{G})$ is finite. ■

This lemma gives us an easy way to prove that some category generated by a finite conditional graph is finite. We are going to apply it here with the following conditional graph $\mathcal{G}$

$$\begin{tikzcd}
& X \arrow[r, bend left, "f"] \arrow[r, shift left, "g"] \arrow[d, "\delta"'] \arrow[ddl, bend right, "\gamma"'] & Y \arrow[l, shift left, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
& W \arrow[r, shift left, "k"] & V \arrow[l, shift left, "w"] \arrow[u, "i"] \arrow[u, bend right, "i'"'] \arrow[dll, shift left, "u"] \\
U \arrow[urr, bend right, "h"] & &
\end{tikzcd}$$

where the conditions are

$$fs = gt = 1_Y,\ \beta i = \beta i' = kw = hu = 1_V,\ \beta g = \beta f = h\gamma = k\delta,$$
$$\gamma s = u\beta,\ \delta t = w\beta,\ ft = i\beta \text{ and } gs = i'\beta.$$

Notice that diagram (16) is build up from $\mathrm{Path}(\mathcal{G})$ by adding some finite (co)limits to it. Due to the equalities $h\gamma = \beta f$ and $hu = 1_V$, $\mathrm{Path}(\mathcal{G})$ is finite if and only if $\mathrm{Path}(\mathcal{G}\backslash\{U\})$ is. Since $k\delta = \beta g$ and $kw = 1_V$, it suffices to consider the category $\mathrm{Path}(\mathcal{G}\backslash\{U, W\})$. Then, with $ft = i\beta$ and $gs = i'\beta$, we only have to prove that $\mathrm{Path}(\mathcal{G}\backslash\{U, W, V\})$ is finite. Since $fs = gt = 1_Y$, if we add some formal arrows $y, y' \colon Y \to Y$ and the conditions $ft = y$ and $gs = y'$, we only have to prove that the category generated by

$$\begin{tikzcd} Y \arrow[loop left, "y"] \arrow[loop right, "y'"] \end{tikzcd}$$

and the equalities coming from $\mathcal{G}$ is finite. But this is obvious since $yy = ftft = i\beta i\beta = i\beta = y$, $yy' = y$, $y'y = y'$ and $y'y' = y'$.

We can thus remove the hypothesis about coproducts in Theorem 2.3.

Theorem 3.3. *Let $n \geqslant 3$ and let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

(i) *$\mathcal{C}$ is a n-permutable category;*
(ii) *if γ and δ in diagram* (5) *are regular epimorphisms, then the comparison morphism λ is also a regular epimorphism;*
(iii) *if γ and δ in diagram* (7) *are regular epimorphisms, then the comparison morphism λ is also a regular epimorphism.*

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) has been proved for Theorem 2.3. Let us prove (iii) $\Rightarrow$ (i). Up to a change of universe, we can suppose $\mathcal{C}$ to be small. Since

$$\text{regular} + \text{(iii)} = \text{regular} + \text{(iii')}$$

is an unconditional exactness property, we know from Theorem 3.1 that $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ is also a regular category which satisfies (iii). Since it has small colimits, we deduce from Theorem 2.3 that it is n-permutable. Since the embedding $\mathcal{C} \hookrightarrow \mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ is full, faithful and preserves finite limits and colimits, $\mathcal{C}$ is also n-permutable. ∎

4. The ternary terms for n-permutable varieties

The stability property for regular epimorphisms given by Theorem 3.3 for n-permutable varieties gives us a diagram that exhibits the existence of the ternary operations which characterise them. In fact, let X be the free algebra on one element and consider diagram (7) with $f = \delta = \nabla_X + 1_X$, $s = (\iota_2\ \iota_3)$, $g = \gamma = 1_X + \nabla_X$, $t = (\iota_1\ \iota_2)$, $\beta = k = h = \nabla_X$, $w = i = \iota_1$ and $u = i' = \iota_2$. The binary terms $p_1(x, y) = x$ and $p_2(x, y) = y$ are such that $(p_1, p_2) \in \mathrm{Eq}(\nabla_X)$. Since λ is surjective, then there exist ternary operations $(w_1, \ldots, w_{n-1}) \in L$ such that $\lambda(w_1, \ldots, w_{n-1}) = (p_1, p_2)$. The identities from Theorem 1.1 then easily follow.

