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Congruences on Dynamic Algebras

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Abstract

The lattice *Cong*$\mathcal{D}$ of all dynamic congruences on a given dynamic algebra $\mathcal{D}$ is presented. Whenever $\mathcal{D}$ is separable with zero we define dynamic ideal on $\mathcal{D}$, given rise to the lattice *Ide*$\mathcal{D}$. The notions of kernel of a dynamic congruence and the congruence generated by a dynamic ideal are introduced to describe a Galois connection between *Cong*$\mathcal{D}$ and *Ide*$\mathcal{D}$. We study conditions under which a dynamic congruence is determined by its kernel.

Keywords: Boolean algebra with operators, dynamic algebra, dynamic congruence, dynamic ideal, Galois connection.

1 Introduction

To enhance the importance of the algebraic structure of the regular part of a dynamic algebra, we introduce the notions of dynamic congruence and dynamic ideal on dynamic algebras viewed as heterogeneous algebras, differing from those presented in dynamic algebras regarded as modal algebras, [6].

2 Dynamic algebras

Dynamic algebras were introduced by Kozen [7] and Pratt [9] to provide models for Propositional Dynamic Logic (*PDL*). Following Pratt a dynamic algebra is a two-sorted algebra $(\mathcal{B}, \mathcal{R}, \langle\rangle)$ verifying a certain set of equations. The Boolean universe $\mathcal{B} = (B, \vee, \sim, 0)$ is a Boolean algebra (an algebra with a binary operation $\vee$, one unary operation $\sim$ and a nullary operation 0, satisfying the usual set of axioms); the regular universe $\mathcal{R} = (R, \cup, ;, ^*)$ (or universe of actions or universe of programs) is an algebra with two binary operations, $\cup$,; and one unary operation *. The heterogeneous operator (diamond) $\langle,\rangle : R \times B \longrightarrow B$ relates both algebras.

Definition 2.1
A *dynamic algebra* $\mathcal{D} = (\mathcal{B} = (B, \vee, \sim, 0), \mathcal{R} = (R, \cup, ;, ^*), \langle, \rangle)$ is an algebra satisfying

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(1) $\mathcal{B}$ is a Boolean algebra
(2a) $\langle a, 0\rangle = 0$
(2b) $\langle a, p \vee q\rangle = \langle a, p\rangle \vee \langle a, q\rangle$
(3) $\langle a \cup b, p\rangle = \langle a, p\rangle \vee \langle b, p\rangle$
(4) $\langle a; b, p\rangle = \langle a, \langle b, p\rangle\rangle$
(5a) $p \vee \langle a; a^*, p\rangle \leq \langle a^*, p\rangle$
(5b) $\langle a^*, p\rangle \leq p \vee \langle a^*, \sim p \wedge \langle a, p\rangle\rangle$
for $p,q \in B$ and $a,b \in R$.

Notes.

1. For p and q elements of the Boolean algebra $\mathcal{B}$, we used $(p \leq q)$ as an abbreviation of $(p \vee q = q)$.
2. Pratt [9] showed that conditions (5a) and (5b) can be replaced by $\langle a^*, p\rangle = min\ a!p$ where $a!p = \{q \in B : p \vee aq \leq q\}$.

Notation. For $a,b \in R$ and $p \in B$ we write ap instead of $\langle a, p\rangle$, ab instead of $a;b$ and use $a \leq b$ whenever ($ap \leq bp$, for every $p \in B$).

Kripke structures, the traditional models for *PDL* [4], were presented in [9] as examples of dynamic algebras. They are defined as follows. The *full Kripke structure* on a given non-empty set S is a triple $(S, \mathcal{P}(S), \mathcal{P}(S \times S))$ where $\mathcal{P}(S)$ is the Boolean algebra of all subsets of S (with the usual set theoretical operations) and $\mathcal{P}(S \times S)$ is the set of all binary relations on S. The operations $;, \cup$ and * are, respectively, the composition, the union and the reflexive transitive closure of binary relations. The diamond operation $\langle a, p\rangle$ (for $a \in \mathcal{P}(\mathcal{S} \times \mathcal{S})$ and $p \in \mathcal{P}(\mathcal{S})$) is defined to be the pre-image of p under a,

$$\{s \in S : (s, s') \in a, \text{for some } s' \in p\}.$$

Kripke structures are defined as the dynamic subalgebras of full Kripke structures. We denote by DA the variety of all dynamic algebras. Pratt [9] proved that

$$DA = HSP(Kri \cup T)$$

where Kri denotes the class of all Kripke structures and T is the class of all dynamic algebras with trivial Boolean part.

Let $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle) \in DA$ and $a, b \in R$. We put $a \cong_{\mathcal{R}} b$, if $ap{=}bp$, for every proposition p in B. The pair $(=, \cong_{\mathcal{R}})$ defines a congruence relation (*cf.* Def.2.3, below) in the dynamic algebra $\mathcal{D}$. A dynamic algebra is *separable* if for any actions a and b in R, such that $ap{=}bp$, for any proposition $p \in B$, then $a{=}b$. For every dynamic algebra $\mathcal{D}$, $\mathcal{D}/(=, \cong_{\mathcal{R}})$ is separable. We use SDA to denote the class of all separable dynamic algebras.

The mixed operator diamond $\langle , \rangle : R \times B \longrightarrow B$ can be seen as

$$\begin{aligned} \langle\rangle : R &\longrightarrow F(B; B) \\ a &\longrightarrow (p \longrightarrow \langle a\rangle p = \langle a, p\rangle) \end{aligned}$$

where $F\ (B;B) = \{f \mid f\colon B \longrightarrow B$ is a mapping$\}$. However, unless $\mathcal{D}$ is separable, $\langle\rangle$ does not define a representation of R since $\langle\rangle$ is not one-to-one. We may have equal mappings $\langle a\rangle$ and $\langle b\rangle$ in $F(B;B)$ with $a \neq b$. Consequently, for $\mathcal{D}$ an arbitrary dynamic algebra, $\{\langle a\rangle : a \in R\}$ is a family of mappings in B indexed by R (where the extensionality axiom is not satisfied). In a separable dynamic algebra, R may be identified with a set of mappings in B, that we will use whenever appropriate. Therefore, a dynamic algebra $\mathcal{D}$, can be viewed as a Boolean algebra with operators [9]. Moreover, defining, for each $a \in R$, $[a]p := \sim\langle\, a, \sim p\rangle$ for every $p \in B$, then $(\mathcal{B}, \{[a]\}_{a\in R})$ is a modal algebra (*cf.* [6]) associated with $\mathcal{D}$. The same modal algebra may be determined by distinct dynamic algebras.

The relation $\leq$ in R, $a \leq b$ whenever ($ap \leq bp$, for every $p \in B$) is reflexive and transitive, but it is not antisymmetric in general (since the conditions $ap\leq bp$ and $bp\leq ap$ give rise to $ap{=}bp$, for every $p \in B$ and not necessarily to $a{=}b$) and, therefore, defines a quasi-order in R. Clearly, in separable dynamic algebras $\leq$ defines a partial order since the antisymmetry is satisfied.

The elementary algebraic theory of dynamic algebras is similar to that of every other algebraic system [10], [11]. We will specify some concepts for later use.

Definition 2.2
Let

$$\mathcal{D} = (\mathcal{B} = (B, \vee, \sim, 0), \mathcal{R} = (R, \cup, ;, *), \langle\rangle) \text{ and}$$
$$\mathcal{D}' = (\mathcal{B}' = (B', \vee', \sim', 0'), \mathcal{R}' = (R', \cup', ;', *'), \langle\rangle')$$

be dynamic algebras and $h_1{:}B \longrightarrow B'$ and $h_2{:}R \longrightarrow R'$ be mappings. The pair $h{=}(h_1,h_2)$ is a *dynamic homomorphism* between the dynamic algebras $\mathcal{D}$ and $\mathcal{D}'$ if h_1 and h_2 preserve the mentioned one-sorted operations and h preserves the mixed operation, *i.e.*, for $p \in B$, $a, b \in R$ we have

1. h_1 is a Boolean homomorphism
2. $h_2(a \cup b){=}h_2(a) \cup' h_2(b)$
3. $h_2(a;b){=}h_2(a);' h_2(b)$
4. $h_2(a^*){=}(h_2(a))^{*\prime}$
5. $h_1(\langle a\rangle\ p){=}\langle h_2(a)\rangle'\ h_1(p)$.

Definition 2.3
Let $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ be a dynamic algebra. The pair $\theta = (\theta_1, \theta_2)$ is a *(dynamic) congruence relation* on $\mathcal{D}$ if θ_1 is a congruence relation on $\mathcal{B}$, θ_2 is a congruence relation on $\mathcal{R}$ and $ap\ \theta_1\ bq$ whenever ($p\ \theta_1\ q$ and $a\ \theta_2\ b$).

In a dynamic algebra $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ we define the congruences Δ_B and ∇_B on B and Δ_R and ∇_R on R as expected:

$$\Delta_B = \{(p, p) : p \in B\} \qquad \nabla_B = \{(p, q) : p, q \in B\}$$
$$\Delta_R = \{(a, a) : a \in R\} \qquad \nabla_R = \{(a, b) : a, b \in R\}.$$

One can easily show that the pairs (Δ_B,Δ_R), (∇_B,∇_R) and (∇_B,Δ_R) are congruences on $\mathcal{D}$, but, in general, (Δ_B,∇_R) is not a congruence on $\mathcal{D}$.

