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A STUDY ON s-IFR AND s-IFRA ORDERINGS AND AN APPLICATION TO PARALLEL SYSTEMS

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Abstract

Many classes of lifetime distributions can be characterized by their ageing properties, which can be described by their survival function or by their failure rate function. These ageing properties will allow us to compare in some sense two distributions within the same family, giving rise to ageing orderings. In this work, we start by presenting two ageing notions related to the monotonicity of the failure rate function and study some of their properties. We also define two ageing orders, the s-IFR and s-IFRA, for which we will present some equivalent conditions and results that facilitate the comparison between two distributions. Furthermore, we present a criterium for the s-IFR order based on the s-IFRA ordering, which will be the "key" result for comparing two parallel systems with two exponentially distributed components. In the last chapter of this monograph, we aim to apply the criterium in order to prove that, in terms of the s-IFRA order, a parallel system with three homogeneous, independent and exponentially distributed components ages faster than a parallel system with three heterogeneous, independent and exponentially distributed components.

Resumo

Muitas classes de distribuições que representam o tempo de vida de um sistema são caracterizadas pelas suas propriedades de envelhecimento, que podem ser descritas através da função de sobrevivência ou da função de taxa de falha. Estas propriedades permitem-nos estabelecer comparações entre distribuições dentro da mesma família, dando origem às ordenações de envelhecimento. Neste trabalho, começamos por apresentar duas noções de envelhecimento, relacionadas com a monotonia da função de taxa de falha, bem como algumas propriedades da mesma. Definimos, também, duas ordenações de envelhecimento, s-IFR e s-IFRA, e apresentamos algumas condições equivalentes e resultados que facilitam a comparação entre duas distribuições. Para além disso, um critério para a ordenação s-IFR é apresentado que será o resultado "chave" para a comparação entre sistemas paralelos com duas componentes com distribuições exponenciais. No último capítulo deste trabalho, o nosso objectivo é provar, em termos da ordem s-IFRA, que um sistema paralelo de três componentes independentes, identica e exponencialmente distribuídas envelhece mais rápido do que um sistema paralelo com três componentes idependentes e exponencialmente distribuídas, com taxas de falha diferentes.

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Chapter 1

Introduction

Ageing and ordering notions between random variables have been quite popular in many areas of probability and statistics, such as, reliability theory, insurance, survival analysis, etc. Ageing notions are usually expressed in terms of the monotonicity of the survival or failure rate functions, while the orderings between the distributions of the variables are defined through relations between this type of functions. The simplest orderings notions are based on a direct comparison between the survival functions or failure rate functions. However, more interesting ordering notions are based on the comparison of the tail (survival) functions through the convexity between the inverse of these functions. These orderings are known as convex orderings and they were introduced by Hardy, Littlewood and Pólya [5]. In Shaked and Shantikumar [9] and in Marshal and Olkin [7], a compilation of orderings and ageing notions, including the convex orderings, can be found, as well as relations between them.

In the case of two random variables that represent the lifetime of two different systems, it is often useful to know which one ages faster. The ageing orderings allow us to compare, in some sense the age of systems, by analysing the behaviour of the distributions associated to their lifetimes. Some of these orderings are based on the behaviour of distributions defined iteratively (see Definition 9), that were studied by Avarous and Meste [3]. These notions can be found in Fagiuoli and Pellerey [4] and in Nanda et. al. [8], where the main concern was establishing different relations between several orderings, with no explicit examples. However, the actual verification of these relations is very difficult, given that they are defined via iterated distributions and, naturally, the computational side of the problem becomes increasingly difficult. This problem was studied in Arab and Oliveira [1] for Weibull and Gamma distributions and for a specific ageing order, the s-IFR order, which compares two distributions through the relative convexity of the survival functions (see Definition 13 for a precise statement). As referred, one of the most important orderings studied by the aforementioned authors is the s-IFR order which is a generalization of the so-called convex transform order (see Definition 3), introduced by Van Zwet [12]. Interestingly, these ideas were used in [12] to introduce and compare skewness between distributions with support in $(0,+\infty)$.

Kochar and Xu [6] proved that a parallel system with exponential component lifetimes with different hazard rates is more skewed than a parallel system with exponential component lifetimes, but with equal hazard rates, according to the convex transform order (or 1-IFR order, which is the same as the IFR order). In other words, they proved that the second system ages faster then the first one, with respect to the IFR order. Arab et. al. [2] extended this result for the s-IFR ordering for parallel systems with two exponentially distributed components.

Our aim is to extend the result established in Arab et. al [2] for parallel systems with three exponentially distributed components. This monograph is structured as follows. In Chapter 2, some stochastic orders and relations between them are presented. We begin by introducing the so called transform orders i.e. the convex, star and superadditive orders. In Section 2.2, a special case of the star order is introduced and a result that relates this and the convex transform orders is established. Finally, in the last section of this chapter, we explore alternative characterizations of monotonicity and convexity described through the study of the sign variation of the relevant functions. The definitions and results presented here are based on Shaked and Shantikumar [9] and on Arab and Oliveira [1].

In Chapter 3, iterated distributions are defined as well as iterated failure rate function. These two definitions give rise to ageing notions that are characterized by the monotonicity of the iterated failure rate function, presented in Section 3.1. In Sections 3.2 and 3.3, hereditary properties of ageing notions are studied. In Section 3.4, we introduce the s-IFR and s-IFRA orderings defined by Nanda et al. [8] and prove some of their properties as well as provide alternative characterizations. We also compare two Weibull distributions using the established characterizations. In Section 3.5, we analyse whether the hereditary property is satisfied by the s-IFR ordering. Finally, in the last section, a criterium for the s-IFR order, through the s-IFRA order is proved, that facilitates in some cases the comparison between two distributions. The definitions and results established in this chapter are mostly based on Arab et. al [2] and on Arab and Oliveira [1].

The last chapter is divided in three sections. In the first one, a comparison between two parallel systems with two exponentially distributed components is presented, according to the s-IFR order. The results proved in this section are based on Arab et. al. [2]. In order to establish a similar result for parallel systems with three exponentially distributed components it is necessary to prove that the criterium presented in Section 3.5 (see Theorem 13 for a precise statement) is valid. Section 4.2 is devoted to the comparison of two parallel systems with three exponentially distributed components according to the s-IFRA order in order to establish a result similar to the one presented in Section 4.1. However, due to the fact that there are 3 components, the difficulty is significantly increased and therefore the desired result was obtained only for a number of cases. Thus, in the last section of Chapter 4, a difficult case, which is not yet solved, is presented, showing the difficulties that can arise. Finally, in Section 4.4, some considerations about future work are presented.

Chapter 2

The transform orders

Throughout this chapter we will assume that X and Y are nonnegative absolutely continuous random variables with distribution functions F and G, and density functions f and g, respectively. Furthermore, we denote by $\mathscr{F}$ the family of distributions such that the distribution function at 0 is equal to 0 and the corresponding density function has support in $[0,+\infty)$. The results presented here are based on [7] and [9].

The ageing orderings presented in this chapter belong to a particular class of stochastic orders, namely the transform orders. They compare random variables with respect to the skewness of the density functions in $\mathscr{F}$. In the sections that follow, the definitions of the different transform orders will be presented as well as their equivalent characterization.

2.1 Transform orders: convex, star and superadditive orders

The notion of skewness is intended to represent the departure of a density from symmetry, where one tail of the density is more "stretched out" than the other. If the mode of such a density is to the left of the center and the right-hand tail is relatively long, then the density is said to be skewed to the right. This is the kind of skewness that is encountered in the family of distributions in $\mathscr{F}$.

There are many ways to measure the skewness of a distribution, however an alternative way of measuring this, is to find an ordering for which, when comparing two distributions, F and G, the inequality "$F \leq G$" captures the meaning of F being less skewed than G. This gives rise to the so-called convex transform order. Since this ordering is defined through the convexity of a given function we start by recalling the notion of a convex function.

Definition 1. *A real-valued function f defined on $[0,+\infty)$ is said to be convex if, for $\alpha \in [0,1]$,*

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \text{ for every } x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Moreover, we also need to recall the definition of quantile through the right continuous inverse of a distribution function, in order to define the convex transform order.

Definition 2. *Let X be a random variable with distribution function F. Then the right continuous inverse of F, F^{-1}, is defined by*

$$F^{-1}(u) = \sup\{x : F(x) \leq u\}, \text{ for every } u \in [0,1].$$

Now, if we consider X to be a nonnegative random variable with distribution function F and $Y = G^{-1}(F(X))$, i.e., another nonnegative random variable with distribution function G, an interesting question would be what properties does $G^{-1} \circ F$ must have so that Y has a more skewed distribution? Let us suppose that the density function of X, f, is made with rubber material and that becomes thinner and thinner toward the right. If we stretch out the right-hand edge of the rubber the density function will change. In fact, f is transformed in another density, g, that has a relatively longer right-hand tail. This flexibility that is required, simply means that the horizontal axis has been transformed by an increasing function $G^{-1} \circ F$, with increments that increase as one moves to the right, that is, $G^{-1}(F(x+\delta)) - G^{-1}(F(x))$ is increasing in $x \geq 0$, for $\delta > 0$. But it is possible to prove that this is equivalent to having that $G^{-1} \circ F$ is convex.

Definition 3. *Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. Then X is said to be smaller than Y in the convex transform order, denoted by $X \leq_c Y$, if $G^{-1}(F(x))$ is convex.*

Therefore, if $X \leq_c Y$, we have that X is less skewed than Y, that is F is less skewed than G. Note that, if $\overline{F}(x) = 1 - F(x)$, denotes the survival function of a random variable with distribution function F, we have that $G^{-1}(F(x)) = \overline{G}^{-1}(\overline{F}(x))$, for $x \geq 0$, so we can also study the convex transform order by analysing the convexity of $\overline{G}^{-1}(\overline{F}(x))$.

Another transform order, the star order, is defined in a very similar fashion to the convex order, but instead of the convexity condition we have the condition introduced in the following definition.

Definition 4. *A real-valued function f defined on $[0,+\infty)$ is said to be starshaped if $\frac{f(x)}{x}$ is increasing for $x > 0$.*

Having introduced this new class of functions we can now define the star order. Note that this order also compares the skewness of two different distributions.

Definition 5. *Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. Then X is said to be smaller than Y in the star order, denoted by $X \leq_* Y$, if $G^{-1}(F(x))$ is starshaped.*

It is easy to prove that, for $F, G \in \mathscr{F}$, $\frac{G^{-1}(F(x))}{x}$ is increasing for $x > 0$ if and only if $G^{-1}(F(\lambda x)) \leq \lambda G^{-1}(F(x))$, for every $\lambda \in [0,1]$. Therefore, we can interpret the star order as a generalization of the convex transform order. A simple characterization of this order is given in the following result.

Theorem 1. *$X \leq_* Y$ if and only if $\dfrac{G^{-1}(u)}{F^{-1}(u)}$ is increasing for $u \in (0,1)$.*

Proof. The conclusion follows immediately by choosing $u = F(x)$ in Definition 5 and by taking into account that F is an increasing function. □

Finally, we present another transform order that also compares the skewness of distributions. However, the particular order will not be part of this study and it is presented here only for the sake of completeness. To define this stochastic order we first need to introduce a new class of functions, the superadditive functions.

Definition 6. *A real-valued function f defined on $[0,+\infty)$ is said to be superadditive if*

$$f(x+y) \geq f(x)+f(y), \text{ for } x \geq 0 \text{ and } y \geq 0.$$

Therefore, the superadditive order is defined as follows.

Definition 7. *Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. Then X is said to be smaller than Y in the superadditive order, denoted by $X \leq_{su} Y$, if $G^{-1}(F(x))$ is supperaditive.*

Having defined the three transform orders, the first question that arises is whether these orders can somehow be related. In fact, if we think about the star and convex transform orders, their relation can easily be revealed as an immediate consequence of the following proposition.

Theorem 2. [7, Proposition 21.A.11] *Let f be a real valued function, such that $f(0) \leq 0$. If f is convex for $x \geq 0$ then f is starshaped.*

Proof. If f is convex for $x \geq 0$ then for all $x_1, x_2 \in [0,\infty)$ and for all $\alpha \in [0,1]$,

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$

Taking $x_2 = 0$ in the previous inequality we obtain $f(\alpha x_1) \leq \alpha f(x_1)$. Since $\alpha \in [0,1]$ and $x \geq 0$, we have that $\alpha x_1 \leq x_1$. Thus,

$$\frac{f(\alpha x_1)}{\alpha x_1} \leq \frac{\alpha f(x_1)}{\alpha x_1} = \frac{f(x_1)}{x_1}.$$

Hence, $\frac{f(x)}{x}$ is increasing for all $x > 0$, that is, f is starshaped. □

Corollary 1. *If $X \leq_c Y$ then $X \leq_* Y$.*

Proof. If $X \leq_c Y$ we have that $G^{-1}(F(x))$ is convex for $x \geq 0$. Moreover, since $F(0) = G(0) = 0$ and both variables have density functions, we can conlude that $G^{-1}(F(0)) = 0$. Hence, $G^{-1}(F(x))$ is starshaped, i.e., $X \leq_* Y$. □

Analogously, we have that the star order implies the superaditive order.