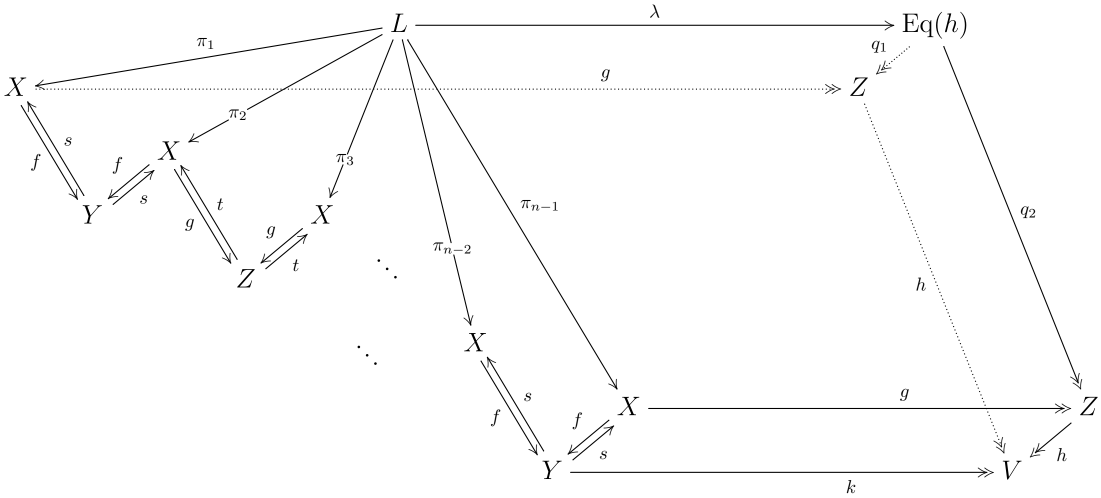
5. Another stability property and the $(n+1)$-ary operations

We saw how the stability properties for regular epimorphisms stated in Theorem 3.3 provided a diagram which exhibits the existence of ternary operations for the varietal case. The following question arises naturally:

> *Is there a (stability) property for n-permutable categories which exhibits the existence of the $(n+1)$-ary operations for the varietal case?*

We conclude this paper with a positive answer; the proof is done separately for $n \geqslant 3$ congruent to 0, 1, 2 and 3, modulo 4.

For $n \geqslant 3$ odd, we consider the diagram



(17)

where the left side is a limit of split epimorphisms and the right quadrangle is the kernel pair of h. We also assume $kf = hg$, which implies the existence of a comparison morphism $\lambda\colon L \to \mathrm{Eq}(h)$ such that $q_1\lambda = g\pi_1$ and $q_2\lambda = g\pi_{n-1}$. The morphisms f, g, h and k are supposed to be regular epimorphisms.

Theorem 5.1. *Let $n = 4l + 3 \geqslant 3$ be congruent to 3 modulo 4 and let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

(i) *$\mathcal{C}$ is an n-permutable category;*

(ii) *for any diagram (17), where h is split by a morphism u such that*

$$uk = (gsft)^l gs = gs(ftgs)^l, \tag{18}$$

then the comparison morphism λ is a regular epimorphism.

Proof: Note that if $n = 3$, then the equivalence between (i) and (ii) is precisely Theorem 1 in [10]. Suppose now that $n = 4l + 3 \geqslant 7$.