As usual, we define $\mathcal{B}/_{\theta_1} = \{[p]_{\theta_1} : p \in B\}$ and $\mathcal{R}/_{\theta_2} = \{[a]_{\theta_2} : a \in R\}$. The structure $\mathcal{D}/_{\theta} = (\mathcal{B}/_{\theta_1}, \mathcal{R}/_{\theta_2}, \langle\rangle)$ is a dynamic algebra (the *quotient dynamic algebra*) with operations defined by

$$[p]_{\theta_1} \vee [q]_{\theta_1} := [p \vee q]_{\theta_1} \qquad [a]_{\theta_2} \cup [b]_{\theta_2} := [a \cup b]_{\theta_2}$$
$$\sim [p]_{\theta_1} := [\sim p]_{\theta_1} \qquad [a]_{\theta_2}; [b]_{\theta_2} := [a; b]_{\theta_2}$$
$$[a]_{\theta_2}[p]_{\theta_1} := [ap]_{\theta_1} \quad \text{and} \quad [a]^*_{\theta_2} := [a^*]_{\theta_2}.$$

The algebra $(\mathcal{B}/_{\theta_1}, \mathcal{R}/_{\theta_2}, \langle\rangle)$ is the homomorphic image of $(\mathcal{B}, \mathcal{R}, \langle\rangle)$ under the natural homomorphism $\nu{=}(\nu_1, \nu_2)$ defined by
$\nu_1 : p \mapsto [p]_{\theta_1}$
$\nu_2 : a \mapsto [a]_{\theta_2}$, $p \in B$, $a \in R$.

Also, associated with each homomorphism, there exists a natural congruence. If $h : \mathcal{D} \longrightarrow \mathcal{D}'$ is a dynamic algebra homomorphism, the kernel relation ker_h=(K_1, K_2) given by

$$\begin{aligned} pK_1 q &\text{ if and only if } h_1(p) = h_1(q) \\ aK_2 b &\text{ if and only if } h_2(a) = h_2(b) \end{aligned}$$

for every $p,q \in B$ and $a,b \in R$, defines a congruence on $\mathcal{D}$.

The homomorphism theorem enable us to assert that every homomorphic image of a dynamic algebra $\mathcal{D}$ is, up to an isomorphism, a quotient algebra [11].

Theorem 2.4
If $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ and $\mathcal{D}'$ are dynamic algebras with $\mathcal{D}'$ homomorphic image of $\mathcal{D}$ under $h : \mathcal{D} \longrightarrow \mathcal{D}'$, then $\mathcal{D}'$ is isomorphic to $\mathcal{D}/_{ker_h}$ under the isomorphism $\psi : \mathcal{D}/_{ker_h} \longrightarrow \mathcal{D}'$ defined by $\psi_1[p]_{K_1} := h_1(p)$ and $\psi_2[a]_{K_2} := h_2(a)$, $p \in B$, $a \in R$.

3 The lattice *CongD*

Denote by $Cong\mathcal{D}$ the set of all dynamic congruences defined on a dynamic algebra $\mathcal{D}$. This set is partially ordered by $(\theta_1, \theta_2) \leq (\gamma_1, \gamma_2)$ if and only if $\theta_1 \subseteq \gamma_1$ and $\theta_2 \subseteq \gamma_2$. Our next aim is to define the lattice $(Cong\mathcal{D}, \wedge_\mathcal{D}, \vee_\mathcal{D})$. Since the intersection, $\theta \cap \gamma$=$(\theta_1 \cap \gamma_1, \theta_2 \cap \gamma_2)$, of any two dynamic congruences θ and γ defined on $\mathcal{D}$ is, itself, a dynamic congruence on $\mathcal{D}$, let $\theta \wedge_\mathcal{D} \gamma := \theta \cap \gamma$. Let us use $\langle\theta\rangle_\mathcal{A}$ to represent the congruence relation generated by the binary relation θ on any (homogeneous or heterogeneous) algebra $\mathcal{A}$, *i.e.*, the intersection of all congruence relations θ' on $\mathcal{A}$ containing θ,

$$\langle\theta\rangle_\mathcal{A} := \cap\{\theta' : \theta' \in Cong\mathcal{A} \text{ and } \theta \subseteq \theta'\}.$$

Now we need to define $\theta \vee_\mathcal{D} \gamma = (\tau_1, \tau_2)$. It is immediate that the regular part of the congruence $\theta \vee_\mathcal{D} \gamma$ can be given by $\tau_2 := \theta_2 \vee_\mathcal{R} \gamma_2 = \langle\theta_2 \cup \gamma_2\rangle_\mathcal{R}$, using the classic definition of supremum of two congruences. As far as the Boolean part, some caution is required. Since the Boolean part must be closed to the diamond operation evolving elements of R, we could be led to think of enlarging $\langle\theta_1 \cup \gamma_1\rangle_\mathcal{B}$ with, for instance, elements of the type (ap,bq) with $(a,b) \in \theta_2$ and $(p,q) \in \gamma_1$. In fact, that is not necessary, as shown below.

Proposition 3.1
Let $\mathcal{D}$ be a dynamic algebra. For $\theta, \gamma \in Cong\mathcal{D}$, $(p, q) \in \langle\theta_1 \cup \gamma_1\rangle_\mathcal{B}$ and $(a, b) \in \langle\theta_2 \cup \gamma_2\rangle_\mathcal{R}$ we have $(ap, bq) \in \langle\theta_1 \cup \gamma_1\rangle_\mathcal{B}$.

Proof. Let $(p, q) \in \langle\theta_1 \cup \gamma_1\rangle_\mathcal{B}$ and $(a, b) \in \langle\theta_2 \cup \gamma_2\rangle_\mathcal{R}$. By [10], we know that there exist natural numbers n, m, a sequence $p_1, p_2, p_3, \ldots, p_n$ in B and a sequence $a_1, a_2, a_3, \ldots, a_m$ in R such that

$$\begin{aligned} &p\theta_1 p_2,\ p_2\gamma_1 p_3,\ p_3\theta_1 p_4, \cdots,\ p_{n-1}\gamma_1 q \\ &a\theta_2 a_2,\ a_2\gamma_2 a_3,\ a_3\theta_2 a_4, \cdots,\ a_{m-1}\gamma_2 b. \end{aligned}$$

Since we always can repeat elements in the sequences, we are allowed to put m=n. So,

$$ap\theta_1 a_2p_2,\ \ a_2p_2\gamma_1 a_3p_3,\ \ a_3p_3\theta_1 a_4p_4, \cdots,\ \ a_{n-1}p_{n-1}\gamma_1 bq.$$

But this means that $(ap, bq) \in \langle\theta_1 \cup \gamma_1\rangle_{\mathcal{B}}$, as required. ■

The structure $(Cong\mathcal{D}, \wedge_{\mathcal{D}}, \vee_{\mathcal{D}})$ where, for every $\theta, \gamma \in Cong\mathcal{D}$, the operations are defined by

$$\theta \wedge_{\mathcal{D}} \gamma = \theta \cap \gamma = (\theta_1 \cap \gamma_1, \theta_2 \cap \gamma_2)$$
$$\theta \vee_{\mathcal{D}} \gamma = \langle\theta \cup \gamma\rangle_{\mathcal{D}} = (\langle\theta_1 \cup \gamma_1\rangle_{\mathcal{B}}, \langle\theta_2 \cup \gamma_2\rangle_{\mathcal{R}})$$

is a lattice, called the *congruence lattice* $Cong\mathcal{D}$ of $\mathcal{D}$. Elements $\theta = (\theta_1, \theta_2)$ and $\gamma = (\gamma_1, \gamma_2)$ of $Cong\mathcal{D}$ are said to be *permutable* or θ and γ *permute* if and only if $\theta \circ \gamma = \gamma \circ \theta$, *i.e.*, $\theta_1 \circ \gamma_1 = \gamma_1 \circ \theta_1$ and $\theta_2 \circ \gamma_2 = \gamma_2 \circ \theta_2$.

A class K of (homogeneous or heterogeneous) algebras is *congruence-permutable* if every algebra A on K is congruence-permutable, *i.e.*, if every pair of congruences on A permutes. Although congruences on Boolean algebras are permutable, it can be easily proved that the variety generated by the class of all separable dynamic algebras, *V(SDA)*, is not congruence-permutable.

Proposition 3.2
The variety *V(SDA)* is a non congruence-permutable class.

Proof. Let $\mathcal{D} = (\mathcal{P}(S), \{a, b, e\}, \langle\rangle)$, for S=$\{1,2\}$, a=$\{(1,1),(2,2)\}$, b=$\{(2,2)\}$ and $e = \emptyset$. For the dynamic congruences α=(α_1, α_2) and β=(β_1, β_2) with $\alpha_1 = \mathcal{P}(S) \times \mathcal{P}(S)$, α_2=$\{(a, a),(b, b),(e, e),(a, b),(b, a)\}$, $\beta_1 = \mathcal{P}(S) \times \mathcal{P}(S)$ and β_2=$\{(a, a),(b, b),(e, e),(b, e),(e, b)\}$ we have $(a, e) \in \alpha_2 \circ \beta_2$ but $(a, e) \notin \beta_2 \circ \alpha_2$. ■

As in the homogeneous case, the supremum of permutable dynamic congruences is given by their composition.