Theorem 3. [7, Proposition 21.A.11] *Let f be a real valued function, such that f is starshaped. Then f is superadditive.*

Proof. As f is starshaped, we have that

$$\frac{f(\alpha(x+y))}{\alpha(x+y)} \leq \frac{f(x+y)}{x+y} = \frac{\alpha f(x+y)}{\alpha(x+y)}, \forall x,y \geq 0, \forall \alpha \in [0,1],$$

that is, $f(\alpha(x+y)) \leq \alpha f(x+y)$. Choosing $\alpha = \frac{x}{x+y}$, the previous inequality can be re-written as $f(x) \leq \frac{x}{x+y} f(x+y)$, while choosing $\alpha = \frac{y}{x+y}$, we have $f(y) \leq \frac{y}{x+y} f(x+y)$. Summing these two inequalities, we have $f(x)+f(y) \leq f(x+y)$, that is, f is superadditive. □

Corollary 2. *If $X \leq_* Y$ then $X \leq_{su} Y$.*

Therefore, we may conclude that the convex transform order, also implies the superadditive order, since the first one implies the star order and this in turn implies the superadditive order. Thus, we have established a relation between all the transform orders.

2.2 DMRL order

We will now introduce another order that compares normalized mean values of the tail probabilities above a given quantile.

Definition 8. *Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. Then X is said to be smaller than Y in the DMRL order, denoted by $X \leq_{DMRL} Y$, if*

$$\frac{\dfrac{1}{E(Y)}\displaystyle\int_{G^{-1}(u)}^{\infty}\overline{G}(x)\,dx}{\dfrac{1}{E(X)}\displaystyle\int_{F^{-1}(u)}^{\infty}\overline{F}(x)\,dx} \quad \textit{is increasing for } u\in[0,1].$$

In what follows, we will denote $\overline{F}_e(t)=\frac{1}{E(X)}\int_t^{\infty}\overline{F}(x)\,dx$ and $\overline{G}_e(t)=\frac{1}{E(Y)}\int_t^{\infty}\overline{G}(x)\,dx$.

Remark 1. *The DMRL order is in fact a special case of the star order. In fact, observe that*

$$\frac{\overline{G}_e\left(G^{-1}(u)\right)}{\overline{F}_e\left(F^{-1}(u)\right)} \text{ is increasing for } u\in[0,1] \Leftrightarrow \frac{\overline{G}_e\left(\overline{G}^{-1}(u)\right)}{\overline{F}_e\left(\overline{F}^{-1}(u)\right)} \text{ is decreasing for } u\in[0,1]$$

But this is equivalent to $\frac{\overline{F}_e(\overline{F}^{-1}(u))}{\overline{G}_e(\overline{G}^{-1}(u))}$ being increasing for $u\in[0,1]$. Taking $u=\overline{G}(x)$ and since $\overline{G}$ is decreasing, we have that $X\leq_{DMRL} Y$ if and only if $\frac{\overline{F}_e(\overline{F}^{-1}(\overline{G}(x)))}{\overline{G}_e(x)}$ is decreasing for $x\geq 0$. Now, observing that $\overline{G}_e^{-1}$ is also decreasing and taking $x=\overline{G}_e^{-1}(y)$, we have that $X\leq_{DMRL} Y$ is equivalent to $\frac{\overline{F}_e(\overline{F}^{-1}(\overline{G}(\overline{G}_e^{-1}(y))))}{y}$ being increasing for $y\in[0,1]$. Therefore, noting that $\overline{G}\circ\overline{G}_e^{-1}$ and $\overline{F}\circ\overline{F}_e^{-1}$ are distribution functions, we may conclude that the DMRL order is a star order.

It is important to mention that, although the convex transform order implies the star order, we cannot conclude immediately that the convex transform order implies the DMRL order. In fact, if Z and W are random variables with distribution functions $\overline{F}\circ\overline{F}_e^{-1}$ and $\overline{G}\circ\overline{G}_e^{-1}$, respectively, we can conclude that $X\leq_{DMRL} Y$ is equivalent to $W\leq_* Z$, but we cannot conclude directly that $X\leq_c Y$ implies $X\leq_{DMRL} Y$. However, these two orders are indeed related as we can see in the following proposition.

Proposition 1. *Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. If $X\leq_c Y$ then $X\leq_{DMRL} Y$.*

Proof. Define $\alpha(x)=G^{-1}(F(x))$, for $x\geq 0$, and $\gamma(u)=\overline{F}_e(\alpha^{-1}(\overline{G}_e^{-1}(u)))$, for $u\in[0,1]$. Differentiating γ, we obtain, $\gamma'(u)=\frac{E(Y)}{E(X)}\left(\frac{d}{dx}\alpha^{-1}(x)\right)_{\overline{G}_e^{-1}(u)}$. Since $X\leq_c Y$, we have that α is convex. Noting that α is increasing, it follows that α^{-1} is concave. In fact, if we denote $a=\alpha^{-1}(\lambda x_1+(1-\lambda)x_2)$ and $b=\lambda\alpha(x_1)+(1-\lambda)\alpha(x_2)$, for $\lambda\in[0,1]$, we have that,

$$\alpha(a)=\lambda x_1+(1-\lambda)x_2=\lambda\alpha(\alpha^{-1}(x_1))+(1-\lambda)\alpha(\alpha^{-1}(x_2))\geq\alpha(\lambda\alpha^{-1}(x_1)+(1-\lambda)\alpha^{-1}(x_2)),$$

since α is convex. Thus, because α is increasing, it follows that $a \geq b$, i.e., α^{-1} is concave. But this is equivalent to $\frac{d}{dx}\alpha^{-1}(x)$ being decreasing. So, $\left(\frac{d}{dx}\alpha^{-1}(x)\right)_{\overline{G}_e^{-1}(u)}$ is increasing and we may conclude that γ is a convex function. But, by Proposition 2, this implies that γ is starshaped, that is,

$$\frac{\gamma(u)}{u} \quad \text{is increasing for } u \in [0,1] \Leftrightarrow \frac{\overline{F}_e\left(F^{-1}\left(G\left(\overline{G}_e^{-1}(u)\right)\right)\right)}{u} \quad \text{is increasing for } u \in [0,1]. \quad (2.1)$$

Now, taking $u = \overline{G}_e(x)$, since $\overline{G}_e$ is decreasing, (2.1) is equivalent to $\frac{\overline{F}_e(F^{-1}(G(x)))}{\overline{G}_e(x)}$ being decreasing for $x \geq 0$. Analogously, taking $x = G^{-1}(y)$, since G is increasing, we have that (2.1) is also equivalent to $\frac{\overline{F}_e(F^{-1}(y))}{\overline{G}_e(G^{-1}(y))}$ being decreasing for $y \in [0,1]$. Hence, the conclusion follows. □

2.3 A different characterization of transform orders

It is obvious that the characterization of these type of orders is mostly based on the study of the monotonicity or convexity of the relevant functions. However, this kind of analysis is not always simple, since the functions may have complicated expressions or may even not have explicit closed representations. So it is of interest to have results that provide us with new tools that facilitate this kind of analysis. In the following we will introduce two results, without proof, that characterize increasing and convex functions, respectively. These results will allow us to characterize in a different way the previous orders as well as the orders that will be presented in the next chapter.

Proposition 2. [1, Lemma 8] *A function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is increasing (resp., decreasing) if and only if for every $a \in \mathbb{R}$, $g(x) - a$ changes sign at most once when x traverses from $-\infty$ to $+\infty$ and if the change occurs it is in the order "$-,+$" (resp., "$+,-$").*

Proposition 3. [1, Theorem 14] *Let f be a continuous function. The function f is convex if and only if for every real numbers a and b, $f(x) - (ax+b)$ changes sign at most twice when x traverses from $-\infty$ to $+\infty$ and if the change of sign occurs twice, it is in the order "$+,-,+$".*

The next proposition provides a characterization of the crossing of two graphical representations. This result will allow us to describe the sign variation of the functions mentioned in Propositions 2 and 3, based on a simpler function.

Proposition 4. [1, Lemma 15] *Let f and g two real-valued functions, and ζ be a strictly increasing (resp., decreasing) and continuous functions defined on the range of f and g. For any real number $c \geq 0$, the functions $f(x) - cg(x)$ and $\zeta(f(x)) - \zeta(cg(x))$ have the same (resp., reverse) sign variation order as x traverses from $-\infty$ to $+\infty$.*

The next result describes the sign variation of a function, after performing integration. This result also provides a way to facilitate the study of the functions when trying to analyse their sign variation.

Proposition 5. [1, Lemma 18] *Let f and g be two real-valued functions defined on $[0,+\infty)$ such that $g(x) = \int_x^\infty f(t)dt$. Assume that, as x traverses from 0 to $+\infty$, $f(x)$ changes sign in one of the following orders "$-,+$" or "$+,-$" or "$+,-,+$" or "$-,+,-,+$". Then $g(x)$, as x traverses from 0 to $+\infty$ has sign variation equal to every possible final part of the sign variation of $f(x)$.*

The results given above lead to the following characterization of the star order.

Theorem 4. *Let X and Y be two nonnegative random variables, with distribution functions F and G respectively. Then, $X \leq_* Y$ if and only if, for every $a \geq 0$, $F(\cdot) - G(a\cdot)$ changes sign at most once and if the change occurs it is in the order "$-, +$".*

Proof. We know that F and G are nonnegative functions, thus we can apply Proposition 2 for $a \geq 0$. Hence, for every $a \geq 0$, $X \leq_* Y$ if and only if $\frac{G^{-1}(F(x))}{x}$ is increasing for $x > 0$, that is, $G^{-1}(F(x)) - ax$ changes sign at most once and if the change occurs it is in the order "$-, +$". As G is an increasing function, by applying Proposition 4 with $G = \zeta$, the result follows. □

Finally, the following theorem is a direct consequence of Proposition 3 and Proposition 4.

Theorem 5. *Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. Then $X \leq_c Y$ if and only if, for every constant a and b, $F(\cdot) - G(a\cdot + b)$ changes sign at most twice and if the changes occurs twice it is in the order "$+, -, +$.*

As an application of the previous results, the next example compares two Weibull distributions, according to the convex transform order. We will denote by $W(\alpha, \theta)$ the Weibull distribution with shape parameter α and scale parameter θ.

Example 1. *Let X and Y be two random variables, with $W(\alpha_1, \theta_1)$ and $W(\alpha_2, \theta_2)$ distributions, respectively, such that $\alpha_1 > \alpha_2$ and $\theta_1, \theta_2 > 0$. In order to prove that $X \leq_c Y$, according to Proposition 3, we need to analyse the sign variation of the function $F(x) - G(ax + b)$, for every constant a and b, and prove that it changes sign at most twice and if the changes occurs twice it is in the order "$+, -, +$". Note that, by Proposition 4, it is possible to conclude that this is equivalent to prove that $H(x) = \overline{G}(x) - \overline{F}(ax + b)$ changes sign at most twice and if the change occurs twice it is in the order "$+, -, +''$, for every constant a and b.*

For the distributions under study, we have that, $\overline{F}(x) = e^{-(\frac{x}{\theta_1})^{\alpha_1}}$ and $\overline{G}(x) = e^{-(\frac{x}{\theta_2})^{\alpha_2}}$. Therefore, we want to analyse the sign variation of the function $H(x) = e^{-(\frac{x}{\theta_2})^{\alpha_2}} - e^{-(\frac{ax+b}{\theta_1})^{\alpha_1}}$. The direct control of the sign variation of H is not easy, but we can use Proposition 4 to transform our function in a function easier to study. Hence, taking $\zeta(x) = \log(x)$, we now want to analyse the sign variation of the function $P(x) = -(\frac{x}{\theta_2})^{\alpha_2} + (\frac{ax+b}{\theta_1})^{\alpha_1}$. Assume that $a, b > 0$. Applying, again, Proposition 4, we know that the sign variation of P is the same as the sign variation of $Q(x) = -\alpha_2 \log(x) + \alpha_2 \log(\theta_2) + \alpha_1 \log(ax + b) - \alpha_1 \log(\theta_1)$. Differentiating Q, we obtain, $Q'(x) = \frac{ax(\alpha_1 - \alpha_2) - \alpha_2 b}{(ax+b)x}$. For $x \geq 0$ we have that $(ax + b)x > 0$. Moreover, $\lim_{x \to 0^+} ax(\alpha_1 - \alpha_2) - \alpha_2 b < 0$ and $\lim_{x \to +\infty} ax(\alpha_1 - \alpha_2) - \alpha_2 b = +\infty$. Hence, Q' changes sign once in the order "$-, +$". Therefore, the monotonicity of Q is $\searrow\nearrow$. Given that, $\lim_{x \to 0^+} Q(x) = \lim_{x \to +\infty} Q(x) = +\infty$, we conclude that the sign variation of Q, which is the same as the sign variation of P and, consequently, of H, is "$+, -, +$". The other cases are proved in an analogous way, and we may conclude, by Proposition 3, that $X \leq_c Y$.

Chapter 3

s-IFR and s-IFRA orderings

In this chapter two ageing notions and the corresponding orders will be discussed. In the sequel we consider X to be a nonnegative random variable with distribution function $F_X \in \mathscr{F}$, survival function (or tail function) $\overline{F}_X = 1 - F_X$ and density function f_X. The definitions and results that follow are based mostly on [1] and [2].

3.1 Iterated Distributions

Let X be a random variable that denotes the lifetime of a system. By applying the iteration procedure that will be defined below, we manage to re-adjust the probabilities of the tails, recursively. Moreover, these probabilities, as it will be seen, correspond to the normalized moments of the residual life of the system.