(i) $\Rightarrow$ (ii) Similarly as for Theorem 2.3, we are going to use the embedding theorem for n-permutable categories, this time using the category $\mathrm{Mod}(\Gamma'_n)$. So let $s' \in S'_n$. If $(a, b) \in \mathrm{Eq}(h)_{s'}$, then we can suppose there exist $x_1 \in X_{s'}$ and $x_{n-1} \in X_{s'}$ such that $g(x_1) = a$ and $g(x_{n-1}) = b$. Since $h(a) = h(b)$, we get

$$kf(x_1) = hg(x_1) = hg(x_{n-1}) = kf(x_{n-1}). \tag{19}$$

We consider the elements

$$\overline{x_i} = v_i^{s'}\big(x_1, sf(x_1), tgsf(x_1), \dots, sf(tgsf)^l(x_1),$$
$$sf(tgsf)^l(x_{n-1}), \dots, sf(x_{n-1}), x_{n-1}\big)$$

in $X_{(s',0)}$, for each $1 \leqslant i \leqslant n-1$. Note that the first half $2l+2$ elements (an even number) is calculated using x_1 and the second half $2l+2$ elements is calculated using x_{n-1}. We want to show that $(\overline{x_1}, \ldots, \overline{x_{n-1}}) \in L_{(s',0)}$ is such that $\lambda(\overline{x_1}, \ldots, \overline{x_{n-1}}) = \alpha^{s'}(a,b)$. Note that, for any $1 \leqslant i \leqslant n-1$,

$$\begin{aligned} f(\overline{x_i}) = v_i^{s'}(&f(x_1), f(x_1), ftgsf(x_1), ftgsf(x_1), \ldots, f(tgsf)^l(x_1), f(tgsf)^l(x_1), \\ &f(tgsf)^l(x_{n-1}), f(tgsf)^l(x_{n-1}), \ldots, f(x_{n-1}), f(x_{n-1})), \\ g(\overline{x_i}) = v_i^{s'}(&g(x_1), gsf(x_1), gsf(x_1), \ldots, gsf(tgsf)^l(x_1), \\ &gsf(tgsf)^l(x_{n-1}), \ldots, gsf(x_{n-1}), gsf(x_{n-1}), g(x_{n-1})), \end{aligned}$$

and

$$\begin{aligned} gsf(tgsf)^l(x_1) &= (gsft)^l gsf(x_1) \overset{(18)}{=} ukf(x_1) \overset{(19)}{=} ukf(x_{n-1}) \\ &= gsf(tgsf)^l(x_{n-1}). \end{aligned} \tag{20}$$

From the identities (2), we obviously have $f(\overline{x_{i-1}}) = f(\overline{x_i})$, when i is even, and to obtain $g(\overline{x_{i-1}}) = g(\overline{x_i})$, when i is odd, we use (20). We also need (20) to get $g(\overline{x_1}) = \alpha^{s'}(g(x_1)) = \alpha^{s'}(a)$ and $g(\overline{x_{n-1}}) = \alpha^{s'}(g(x_{n-1})) = \alpha^{s'}(b)$. Therefore,

$$(a,b) \overset{(2)}{=} \pi^{s'}(\alpha^{s'}(a,b)) = \pi^{s'}(\lambda(\overline{x_1}, \ldots, \overline{x_{n-1}})) \in \mathrm{Im}(\lambda)_s$$

and λ is a regular epimorphism.

(ii) $\Rightarrow$ (i) The proof is similar to the one of (iii) $\Rightarrow$ (i) in Theorem 3.3. First we prove it using binary coproducts and Theorem 5 of [24] (characterising n-permutable categories with binary coproducts through $(n+1)$-ary approximate Hagemann-Mitschke co-operations). For an arbitrary object X, we apply the assumption to diagram (17) with

$$\begin{aligned} f &= (2l+2)\nabla_X \colon (4l+4)X \to (2l+2)X, \\ s &= \big(\iota_2\ \iota_4\ \cdots\ \iota_{2l}\ \iota_{2l+2}\ \iota_{2l+3}\ \iota_{2l+5}\ \cdots\ \iota_{4l+3}\big), \\ g &= 1_X + (2l+1)\nabla_X + 1_X \colon (4l+4)X \to (2l+3)X, \\ t &= \big(\iota_1\ \iota_3\ \cdots\ \iota_{2l+1}\ \iota_{2l+3}\ \iota_{2l+4}\ \iota_{2l+6}\ \cdots\ \iota_{4l+4}\big), \\ k &= \nabla_X^{2l+2} \colon (2l+2)X \to X, \\ h &= \nabla_X^{2l+3} \colon (2l+3)X \to X \end{aligned}$$

and $u = \iota_{l+2}$, where $\nabla_X^i \colon iX \to X$ denotes the codiagonal.