A class K of (homogeneous or heterogeneous) algebras is *congruence-distributive* if, for every algebra $\mathcal{A}$ on K, the lattice $Cong\mathcal{A}$ is distributive.

Proposition 3.3
The variety *DA* is a non distributive-congruence class.

Proof. Let $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ be the dynamic algebra, where $\mathcal{B}$ is the trivial Boolean algebra B=$\{0\}$, $\mathcal{R}$ is defined (*cf.* [5, Fig.3]) by $R = \{a, b, c, d, \mathrm{o}\}$ with

$$\begin{array}{lllll}
a \cup a = a & b \cup b = b & c \cup c = c & d \cup d = d & \mathrm{o} = \mathrm{o} \\
a \cup b = c & b \cup c = c & c \cup d = d & d \cup \mathrm{o} = d & \\
a \cup c = c & b \cup d = d & c \cup \mathrm{o} = c & & \\
a \cup d = d & b \cup \mathrm{o} = b & & & \\
a \cup \mathrm{o} = a & & & & \\
m; n = \mathrm{o} & & & & \\
d; l = l; d = l & & & & \\
l^* = l, & & & &
\end{array}$$

and, obviously, $\langle l,0\rangle$=0, for every $l \in R$ and $m,n \in \{a,b,c,\mathrm{o}\}$.

Let $\alpha{=}(\alpha_1,\alpha_2)$, $\beta{=}(\beta_1,\beta_2)$ and $\gamma{=}(\gamma_1,\gamma_2)$ be dynamic congruences defined by
$\alpha_1{=}\beta_1{=}\gamma_1{=}\{(0,0)\}{=}\Delta_B$
$\alpha_2 = \{(a, a), (b, b), (c, c), (d, d), (\mathrm{o}, \mathrm{o}), (b, c), (c, b)\}$
$\beta_2 = \{(a, a), (b, b), (c, c), (d, d), (\mathrm{o}, \mathrm{o}), (a, \mathrm{o}), (\mathrm{o}, a), (b, c), (c, b)\}$
$\gamma_2 = \{(a, a), (b, b), (c, c), (d, d), (\mathrm{o}, \mathrm{o}), (a, c), (c, a), (b, \mathrm{o}), (\mathrm{o}, b)\}$
We have $\alpha\vee(\beta\wedge\gamma) = \alpha$ and $(\alpha\vee\beta)\wedge(\alpha\vee\gamma) = \beta$. So *DA* is a non distributive-congruence variety. ■

A variety *V* of (homogeneous or heterogeneous) algebras is *arithmetical* if, each *A* in *V* is congruence-permutable and congruence-distributive. It is well known, (*cf.* [12]), that for any class *K* of homogeneous algebras with a common discriminator term, *V(K)* is arithmetical. Adaptations of this results can be established for any class *K* of dynamic algebras.

Definition 3.4
If $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ is a dynamic algebra, a pair of terms on the language of $\mathcal{D}$,$(t_1(x, y, z)$, $t_2(x', y', z'))$, is said to be a *pair of discriminator terms* whenever, for all $p, q, r \in B$ and $a, b, c \in R$

$$t_1(p, q, r) = \begin{cases} r & \text{if } p = q \\ p & \text{if } p \neq q \end{cases}$$
$$t_2(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{if } a \neq b, \end{cases}$$

i.e., $t_1(x, y, z)$ is a discriminator term on $\mathcal{B}$ and $t_2(x',y',z')$ is a discriminator term on $\mathcal{R}$.

Lemma 3.5
Let *K* be a class of dynamic algebras with a common pair of discriminator terms. Then *V(K)* is arithmetical.

Proposition 3.6
Let *K* be the class of the simple algebras of *SDA*. Then *V(K)* is a proper subvariety of *V(SDA)* (and, consequently of *DA*).

Proof. Since simple separable dynamic algebras are of the type $(B, \{\exists_s\}, \langle\rangle)$, for *B* arbitrary Boolean algebra with the singular element $\exists_s$ of the regular part defined by $\exists_s(p) = 1$, for all p$\neq$ 0, p $\in B$ and $\exists_s(0)$=0, [8], the pair of terms (t_1, t_2) given by

$$t_1(x, y, z) = (\exists_s(x + y) \wedge x) \vee (\sim \exists_s(x + y) \wedge z)$$
$$t_2(x', y', z') = x'$$

where the symbol + denotes the symmetric Boolean difference operation, establishes a pair of discriminator terms on the class *K* (*cf.* [12] for t_1). Therefore *V(K)* is an arithmetical variety. But *DA* and *V(SDA)* are non congruence-permutable varieties. ■

4 Separating congruence

Let $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ be a dynamic algebra. In this paragraph we are going to show that, corresponding to a dynamic congruence $\theta{=}(\theta_1{,}\theta_2)$ on $\mathcal{D}$, there exists always a dynamic congruence ϕ on $\mathcal{D}$ with the same Boolean part such that $\mathcal{D}/\phi$ is separable (*cf.* Prop.4.2). We also verify that, in general, θ_1 does not determine θ_2. For example, if $\mathcal{D}/\theta$ is not separable and γ=(θ_1,γ_2) is given by $a\gamma_2\, b$ if and only if $ap\theta_1 bp$, for all $p \in B$, then γ is a congruence on $\mathcal{D}$, distinct from θ (*cf.* Prop.4.2). That is the case of Ex.5.13.

There are Boolean congruences on $\mathcal{D}$ that are not the Boolean part of any of its dynamic congruences. Let $\mathcal{B}$ the Boolean algebra with four elements $\{0,\ p,\ q, 1\}$ and R={a} where a is defined by $a0 = 0$ and $ap = aq = a1 = 1$. Since $I_1 = \{0,\ p\}$ is a Boolean ideal on $\mathcal{B}$, we can affirm that θ_1 defined by, for $s,r \in B$, $(s,r) \in \theta_1$ iff $s \vee i = r \vee i$, for some $i \in I_1$, is a Boolean congruence on $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ with congruence classes $[0]_{\theta_1}$={0, p} and $[q]_{\theta_1}$={q, 1}. But *Cong*$\mathcal{R}$ is the singular set $\{\Delta_R = \{(a, a)\}\}$ and (θ_1, Δ_R) is not a dynamic congruence on $\mathcal{D}$, since $(0, p) \in \theta_1$ and $(a, a) \in \Delta_{\mathrm{R}}$ but $(a0,ap) \notin \theta_1$. Therefore, the dynamic algebra $\mathcal{D}$ has a Boolean congruence θ_1 which is not the Boolean part of any of its dynamic congruences.

A Boolean congruence θ_1 is called *dynamical* on $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ whenever there exists a congruence θ_2 on $\mathcal{R}$ such that $(\theta_1,\ \theta_2)$ is a congruence on $\mathcal{D}$.

A dynamic algebra $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ determines, as previously seen, a modal algebra $(\mathcal{B}, \{[a] : a \in R\})$. In any modal algebra, $(\mathcal{B}, \{f_i\}_{i<k})$ (where k denotes an arbitrary ordinal), there exists an isomorphism between the lattice of the *open filters* (filters F on $\mathcal{B}$ satisfying condition: if $p \in F$ then $f_i\ p \in F$, for every $i<k$), and the lattice of the modal congruences,(*cf.* [6 Thm.4.1.10]). In this context since a dynamical congruence is a modal congruence, we can affirm that to each dynamical congruence corresponds a unique open filter on $\mathcal{B}$.

Proposition 4.1
Let θ_1 be a Boolean congruence on $\mathcal{D}$. Then θ_1 is dynamical on $\mathcal{D}$ if and only if $(\theta_1,\ \Delta_{\mathcal{R}})$ is a dynamic congruence on $\mathcal{D}$.

Proposition 4.2
Let θ_1 be a dynamical congruence on $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ and let ϕ=(ϕ_1, ϕ_2) be defined by

$$\phi_1 = \theta_1$$
$$\phi_2 = \{(a,\, b) \in R \times R : [ap]_{\theta_1} = [bp]_{\theta_1},\ \text{for every } p \in B\}.$$

Then

1. ϕ is the greatest congruence on $\mathcal{D}$ having θ_1 as Boolean part and $\mathcal{D}/\phi$ is separable;
2. $(\theta_1,\ \Delta_{\mathcal{R}})$ is the smallest dynamic congruence on $\mathcal{D}$ having θ_1 as Boolean part.

Proof. Let $(\theta_1, \theta_2) \in$ *Cong*$\mathcal{D}$. We show that ϕ defines a congruence on $\mathcal{D}$.