Definition 9. *For each $x \geq 0$, define*

$$\overline{T}_{X,0}(x) = f_X(x) \quad \text{and} \quad \widetilde{\mu}_{X,0} = \int_0^\infty \overline{T}_{X,0}(t)\,dt = 1.$$

For each $s \geq 1$, we define the s-iterated distribution, $T_{X,s}$, by their tails $\overline{T}_{X,s} = 1 - T_{X,s}$ as follows,

$$\overline{T}_{X,s}(x) = \frac{1}{\widetilde{\mu}_{X,s-1}} \int_x^\infty \overline{T}_{X,s-1}(t)\,dt, \text{ where } \widetilde{\mu}_{X,s-1} = \int_0^\infty \overline{T}_{X,s-1}(t)\,dt,$$

assuming that these integrals are finite. Moreover, we extend the domain of definition of each $\overline{T}_{X,s}$ defining $\overline{T}_{X,s}(x) = 1$, for $x < 0$.

Remark 2. *For $s = 1$, $\overline{T}_{X,1}(x) = \overline{F}_X$.*

Although this function is defined in an iterated way, it can also be expressed by means of a closed formula.

Lemma 1. [1, Lemma 2] *The s-iterated distribution, $\overline{T}_{X,s}$ may be represented as*

$$\overline{T}_{X,s}(x) = \frac{1}{\prod_{j=1}^{s-1} \widetilde{\mu}_{X,j}} \int_x^\infty \frac{(t-x)^{s-1}}{(s-1)!} f_X(t)dt. \tag{3.1}$$

Proof. From Definition 9,

$$\overline{T}_{X,s}(x) = \frac{1}{\widetilde{\mu}_{X,s-1}} \int_x^{+\infty} \overline{T}_{X,s-1}(t)\,dt = \frac{1}{\widetilde{\mu}_{X,s-1}} \int_x^{+\infty} \frac{1}{\widetilde{\mu}_{X,s-2}} \int_t^{+\infty} \overline{T}_{X,s-2}(u)\,dudt.$$

Reversing the order of integration, we have that

$$\overline{T}_{X,s}(x) = \frac{1}{\widetilde{\mu}_{X,s-1}\widetilde{\mu}_{X,s-2}} \int_x^{+\infty} \int_x^{u} \overline{T}_{X,s-2}(u)\,dtdu = \frac{1}{\widetilde{\mu}_{X,s-1}\widetilde{\mu}_{X,s-2}} \int_x^{+\infty} (u-x)\overline{T}_{X,s-2}(u)du.$$

Repeating this process k times we obtain

$$\overline{T}_{X,s}(x) = \frac{1}{\prod_{j=1}^{k} \widetilde{\mu}_{X,s-j}} \int_x^{+\infty} \frac{(t-x)^{k-1}}{(k-1)!} \overline{T}_{X,s-k}(t)\,dt. \tag{3.2}$$

For $k = s$ and taking into account that $\widetilde{\mu}_{X,0} = 1$, we conclude that

$$\overline{T}_{X,s}(x) = \frac{1}{\prod_{j=1}^{s-1} \widetilde{\mu}_{X,s-j}} \int_x^{+\infty} \frac{(t-x)^{s-1}}{(s-1)!} \overline{T}_{X,0}(t)\,dt = \frac{1}{\prod_{j=1}^{s-1} \widetilde{\mu}_{X,j}} \int_x^{+\infty} \frac{(t-x)^{s-1}}{(s-1)!} f_X(t)\,dt.$$

□

This closed representation will allow us to introduce and use a method that permits the derivation of properties for specific families of distributions.

Remark 3. *From (3.1) we can obtain another closed representation for $\overline{T}_{X,s}$. In fact, choosing $x = 0$ and taking into account that $\overline{T}_{X,s}(0) = 1$, we have that*

$$E(X^{s-1}) = (s-1)! \prod_{j=0}^{s-1} \widetilde{\mu}_{X,j}.$$

Hence, replacing this expression in (3.1) it follows that

$$\overline{T}_{x,s}(x) = \frac{1}{E(X^{s-1})} \int_x^{\infty} f_X(t)(t-x)^{s-1} dt. \tag{3.3}$$

This may also be rewritten as,

$$\overline{T}_{x,s}(x) = \frac{1}{E(X^{s-1})} E((X-x)_+^{s-1}.$$

We will be using these iterated distributions to establish ageing properties of distributions and ageing relations between different distributions within the same family.

We now discuss some definitions of ageing notions. Many ageing notions of a distribution are based on the failure rate function of a distribution, given by $\frac{f_X(x)}{\overline{F}_X(x)} = \frac{\overline{T}_{X,0}(x)}{\overline{T}_{X,1}(x)}$. The failure rate of a system indicates the frequency with which that system fails. Since the iterated distributions have been defined, it becomes natural to define the iterated failure rate.

Definition 10. *For each $s \geq 1$ and $x \geq 0$ we define the s-iterated failure rate function as*

$$r_{X,s}(x) = \frac{\overline{T}_{X,s-1}(x)}{\int_x^\infty \overline{T}_{X,s-1}(t)dt} = \frac{\overline{T}_{X,s-1}(x)}{\tilde{\mu}_{X,s-1}\overline{T}_{X,s}(x)}.$$

The next lemma provides an example of computation of the iterated distribution and failure rate functions.

Lemma 2. [8, Theorem 2.7] *Let X be a random variable with exponential distribution with hazard rate λ. Then, for $s \geq 1$, $\overline{T}_{X,s} = e^{-x\lambda}$ and $r_{X,s}(x) = \lambda$.*

The monotonicity of the failure rate function is a relevant property satisfied by many distributions. Given that the monotonicity of the failure rate of a distribution is expressed by the monotonicity of $r_{X,1}$, we may extend this monotonicity notion to the s-iterated failure rate.

Definition 11. *Let X be a nonnegative random variable with survival function $\overline{F}_X$ and density function f_X.*

1. *Then X is said to be IFR (resp., DFR) if $\overline{F}_X$ is logconcave (resp., logconvex), i.e., if $\log(\overline{F}_X)$ is concave (resp., $\log(\overline{F}_X)$ is convex).*
2. *Then X is said to be IFRA (resp., DFRA) if $-\log(\overline{F}(x))$ is starshaped (resp., antisharped, that is $-\frac{\log(\overline{F}(x))}{x}$ is deacresing for $x > 0$).*
3. *Then, for $s = 1, 2, \ldots$, X is said to be s-IFR (resp., s-DFR) if $r_{X,s}$ is increasing (resp., decreasing), for $x \geq 0$.*
4. *Then, for $s = 1, 2, \ldots$, X is said to be s-IFRA (resp., s-DFRA) if $\frac{1}{x}\int_0^x r_{X,s}(t)\,dt$ is increasing (resp., decreasing), for $x > 0$.*

It is easy to see that that the statements above can be considered as generalizations of the monotonicity of the failure rate. In fact, having that $\log(\overline{F}_X)$ to be concave is equivalent of having that its derivative, $-\frac{f_X}{\overline{F}_X}$, is decreasing. Moreover, $-\log(\overline{F}(x)) = \int_0^x \frac{f(t)}{\overline{F}(t)}\,dt$. Hence, $-\log(\overline{F}(x))$ is starshaped if and only if $\frac{1}{x}\int_0^x \frac{f(t)}{\overline{F}(t)}\,dt$ is increasing, for $x > 0$.

Most of the times, studying the monotonicity of these functions is not a simple task, since the expression of $\overline{T}_{X,s}$ does not always have an explicit closed representation or a manageable one. Nevertheless, we are able to prove that the Weibull distribution is 1-IFR, as its distribution and density functions have explicit closed formulas.

Theorem 6. *Let X be a nonnegative random variable with Weibull distribution with shape parameter α and scale parameter θ and $s \geq 1$ an integer. If $\alpha \geq 1$ (resp., $\alpha < 1$), then X is 1-IFR (resp., 1-DFR).*

Proof. We have that, for the Weibull distribution,

$$r_{X,1}(x) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1}.$$

So if $\alpha \geq 1$ we have that $r_{X,1}$ is increasing, that is, X is s-IFR, and if $\alpha < 1$ we have that $r_{X,1}$ is decreasing, that is, X is s-DFR. □

3.2 Hereditary of the s-IFR monotonicity

Once proved the monotonicity of the failure rate for a family of distributions, it is natural to try to prove the same for the s-iterated failure rate. The following definitions and results from Fagiuoli and Perelley [4] will allow us to establish an hereditary property for the s-IFR monotonicity.

Definition 12. [4, Definition 2.10] *Let Y be a nonnegative random variable with distribution function F_Y and density function f_Y. Then X is said to be smaller than Y in the s-FR order if $\frac{\overline{T}_{Y,s}(x)}{\overline{T}_{X,s}(x)}$ is increasing for $x \geq 0$ and it is denoted by $X \leq_{s-FR} Y$.*

Proposition 6. [4, Theorem 3.4] *Let Y be a nonnegative random variable with distribution function F_Y and density function f_Y. If $X \leq_{s-FR} Y$ then $X \leq_{(s+1)-FR} Y$, for every $s \geq 0$.*

Proof. We have that $X \leq_{s-FR} Y$ if and only if $\frac{\overline{T}_{Y,s}(x)}{\overline{T}_{X,s}(x)}$ is increasing for $x \geq 0$. By Proposition 2, this is equivalent to having that $\overline{T}_{Y,s} - a\overline{T}_{X,s}$ changes sign at most once and if the change occurs it is in the order "$-,+$", for every $a \in \mathbb{R}$. Given that

$$\overline{T}_{Y,s+1}(x) - a\overline{T}_{X,s+1}(x) = \frac{1}{\widetilde{\mu}_{Y,s}} \int_x^{+\infty} \left(\overline{T}_{Y,s}(t) - a\frac{\widetilde{\mu}_{Y,s}}{\widetilde{\mu}_{X,s}} \overline{T}_{X,s}(t) \right) dt,$$

by Proposition 5 we have that $\overline{T}_{Y,s+1}(x) - a\overline{T}_{X,s+1}(x)$ changes sign at most once and if the change occurs it is in the order "$-,+$". Again, by Proposition 2, we conclude that $\frac{\overline{T}_{Y,s+1}(x)}{\overline{T}_{X,s+1}(x)}$ is increasing, that is, $X \leq_{(s+1)-FR} Y$. □

Proposition 7. [4, Theorem 4.3] *Let X be a random variable representing the lifetime of a system and let $X_t = X - t/X > t$ denote the residual lifetime of that system at age t, for $t \geq 0$. Then $X \geq_{s-FR} X_t$ (resp., $X \leq_{s-FR} X_t$), for every $t \geq 0$, if and only if X is s-IFR (resp., X is s-DFR).*

We may now establish the hereditary property of the monotonicity of the iterated failure rate.

Lemma 3. [1, Lemma 6] *Let X be a nonnegative random variable with moment of order $s \geq 1$. If X is s-IFR (resp., s-DFR) then X is $(s+1)$-IFR (resp., $(s+1)$-DFR).*

Proof. By Proposition 7, X is s-IFR (resp., s-DFR) if and only if $X \geq_{s-FR} X_t$ (resp., $X \leq_{s-FR} X_t$), for every $t \geq 0$. But $X \geq_{s-FR} X_t$ (resp., $X \leq_{s-FR} X_t$) implies that $X \geq_{(s+1)-FR} X_t$ (resp., $X \leq_{(s+1)-FR} X_t$) for every $s \geq 0$, by Proposition 6. Again, by Proposition 7 we conclude that X is (s+1)-IFR (resp., (s+1)-DFR). □

After establishing the hereditary monotonicity of the s-IFR property in Lemma 3, it is easy to describe the iterated failure rate monotonicity of the Weibull distributions.

Theorem 7. *Let X be a nonnegative random variable with Weibull distribution with shape parameter α and scale parameter θ, and $s \geq 1$ an integer. If $\alpha \geq 1$ (resp., $\alpha < 1$), then X is s-IFR (resp., s-DFR).*

Proof. By Lemma 3 it is enough to prove the result for $s = 1$. But that was already proved in Theorem 6. □

The previous lemma implies that for most distributions it is sufficient to verify the 1-IFR or 1-DFR property. However, there are cases for which this property only becomes true after a few iterations, that is, the lower monotonicity order does not hold in either direction. Exhibiting distributions that do not have lower iterated monotonicity but verify it after a few iteration steps, usually requires a suitable modification of known families of distributions. In what follows we provide an example of such modification, using fattened tail Pareto distributions.