We must check that $uk = (gsft)^l gs$. We are actually going to check it in the dual category $\mathcal{C}^{\mathrm{op}}$ using a set-theoretical argument. For that, we consider

$(x_1, \dots, x_{2l+3})$ in the product X^{2l+3} and the effect of applying the first $gsft$:

$$(x_1, \dots, x_{2l+3}) \xmapsto{gsft} (x_2, \dots, x_{l+1}, x_{l+2}, x_{l+2}, x_{l+2}, x_{l+3}, \dots, x_{2l+2}).$$

The first and last entries disappear while the middle one appears three times. Thus

$$(x_1, \dots, x_{2l+3}) \xmapsto{(gsft)^l} (x_{l+1}, x_{l+2}, \dots, x_{l+2}, x_{l+3}) \xmapsto{gs} (x_{l+2}, \dots, x_{l+2}).$$

The rest of the proof using binary coproducts then goes similarly as in Theorem 2.3.

To prove that we can remove this assumption on coproducts, we use the same technique as in Section 3. The only non-obvious part is to prove that the category underlying the shape of the diagram used to build diagram (17) together with the section u of h as in (ii) is finite. The corresponding conditional graph $\mathcal{G}$ is given by

$$\begin{tikzcd}
X \arrow[r, shift right, "g"'] \arrow[d, shift right, "f"'] & Z \arrow[l, shift right, "t"'] \arrow[d, shift right, "h"'] \\
Y \arrow[u, shift right, "s"'] \arrow[r, "k"'] & V \arrow[u, shift right, "u"']
\end{tikzcd}$$

with the conditions $fs = 1_Y$, $gt = 1_Z$, $hu = 1_V$, $hg = kf$ and $uk = (gsft)^l gs$. Obviously, we can add some arrows $y\colon Y \to Z$ and $z\colon Z \to Y$ together with the conditions $y = gs$ and $z = ft$ to form a conditional graph $\mathcal{G}'$ such that $\mathrm{Path}(\mathcal{G}) \cong \mathrm{Path}(\mathcal{G}')$. Since $fs = 1_Y$, $gs = y$, $ft = z$ and $gt = 1_Z$, applying Lemma 3.2, $\mathrm{Path}(\mathcal{G}')$ is finite if and only if $\mathrm{Path}(\mathcal{G}' \backslash X)$ is. Again, we can add an arrow $x\colon Z \to Z$ and the condition $x = yz$ to $\mathcal{G}' \backslash X$ to obtain the conditional graph $\mathcal{G}''$ such that $\mathrm{Path}(\mathcal{G}'') \cong \mathrm{Path}(\mathcal{G}' \backslash X)$.

$$\begin{tikzcd}
 & Z \arrow[dl, shift right, "z"'] \arrow[d, shift right, "h"'] \arrow[loop right, "x"] \\
Y \arrow[ur, shift right, "y"'] \arrow[r, "k"'] & V \arrow[u, shift right, "u"']
\end{tikzcd}$$

Since $kz = kft = hgt = h$ and $yz = x$, we only need to show that $\mathrm{Path}(\mathcal{G}'' \backslash Y)$ is finite. Again, since $uh = ukft = (gsft)^l gsft = (gsft)^{l+1} = (yz)^{l+1} = x^{l+1}$, it remains to show that $\mathrm{Path}(\mathcal{G}'' \backslash \{Y, V\})$ is finite. For that, it is enough to observe that

$$\begin{aligned} x^{2l+2} &= (gsft)^{2l+2} = (gsft)^l gsft(gsft)^{l+1} \\ &= ukft(gsft)^{l+1} = uh(gsft)^l gsft \\ &= uhukft = ukft = (gsft)^{l+1} = x^{l+1}. \end{aligned}$$ ■