Trivially, ϕ_2 is an equivalence relation. Let (a, b), $(c, d) \in \phi_2$. Then $[ap]_{\theta_1}$=$[bp]_{\theta_1}$ and $[cp]_{\theta_1}$=$[dp]_{\theta_1}$, for every $p \in B$. Immediately
$[(a \cup c)p]_{\theta_1} = [ap \vee cp]_{\theta_1} = [ap]_{\theta_1} \vee [cp]_{\theta_1} = [bp]_{\theta_1} \vee [dp]_{\theta_1} = [bp \vee dp]_{\theta_1} = [(b \cup d)p]_{\theta_1}$,
$[(ac)p]_{\theta_1} = [a(cp)]_{\theta_1} = [a]_{\theta_2}[cp]_{\theta_1} = [a]_{\theta_2}[dp]_{\theta_1} = [a(dp)]_{\theta_1} = [b(dp)]_{\theta_1} = [(bd)p]_{\theta_1}$,
$[a^*p]_{\theta_1} = [a^*]_{\theta_2}[p]_{\theta_1} = [a]^*_{\theta_2}[p]_{\theta_1} = min\{[q]_{\theta_1} : [p]_{\theta_1} \vee [a]_{\theta_2}[q]_{\theta_1} \le [q]_{\theta_1}\}$
$= min\{[q]_{\theta_1} : [p]_{\theta_1} \vee [b]_{\theta_2}[q]_{\theta_1} \le [q]_{\theta_1}\} = [b]^*_{\theta_2}[p]_{\theta_1} = [b^*]_{\theta_2}[p]_{\theta_1} = [b^*p]_{\theta_1}$, for every $p \in B$, and hence, $(a \cup c, b \cup d)$, (ac, bd) and $(a^*, b^*) \in \phi_2$.

Let $(a, b) \in \phi_2$ and $(p, q) \in \phi_1$. We have $[a]_{\theta_2}[p]_{\theta_1}$=$[b]_{\theta_2}[p]_{\theta_1}$. Since $[p]_{\theta_1}$ =$[q]_{\theta_1}$ then $[a]_{\theta_2}[p]_{\theta_1}$=$[b]_{\theta_2}[q]_{\theta_1}$, and consequently, $[ap]_{\theta_1}$=$[bq]_{\theta_1}$. Therefore, $(ap, bq) \in \phi_1$.

It is clear that for any dynamic congruence α=(θ_1, α_2), we have, necessarily, $\alpha \subseteq \phi$.

Now suppose that $[a]_{\phi_2}[p]_{\phi_1} = [b]_{\phi_2}[p]_{\phi_1}$, for every $p \in B$. Then, for every $p \in B$,

$[ap]_{\phi_1} = [bp]_{\phi_1}$

$[ap]_{\theta_1} = [bp]_{\theta_1}$.

Consequently, $(a, b) \in \phi_2$, *i.e.*, $[a]_{\phi_2}$=$[b]_{\phi_2}$, establishing the separability of $\mathcal{D}/\phi$. ■

Definition 4.3
Let θ_1 and ϕ be as defined at Prop. 4.2. The relation ϕ is called the *separating congruence* of any $\theta \in Cong\mathcal{D}$ having θ_1 as Boolean part (or simply a *separating congruence*).

For any dynamic algebra $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$, $(\Delta_{\mathcal{B}}, \cong_{\mathcal{R}})$ (*cf.* defined at §1) is the separating congruence of $(\Delta_{\mathcal{B}}, \Delta_{\mathcal{R}})$. Immediately, if $\mathcal{D}$ is separable, then $(\Delta_{\mathcal{B}}, \Delta_{\mathcal{R}})$ is the unique dynamic congruence having $\Delta_{\mathcal{B}}$ as Boolean part.

Proposition 4.4
Let θ be a dynamic congruence on a dynamic algebra $\mathcal{D}$. Then $\mathcal{D}/\theta$ is separable if and only if θ is a separating congruence on $\mathcal{D}$.

5 Dynamic Congruences and Dynamic Ideals

From now on we are going to use only separable dynamic algebras $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ where the regular part contains an element o satisfying $\mathrm{o}p = 0$, for every $p \in B$, and therefore, satisfying $\mathrm{o}a = \mathrm{o}$, $a\mathrm{o} = \mathrm{o}$ and $\mathrm{o} \cup a = a$, $a \cup \mathrm{o} = a$, for every $a \in R$. Such an algebra is called a *separable dynamic algebra with zero*.

We notice that every separable dynamic algebra $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ can be enriched with a *zero*, by simply adding elements o, id and $a \cup id$, for every $a \in R$, on the regular part of $\mathcal{D}$ (where id is defined by $\langle id\rangle\ p = p$, for every $p \in B$).

Usually, the notion of *ideal* in a given class of algebras is established so that the zero-classes of congruence relations are easily seen to be ideals. We will show, in this paragraph, that the class of separable dynamic algebras with zero is not a class with ideal determined congruences, [3], *i.e.*, each *ideal* is not the zero-class of a unique congruence relation.

Definition 5.1
A (*dynamic*) *ideal* on a dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ is a pair $I = (I_1, I_2)$ satisfying the following conditions

1. I_1 is a Boolean ideal on $\mathcal{B}$;
2. if $p \in I_1$ and $a \in R$ then $ap \in I_1$;
3. (a) $I_2 \neq \emptyset$;
   (b) if $a, b \in I_2$, $c, d \in R$ and $d \leq a$ then $a \cup b$, ac, ca, $d \in I_2$;
4. if $a \in I_2$ and $p \in B$ then $ap \in I_1$.

Such a subset I_2 of R satisfying condition (3) is called an *(action) ideal* of R.

By definition of dynamic ideal, since $\mathrm{o} \in R$ then $\mathrm{o} \in I_2$ ($\mathrm{o} = \mathrm{o}\, j \in I_2$, for some $j \in I_2 \neq \emptyset$). We denote by $Ide\mathcal{D}$ the set of all ideals on a dynamic algebra with zero, $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$.

Similarly to the case of the congruences, in a separable dynamic algebra with zero $\mathcal{D}$, there are Boolean ideals that are not the Boolean part of any dynamic ideal on $\mathcal{D}$. A Boolean ideal I_1 is called *dynamical* on $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ if there exists an action ideal I_2 on $\mathcal{R}$ such that (I_1, I_2) is a dynamic ideal on $\mathcal{D}$.

Proposition 5.2
Let I_1 be a Boolean ideal on a separable dynamic algebra with zero, $\mathcal{D}$. Then I_1 is a dynamical ideal on $\mathcal{D}$ if and only if $(I_1, \{\mathrm{o}\})$ is a dynamic ideal.

We will present a separable dynamic algebra with zero where there exists a Boolean ideal I_1 which is not dynamical. Let $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ be the dynamic algebra with $\mathcal{B}$ the Boolean algebra with four elements $\{0, p, q, 1\}$ and $R= \{a, \mathrm{o}, id\}$ where a, o and id are defined by $a0=0$, $ap=aq=a1=1$, $\langle \mathrm{o}\rangle s=0$ and $\langle id\rangle s=s$, for every $s \in B$. The pair $(I_1, \{\mathrm{o}\})$ with I_1 the Boolean ideal $\{0, p\}$, is not a dynamic ideal, since $a \in R$, $p \in I_1$ but $ap = 1 \notin I_1$.

Proposition 5.3
Let I_1 be a dynamical ideal on a separable dynamic algebra with zero. Then

1. $(I_1, \{\mathrm{o}\})$ is the smallest dynamic ideal having I_1 as Boolean part;
2. $(I_1, \{a\colon ap \in I_1, \text{ for every } p \in B\})$ is the greatest dynamic ideal having I_1 as Boolean part.

Definition 5.4
Let I_1 be a dynamical ideal on a separable dynamic algebra with zero, $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$, and let $F{=}(F_1, F_2)$ be defined by

$$F_1 := I_1$$
$$F_2 := \{a\colon ap \in I_1, \text{ for every } p \in B\}.$$

We say that F is the *determining ideal* of any $I \in Ide\mathcal{D}$ having I_1 as Boolean part (or simply, a *determining ideal*).

Definition 5.5
If $\theta = (\theta_1, \theta_2) \in Cong\mathcal{D}$ where $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ is a separable dynamic algebra with zero, we say that $\mathcal{I}(\theta) := \mathcal{I}^\theta = (\mathcal{I}_1^\theta, \mathcal{I}_2^\theta)$ defined by $\mathcal{I}_1^\theta := \{p \in B : p\,\theta_1 0\} = [0]_{\theta_1}$
$\mathcal{I}_2^\theta := \{a \in R : a\,\theta_2 \mathrm{o}\} = [\mathrm{o}]_{\theta_2}$
is the *kernel of the congruence* θ.

Proposition 5.6
The kernel of a congruence θ, $\mathcal{I}(\theta)$, on a separable dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$, is an ideal on $\mathcal{D}$.