Example 2. [1, Example 9] *Consider a random variable X with density* $f_X(x) = c' \frac{\log(x+c)}{(x+c)^3}$ *for* $x > 0$*, where* $c, c' \in \mathbb{R}$*. Integrating we have that,*

$$\overline{T}_{X,1}(x) = c' \frac{2\log(x+c)+1}{4(x+c)^2} \quad \textit{and} \quad \overline{T}_{X,2}(x) = \frac{c}{2\log c+3} \frac{2\log(x+c)+3}{x+c}.$$

Thus,

$$r_{X,1}(x) = \frac{4\log(x+c)}{(x+c)(2\log(x+c)+1)} \quad \textit{and} \quad r_{X,2}(x) = \frac{2\log(x+c)+1}{(x+c)(2\log(x+c)+3)}.$$

We can verify that if $c \in (e^{-1+\frac{\sqrt{5}}{2}}, e^{\frac{1}{2}})$*,* $r_{X,1}$ *is not monotone, while* $r_{X,2}$ *is decreasing. In fact, differentiating* $r_{X,1}$ *we have that*

$$r'_{X,1}(x) = \frac{-\log(x+c) - 2\left(\log(x+c)\right)^2 + 1}{(x+c)^2(2\log(x+c)+1)^2}.$$

Therefore, $r'_{X,1}(x) = 0$ *if and only if* $x = e^{-1} - c$ *or* $x = e^{\frac{1}{2}} - c$*. So, for* $c \in (e^{-1+\frac{\sqrt{5}}{2}}, e^{\frac{1}{2}})$ *the roots of* $r'_{X,1}$ *are positive and* $r_{X,1}$ *is decreasing at first, until it reaches its minimum, then, it starts increasing, and finally, after reaching its maximum, it decreases again. Differentiating now* $r_{X,2}$ *we have that*

$$r'_{X,2}(x) = \frac{-8\log(x+c) - 4\left(\log(x+c)\right)^2 + 1}{(x+c)^2(2\log(x+c)+3)^2}.$$

Hence, $r'_{X,2}(x) = 0$ *if and only if* $x = e^{-1-\frac{\sqrt{5}}{2}} - c$ *or* $x = e^{-1+\frac{\sqrt{5}}{2}} - c$*. For* $c \in (e^{-1+\frac{\sqrt{5}}{2}}, e^{\frac{1}{2}})$ *one of the roots is negative and the other one is positive. Since we are only interested in nonnegative values, we can conclude that* $r_{X,2}$ *is decreasing for* $x > 0$*. Thus, we obtain that the 1-IFR or 1-DFR properties are not satisfied but the 2-DFR is. One can see that the density considered has finite expectation but the second order moment does not exist. Thus, by Remark 3, the 3-iterated distribution is no longer definable. So, if we want to obtain examples of distributions that have higher orders finite moments, we can consider densities of the form* $c' \frac{\log(x+c)}{(x+c)^\alpha}$*, for* $\alpha > 0$ *sufficiently large. In our case, if we want the 3-iterated distribution to be definable, we may choose* $f(x) = c' \frac{log(x+c)}{(x+c)^4}$*. It is easy to see that the expected value and the second order moment are finite. Hence, the 3-iterated distribution is definable. Analogously to the previous case, it is not possible to define the 4-iterated distribution. In this case, we would need to consider* $f(x) = c' \frac{\log(x+c)}{(x+c)^5}$*. If we do this successively, it is possible to see, that we are able to define* $\overline{T}_{X,s}$ *if we choose densities of the type* $c' \frac{\log(x+c)}{(x+c)^{s+1}}$*. In the same way as before, we can find an interval for c where* $r_{X,s}$ *is not monotone for a small s, but, for s sufficiently large,* $r_{X,s}$ *is.*

There are families of distributions whose survival functions are not always expressed by a closed explicit representation. So, it is difficult to analyse the monotonicity of the iterated failure rate directly, as it was done for Weibull distributions. Therefore, one way to overcome this obstacle is by applying the characterization of increasing functions stated in Proposition 2, in order to prove the increasing property of the iterated failure rate function.

Theorem 8. *Let X be a nonnegative random variable with distribution $\Gamma(\alpha,\theta)$, and $s \geq 1$ an integer. If $\alpha \geq 1$ (resp., $\alpha < 1$) then X is s-IFR (resp., s-DFR).*

Proof. Again, by Lemma 3 it is enough to prove the result for $s = 1$, that is, it is enough to prove that $r_{X,1}(x) = \frac{f_X(x)}{\overline{F}_X}$ is either increasing or decreasing for $x \geq 0$. Since, in general, $\overline{F}_X$ does not have an explicit closed representation, for this family of distributions, we are going to use Proposition 2 to prove the monotonicity of the failure rate. As f_X and $\overline{F}_X$ are nonnegative, we only need to consider the case where $a > 0$, while applying Proposition 2. Hence, we want to analyse the sign variation of $\frac{f_X(x)}{\overline{F}_X(x)} - a$, for every $a > 0$. Observe that studying the sign of $\frac{f_X(x)}{\overline{F}_X(x)} - a$ is the same that as describing the sign of $H(x) = f_X(x) - a\overline{F}_X(x)$, for $x \geq 0$. Let us consider the case $\alpha \geq 1$. It is easy to observe that $H(0) = -a < 0$ and $\lim_{x\to\infty} H(x) = 0$. Furthermore, differentiating H, we have that

$$H'(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^{\alpha+1}\Gamma(\alpha)}\left((a\theta-1)x - \theta(1-\alpha)\right).$$

The sign of H is determined by $\ell(x) = ((a\theta-1)x - \theta(1-\alpha))$, since $x \geq 0$. For $x = 0$ we have that $\ell(0) = -\theta(1-\alpha)$. Given that $\alpha \geq 1$, we conclude that $\ell(0) > 0$. Consequently, if $a\theta - 1 > 0$ then $\ell(x) > 0$, for every $x \geq 0$. Hence, H' is positive and H is increasing. Knowing that $H(0) < 0$ and $\lim_{x\to\infty} H(x) = 0$, we obtain that the sign of H is "$-$". If $a\theta - 1 < 0$, the sign variation of ℓ, for $x \geq 0$, is "$+,-$". Thus, H' changes sign exactly once, that is, H has a maximum. Since $H(0) < 0$ and $\lim_{x\to\infty} H(x) = 0$, we have that H has a sign variation in the order "$-,+$". By Proposition 2 the conclusion follows. In the case $\alpha < 1$, taking into account that $\lim_{x\to 0} H(x) = +\infty$, the proof is analogous. □

3.3 Non-hereditary of the s-IFRA monotonicity

Having established the hereditary property of the s-IFR monotonicity, it becomes natural to ask ourselves if the s-IFRA monotonicity also has the same property. Unlike the s-IFR monotonicity, the s-IFRA monotonicity does not have the hereditary property. This will be proved using parallel systems, with exponentially distributed components. The lifetime of these kind of systems is expressed as the maximum of the lifetimes of each component, hence, the distribution functions of the system lifetime is expressed as a linear combination of exponential terms. In order to prove the non-hereditary of the s-IFRA monotonicity it is important to be able to count the roots of such expressions. Therefore, the following result, which first appeared in Tossavainen [11] and then it was also proved in Shestopaloff [10], is presented without proof.

Proposition 8. [10, Theorem 1] *Let $n \geq 0$, $p_0 > p_1 > \ldots > p_n > 0$, and $\alpha_j \neq 0$, $j = 0,1,\ldots,n$, be real numbers. Then the function $f(t) = \sum_{j=0}^{n} \alpha_j p_j^t$ has no real zeros if $n = 0$, and for $n \geq 1$ has at most as many real zeros as there are sign changes in the sequence of coefficients $\alpha_0,\alpha_1,\alpha_2,\ldots,\alpha_n$.*

Proposition 9. [2, Proposition 3.1] *Let Y_1 and Y_2 be independent exponential random variables with hazard rate* 1 *and* $\lambda \neq 1$*, respectively, and define* $Y = \max(Y_1, Y_2)$*. Then Y is 1-IFRA, but it is neither 2-IFRA nor 2-DFRA. Moreover, there exists $s_0 > 2$ such that Y is s-DFR for every $s \geq s_0$.*

Proof. Since Y_1 and Y_2 have exponential distributions, applying Lemma 2, we have that $\tilde{\mu}_{Y_1,s} = 1$ and $\tilde{\mu}_{Y_2,s} = \frac{1}{\lambda}$. Additionally, taking into account (3.3), we may conclude that,

$$\overline{T}_{Y,s} = \frac{1}{c(s,\lambda)}\left(e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-1}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-1}}\right),$$

where, $c(s,\lambda) = 1 + \frac{1}{\lambda^{s-1}} + \frac{1}{(\lambda+1)^{s-1}}$. To prove that Y is $1-IFRA$ we need to verify that $-t_1(x) = \frac{\log(\overline{T}_{Y,1}(x))}{x}$ is decreasing for $x > 0$, i.e., that $\frac{\overline{T}^{-1}_{Y_1,1}(\overline{T}_{Y,1}(x))}{x}$ is decreasing, for $x > 0$. By Proposition 2, this means that we need to prove that $H(x) = e^{-x} + e^{-\lambda x} - e^{-(\lambda+1)x} - e^{ax}$ changes sign at most once, and if the change occurs it is in the order "$+,-$", when x traverses from $-\infty$ to $+\infty$. Observe that, for $x \geq 0$, $\overline{T}_{Y,1}(x) \geq e^{-x}$, so it is sufficient to consider the case $a < 1$. In addition, we only need to consider the case where $\lambda > 1$. In fact, if we consider $\lambda < 1$ it is enough to divide the variables by λ, since the order relation does not change. So, by Proposition 8, we may conclude that H has at most two roots. Moreover, $\lim_{x\to+\infty} H(x) = 0^-$, $H(0) = 0$ and $H'(0) = a > 0$. Therefore, the second root exists and it is positive. Consequently, the sign variation of H is "$+,-$", meaning that Y is 1-IFRA. To prove that Y is neither 2-IFRA nor 2-DFRA, we need to verify that $t_2(x) = -\frac{\log(\overline{T}_{Y,2}(x))}{x}$ is not monotone. We can see that $\lim_{x\to 0} t_2(x) = \frac{1}{1+\frac{1}{\lambda}-\frac{1}{\lambda+1}} < 1$ and $\lim_{x\to+\infty} t_2(x) = 1$, since $\lambda > 1$. Now, observe that

$$t_2(x) = 1 \Leftrightarrow P(x) = \left(\frac{1}{\lambda+1} - \frac{1}{\lambda}\right)e^{-x} + \frac{e^{-\lambda x}}{\lambda} - \frac{e^{-x(\lambda+1)}}{\lambda+1} = 0.$$

By Proposition 8, we have that P has at most two real roots. Given that $P(0) = 0$, $P'(0) = \frac{1}{\lambda} - \frac{1}{\lambda+1} > 0$ and $\lim_{x\to+\infty} P(x) = 0^-$, the second root exists and is strictly positive. Thus, since $\lim_{x\to 0} t_2(x) < 1$, $\lim_{x\to+\infty} t_2(x) = 1$ and exists $x > 0$ such that $t_2(x) = 1$, we conclude that $t_2(x)$ is not monotone, and, therefore, the second statement is proved. For the final statement we want to prove that $r_{Y,s}$ is decreasing, for $x \geq 0$. But, the monotonicity of $r_{Y,s}$ coincides with the monotonicity of

$$N(x) = \frac{e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-2}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-2}}}{e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-1}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-1}}}.$$

Differentiating N, we have that the sign of N' is the same as the sign of its numerator,

$$Q(x) = -\frac{(\lambda-1)^2}{\lambda^{s-1}}e^{-(\lambda+1)x} + \frac{\lambda^2}{(\lambda+1)^{s-1}}e^{-(\lambda+2)x} + \frac{1}{(\lambda^2+\lambda)^{s-1}}e^{-(2\lambda+1)x}.$$

For both, $\lambda > 1$ and $\lambda < 1$, it is easy to see, by Proposition 8, that Q has at most one real root. Furthermore,

$$\lim_{x\to+\infty} Q(x) = 0^- \quad \text{and} \quad Q(0) = \frac{\lambda^{s+1} + 1 - (\lambda-1)^2(\lambda+1)^{s-1}}{(\lambda(\lambda+1))^{s-1}}.$$

So, if $Q(0) > 0$, Q has sign variation "$+,-$", when $x \geq 0$, and if $Q(0) < 0$, Q is negative, for $x \geq 0$. Since, $Q(0)$ as a function of s becomes negative, as the numerator has a negative coefficient for λ^s, the largest power in that expression, there exists $s_0 > 2$, such that for $s \geq s_0$, $Q(0)$ is negative, and, consequently $r_{Y,s}$ is decreasing. □

3.4 Iterated failure rate ordering

We finally present the s-IFR and s-IFRA orderings. These orders compare different distributions with respect to their iterated failure rate.

Definition 13. *Let X and Y be two random variables with distribution functions F_X, $F_Y \in \mathscr{F}$, respectively, and $s \geq 1$ an integer. Then*

1. *X (or its distribution F_X) is said to be smaller than Y (or its distribution F_Y) in the s-IFR order if $c_s(x) = \overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x))$ is convex and it is denoted by $X \leq_{s-IFR} Y$ (or $F_X \leq_{s-IFR} F_Y$).*

2. *X (or its distribution F_X) is said to be smaller than Y (or its distribution F_Y) in the s-IFRA order if $t_s(x) = \frac{1}{x}c_s(x)$ is increasing and it is denoted by $X \leq_{s-IFR} Y$ (or $F_X \leq_{s-IFR} F_Y$).*

As it was previously mentioned, the s-IFR order is a convex transform order and the s-IFRA order is a star order, since saying that t_s is increasing is the same as saying that c_s is starshaped. Furthermore, since the convex transform order implies the star order, we may conclude that the s-IFR order implies the s-IFRA order.

It is possible to define an equivalence relation for the s-IFR order.

Definition 14. [8, Definition 2.2] *Let X and Y be two random variables with distribution functions F_X, $F_Y \in \mathscr{F}$, respectively. Then X and Y are said to be s-IFR equivalent if there exists a constant $k > 0$ such that $F_X(x) = F_Y(kx)$, for all $x \geq 0$ and we denote this relation by $X \sim_{s-IFR} Y$ (or $F_X \sim_{s-IFR} Y$).*

The above equivalence relation was instrumental for Nanda et al [8] to prove that both of these orders are in fact partial orderings of the equivalence classes of $\mathscr{F}$.