Remark 5.2. If $\mathcal{V}$ is an n-permutable variety of universal algebras ($n = 4l+3$), the above stability property exhibits the $(n+1)$-ary operations from Theorem 1.2. Indeed, let X be the free algebra on one element and consider diagram (17) together with the splitting u of h as given in the proof of (ii) $\Rightarrow$ (i). The $(2l+3)$-ary terms p_1 and p_{2l+3} given by the first and the $(2l+3)$-th projections, respectively, are such that $(p_1, p_{2l+3}) \in \mathrm{Eq}(\nabla_X^{2l+3})$. Since λ is a surjection, by assumption, there exist $(4l+4) = (n+1)$-ary terms $(v_1, \ldots, v_{n-1}) \in L$ such that $\lambda(v_1, \ldots, v_{n-1}) = (p_1, p_{2l+3})$. It then easily follows that $v_1, \ldots, v_{n-1}$ satisfy the identities of Theorem 1.2.

Theorem 5.3. *Let $n = 4l+1 \geqslant 5$ be congruent to 1 modulo 4 and let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

(i) *$\mathcal{C}$ is an n-permutable category;*

(ii) *for any diagram* (17) *where k is split by a morphism w such that*

$$wk = (ftgs)^l, \tag{21}$$

the comparison morphism λ is a regular epimorphism.

Proof: The proof is similar to that of Theorem 5.1, where now

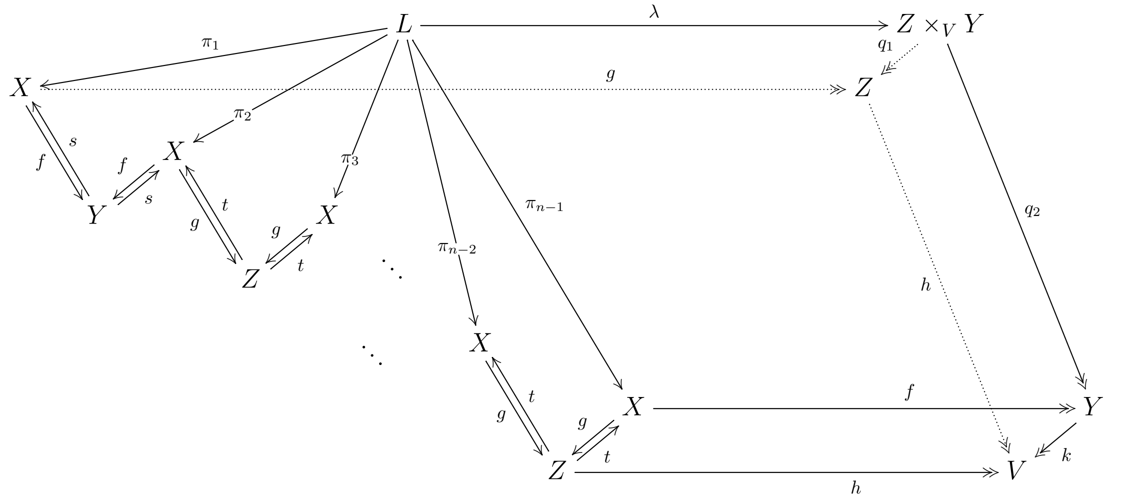
$$\overline{x_i} = v_i^{s'}\big(x_1, sf(x_1), tgsf(x_1), \ldots, (tgsf)^l(x_1), \\ (tgsf)^l(x_{n-1}), \ldots, sf(x_{n-1}), x_{n-1}\big)$$

is used for the implication (i) $\Rightarrow$ (ii). For the converse implication (ii) $\Rightarrow$ (i), we consider

$$\begin{aligned} f &= (2l+1)\nabla_X \colon (4l+2)X \to (2l+1)X, \\ s &= \big(\iota_2\ \iota_4\ \cdots\ \iota_{2l}\ \iota_{2l+2}\ \iota_{2l+3}\ \iota_{2l+5}\ \cdots\ \iota_{4l+1}\big), \\ g &= 1_X + (2l)\nabla_X + 1_X \colon (4l+2)X \to (2l+2)X, \\ t &= \big(\iota_1\ \iota_3\ \cdots\ \iota_{2l-1}\ \iota_{2l+1}\ \iota_{2l+2}\ \iota_{2l+4}\ \cdots\ \iota_{4l+2}\big), \\ k &= \nabla_X^{2l+1} \colon (2l+1)X \to X, \\ h &= \nabla_X^{2l+2} \colon (2l+2)X \to X \end{aligned}$$