Proof.
1. We know that $\mathcal{I}_1^\theta$ is a Boolean ideal .
2. If $a \in R$ and $p \in \mathcal{I}_1^\theta$, then $p\,\theta_1 0$ and $a\theta_2 a$. Therefore, $(ap)\theta_1(a0)$, *i.e.*, $(ap)\theta_1 0$. So $ap \in \mathcal{I}_1^\theta$.
3. (a) $\mathcal{I}_2^\theta \neq \emptyset$, since $\mathrm{o} \in \mathcal{I}_2^\theta$.
   (b) If $a, b \in \mathcal{I}_2^\theta$, $c, d \in R$ and $d \leq a$ then $a \cup b$, ac, ca, $d \in \mathcal{I}_2^\theta$. In fact, if $a, b \in \mathcal{I}_2^\theta$ then $a\,\theta_2\mathrm{o}$ and $b\,\theta_2\mathrm{o}$, and therefore, $(a \cup b)\theta_2(\mathrm{o} \cup \mathrm{o})$, *i.e.*, $(a \cup b)\theta_2\mathrm{o}$. Thus $a \cup b \in \mathcal{I}_2^\theta$. Since $c\,\theta_2 c$ and $a\,\theta_2\mathrm{o}$ then $(ac)\theta_2(\mathrm{o}\,c)$ and $(ca)\theta_2(c\,\mathrm{o})$, *i.e.*, $(ac)\theta_2\,\mathrm{o}$ and $(ca)\theta_2\,\mathrm{o}$. To this extend ac and ca are elements of $\mathcal{I}_2^\theta$. Since $d\,\theta_2 d$ and $a\,\theta_2\mathrm{o}$ then $(d \cup a)\,\theta_2\,(d \cup \mathrm{o})$, and therefore, $a\,\theta_2 d$ (from $d \leq a$ we get $d \cup a{=}a$). But $a\,\theta_2\mathrm{o}$ and so $d\theta_2\mathrm{o}$. Therefore, $d \in \mathcal{I}_2^\theta$.
4. Let $a \in \mathcal{I}_2^\theta$ and $p \in B$. Then $a\,\theta_2\mathrm{o}$ and $p\,\theta_1 p$, and therefore, $(ap)\theta_1\mathrm{o}p$, *i.e.*, $(ap)\theta_1 0$. Immediately, $ap \in \mathcal{I}_1^\theta$. ■

Definition 5.7
The *kernel* of a dynamic homomorphism $h : \mathcal{D} \longrightarrow \mathcal{D}'$, between dynamic algebras with zero, is the pair

$$(\{p \in B : h_1(p) = 0\}, \{a \in R : h_2(a) = \mathrm{o}\}).$$

Proposition 5.8
The kernel of a dynamic homomorphism $h : \mathcal{D} \longrightarrow \mathcal{D}'$, between separable dynamic algebras with zero, is a dynamic ideal on $\mathcal{D}$.

Definition 5.9
If $I = (I_1, I_2)$ is a dynamic ideal on a separable dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$, we define $\mathcal{C}(I) := \mathcal{C}^I = (\mathcal{C}_1^I, \mathcal{C}_2^I)$ by

$p\,\mathcal{C}_1^I q$ if and only if $p \vee i = q \vee i$, for some $i \in I_1$,

$a\,\mathcal{C}_2^I b$ if and only if $a \cup j = b \cup j$, for some $j \in I_2$,

for $p,\ q \in B$ and $a,\ b \in R$.

Proposition 5.10
If $I = (I_1, I_2)$ is a dynamic ideal on a separable dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$, then $\mathcal{C}(\mathcal{I})$ is a congruence relation on $\mathcal{D}$.

Proof. (i) The relation $\mathcal{C}_1^I$ is a congruence relation on $\mathcal{B}$ (a known result in Boolean algebras).

(ii) Let us prove that $\mathcal{C}_2^I$ is an equivalence relation on $\mathcal{R}$.

(a) For every $a \in R$, $a\,\mathcal{C}_2^I a$ since $a \cup \mathrm{o} = a \cup \mathrm{o}$, with $\mathrm{o} \in I_2$.

(b) Obviously, for $a,\ b \in R$, if $a\,\mathcal{C}_2^I b$ then $b\,\mathcal{C}_2^I a$.

(c) For $a,\ b \in R$, if $a\,\mathcal{C}_2^I b$ and $b\,\mathcal{C}_2^I c$ then $a\,\mathcal{C}_2^I c$. In fact, if $a\,\mathcal{C}_2^I b$ *and* $b\,\mathcal{C}_2^I c$ then $a \cup j = b \cup j$ and $b \cup k = c \cup k$ for some j and k, both in I_2. Therefore,

$$\begin{aligned} a \cup (j \cup k) &= b \cup j \cup k \\ &= c \cup k \cup j \\ &= c \cup (j \cup k). \end{aligned}$$

Since $j \cup k \in I_2$, then $a\,\mathcal{C}_2^I c$.

(iii) Now we prove that, for $a,\ b \in R$ and $p,\ q \in B$, if $a\,\mathcal{C}_2^I b$ and $p\,\mathcal{C}_1^I q$ *then* $(ap)\mathcal{C}_1^I(bq)$. Let us assume that $p\,\mathcal{C}_1^I q$ *and* $a\,\mathcal{C}_2^I b$. Then $p \vee i = q \vee i$ for some $i \in I_1$ and $a \cup j = b \cup j$ for some $j \in I_2$ (and therefore $aq \vee jq = bq \vee jq$). Since $p \vee i = q \vee i$ then $ap \vee ai = aq \vee ai$ and moreover $ap \vee ai \vee jq = aq \vee ai \vee jq$. So

$$\begin{aligned} ap \vee (ai \vee jq) &= aq \vee jq \vee ai \\ &= bq \vee jq \vee ai \\ &= bq \vee (ai \vee jq). \end{aligned}$$

Since $ai \vee jq \in I_1$ then $(ap)\mathcal{C}_1^I(bq)$.

(iv) It remains to be proved that, for $a,\ b \in R$, if $a\,\mathcal{C}_2^I b$ and $c\,\mathcal{C}_2^I d$ then $(a \cup c)\mathcal{C}_2^I(b \cup d)$, $(ac)\mathcal{C}_2^I(bd)$ *and* $(a^*)\mathcal{C}_2^I(b^*)$. Let us admit that $a\,\mathcal{C}_2^I b$ and $c\,\mathcal{C}_2^I d$. Then there exist j, k in I_2 such that $a \cup j = b \cup j$ and $c \cup k = d \cup k$.

(a) So, $a \cup j \cup c \cup k = b \cup j \cup d \cup k$, and therefore, $(a \cup c) \cup (j \cup k) = (b \cup d) \cup (j \cup k)$. Since $j \cup k \in I_2$ then $(a \cup c)\mathcal{C}_2^I(b \cup d)$.

(b) From $c \cup k = d \cup k$ we get $a(c \cup k) = a(d \cup k)$, *i.e.*, $ac \cup ak = ad \cup ak$. (We note that we are working with a separable dynamic algebra , and, therefore, we are allowed to say that $a(b \cup c) = ab \cup ac$, for every $a,\ b,\ c \in R$.)

Hence,

$$\begin{aligned} ac \cup (ak \cup jd) &= ad \cup ak \cup jd \\ &= (a \cup j)d \cup ak \\ &= (b \cup j)d \cup ak \\ &= bd \cup (jd \cup ak). \end{aligned}$$

From $jd \cup ak \in I_2$ we get $(ac)\mathcal{C}_2^I(bd)$.

(c) Since $a \cup j = b \cup j$ then $ap \vee jp = bp \vee jp$, for every $p \in B$. In particular, for b^*p we have $a(b^*p) \vee j(b^*p) = b(b^*p) \vee j(b^*p)$. Then $ab^*p \vee jb^*p \vee a^*jb^*p = bb^*p \vee jb^*p \vee a^*jb^*p$.

Condition $(5a)$ of dynamic algebras yields $q \leq q \vee aa^*q \leq a^*q$ and consequently $q \vee a^*q = a^*q$, for every $q \in B$. For $q = jb^*p$, we get $jb^*p \vee a^*jb^*p = a^*jb^*p$.

Therefore

$$ab^*p \vee a^*jb^*p = bb^*p \vee a^*jb^*p. \tag{5.1}$$

But
$$\begin{aligned} a(b^*p \vee a^*jb^*p) &= ab^*p \vee aa^*jb^*p \\ &\leq ab^*p \vee a^*jb^*p \qquad (by\ (5a)) \\ &= bb^*p \vee a^*jb^*p \qquad (by\ (5.1)) \\ &\leq b^*p \vee a^*jb^*p. \qquad (by\ (5a)) \end{aligned}$$

If we take $r := b^*p \vee a^*jb^*p$, we have $ar \leq r$. Since $p \leq b^*p$ (for every $p \in B$) and $b^*p \leq r$ then $p \leq r$. Therefore $p \vee ar \leq r$, and so $r \in a!p$, and consequently, $a^*p \leq r$, *i.e.*, $a^*p \leq b^*p \vee a^*jb^*p$. Interchanging a and b, we get $b^*p \leq a^*p \vee b^*ja^*p$. Consequently, we have $a^*p \leq b^*p \vee a^*jb^*p \vee b^*ja^*p$ and $b^*p \leq a^*p \vee b^*ja^*p \vee a^*jb^*p$. Letting $l := a^*jb^* \cup b^*ja^*$ we get $a^*p \leq b^*p \vee lp$ and $b^*p \leq a^*p \vee lp$, and hence $a^*p \vee lp \leq b^*p \vee lp$ and $b^*p \vee lp \leq a^*p \vee lp$. Therefore, $a^*p \vee lp = b^*p \vee lp$, for every $p \in B$. So, there exists $l \in I_2$ such that $a^* \cup l = b^* \cup l$ which means that $(a^*)\mathcal{C}_2^I(b^*)$. ■

Proposition 5.11
If $I{=}(I_1, I_2)$ is an ideal on a separable dynamic algebra with zero, then $\mathcal{I}(\mathcal{C}(I)) = I$.