Theorem 9. [8, Theorem 2.1] *Let X and Y be two random variables with distribution functions F_X, $F_Y \in \mathscr{F}$, respectively.*

1. *The relationship $F_X \leq_{s-IFR} F_Y$ is a partial order relation on the equivalence classes of $\mathscr{F}$ with respect to $\sim_{s-IFR}$.*

2. *The relationship $F_X \leq_{s-IFRA} F_Y$ is a partial order relation on the equivalence classes of $\mathscr{F}$ with respect to $\sim_{s-IFR}$.*

Corollary 3. [1, Corollary 1] *Let X and Y be two nonnegative random variables with distribution functions $F_X, F_Y \in \mathscr{F}$, $s \geq 1$ an integer and $\alpha_1, \alpha_2 > 0$. Then $X \leq_{s-IFR} Y$ if and only if $\alpha_1 X \leq_{s-IFR} \alpha_2 Y$.*

Proof. We have that $F_X(x) = P(X \leq x) = P(\alpha_1 X \leq \alpha_1 x) = F_{\alpha_1 X}(\alpha_1 x)$, for every $x \geq 0$. So, $X \sim_{s-IFR} \alpha_1 X$. Analogously, $Y \sim_{s-IFR} \alpha_2 Y$. By Theorem 9, the s-IFR order defines a partial order relation on the equivalence classes of $\mathscr{F}$, hence $\alpha_1 X \leq_{s-IFR} \alpha_2 Y$. □

So, when comparing parametric distributions with a scale parameter, we may assume that this parameter is equal to 1.

The exponential distribution plays an important role when dealing with ageing notions. In fact, it is possible to establish a relation between the s-IFR or s-IFRA orders and s-IFR or s-IFRA monotonicity, when one of the variables has an exponential distribution.

Theorem 10. [8, Theorem 2.2 and Theorem 3.3] *Let X be a random variable with distribution function $F_X \in \mathscr{F}$ and Y with exponential distribution with hazard rate λ. Then,*

1. *$X \leq_{s-IFR} Y$ (resp., $Y \leq_{s-IFR} X$) if and only if X is s-IFR (resp., X is s-DFR).*
2. *$X \leq_{s-IFRA} Y$ (resp., $Y \leq_{s-IFRA} X$) if and only if X is s-IFRA (resp., X is s-DFRA).*

Proof. We will only present the prove for the case where $X \leq_{s-IFR} Y$. By Definition 13, $X \leq_{s-IFR} Y$ if and only if $\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x))$ is convex. Since Y is random variable with exponential distribution, applying Lemma 2, we have that $\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x)) = -\frac{1}{\lambda}\log(\overline{T}_{X,s}(x))$. Therefore, $X \leq_{s-IFR} Y$ if and only if $-\frac{1}{\lambda}\log(\overline{T}_{X,s}(x))$ is convex, i.e., $\frac{1}{\lambda}\log(\overline{T}_{X,s}(x))$ is concave. We know that a function is concave if and only if its derivative is decreasing, hence $X \leq_{s-IFR} Y$ if and only if $-\frac{\overline{T}_{X,s-1}(x)}{\overline{T}_{X,s}}$ is decreasing. But the last expression is equivalent to having that $r_{X,s}$ is increasing, that is, X is s-IFR. □

The characterization of convex and increasing functions through their sign variation provides an alternative way to characterize both the s-IFR order and s-IFRA order.

Theorem 11. [8, Proposition 2.1 and Proposition 3.1] *Let X and Y be two random variables with distribution functions $F_X, F_Y \in \mathscr{F}$. Then*

1. *$X \leq_{s-IFR} Y$ if and only if for any real numbers a and b, $\overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax+b)$ changes sign at most twice and if the change occurs twice it is in the order "$+,-,+$", as x traverses from 0 to $+\infty$.*
2. *$X \leq_{s-IFRA} Y$ if and only if for any real number a, $\overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax)$ changes sign at most once and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$.*

Proof. By Proposition 3, $X \leq_{s-IFR} Y$ if and only if for any real numbers a and b, $\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x)) - (ax+b)$ changes sign at most twice and if the change occurs twice it is in the order "$+,-,+$". Since, $\overline{T}_{Y,s}$ is decreasing, applying Proposition 4, we have that $X \leq_{s-IFR} Y$ if and only if $\overline{T}_{X,s}(x) - \overline{T}_{Y,s}(ax+b)$ changes sign at most twice and if the change occurs twice it is in the order "$-,+,-$". But this is equivalent to having that $\overline{T}_{Y,s}(ax+b) - \overline{T}_{X,s}(x)$ changes sign at most twice and if the change occurs twice it is in the order "$+,-,+$". Taking $y = ax+b$, it follows that $X \leq_{s-IFR} Y$ if and only if $\overline{T}_{Y,s}(y) - \overline{T}_{X,s}(\frac{y}{a} - \frac{b}{a})$ changes sign at most twice and if the change occurs twice it is in the order "$+,-,+$". Given that this happens for every $a, b \in \mathbb{R}$, the conclusion follows. The case for the s-IFRA ordering is similar to the previous, but we consider $b = 0$ and take into account Proposition 2. □

Remark 4. *When applying Theorem 11 we only need to consider the case $a > 0$. In fact, considering the s-IFR order and denoting by $V_s(x) = \overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax+b)$, for every $a, b \in \mathbb{R}$, we have that*

$V_s(0) = 1 - \overline{T}_{X,s}(b)$ and from the definition of the iterated tails,

$$V_s'(x) = -\frac{1}{\tilde{\mu}_{Y,s-1}}\overline{T}_{Y,s-1}(x) + \frac{a}{\tilde{\mu}_{X,s-1}}\overline{T}_{X,s-1}(ax+b).$$

If $a<0$, since $\overline{T}_{Y,s}$ and $\overline{T}_{X,s}$ are nonnegative for every integer ≥ 1, we have that $V_s'(x) \leq 0$. Moreover, by Definition 9, $\lim_{x\to\infty} V_s(x) = -1$. Thus the sign variation of V_s is "$+,-$" when $b>0$ and "$-$" when $b \leq 0$. Therefore, for $a<0$, the condition described in Theorem 11 is always satisfied. For the s-IFRA order the conclusion is the same, since the only difference is that we consider $b=0$, and therefore, $V_s(0)=0$ and V_s is negative for $a<0$.

The above characterization requires explicit expressions of the tails of the iterated distributions, which are often not available. The following theorem, provides a complement for the previous result, when the expressions of the distributions functions are not available.

Theorem 12. [1, 2, Theorem 19 and Theorem 2.4] *Let X and Y be two random variables with densities f_X, f_Y and distribution functions $F_X, F_Y \in \mathscr{F}$, respectively. Define, for some positive integer $k \leq s$, and real numbers a and b, the function*

$$H_k(x) = \frac{1}{\prod_{j=1}^k \tilde{\mu}_{Y,s-j}}\overline{T}_{Y,s-k}(x) - \frac{a^k}{\prod_{j=1}^k \tilde{\mu}_{X,s-j}}\overline{T}_{X,s-k}(ax+b).$$

1. *If H_k changes sign at most twice, for every $a>0$ and $b\in\mathbb{R}$, and if the change occurs twice it is in the order "$+,-,+$", as x traverses from 0 to $+\infty$, then $X \leq_{s-IFR} Y$.*
2. *If H_k changes sign at most once, for every $a>0$ and $b=0$, and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$, then $X \leq_{s-IFRA} Y$.*

Proof. We only need to prove the result for the s-IFR order, since for the s-IFRA the proof is analogous. From (3.2) we have that

$$V_s(x) = \frac{1}{\prod_{j=1}^k \widetilde{\mu}_{Y,s-j}}\int_x^{+\infty}\frac{(t-x)^{k-1}}{(k-1)!}\overline{T}_{Y,s-k}(t)\,dt - \frac{1}{\prod_{j=1}^k \widetilde{\mu}_{X,s-j}}\int_{ax+b}^{+\infty}\frac{(t-(ax+b))^{k-1}}{(k-1)!}\overline{T}_{X,s-k}(t)\,dt.$$

Taking $y = \frac{t-b}{a}$ in the second integral, we obtain

$$V_s(x) = \frac{1}{\prod_{j=1}^k \widetilde{\mu}_{Y,s-j}}\int_x^{+\infty}\frac{(t-x)^{k-1}}{(k-1)!}\overline{T}_{Y,s-k}(t)\,dt - \frac{a^k}{\prod_{j=1}^k \widetilde{\mu}_{X,s-j}}\int_x^{+\infty}\frac{(y-x)^{k-1}}{(k-1)!}\overline{T}_{X,s-k}(ay+b)\,dt$$

Hence,

$$V_s(x) = \int_x^{+\infty}\frac{(t-x)^{k-1}}{(k-1)!}H_k(t)\,dt = \int_x^{+\infty}\int_x^t (k-1)\,(u_1-x)^{k-2}\,\frac{H_k(t)}{(k-1)!}\,du_1 dt.$$

Reversing the order of integration and repeating this process $k-1$ times, we have that

$$V_s(x) = \int_x^{+\infty}\int_{u_1}^{+\infty}(k-1)\,(u_1-x)^{k-2}\,\frac{H_k(t)}{(k-1)!}\,dt du_1 = \cdots = \int_x^{+\infty}\int_{u_1}^{+\infty}\cdots\int_{u_{k-1}}^{+\infty}H_k(t)\,dt du_{k-1}\cdots du_1. \tag{3.4}$$

Applying Proposition 5, iteratively, we conclude that V_s has sign variation equal to "$+,-,+$" or "$-,+$" or "$+$", i.e., V_s changes sign at most twice and if the change occurs twice it is in the order "$+,-,+$". From Theorem 11 we conclude that $X \leq_{s-IFR} Y$. □

As we can see, it is enough to study the sign variation of H_k, when we are analysing the sign variation of V_s.

As mentioned before, the expressions of the iterated tails are not always explicit. Therefore, in most cases we are interested in applying the previous theorem for $k = s$, in which case we are working with density functions, or $k = s-1$, in which we are using survival functions. Taking now into account Remark 3, we have that

$$\frac{1}{(s-1)!}H_s(x) = \frac{1}{E(Y^{s-1})}f_Y(x) - \frac{a^s}{E(X^{s-1})}f_X(ax+b)$$

and

$$\frac{1}{(s-1)!}H_{s-1}(x) = \frac{1}{E(Y^{s-1})}\overline{F}_Y(x) - \frac{a^{s-1}}{E(X^{s-1}}\overline{F}_X(ax+b).$$

Note that the factor $\frac{1}{(s-1)!}$ has no influence on the sign variation. Using Proposition 4 with $\zeta(x) = \log x$, we can have a similar result to Theorem 12. This transformation is useful, since in most cases it is difficult to analyse the sign variation of H_s and H_{s-1}.

Corollary 4. [2, Theorems 2.3 and 2.4] *Let X and Y be two random variables with densities f_X and f_Y and distribution functions $F_X, F_Y \in \mathscr{F}$, respectively. Define, for every constants a and b, the functions,*

$$P_s(x) = \log(f_Y(x)) - \log(f_X(ax+b)) + \log\left(\frac{E(X^{s-1})}{a^s E(Y^{s-1})}\right)$$

and

$$P_{s-1}(x) = \log(\overline{F}_Y(x)) - \log(\overline{F}_X(ax+b)) + \log\left(\frac{E(X^{s-1})}{a^s E(Y^{s-1})}\right).$$

1. *If for every $a > 0$ and $b \in \mathbb{R}$, either of the functions, P_s or P_{s-1} changes sign at most twice when x traverses from 0 to $+\infty$, and if the change occurs twice it is in the order "$+,-,+$", then $X \leq_{s-IFR} Y$.*

2. *If for every $a > 0$ and $b = 0$, either of the functions, P_s or P_{s-1} changes sign at most once when x traverses from 0 to $+\infty$, and if the change occurs it is in the order "$-,+$", then $X \leq_{s-IFRA} Y$.*

Having established these results we are able to compare two Weibull distributions. Observe that this is, in fact, a generalization of Example 1.

Proposition 10. [1, Proposition 3] *Let X and Y be two random variables with $W(\alpha_1, \theta_1)$ and $W(\alpha_2, \theta_2)$ distributions, respectively, and let $\alpha_1 > \alpha_2 > 1$ and $\theta_1, \theta_2 > 0$. Then $X \leq_{s-IFR} Y$.*

Proof. In order to prove that $X \leq_{s-IFR} Y$ we need to study the sign variation of H_{s-1}, which is equivalent, by Proposition 4, to study the sign variation of P_{s-1}. Note that, as proved in Corollary 3, we may consider $\theta_1 = \theta_2 = 1$. Hence,

$$P_{s-1}(x) = -x^{\alpha_2} + (ax+b)^{\alpha_1} + \log\left(\frac{E(Y^{s-1})}{a^{s-1}E(X^{s-1})}\right), \text{ for } a > 0 \text{ and } b \in \mathbb{R}.$$

Differentiating this expression, we have

$$P'_{s-1}(x) = -\alpha_2 x^{\alpha_2 - 1} + a\alpha_1 (ax+b)^{\alpha_1 - 1}.$$

The direct control of the sign variation of this function is difficult, so we will apply Proposition 4, with $\zeta(x) = \log(x)$, one more time, to P'_{s-1}. Thus, analysing the sign variation of P'_{s-1} is the same as studying the sign variation of

$$Q_{s-1}(x) = \log(a\alpha_1) + (\alpha_1 - 1)\log(ax+b) - \log(\alpha_2) - (\alpha_2 - 1)\log(x).$$

Differentiating Q_{s-1}, we obtain,

$$Q'_{s-1}(x) = \frac{a(\alpha_1 - 1)x - (\alpha_2 - 1)(ax+b)}{(ax+b)x}$$

Now, as we only need to analyse the sign variation for $a > 0$, we will separate the cases where $b > 0$ and $b \leq 0$.