and $w = \iota_{l+1}$. ■

For $n \geqslant 4$ even, we consider the diagram



(22)

where the left side is a limit of split epimorphisms and the right quadrangle is a pullback of regular epimorphisms. We also assume $kf = hg$, which implies the existence of a comparison morphism $\lambda \colon L \to Z \times_V Y$ such that $q_1\lambda = g\pi_1$ and $q_2\lambda = f\pi_{n-1}$.

Theorem 5.4. *Let $n = 4l \geqslant 4$ be divisible by 4 and let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

(i) *$\mathcal{C}$ is an n-permutable category;*
(ii) *for any diagram (22), where k is split by a morphism w such that*

$$wk = (ftgs)^l, \tag{23}$$

then the comparison morphism λ is a regular epimorphism.

Proof: The proof is similar to that of Theorem 5.1, where now

$$\overline{x_i} = v_i^{s'}\big(x_1, sf(x_1), tgsf(x_1), \ldots, (tgsf)^l(x_1), (tgsf)^l(x_{n-1}), \ldots, sf(x_{n-1})\big)$$

is used for the implication (i) $\Rightarrow$ (ii). For the converse implication (ii) $\Rightarrow$ (i), we consider

$$\begin{aligned}
f &= (2l)\nabla_X + 1_X \colon (4l+1)X \to (2l+1)X,\\
s &= \big(\iota_2\ \iota_4\ \cdots\ \iota_{2l-2}\ \iota_{2l}\ \iota_{2l+1}\ \iota_{2l+3}\ \cdots\ \iota_{4l+1}\big),\\
g &= 1_X + (2l)\nabla_X \colon (4l+1)X \to (2l+1)X,\\
t &= \big(\iota_1\ \iota_3\ \cdots\ \iota_{2l-1}\ \iota_{2l+1}\ \iota_{2l+2}\ \iota_{2l+4}\ \cdots\ \iota_{4l}\big),\\
k &= \nabla_X^{2l+1} \colon (2l+1)X \to X,\\
h &= \nabla_X^{2l+1} \colon (2l+1)X \to X
\end{aligned}$$

and $w = \iota_{l+1}$. ■

Theorem 5.5. *Let $n = 4l + 2 \geqslant 6$ be congruent to 2 modulo 4 and let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

(i) *$\mathcal{C}$ is an n-permutable category;*

(ii) *for any diagram (22), where h is split by a morphism u such that*

$$uk = (gsft)^l gs = gs(ftgs)^l, \tag{24}$$

then the comparison morphism λ is a regular epimorphism.

Proof: The proof is similar to that of Theorem 5.1, where now

$$\overline{x_i} = v_i^{s'}(x_1, sf(x_1), tgsf(x_1), \ldots, (tgsf)^l(x_1), sf(tgsf)^l(x_1),\\ sf(tgsf)^l(x_{n-1}), \ldots, sf(x_{n-1}))$$

is used for the implication (i) $\Rightarrow$ (ii). For the converse implication (ii) $\Rightarrow$ (i), we consider

$$\begin{aligned}
f &= (2l+1)\nabla_X + 1_X \colon (4l+3)X \to (2l+2)X,\\
s &= \big(\iota_2\ \iota_4\ \cdots\ \iota_{2l}\ \iota_{2l+2}\ \iota_{2l+3}\ \iota_{2l+5}\ \cdots\ \iota_{4l+3}\big),\\
g &= 1_X + (2l+1)\nabla_X \colon (4l+3)X \to (2l+2)X,\\
t &= \big(\iota_1\ \iota_3\ \cdots\ \iota_{2l-1}\ \iota_{2l+1}\ \iota_{2l+2}\ \iota_{2l+4}\ \cdots\ \iota_{4l+2}\big),\\
k &= \nabla_X^{2l+2} \colon (2l+2)X \to X,\\
h &= \nabla_X^{2l+2} \colon (2l+2)X \to X
\end{aligned}$$

and $u = \iota_{l+2}$. ■

Remark 5.6. As for the case $n = 4l + 3$, the above characterisations of n-permutable categories for the cases $n = 4l$, $4l + 1$ and $4l + 2$ also give rise to the $(n + 1)$-ary operations in the case of an n-permutable variety of universal algebras.