Proof. We pretend to show that $[0]_{\mathcal{C}_1^I} = I_1$ and $[o]_{\mathcal{C}_2^I} = I_2$. The first assertion is a well known Boolean fact.

(i) We begin by proving that $[\mathrm{o}]_{\mathcal{C}_2^I} \subseteq I_2$. For $a \in [\mathrm{o}]_{\mathcal{C}_2^I}$ we have $a \cup j = \mathrm{o} \cup j$, for some $j \in I_2$. Then, $a \cup j = j$, and therefore, $a \leq j$. Since $j \in I_2$ then $a \in I_2$.

(ii) Now, we prove that $I_2 \subseteq [\mathrm{o}]_{\mathcal{C}_2^I}$. Let $a \in I_2$. From $a \cup a = \mathrm{o} \cup a$ we get $a \in [\mathrm{o}]_{\mathcal{C}_2^I}$. ■

Proposition 5.12
On a separable dynamic algebra with zero, a dynamic ideal is a determining ideal if and only if it is the kernel of a separating congruence.

Proof. Let $I{=}(I_1, I_2)$ be a dynamic ideal on a separable dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$. By Prop.5.11 there exists a dynamic congruence θ such that $I = \mathcal{I}(\theta)$. Let ϕ be the separating congruence of θ. We have $\phi_1{=}\theta_1$ and $\phi_2{=}\{(a,b){:}ap\theta_1 bp$ for every $p \in B\}$. So, $\mathcal{I}_1^\phi = \mathcal{I}_1^\theta = I_1$ and $\mathcal{I}_2^\phi = \{a : a\phi_2\mathrm{o}\}$. Then, we have to prove that $\{a : a\,\phi_2\mathrm{o}\} = \{a : ap \in I_1$, for every $p \in B\}$. But, for every $a \in R$, we have $a\,\phi_2\mathrm{o}$ if and only if $ap\,\theta_1 0$, for every $p \in B$, *i.e.*, $ap \in [0]_{\theta 1} = I_1$. ■

As the following example will show, there exist separable dynamic algebras with zero where dynamic congruences may or may not have the same kernel of its separating congruences. So, we can have different congruences with the same kernel. It is a well known fact that every Boolean congruence is determined by its kernel. That is not the case for dynamic congruences.

Example 5.13
Let $\mathcal{D}_1 = (\mathcal{P}(S), \{a, b, c, d, \mathrm{o}\}, \langle\rangle)$, for $S{=}\{1,2,3\}$, $a{=}\{(1,1)\}$, $b{=}\{(2,2)\}$, $c{=}\{(1, 1), (2, 2)\}$, $d{=}\{(1, 1), (2, 2), (3, 3)\}$ and $\mathrm{o} = \emptyset$. We begin with a short study on the

structure $\mathcal{D}_1$, that will be useful later on. We note that $\mathcal{D}_1$ is a well defined Kripke structure. In fact,

$$\begin{array}{lllll}
a\{1\}=\{1\} & b\{1\}=\emptyset & c\{1\}=\{1\} & d\{1\}=\{1\} & o\{1\}=\emptyset \\
a\{2\}=\emptyset & b\{2\}=\{2\} & c\{2\}=\{2\} & d\{2\}=\{2\} & o\{2\}=\emptyset \\
a\{3\}=\emptyset & b\{3\}=\emptyset & c\{3\}=\emptyset & d\{3\}=\{3\} & o\{3\}=\emptyset
\end{array}$$

$$\begin{array}{lllll}
a\cup a=a & b\cup b=b & c\cup c=c & d\cup d=d & \mathrm{o}\cup\mathrm{o}=\mathrm{o} \\
a\cup b=c & b\cup c=c & c\cup d=d & d\cup\mathrm{o}=d & \\
a\cup c=c & b\cup d=d & c\cup\mathrm{o}=c & & \\
a\cup d=d & b\cup\mathrm{o}=b & & & \\
a\cup\mathrm{o}=a & & & &
\end{array}$$

$$\begin{array}{lllll}
aa=a & ba=\mathrm{o} & ca=a & da=a & \mathrm{o}a=\mathrm{o} \\
ab=\mathrm{o} & bb=b & cb=b & db=b & \mathrm{o}b=\mathrm{o} \\
ac=a & bc=b & cc=c & dc=c & \mathrm{o}c=\mathrm{o} \\
ad=a & bd=b & cd=c & dd=d & \mathrm{o}d=\mathrm{o} \\
a\mathrm{o}=\mathrm{o} & b\mathrm{o}=\mathrm{o} & c\mathrm{o}=\mathrm{o} & d\mathrm{o}=\mathrm{o} & \mathrm{oo}=\mathrm{o}
\end{array}$$

$$\begin{array}{lllll}
a^*=d & b^*=d & c^*=d & d^*=d & 0^*=d
\end{array}$$

Let $\theta=\mathcal{C}(I)$ and $\beta=\mathcal{C}(J)$ with I=($\{\emptyset, \{1\}, \{3\}, \{1,3\}\}$, $\{\mathrm{o}, a\}$) and J=($\{\emptyset, \{1\}\}$, $\{\mathrm{o}\}$). Since, I and J are dynamic ideals on $\mathcal{D}_1$ then θ and β are congruences with classes

$$\begin{array}{ll}
[\emptyset]_{\theta_1}=\{\emptyset,\{1\},\{3\},\{1,3\}\} & [\emptyset]_{\beta_1}=\{\emptyset,\{1\}\} \\
[\{2\}]_{\theta_1}=\{\{2\},\{1,2\},\{2,3\},\{1,2,3\}\} & [\{2\}]_{\beta_1}=\{\{2\},\{1,2\}\} \\
[\mathrm{o}]_{\theta_2}=\{\mathrm{o},\, a\} & [\{3\}]_{\beta_1}=\{\{3\},\{1,3\}\} \\
[b]_{\theta_2}=\{b,\, c\} & [\{2,3\}]_{\beta_1}=\{\{2,3\},\{1,2,3\}\} \\
[d]_{\theta_2}=\{d\}. & [m]_{\beta_2}=\{m\}, \text{ for every } m\in\{a,b,c,d,\mathrm{o}\}.
\end{array}$$

The separating congruences of θ and β are, respectively, δ and ϕ with classes

$$\begin{array}{ll}
[t]_{\delta_1}=[t]_{\theta_1}, \text{ for every } t\in\mathcal{P}(S) & [t]_{\phi_1}=[t]_{\beta_1}, \text{ for every } t\in\mathcal{P}(S) \\
[\mathrm{o}]_{\delta_2}=\{\mathrm{o},\, a\} & [\mathrm{o}]_{\phi_2}=\{\mathrm{o},\, a\} \\
[b]_{\delta_2}=\{b,\, c,\, d\} & [b]_{\phi_2}=\{b,\, c\} \\
 & [d]_{\phi_2}=\{d\}
\end{array}$$

So, while the kernel of β is ($\{\emptyset, \{1\}\}$, $\{\mathrm{o}\}$) and the kernel of ϕ is ($\{\emptyset, \{1\}\}$, $\{\mathrm{o}, a\}$), the congruences θ and δ have the same kernel ($\{\emptyset, \{1\}, \{3\}, \{1,3\}\}$, $\{\mathrm{o},a\}$).

To any dynamic ideal I, since $\mathcal{I}(\mathcal{C}(I)) = I$, the class of all dynamic congruences having I as its kernel is not empty. On the other hand, the intersection of all these dynamic congruences is always a congruence with kernel equal to I. Therefore, we can establish the following definition.

Definition 5.14
The *congruence generated* by a dynamic ideal I, is the smallest congruence with kernel equal to I.

Proposition 5.15
Let I=(I_1, I_2) be an ideal on a separable dynamic algebra with zero. Then $\mathcal{C}(I)$ is the congruence generated by I. In particular, $\mathcal{C}(\mathcal{I}(\theta)) \subseteq \theta$, for every dynamic congruence θ.

Proof. Let I=(I_1, I_2) be an ideal on a separable dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$. Let $\delta \in Cong\mathcal{D}$ such that $\mathcal{I}(\delta) = I$. If $(a, b) \in \mathcal{C}_2^I$, then $a \cup j = b \cup j$, for some $j \in I_2$. By assumption, $I_2 = \mathcal{I}_2^\delta$, and so, $a \cup j = b \cup j$, for some $j \in \mathcal{I}_2^\delta$, *i.e.*, $a \cup j = b \cup j$, for some j such that $j\, \delta_2 \text{o}$. Since $j \delta_2 \text{o}$ we have $(a \cup j)\, \delta_2\, (a \cup \text{o})$ and $(b \cup j)\, \delta_2\, (b \cup \text{o})$. Then $(a \cup j)\, \delta_2 a$ and $(b \cup j)\, \delta_2 b$, and from $a \cup j = b \cup j$, we get $a\, \delta_2 b$, *i.e.*, $(a, b) \in \delta_2$. Similarly, we prove that $\mathcal{C}_1^I \subseteq \delta_1$. (Actually, $\mathcal{C}_1^I = \delta_1$, a known result in Boolean algebras.) ■

We could be led to ask if, for every congruence θ=(θ_1, θ_2) on a separable dynamic algebra with zero, we have $\mathcal{C}(\mathcal{I}(\theta)) = \theta$. In general, that is not the fact.