Assume, that $b > 0$. Here, we need to study the sign variation for $x \geq 0$. We have that, for $x \geq 0$, the denominator of $Q'_{s-1}(x)$ is positive and $\lim_{x \to 0^+} a(\alpha_1 - 1)x - (\alpha_2 - 1)(ax+b) = (1 - \alpha_2)b < 0$. Thus, since $\lim_{x \to +\infty} a(\alpha_1 - 1)x - (\alpha_2 - 1)(ax+b) = +\infty$, the sign variation of Q'_{s-1} is "$-,+$" and, consequently, the monotonicity of Q_{s-1} is, in the same interval, $\searrow\nearrow$. Since, $\lim_{x \to 0^+} Q_{s-1} = +\infty$ and $\lim_{x \to +\infty} Q_{s-1} = +\infty$ and applying Proposition 2, we have that P'_{s-1}, which has the same sign variation of Q_{s-1}, changes sign at most twice and if the change occurs it is in the order "$+,-,+$". Hence, the monotonicity of P_{s-1} is $\nearrow\searrow\nearrow$. Again, by Proposition 4 and , given that $\alpha_1 > \alpha_2$ implies that $\lim_{x \to +\infty} = +\infty$, we conclude that the most sign varying possibility for P_{s-1} is "$-,+,-,+$". Since, P_{s-1} has the same sign variation has H_{s-1}, taking into account (3.4) and Proposition 5, we conclude that V_s changes sign in one of the following orders: "$-,+,-,+$" or "$+,-,+$" or "$-,+$" or "$+$". Remembering that $V_s(0) \geq 0$, it follows that the possible choices for the sign variation of V_s are "$+,-,+$" or "$+$".

Assume, now, that $b \leq 0$. For $x \leq -\frac{b}{a}$, $V_s(x) = \overline{T}_{Y,s}(x) - 1 \leq 0$, since $ax + b \leq 0$. Hence, we only need to study the sign variation of V_s for $x \in (-\frac{b}{a}, +\infty)$. Given that $b \leq 0$ and $x > -\frac{b}{a} > 0$ we have that $Q'_{s-1}(x) > 0$, thus Q_{s-1} is increasing in $(0, +\infty)$. As $\alpha_1 > \alpha_2 > 1$, it follows that $\lim_{x \to (-\frac{b}{a})^+} Q_{s-1}(x) = -\infty$ and $\lim_{x \to +\infty} Q_{s-1}(x) = +\infty$, hence, the sign variation of P'_{s-1} which coincides with the sign variation of Q_{s-1} is "$-,+$", thus the monotonicity of P_{s-1} is $\searrow\nearrow$. By Proposition 2 and by the fact that $\lim_{x \to +\infty} P_{s-1}(x) = +\infty$, it follows that if $P_{s-1}(-\frac{b}{a}) > 0$ the most sign varying possibility of P_{s-1} is "$+,-,+$", while if $P_{s-1}(-\frac{b}{a}) < 0$, P_{s-1} changes sign at most once in the order "$-,+$". Taking into account that the sign variation of P_{s-1} is equal to the sign variation of H_{s-1}, (3.4) and Proposition 5, V_s changes sign, in $(-\frac{b}{a}, +\infty)$, in one of the following orders: "$+,-,+$" or "$-,+$" or "$+$". Since, $V_s(-\frac{b}{a}) \leq 0$ we have that the sign variation of V_s is "$-,+$".

Therefore, we may conclude that the possible sign variations for V_s, as x traverses from 0 to $+\infty$, are "$+,-,+$" or "$-,+$" or "$+$". By Theorem 11, it follows that $X \leq_{s-IFR} Y$. □

Proposition 11. [1, Proposition 4] *Let X and Y be two random variables with $W(\alpha_1, \theta_1)$ and $W(\alpha_2, \theta_2)$ distributions, respectively, and let $\alpha_1 > 1 > \alpha_2 > 0$ and $\theta_1, \theta_2 > 0$. Then $X \leq_{s-IFR} Y$.*

Proof. Since $\alpha_1 > 1$, by Theorem 7, we have that X is s-IFR. But, taking into account Theorem 10, this is equivalent to have $X \leq_{s-IFR} Z$, where Z has an exponential distribution. Analogously, by Theorem 7, given that $\alpha_2 < 1$, we have that Y is s-DFR. Again, by Theorem 10, Y is s-DFR if and only if $Z \leq_{s-IFR} Y$. Since, $\leq_{s-IFR}$ is a partial ordering, it is transitive. Hence, $X \leq_{s-IFR} Z$ and $Z \leq_{s-IFR} Y$ implies that $X \leq_{s-IFR} Y$. □

Analogous results for the Gamma family of distributions were proved in Arab and Oliveira [1]. However, the technicalities are much longer to handle so we will not include their proof here.

3.5 Non-hereditary of the s-IFR ordering

We now have a look at the hereditary properties of the s-IFR ordering. Unlike the s-IFR monotonicity, we will see that the ordering relation is not an hereditary property. In order to prove that, we need to recall Definition 8. Observe that, saying that $\frac{\overline{G}_e(\overline{G}^{-1}(u))}{\overline{F}_e(\overline{F}^{-1}(u))}$ (defined on Page 6) is decreasing is equivalent to saying that $d(x) = \frac{\overline{T}_{Y,2}(\overline{T}_{Y,1}^{-1}(x))}{\overline{T}_{X,2}(\overline{T}_{X,1}^{-1}(x))}$ is decreasing. Moreover, note that having $X \leq_c Y$ is the same as having $X \leq_{1-IFR} Y$, and, therefore, Proposition 1 relates the 1-IFR and DMRL orders.

Nanda et. al [8] mention, without proof, that the DMRL order is equivalent to the 2-IFR order. However, this is true only under the condition that one of the random variables is exponentially distributed. In fact, this equivalence is, in general, not true.

Proposition 12. [2, Proposition 5.1] *Let X be a random variable with distribution function $F_X \in \mathscr{F}$ and Y a random variable with exponential distribution. Then $X \leq_{2-IFR} Y$ if and only if $X \leq_{DMRL} Y$.*

Proof. Taking into account that the exponential distribution has a scale parameter, it is enough to consider the case where Y has mean 1. Then, $\overline{T}_{Y,2}(x) = \overline{T}_{Y,1}(x) = e^{-x}$, by Lemma 2. Hence, $X \leq_{DMRL} Y$ is equivalent to $d(x) = \frac{x}{\overline{T}_{X,2}(\overline{T}_{X,1}^{-1}(x))}$ being decreasing. On the other hand, $X \leq_{2-IFR} Y$ if and only if $c_2(x) = \overline{T}_{Y,2}^{-1}(\overline{T}_{X,2}(x))$ is convex, or equivalently, if and only if c_2' is increasing. Differentiating, we obtain $c_2'(x) = \frac{\overline{T}_{X,1}(x)}{\mu_{X,1}\overline{T}_{Y,1}(\overline{T}_{Y,2}^{-1}(\overline{T}_{X,2}(x)))} = \frac{\overline{T}_{X,1}(x)}{\mu_{X,1}\overline{T}_{X,2}}$. Hence, having that c_2' is increasing is the same as having that $c(x) = \frac{\overline{T}_{X,1}(x)}{\overline{T}_{X,2}}$ is increasing. Observe now that, since $\overline{T}_{X,1}^{-1}$ is decreasing,, $c(\overline{T}_{X,1}^{-1}(x))$ is decreasing. But, $c(\overline{T}_{X,1}^{-1}(x)) = d(x)$, so the conclusion follows. □

As a consequence of this proposition we have the hereditary property of the s-IFR ordering, when one of the random variables is exponentially distributed.

Corollary 5. [2, Corollary 5.1] *Let X be a random variable with distribution function $F_X \in \mathscr{F}$ and Y a random variable with exponential distribution. If, for some $s \geq 1$, $X \leq_{s-IFR} Y$, then $X \leq_{(s+1)-IFR} Y$.*

Proof. Define X_1^* with tail $\overline{T}_{X,s}$ and Y_1^* with tail $\overline{T}_{Y,s}$. So, $X \leq_{s-IFR} Y$ is equivalent to having that $X_1^* \leq_{1-IFR} Y_1^*$. By Proposition 1, this implies that $X_1^* \leq_{DMRL} Y_1^*$, which is equivalent to $X_1^* \leq_{2-IFR} Y_1^*$, by Proposition 12. But this is just a rewriting for $X \leq_{(s+1)-IFR} Y$. □

However, the same hereditary does not hold when comparing random variables with other distributions.

Proposition 13. [2, Proposition 5.2] *Neither the 1-IFR or the DMRL orders imply the 2-IFR order.*

Proof. Given $\alpha_1, \alpha_2 > 0$, we say that X has a branched Pareto distribution, $X \sim BP(\alpha_1, \alpha_2)$, if its survival function is

$$\overline{T}_{X,1}(x) = \frac{\alpha_1^2}{(x+\alpha_1)^2}\mathbb{1}_{[0,\alpha_1]}(x) + \frac{(\alpha_1+\alpha_2)^2}{4(x+\alpha_2)^2}\mathbb{1}_{(\alpha_1,+\infty)}(x).$$

Integrating, we obtain

$$\overline{T}_{X,2}(x) = \frac{4}{3\alpha_1+\alpha_2}\left(\frac{\alpha_1^2}{x+\alpha_1}+\frac{\alpha_2-\alpha_1}{4}\right)\mathbb{1}_{[0,\alpha_1]}(x) + \frac{(\alpha_1+\alpha_2)^2}{(3\alpha_1+\alpha_2)(x+\alpha_2)}\mathbb{1}_{(\alpha_1,+\infty)}(x),$$

$$\overline{T}^{-1}_{X,1}(x) = \left(\frac{\alpha_1+\alpha_2}{2\sqrt{x}}-\alpha_1\right)\mathbb{1}_{[0,\frac{1}{4}]}(x) + \left(\frac{\alpha_1}{\sqrt{x}}-\alpha_1\right)\mathbb{1}_{(\frac{1}{4},+\infty)}(x),$$

$$\overline{T}^{-1}_{X,2}(x) = \left(\frac{(\alpha_1+\alpha_2)^2}{(3\alpha_1+\alpha_2)x}-\alpha_2\right)\mathbb{1}_{[0,\frac{\alpha_1+\alpha_2}{3\alpha_1+\alpha_2}]}(x) + \left(\frac{4\alpha_1^2}{(3\alpha_1+\alpha_2)x-(\alpha_2-\alpha_1)}-\alpha_1\right)\mathbb{1}_{(\frac{\alpha_1+\alpha_2}{3\alpha_1+\alpha_2},+\infty)}(x).$$

Moreover,

$$\overline{T}_{X,2}(\overline{T}^{-1}_{X,1}(x)) = \frac{2(\alpha_1+\alpha_2)\sqrt{x}}{3\alpha_1+\alpha_2}\mathbb{1}_{[0,\frac{1}{4}]}(x) + \frac{4}{3\alpha_1+\alpha_2}\left(\alpha_1\sqrt{x}+\frac{\alpha_2-\alpha_1}{4}\right)\mathbb{1}_{(\frac{1}{4},+\infty)}(x),$$

and

$$\overline{T}_{X,1}(\overline{T}^{-1}_{X,2}(x)) = \frac{(3\alpha_1+\alpha_2)^2}{4(\alpha_1+\alpha_2)^2}x^2\mathbb{1}_{[0,\frac{\alpha_1+\alpha_2}{3\alpha_1+\alpha_2}]}(x) + \frac{((3\alpha_1+\alpha_2)x-(\alpha_2-\alpha_1))^2}{16\alpha_1^2}\mathbb{1}_{(\frac{\alpha_1+\alpha_2}{3\alpha_1+\alpha_2},+\infty)}(x).$$

Considering now that $X \sim BP(5,10)$ and $Y \sim BP(2,6)$, we find

$$d(x) = \frac{10}{9}\mathbb{1}_{[0,\frac{1}{4})}(x) + \frac{25(2\sqrt{x}+1)}{12(5\sqrt{x}+\frac{5}{4})}\mathbb{1}_{[\frac{1}{4},+\infty)}(x).$$

It is easy to see that this function is decreasing. In fact, for $x \in [0,\frac{1}{4})$, d is constant. For $x \in [\frac{1}{4},+\infty)$, we have that the sign of the derivative of d is given by the sign of $N(x) = -\frac{5}{4\sqrt{x}}$, which is negative for $x > 0$. Hence, d is decreasing. Furthermore, based on what we have seen before, we have that c_2 is convex if and only if $c_2'(\overline{T}^{-1}_{X,2}(x))$ is decreasing. Note that,

$$c_2'(\overline{T}^{-1}_{X,2}(x)) = \frac{81}{100}\mathbb{1}_{[0,\frac{3}{5}]}(x) + \frac{9x^2}{(5x-1)^2}\mathbb{1}_{(\frac{3}{5},\frac{2}{3}]}(x) + \frac{4(3x-1)^2}{(5x-1)^2}\mathbb{1}_{(\frac{2}{3},1]}(x).$$

This function is not monotone: for $x \in [0,\frac{3}{5}]$, $c_2'(\overline{T}^{-1}_{X,2}(x))$ is constant; for $x \in (\frac{3}{5},\frac{2}{3}]$, differentiating we can see that $c_2'(\overline{T}^{-1}_{X,2}(x))$ is increasing; and for $x \in (\frac{2}{3},1]$, differentiating again this function, we may conclude that $c_2'(\overline{T}^{-1}_{X,2}(x))$ is decreasing. Therefore, c_2 is not convex. Hence, we have proved that the DMRL order does not imply the 2-IFR order. Finally, in order to prove that the 1-IFR order does not imply the 2-IFR, we need to prove that c_1 is a convex function. In fact,

$$c_1(x) = \left(\frac{2(x+5)}{5}-2\right)\mathbb{1}_{[0,5]}(x) + \left(\frac{8(x+10)}{15}-6\right)\mathbb{1}_{(5,+\infty)}(x),$$

which is clearly convex. □

3.6 A criterium for the s-IFR ordering

It is obvious that when studying the sign variation of P_{s-1} in Proposition 10, significant difficulties arise for the case where $b<0$. Therefore, it would be useful if we only needed to verify the sign variation of these functions for $a>0$ and $b\geq 0$. In fact, with the help of the s-IFRA ordering a simplification of this analysis may be obtained.