References

[1] M. Barr, *Exact categories*, Lecture Notes in Math vol. 236 Springer (1971) 1–120.
[2] M. Barr, *Representation of categories*, J. Pure Appl. Algebra 41 (1986) 113–137.
[3] F. Borceux, *Handbook of Categorical Algebra 1*, Cambridge University Press (1994).
[4] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, Lecture Notes Math. vol. 1488 Springer (1991) 43–62.
[5] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Structures 4 (1996) 307–327.
[6] D. Bourn, Z. Janelidze, *Approximate Mal'tsev operations*, Theory Appl. Categ. 21 (2008) 152–171.
[7] A. Carboni, G.M. Kelly, M.C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Structures 1, no. 4 (1993) 385-421.
[8] A. Carboni, J. Lambek, M.C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra 69 (1991) 271–284.
[9] A. Carboni, M.C. Pedicchio, N. Pirovano, *Internal graphs and internal groupoids in Mal'tsev categories*, Canad. Math. Soc. Conf. Proc. 13 (1992) 97–109.
[10] M. Gran, D. Rodelo, *A new characterisation of Goursat categories*, Appl. Categ. Structures 1, no. 20 (2012) 229-238.
[11] M. Gran, D. Rodelo, *The Cuboid Lemma and Mal'tsev categories*, Appl. Categ. Structures Vol. 20 (2012) 229-238.
[12] M. Gran, D. Rodelo, *Beck-Chevalley condition and Goursat categories*, J. Pure Appl. Algebra 221 (2017) 2445–2457.
[13] J. Hagemann, A. Mitschke, *On n-permutable congruences*, Alg. Universalis 3 (1973) 8-12.
[14] P.-A. Jacqmin, *An embedding theorem for regular Mal'tsev categories*, submitted.
[15] P.-A. Jacqmin, *Embedding theorems in non-abelian categorical algebra*, PhD thesis, Université catholique de Louvain (2016).
[16] P.-A. Jacqmin, Z. Janelidze, *Unconditional exactness properties*, in preparation.
[17] P.-A. Jacqmin, Z. Janelidze, *Closedness properties of internal relations VII: Embedding theorems*, in preparation.
[18] Z. Janelidze, *Subtractive categories*, Appl. Categ. Structures 13 (2005) 343–350.
[19] Z. Janelidze, *Closedness properties of internal relations I: A unified approach to Mal'tsev, unital and subtractive categories*, Theory Appl. Categ. 16 (2006) 236–261.
[20] Z. Janelidze, *Closedness properties of internal relations V: Linear Mal'tsev conditions*, Alg. Universalis 58 (2008) 105–117.
[21] Z. Janelidze, *Closedness properties of internal relations VI: Approximate operations*, Cah. Topol. Géom. Différ. Catég. 50 (2009) 298–319.
[22] Z. Janelidze, D. Rodelo, T. Van der Linden, *Hagemann's theorem for regular categories*, J. Homotopy Relat. Structures 9(1) (2014) 55-66.
[23] A. I. Mal'tsev, *On the general theory of algebraic systems*, Mat. Sbornik N. S. 35 no. 6 (1954) 3–20.
[24] D. Rodelo, T. Van der Linden, *Approximate Hagemann–Mitschke co-operations*, Appl. Categ. Structures 22 (2014) 1009-1020.

Pierre-Alain Jacqmin
Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, chemin du cyclotron 2 bte L7.01.02, B–1348 Louvain-la-Neuve, Belgium

E-mail address: pierre-alain.jacqmin@uclouvain.be

Diana Rodelo
CMUC, Department of Mathematics, University of Coimbra, 3001–501 Coimbra, Portugal and
Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005–139 Faro, Portugal

E-mail address: drodelo@ualg.pt