Example 5.16
Recovering Ex.5.13, we have $\mathcal{C}(\mathcal{I}(\delta)) \neq \delta$, since $\mathcal{I}_2^\delta = \{\text{o}, a\} = [\text{o}]_{\theta_2}$, and therefore, $\mathcal{C}_2^{\mathcal{I}(\delta)} = \theta_2 \neq \delta_2$.

It is immediate to establish, as a consequence of Prop.5.11, a necessary and sufficient condition on θ to have the equality $\mathcal{C}(\mathcal{I}(\theta)) = \theta$.

Proposition 5.17
Let $\theta = (\theta_1, \theta_2) \in Cong\mathcal{D}$ where $\mathcal{D}$ is a separable dynamic algebra with zero. Then $\mathcal{C}(\mathcal{I}(\theta)) = \theta$ if and only if $\theta = \mathcal{C}(I)$ for some ideal I on $\mathcal{D}$, i.e., $\mathcal{C}(\mathcal{I}(\theta)) = \theta$ if and only if $\theta \in \mathcal{C}(\mathcal{I}de\mathcal{D})$.

6 The lattice *Ide*$\mathcal{D}$ and a Galois Connection between *Cong*$\mathcal{D}$ and *Ide*$\mathcal{D}$

We intend to insert a lattice structure into $Ide\mathcal{D}$, the set of all dynamic ideals on a separable dynamic algebra with zero, $\mathcal{D}$. To do so we need to define, for arbitrary dynamic ideals I and J, $I \wedge_\mathcal{D} J$ and $I \vee_\mathcal{D} J$. It is immediate to put $I \wedge_\mathcal{D} J = (I_1 \cap J_1, I_2 \cap J_2)$. Once again the disjunction requires some attention. Surprisingly, we are able to prove the following proposition. We notice that we use the fact that the subset of actions is a union of action ideals and not an arbitrary subset of actions.

We denote by $\langle X \rangle_\mathcal{A}$ the ideal generated by a subset X of any (homogeneous or heterogeneous) algebra $\mathcal{A}$, *i.e.*, the intersection of all ideals I on $\mathcal{A}$ containing X,

$$\langle X \rangle_\mathcal{A} := \cap\{I : I \textit{ ideal on } A \text{ and } X \subseteq I\}.$$

Proposition 6.1

Let I=(I_1, I_2) and J=(J_1, J_2) be elements of $Ide\mathcal{D}$, with $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ a separable dynamic algebra with zero. We have:

$$\langle I_1 \cup J_1 \rangle_\mathcal{B} = \{p \in B : p \leq p_1 \vee \cdots \vee p_n, \text{ for some integer } n \geq 1 \text{ and some } p_i \in I_1 \cup J_1, i = 1, \cdots, n\}$$

$$\langle I_2 \cup J_2 \rangle_\mathcal{R} = \{a \in R : a \leq a_1 \cup \cdots \cup a_n, \text{ for some integer } n \geq 1 \text{ and some } a_i \in I_2 \cup J_2, i = 1, \cdots, n\}$$

PROOF. We only have to prove the second identity since the first is a well known Boolean result [2]. We want to show that, for $X=\{a \in R: a \leq a_1 \cup \ldots \cup a_n$ for some integer $n \geq 1$ and some $a_i \in I_2 \cup J_2,\ i=1,\ldots,n\}$, we have:

(i) X is an (action) ideal of $\mathcal{R}$;
(ii) $I_2 \cup J_2 \subseteq X$;
(iii) if Y is an (action) ideal of $\mathcal{R}$ and $I_2 \cup J_2 \subseteq Y$, then $X \subseteq Y$.

Notice that X is nonempty, since $o \leq a_i$ for any $a_i \in I_2 \cup J_2$. Now, let $a,b \in X$, $c,d \in R$ and $d \leq a$. Then $a \leq a_1 \cup \ldots \cup a_n$, for some integer $n \geq 1$ and some $a_i \in I_2 \cup J_2$, $i=1,\ldots,n$, and $b \leq b_1 \cup \ldots \cup b_m$ for some integer $m \geq 1$ and some $b_j \in I_2 \cup J_2$, $j=1,\ldots,m$.

So, $a \cup b \leq a_1 \cup \ldots \cup a_n \cup b_1 \cup \ldots \cup b_m$ $d \leq a \leq a_1 \cup \ldots \cup a_n$, $ca \leq ca_1 \cup \ldots \cup ca_n$ and $ac \leq a_1 c \cup \ldots \cup a_n c$. Since for $i=1,\ldots,n$ and $j=1,\ldots,m$, a_i, b_j, ca_i and $a_i c$ are elements of $I_2 \cup J_2$, we get $a \cup b$, d, ca and ac in $\langle I_2 \cup J_2\rangle_{\mathcal{R}}$. Therefore, X is an (action) ideal of $\mathcal{R}$. It is straightforward that $I_2 \cup J_2 \subseteq X$. Let $a \in X$ and Y be an (action) ideal of $\mathcal{R}$ such that $I_2 \cup J_2 \subseteq Y$. Then $a \leq a_1 \cup \ldots \cup a_n$, for some integer $n \geq 1$ and some $a_i \in I_2 \cup J_2 \subseteq Y$, $i=1,\ldots,n$. But Y is an (action) ideal of $\mathcal{R}$ and $a_i \in Y$ for $i=1,\ldots,n$ so $a_1 \cup \ldots \cup a_n \in Y$. Therefore $a \leq s$ with $s = a_1 \cup \ldots \cup a_n \in Y$. Since Y is an (action) ideal of $\mathcal{R}$ we get $a \in Y$. ■

For $\mathcal{D}=(\mathcal{B},\mathcal{R},\langle\rangle)$ dynamic algebra, $I \subseteq B$ and $J \subseteq R$ we write JI to represent the set $JI := \{ap : a \in J \text{ and } p \in I\}$.

PROPOSITION 6.2
For $I=(I_1,I_2)$ and $J=(J_1,J_2)$ elements of $Ide\mathcal{D}$, with $\mathcal{D}=(\mathcal{B},\mathcal{R},\langle\rangle)$ a separable dynamic algebra with zero, we have

$$R(\langle I_1 \cup J_1\rangle_{\mathcal{B}}) \subseteq \langle I_1 \cup J_1\rangle_{\mathcal{B}},$$
$$(\langle I_2 \cup J_2\rangle_{\mathcal{R}})B \subseteq \langle I_1 \cup J_1\rangle_{\mathcal{B}}.$$

PROOF. Let $a \in R$ and $p \in \langle I_1 \cup J_1\rangle_{\mathcal{B}}$. By Prop.6.1, there exist an integer $n \geq 1$ and $p_i \in I_1 \cup J_1$, $i=1,\ldots,n$ such that $p \leq p_1 \vee \ldots \vee p_n$. Therefore $ap \leq ap_1 \vee \ldots \vee ap_n$ and, since either $p_i \in I_1$ or $p_i \in J_1$ for $i=1,\ldots,n$, we have either $ap_i \in I_1$ or $ap_i \in J_1$ and then $ap \in \langle I_1 \cup J_1\rangle_{\mathcal{B}}$.

Now, let $d \in \langle I_2 \cup J_2\rangle_{\mathcal{R}}$ *and* $q \in B$. By Prop.6.1, there exist an integer $m \geq 1$ and $d_i \in I_2 \cup J_2$, $i=1,\ldots,m$ such that $d \leq d_1 \cup \ldots \cup d_m$. Therefore $dq \leq d_1 q \vee \ldots \vee d_m q$ and, since either $d_i \in I_2$ or $d_i \in J_2$ for $i=1,\ldots,m$, we have either $d_i q \in I_1$ or $d_i q \in J_1$ and then $dq \in \langle I_1 \cup J_1\rangle_{\mathcal{B}}$. ■

Therefore, we are able to establish that the structure $\mathcal{I}de\mathcal{D}=(Ide\mathcal{D},\wedge_{\mathcal{D}},\vee_{\mathcal{D}})$ where, for every $I=(I_1,I_2)$, $J=(J_1,J_2) \in Ide\mathcal{D}$, the operations are defined by

$$I \wedge_{\mathcal{D}} J := I \cap J = (I_1 \cap J_1, I_2 \cap J_2)$$
$$I \vee_{\mathcal{D}} J := \langle I \cup J\rangle_{\mathcal{D}} = (\langle I_1 \cup J_1\rangle_B, \langle I_2 \cup J_2\rangle_R)$$

is a lattice, called the *lattice of ideals* of $\mathcal{D}$.