Theorem 13. [2, Theorem 6.1] *Let X and Y be random variables with distribution functions $F_X,F_Y\in\mathscr{F}$, respectively. If $X\leq_{s-IFRA}Y$ and the criterium from Theorem 11 is satisfied for $b\geq 0$, then $X\leq_{s-IFR}Y$.*

Proof. In order to prove that $X\leq_{s-IFR}Y$, we need to prove that $c_s(x)=\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x))$ is convex, or equivalently, that $c_s^{-1}(x)=\overline{T}_{X,s}^{-1}(\overline{T}_{Y,s}(x))$ is concave. But this is equivalent to prove that $V_s(x)=\overline{T}_{X,s}^{-1}(\overline{T}_{Y,s}(x))-(ax+b)$ has, at most, the sign variation "$-,+,-$", for every real numbers a and b. On the other hand, $X\leq_{s-IFRA}Y$ means that $\frac{c_s(x)}{x}$ is increasing for $x>0$, or equivalently, that $\frac{c_s^{-1}(x)}{x}$ is decreasing for $x>0$. Observe now that the sign variation of V_s is the same as $\frac{V_s(x)}{x}=\left(\frac{\overline{T}_{X,s}^{-1}(\overline{T}_{Y,s}(x))}{x}-a\right)-\frac{b}{x}$, for $x>0$. The expression in parenthesis is decreasing, given that $X\leq_{s-IFRA}Y$. For $b<0$, $\frac{b}{x}$ is increasing. Thus, $\frac{V_s(x)}{x}$ has at most one root and, therefore, changes sign at most once for $b<0$. Since Theorem 11 is verified for $b<0$ and, by the hypothesis, for $b\geq 0$, we have that $X\leq_{s-IFR}Y$. □

We may now complete the comparison within Weibull distributions when the shape parameters are less than 1.

Proposition 14. *Let X and Y be two random variables with $W(\alpha_1,\theta_1)$ and $W(\alpha_2,\theta_2)$ distributions, respectively, and let $1>\alpha_1>\alpha_2>0$ and $\theta_1,\theta_2>0$. Then $X\leq_{s-IFR}Y$.*

Proof. We will begin by proving the s-IFRA ordering. For that we consider $a>0$ and $b=0$ on the definition of $V_s(x)=\overline{T}_{Y,s}(x)-\overline{T}_{X,s}(ax+b)$ and of the expressions computed in the course of the proof of Proposition 10. We want to prove that V_s changes sign at most once and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$. Since, $\alpha_1>\alpha_2$, we have that $Q_{s-1}'(x)>0$, for $x>0$. Hence, Q_{s-1} is increasing and, given that $\lim_{x\to 0^+}Q_{s-1}(x)=-\infty$ and $\lim_{x\to+\infty}Q_{s-1}(x)=+\infty$, has sign variation "$-,+$". Therefore, P_{s-1} has monotonicity "$\searrow\nearrow$". Moreover, $\lim_{x\to+\infty}P_{s-1}(x)=+\infty$, which implies that the most sign varying possibility for P_{s-1} is "$+,-,+$". Thus, by applying Proposition 5 and (3.4), we may conclude that V_s has the possible sign variations, "$+,-,+$" or "$-,+$" or "$+$". But $V_s(0)=0$ so, V_s, at most, has sign variation "$-,+$". Thus, $X\leq_{s-IFRA}Y$. Now, in order to apply Theorem 13, we only need to prove that V_s changes sign at most twice in the order "$+,-,+$", as x traverses from 0 to $+\infty$, for every $a>0$ and $b>0$ (since when $b=0$ we already showed that V_s changes sign at most once in the order "$-,+$"). Note that, now we consider again the expressions of of Q_{s-1}', P_{s-1}' and P_{s-1} in Proposition 10. Taking into account that $\alpha_2<1$, we may conclude that $Q_{s-1}'(x)>0$ for $x>0$. Furthermore, $\lim_{x\to 0^+}Q_{s-1}(x)=-\infty$ and $\lim_{x\to+\infty}Q_{s-1}(x)=+\infty$. Hence, since Q_{s-1} is increasing, we have that Q_{s-1} has sign variation "$-,+$". Thus, P_{s-1} has monotonicity

"$\searrow\nearrow$". Given that $\lim_{x\to+\infty} P_{s-1}(x) = +\infty$, it follows that the most sign varying possibility for P_{s-1} is "$+,-,+$". But, by Proposition 5 and (3.4), it follows that this is also the most sign varying possibility for V_s. By Theorem 13 we may conclude that $X \leq_{s-IFR} Y$. □

Once again, it is possible to establish a similar result for Gamma distributions, but the proof is omitted for brevity. The interested reader is referred to Arab et. al [2] for a detailed proof.

Chapter 4

Failure rate properties of parallel systems

As referred in Section 3.3, the lifetime of a parallel system is expressed as the maximum of the lifetimes of each component. Kochar and Xu [6] proved that a parallel system where the components have the same exponential distribution ages faster, with respect to the 1-IFR order, than a same sized parallel system where the components have exponential lifetimes with different hazard rates. We will apply the results established so far to extend this result to the s-IFR ordering.

4.1 Failure rate ordering of parallel systems with two exponentially distributed components

We now apply the criterium introduced in Theorem 13 to compare two parallel systems with exponential distributed lifetimes, with respect to the s-IFR ordering. Throughout this section we will consider the following random variables

$$\begin{cases} X = \max(X_1, X_2), & \text{where } X_1 \text{ and } X_2 \text{ are independent random variables,} \\ & \text{exponentially distributed with hazard rate } 1 \\ Y = \max(Y_1, Y_2), & \text{where } Y_1 \text{ and } Y_2 \text{ are independent random variables,} \\ & \text{exponentially distributed with hazard rate } 1 \text{ and } \lambda > 1, \text{ respectively.} \end{cases} \tag{4.1}$$

Remark 5. *Assume that Y_1 and Y_2 are two random variables with exponential distributions with hazard rates λ_1 and λ_2, respectively. Since λ_1 and λ_2 are scale parameters, we can normalize the variables by dividing all the parameters by λ_1 and take $\lambda = \frac{\lambda_2}{\lambda_1}$, where λ is assumed to be a constant larger than 1. So it is always possible to reduce a general case to the present case.*

As it was stated in Proposition 9, we have that $\overline{T}_{Y,s} = \frac{1}{c(s,\lambda)}\left(e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-1}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-1}}\right)$, where, $c(s,\lambda) = 1 + \frac{1}{\lambda^{s-1}} + \frac{1}{(\lambda+1)^{s-1}}$. Similarly, we have that $\overline{T}_{X,s} = \frac{2^{s-1}}{2^s-1}\left(2e^{-x} - \frac{e^{-2x}}{2^{s-1}}\right)$. We want to prove that $X \leq_{s-IFR} Y$, for every $s \geq 1$. However, we need to prove a few auxiliary results.

Proposition 15. [2, Proposition 7.1] *Let X and Y be two random variables defined as in 4.1. For every $s \geq 1$ and $x \geq 0$, we have that $\overline{T}_{X,s}(x) \geq \overline{T}_{Y,s}(x)$.*

Proof. Define

$$U_s(x) = \overline{T}_{X,s}(x) - \overline{T}_{Y,s}(x) = e^{-x}\left(\frac{2^s}{2^s-1} - \frac{1}{c(s,\lambda)}\right) - \frac{e^{-2x}}{2^s-1} - \frac{e^{-\lambda x}}{c(s,\lambda)\lambda^{s-1}} + \frac{e^{-(\lambda+1)x}}{c(s,\lambda)(\lambda+1)^{s-1}}.$$

Since we are considering $\lambda > 1$ we have that the sign of the coefficients of U_s, after ordering them decreasingly with respect to the exponents, are "$+,-,-,+$". So, U_s has at most 2 real roots, by Proposition 8. One of the roots is easily located, since $U_s(0) = 0$. Note also that the sign pattern of the coefficients implies that $\lim_{x\to-\infty} U_s(x) = +\infty$ and $\lim_{x\to+\infty} U_s(x) = 0^+$. In order to locate the other root, we need to study the sign variation of the derivatives of orders $k < s$ and s of U_s. Differentiating U_s we have

$$U_s^{(k)}(x) = (-1)^k\left[\frac{2^s e^{-x} - 2^k e^{-2x}}{2^s-1} - \frac{1}{c(s,\lambda)}\left(e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-1-k}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-1-k}}\right)\right],$$

and

$$U_s^{(s)}(x) = (-1)^s\left[\frac{2^s e^{-x} - 2^s e^{-2x}}{2^s-1} - \frac{1}{c(s,\lambda)}\left(e^{-x} + \lambda e^{-\lambda x} - (\lambda+1)e^{-(\lambda+1)x}\right)\right].$$

It is easy to see that $U_s^{(s)}(0) = 0$. Moreover, if k is even we have that the sign pattern given by the coefficients of $U_s^{(k)}$ is "$+,-,-,+$", implying that $U_s^{(k)}$ has at most two real roots, $\lim_{x\to-\infty} U_s^{(k)} = +\infty$ and $\lim_{x\to+\infty} U_s^{(k)} = 0^+$. If k is odd then the signs of the coefficients of $U_s^{(k)}$ are "$-,+,+,-$". Therefore, $U_s^{(k)}$ has again at most two real roots, $\lim_{x\to-\infty} U_s^{(k)} = -\infty$ and $\lim_{x\to+\infty} U_s^{(k)} = 0^-$. In order to analyse the sign variation of these functions we need to separate into two cases, as done in the analysis of the signs of the coefficients of $U_s^{(k)}$.

s even: The sign pattern given by the coefficient of $U_s^{(s)}$ is "$+,-,-,+$". Therefore, by Proposition 8, this function has at most two real roots and the behaviour of $U_s^{(s)}$ is the same as the behaviour of $U_s^{(k)}$, for k even, when $x \to \pm\infty$. As $U_s^{(s)}(0) = 0$ we only need to locate one more root. Observe that, due to the limits computed above and the maximum number of possible roots, $U_s^{(s)}$ has one of the following sign variations, "$+$" or "$-,+$". Additionally, $U_s^{(s-1)}(0) = \frac{2^{s-1}-2^s}{2^s-1} + \frac{1}{c(s,\lambda)}$, whose sign depends on s and λ. Hence, we need to analyse what happens to the sign variation of $U_s^{(s-1)}$ when $U_s^{(s-1)}(0)$ is positive or negative.

sign variation of $U_s^{(s)}$	"$-,+$"		"$+$"	
monotonicity of $U_s^{(s-1)}$	"$\searrow\nearrow$"		"$\nearrow$"	
$U_s^{(s-1)}(0)$	positive	negative	positive	negative
sign variation of $U_s^{(s-1)}$ in $(0,+\infty)$	"$+,-$"	"$-$"	not possible	"$-$"

Thus, there are only two possible sign variations for $U_s^{(s-1)}$, when $x \in (0,+\infty)$: "$+,-$" or "$+$". Since, $U_s^{(s-2)}(0) = \frac{2^s-2^{s-2}}{2^s-1} - \frac{1}{c(s,\lambda)}(1 + \frac{1}{\lambda} - \frac{1}{\lambda+1})$, again we cannot decide about its sign. So, when analysing the sign variation of $U_s^{(s-2)}$ we need to consider both possibilities.

sign variation of $U_s^{(s-1)}$	"$+,-$"		"$-$"	
monotonicity of $U_s^{(s-2)}$	"$\nearrow\searrow$"		"$\searrow$"	
$U_s^{(s-2)}(0)$	positive	negative	positive	negative
sign variation of $U_s^{(s-2)}$ in $(0,+\infty)$	"$+$"	"$-,+$"	"$+$"	not possible

So, $U_s^{(s-2)}$ has the same possible sign variations as U_s, in $(0,+\infty)$. Repeating now this process, we are able to conclude that the sign variation of $U_s^{'}$ has the same behaviour as the sign variation of $U_s^{(s-1)}$. Remembering that $U_s(0) = 0$ and that $\lim_{x\to+\infty} U_s(x) = 0^+$, we have the following sign variation table:

sign variation of $U_s^{'}$	"$+,-$"	"$-$"
monotonicity of U_s	"$\nearrow\searrow$"	"$\searrow$"
sign variation of U_s in $(0,+\infty)$	"$+$"	not possible

Therefore, we may conclude that $U_s(x) \geq 0$, for $x \geq 0$.

s odd: In this case, we have that the sign pattern given by the coefficients of $U_s^{(s)}$ is "$-,+,+,-$". Thus, $\lim_{x\to-\infty} U_s^{(s)} = -\infty$, $\lim_{x\to+\infty} U_s^{(s)} = 0^-$ and $U_s^{(s)}$ has at most two real roots. Given that $U_s^{(s)}(0) = 0$, we may conclude that the sign variation of $U_s^{(s)}$ is at most "$+,-$", when $x \in (0,+\infty)$. But this coincides with the case $s-1$, when s even, so we can repeat the previous arguments to conclude that $U_s(x) \geq 0$, for $x \geq 0$.