DEFINITION 6.3
[1] A *Galois-connection* between the partially-ordered sets $(A,\leq_A)$ and $(B,\leq_B)$ is a pair of functions $(\mathcal{C},\mathcal{I})$, $\mathcal{C}: A \longrightarrow B$, $\mathcal{I}: B \longrightarrow A$ satisfying, for all $x \in A$ and $y \in B$, the following restriction

$$\mathcal{C}(x) \leq_B y \text{ iff } x \leq_A \mathcal{I}(y).$$

THEOREM 6.4
The pair of maps $\mathcal{C} : Ide\mathcal{D} \longrightarrow Cong\mathcal{D}$ (that for each $I \in Ide\mathcal{D}$ assigns the congruence $\mathcal{C}(I)$) and $\mathcal{I} : Cong\mathcal{D} \longrightarrow Ide\mathcal{D}$ (that for each $\theta \in Cong\mathcal{D}$ assigns the ideal $\mathcal{I}(\theta)$) defines a Galois-connection $(\mathcal{C}, \mathcal{I})$, between the lattices $Ide\mathcal{D}$ and $Cong\mathcal{D}$.

PROOF. Let I be in $Ide\mathcal{D}$ and θ be in $Cong\mathcal{D}$. If $\mathcal{C}(I) \subseteq \theta$ then $\mathcal{I}(\mathcal{C}(I)) \subseteq \mathcal{I}(\theta)$, and by Prop.5.11 we have $I \subseteq \mathcal{I}(\theta)$. Now, if $I \subseteq \mathcal{I}(\theta)$ then $\mathcal{C}(I) \subseteq \mathcal{C}(\mathcal{I}(\theta))$. By Prop.5.15 we conclude $\mathcal{C}(I) \subseteq \theta$. ■

As an immediate consequence of 7.33 of [1] we have the following results.

COROLLARY 6.5
Let $\mathcal{D}$ be a separable dynamic algebra with zero. Then
1. for each $I \in Ide\mathcal{D}$, $\mathcal{C}(I)$ is the least congruence θ such that $I \subseteq \mathcal{I}(\theta)$;
2. for each $\theta \in Cong\mathcal{D}$, $\mathcal{I}(\theta)$ is the greatest ideal I such that $\mathcal{C}(I) \subseteq \theta$.

7 Additive Ideals

We already know that, in general, there are several dynamic congruences with the same kernel. The aim of this section is to study conditions under which a dynamic congruence is determined by its kernel. For that purpose we introduce the concept of *additive ideal*.

DEFINITION 7.1
An ideal I=(I_1, I_2) on a separable dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ is said to be an *additive ideal* if, for every a,$b \in R$ such that ap+$bp \in I_1$, for every $p \in B$, there exists a $c \in I_2$ such that $ap + bp = cp$, for every $p \in B$.

On a separable dynamic algebra with zero, the pair $(\{0\}, \{\text{o}\})$ is an additive ideal. Actually, it is the unique ideal with $\{0\}$ as Boolean part.

The following table distinguishes the ideals on the dynamic algebra $\mathcal{D}_1$, given at Ex.5.13, into additive and non-additive ideals.

Additive Ideals $I = (I_1, I_2)$
$I_1 = \{\emptyset\}$; $I_2 = \{\text{o}\}$
$I_1 = \{\emptyset, \{1\}\}$; $I_2 = \{\text{o}, a\}$
$I_1 = \{\emptyset\ \{2\}\}$; $I_2 = \{\text{o}, b\}$
$I_1 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$; $I_2 = \{\text{o}, a, b, c\}$

Non-Additive Ideals $I = (I_1, I_2)$
$I_1 = \{\emptyset, \{l\}\}$; $I_2 = \{\text{o}\}$
$I_1 = \{\emptyset, \{2\}\}$; $I_2 =\{\text{o}\}$
$I_1 = \{\emptyset, \{3\}\}$; $I_2 = \{\text{o}\}$
$I_1 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$; $I_2 = \{\text{o}\}$
$I_1 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$; $I_2 = \{\text{o}, a\}$
$I_1 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$; $I_2 = \{\text{o}, b\}$
$I_1 = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$; $I_2 = \{\text{o}\}$
$I_1 = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$; $I_2 = \{\text{o}, a\}$
$I_1 = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$; $I_2 = \{\text{o}\}$
$I_1 = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$; $I_2 = \{\text{o}, b\}$
$I_1 = \mathcal{P}(S)$; (S); $I_2 = \{\text{o}\}$
$I_1 = \mathcal{P}(S)$; $I_2 = \{\text{o}, a\}$
$I_1 = \mathcal{P}(S)$; $I_2 = \{\text{o}, b\}$
$I_1 = \mathcal{P}(S)$; $I_2 = \{\text{o}, a, b, c\}$
$I_1 = \mathcal{P}(S)$; $I_2 = \{\text{o}, a, b, c, d\}$

THEOREM 7.2
Let $\theta = (\theta_1, \theta_2) \in Cong\mathcal{D}$ where $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$ is a separable dynamic algebra with zero. If $\mathcal{I}(\theta)$ is an additive ideal, then

1. $\mathcal{C}(\mathcal{I}(\theta)) = \theta$;
2. θ is determined by its kernel.

PROOF. (1) We intend to show that $\mathcal{C}(\mathcal{I}(\theta)) = \theta$, *i.e.*, that $\mathcal{C}_1^{\mathcal{I}(\theta)} = \theta_1$ and $\mathcal{C}_2^{\mathcal{I}(\theta)} = \theta_2$. We are going to prove only that $\mathcal{C}_2^{\mathcal{I}(\theta)} = \theta_2$ (since $\mathcal{C}_1^{\mathcal{I}(\theta)} = \theta_1$ is a known result in Boolean algebras).

(i) We begin by showing that $\mathcal{C}_2^{\mathcal{I}(\theta)} \subseteq \theta_2$. Let $(a, b) \in \mathcal{C}_2^{\mathcal{I}(\theta)}$. Then $a \cup j = b \cup j$, for some $j \in \mathcal{I}_2^{\theta}$. Since $j\, \theta_2 \mathrm{o}$ then $(a\cup j)\ \theta_2\ (a\cup \mathrm{o})$ and $(b\cup \mathrm{j})\theta_2\ (b\cup \mathrm{o})$, and therefore, $(a\cup j)\theta_2 a$ and $(b\cup j)\theta_2 b$. Since $a\cup j = b\cup j$ then $a\ \theta_2 b$, *i.e.*, $(a,b) \in \theta_2$.

(ii) Now we prove that $\theta_2 \subseteq \mathcal{C}_2^{\mathcal{I}(\theta)}$. For $(a, b) \in \theta_2$ we have $ap\theta_1 bp$, for every $p \in B$. And therefore, $(ap+bp)\ \theta_1 0$, *i.e.*, $ap + bp \in [0]_{\theta_1} = \mathcal{I}_1^{\theta}$. Since $\mathcal{I}(\theta)$ is an additive ideal, there exists a $c \in \mathcal{I}_2^{\theta}$ such that $ap + bp = cp$, for every $p \in B$. Therefore $ap \vee cp = bp \vee cp$, and so, $(a\cup c)p = (b\cup c)p$, for every $p \in B$. The separability of $\mathcal{D}$ brings $a\cup c = b\cup c$, with $c \in \mathcal{I}_2^{\theta}$, *i.e.*, $(a, b) \in \mathcal{C}_2^{\mathcal{I}(\theta)}$.

(2) We have to prove that θ is the unique congruence having $\mathcal{I}(\theta)$ as kernel. Let δ be a congruence on $\mathcal{D}$ such that $\mathcal{I}(\delta) = \mathcal{I}(\theta)$. Then $\mathcal{I}(\delta)$ is an additive ideal and, by (1), we have $\delta = \mathcal{C}(\mathcal{I}(\delta)) = \mathcal{C}(\mathcal{I}(\theta)) = \theta$. ■

A counterexample to the reciprocal result of Th.7.2(1) is given by a congruence generated by a non-additive ideal.

PROPOSITION 7.3
Let $I = (I_1, I_2)$ and $I' = (I_1, I_2')$ be ideals on a separable dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$. If I is an additive ideal on $\mathcal{D}$ then

1. $I' \subseteq I$;
2. I is a determining ideal on $\mathcal{D}$.

PROOF. 1. For $a \in I_2'$, we have $ap \in I_1$, for every $p \in B$. Since I is additive and $ap + \mathrm{o}p \in I_1$, for every $p \in B$ (we recall that $ap + \mathrm{o}p = ap$), there exists a $c \in I_2$ such that $ap + \mathrm{o}p = cp$, for every $p \in B$. Therefore $ap = cp$, for every $p \in B$ and, since $\mathcal{D}$ is separable, we have $a = c$, showing that $a \in I_2$.

2. By 1., if there exists an additive ideal with Boolean part I_1, then it is the greatest (and the unique) additive ideal with Boolean part I_1. By Prop.5.3 the determining ideal of any ideal having I_1 as Boolean part is the greatest ideal with this property. ■

By recalling Ex.5.13, $(\{\emptyset, \{1\}, \{3\}, \{1,3\}\}, \{\mathrm{o}, \mathrm{a}\})$ is an example of a non-additive determining ideal. Therefore, a determining ideal is not necessarily an additive ideal.

COROLLARY 7.4
If $I = (I_1, I_2)$ is an additive ideal on a separable dynamic algebra with zero $\mathcal{D} = (\mathcal{B}, \mathcal{R}, \langle\rangle)$, then $\mathcal{C}(I)$ is a separating congruence.

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