□

Corollary 6. [2, Corollary 7.1] *Let X and Y be two random variables defined as in* (4.1). *Then* $\frac{\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x))}{x} \leq 1$, *for every* $x > 0$.

Proof. Taking into account what was proved in Proposition 15 and that $\overline{T}_{Y,s}$ is decreasing, the conclusion follows. □

Proposition 16. [2, Proposition 7.2] *Let X and Y be two random variables defined as in* (4.1). *For every $s \geq 1$, $X \leq_{s-IFRA} Y$.*

Proof. We need to prove that $t_s(x) = \frac{\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x))}{x}$ is increasing, for $x > 0$, or, equivalently, that the sign variation in $(0,+\infty)$ of $t_s(x) - a$ is, at most, "$-,+$", for $a > 0$. But, by Corollary 6, we only need to consider the case where $0 < a \leq 1$. So, proving that $X \leq_{s-IFRA} Y$ is equivalent to proving that $\overline{T}_{X,s}(x) - \overline{T}_{Y,s}(ax)$ changes sign at most once for $0 < a \leq 1$, and if the change occurs it is in the order "$+,-$", as x traverses from 0 to $+\infty$. But, this is still equivalent to proving that the sign variation of $V_s(x) = \overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax)$ is, at most, "$-,+$", for $a > 1$. Writing the expression of V_s explicitly, we have

$$V_s(x) = \frac{1}{c(s,\lambda)}\left(e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-1}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-1}}\right) - \frac{2^s e^{-ax} - e^{-2ax}}{2^s - 1}.$$

The arguments to characterize the sign variation of V_s will be analogous to what was done in Proposition 15, so we need the derivatives of orders $k < s$ and s of V_s. Thus,

$$V_s^{(k)}(x) = (-1)^k\left[\frac{1}{c(s,\lambda)}\left(e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-1-k}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-1-k}}\right) - \frac{2^s a^k e^{-ax} - 2^k a^k e^{-2ax}}{2^s - 1}\right],$$

and

$$V_s^{(s)}(x) = (-1)^s \left[\frac{1}{c(s,\lambda)} \left(e^{-x} + \lambda e^{-\lambda x} - (\lambda+1)e^{-(\lambda+1)x} \right) - \frac{2^s a^s e^{-ax} - 2^s a^s e^{-2ax}}{2^s - 1} \right].$$

Note that $V_s^{(s)}(0) = 0$. In order to apply Proposition 8, we need to order decreasingly, with respect to the exponents, the exponential terms in the previous functions, which means we need to separate our analysis into several cases, depending on the location of a with respect to λ.

Case 1: $1 < a < 2a < \lambda < \lambda + 1$. In this case, we have that the sign of the coefficients of V_s are "$+,-,+,+,-$", implying that V_s has at most three real roots. Additionally, $\lim_{x\to-\infty} V_s(x) = -\infty$, $\lim_{x\to+\infty} V_s(x) = 0^+$ and $V_s(0) = 0$. Observe that when k is even we have that the signs of the coefficients of $U_s^{(k)}$ are "$+,-,+,+,-$". By Proposition 8, we conclude that $V_s^{(k)}(x)$ has at most three real roots. Furthermore, $\lim_{x\to-\infty} V_s^{(k)}(x) = -\infty$ and $\lim_{x\to+\infty} V_s^{(k)}(x) = 0^+$. When k is odd, the sign pattern given by the coefficients of $V_s^{(k)}(x)$ is "$-,+,-,-,+$. Hence, $\lim_{x\to-\infty} V_s^{(k)}(x) = +\infty$, $\lim_{x\to+\infty} V_s^{(k)}(x) = 0^-$ and $V_s^{(k)}(x)$ has at most three real roots. As it was done previously, we need to separate our analysis into two cases.

s even: The sign pattern given by the coefficients of $V_s^{(s)}$ is "$+,-,+,+,-$", which implies that $\lim_{x\to-\infty} V_s^{(s)}(x) = -\infty$, $\lim_{x\to+\infty} V_s^{(s)}(x) = 0^+$ and $V_s^{(s)}$ has at most three real roots. Hence, the most sign varying in $(0,+\infty)$ case for $V_s^{(s)}$ is "$+,-,+$". Given that $V_s^{(s-1)}(0) = \frac{2^{s-1}a^{s-1}}{2^s-1} - \frac{1}{c(s,\lambda)}$, we cannot determine its sign, so we need to study the case where it is positive and negative as it is presented in the table bellow.

sign variation of $V_s^{(s)}$	"$+,-,+$"		"$-,+$"		"$+$"	
monotonicity of $V_s^{(s-1)}$	"$\nearrow\searrow\nearrow$"		"$\searrow\nearrow$"		"$\nearrow$"	
$V_s^{(s-1)}(0)$ sign variation of $V_s^{(s-1)}$ in $(0,+\infty)$	positive "$+,-$"	negative "$-,+,-$"	positive "$+,-$"	negative "$-$"	positive not possible	negative "$-$"

Now, proceeding with the analysis of $V_s^{(s-2)}$, we have that $V_s^{(s-2)}(0) = \frac{1}{c(s,\lambda)}(1 + \frac{1}{\lambda} + \frac{1}{\lambda+1}) - \frac{2^s a^{s-2} - 2^{s-2} a^{s-2}}{2^s-1}$. So, once more we cannot determine the sign of the derivative of order $s-2$ of V_s at 0. By analysing the possible cases we obtain

sign variation of $V_s^{(s-1)}$	"$-,+,-$"		"$+,-$"		"$-$"	
monotonicity of $V_s^{(s-2)}$	"$\searrow\nearrow\searrow$"		"$\nearrow\searrow$"		"$\searrow$"	
$V_s^{(s-2)}(0)$ sign variation of $V_s^{(s-2)}$ in $(0,+\infty)$	positive "$+,-,+$"	negative "$-,+$"	positive "$+$"	negative "$-,+$"	positive "$+$"	negative not possible

We may conclude that the most sign varying possibility for $V_s^{(s-2)}$ is the same as for $V_s^{(s)}$. Hence, we may resort on the same argument to conclude that the most sign varying in $(0,+\infty)$ case for V_s' is "$-,+,-$".

sign variation of $V_s^{'}$	"$-,+,-$"	"$-,+$"	"$-$"
monotonicity of V_s	"$\searrow\nearrow\searrow$"	"$\searrow\nearrow$"	"$\searrow$"
sign variation of U_s in $(0,+\infty)$	"$-,+$"	"$+$"	not possible

We may conclude that V_s changes sign at most once in $(0,+\infty)$, and if the change occurs it is in the order "$-,+$".

s odd: The sign of the coefficients of $V_s^{(s)}$ are "$-,+,-,-,+$". Thus, $\lim_{x\to\infty} V_s^{(s)}(x) = +\infty$, $\lim_{x\to+\infty} V_s^{(s)}(x) = 0^-$ and $V_s^{(s)}$ has at most three real roots. This implies that the sign variation of $V_s^{(s)}$ in $(0,+\infty)$ is, at most, $-,+,-$, which corresponds to the behaviour of the derivative of order $s-1$ when s is even. So repeating the previous arguments, the conclusion still holds.

Case 2: $1 < a < \lambda < 2a < \lambda+1$. The sign pattern of the coefficients of V_s coincides with the one observed in the previous case, so the result also holds.

Case 3: $1 < a < \lambda < \lambda+1 < 2a$. After ordering decreasingly, with respect to the exponents, the exponential terms in V_s we have that the sign pattern of the coefficients is "$+,-,+,-,+$". Hence, by Proposition 8, V_s has at most four real roots. Once again, since $V_s(0) = 0$, we only need to locate the other three roots. Given the number of possible roots, a direct usage of the previous arguments does not allow to conclude about a sign variation compatible with the s-IFRA order. However, note that $a > \frac{\lambda+1}{2}$, so, $V_s(x) > \overline{T}_{Y,s}(x) - \overline{T}_{X,s}(\frac{\lambda+1}{2}x) = V_{s,*}(x)$, for every fixed $x \geq 0$. So, if we prove that $V_{s,*} \geq 0$, for every $x \geq 0$, it follows that $V_s(x) \geq 0$, for every $x \geq 0$. Rewriting $V_{s,*}$ with the exponentials already ordered decreasingly with respect to the exponents, we obtain,

$$V_{s,*}(x) = \frac{e^{-x}}{c(s,\lambda)} - \frac{2^s}{2^s-1} e^{-\frac{\lambda+1}{2}x} + \frac{1}{c(s,\lambda)\lambda^{s-1}} e^{-\lambda x} + \left(\frac{1}{2^s-1} - \frac{1}{c(s,\lambda)(\lambda+1)^{s-1}}\right) e^{-(\lambda+1)x}.$$

The coefficient of $e^{-(\lambda+1)x}$ is positive, since, the sign of $\frac{1}{2^s-1} - \frac{1}{c(s,\lambda)(\lambda+1)^{s-1}}$ is the same as the sign of $A(s,\lambda) = c(s,\lambda)(\lambda+1)^{s-1} - 2^s - 1$. If we differentiate $A(s,\lambda)$ as a function of λ, we can see that this function is increasing. Noting that $A(s,1) = 0$, it follows that the coefficient is positive. Therefore, the sign pattern is "$+,-,+,+$". Hence, besides $V_{s,*}(0) = 0$, we have that $V_{s,*}$ has at most two real roots, $\lim_{\to-\infty} V_{s,*}(x) = +\infty$ and $\lim_{x\to+\infty} V_{s,*}(x) = 0^+$. Observe now that,

$$V_{s,*}^{(k)}(x) = (-1)^k \left[\frac{1}{c(s,\lambda)} e^{-x} - \frac{2^{s-k}}{2^s-1}(\lambda+1)^k e^{-\frac{\lambda+1}{2}x} + \frac{1}{c(s,\lambda)\lambda^{s-1-k}} e^{-\lambda x} \right.$$

$$\left. + \left(\frac{1}{2^s-1} - \frac{1}{c(s,\lambda)(\lambda+1)^{s-1}}\right)(\lambda+1)^k e^{-(\lambda+1)x}\right],$$

$$V_{s,*}^{(s)}(x) = (-1)^s \left[\frac{1}{c(s,\lambda)} e^{-x} - \frac{(\lambda+1)^s}{2^s - 1} e^{-\frac{\lambda+1}{2}x} + \frac{\lambda}{c(s,\lambda)} e^{-\lambda x} \right.$$

$$\left. + \left(\frac{1}{2^s - 1} - \frac{1}{c(s,\lambda)(\lambda+1)^{s-1}} \right) (\lambda+1)^s e^{-(\lambda+1)x} \right].$$

Note that, when k is even, we have that the sign pattern given by the coefficients of $V_{s,*}^{(k)}$ is "$+,-,+,+$". Hence, $\lim_{x\to-\infty} V_{s,*}^{(k)}(x) = +\infty$, $\lim_{x\to+\infty} V_{s,*}^{(k)}(x) = 0^+$ and $V_{s,*}^{(k)}$ has at most two real roots. When k is odd the sign pattern is "$-,+,-,-$", and, therefore, the sign of the limits is reversed, when $x \to \pm\infty$. Now, we need to analyse what happens for s odd or even.

s even: Repeating the arguments above, the sign pattern of the coefficients of $V_{s,*}^{(s)}$ is the same as for $V_{s,*}$, so the limits when $x \to \pm\infty$ coincide. Moreover, $V_{s,*}^{(s)}(0) = 0$, so we only need to locate one more root.

sign variation of $V_{s,*}^{(s)}$	"$-,+$"		"$+$"	
monotonicity of $V_{s,*}^{(s-1)}$	"$\searrow\nearrow$"		"$\nearrow$"	
$V_{s,*}^{(s-1)}(0)$	positive	negative	positive	negative
sign variation of $V_{s,*}^{(s-1)}$ in $(0,+\infty)$	"$+,-$"	"$-$"	not possible	"$-$"

Thus, the possibilities for the sign variation of $V_{s,*}^{(s-1)}$, in $(0,+\infty)$ are "$+,-$" or "$-$". Doing the same analysis for $V_{s,*}^{(s-2)}$, we have

sign variation of $V_{s,*}^{(s-1)}$	"$+,-$"		"$-$"	
monotonicity of $V_{s,*}^{(s-2)}$	"$\nearrow\searrow$"		"$\searrow$"	
$V_{s,*}^{(s-2)}(0)$	positive	negative	positive	negative
sign variation of $V_{s,*}^{(s-2)}$ in $(0,+\infty)$	"$+$"	"$-,+$"	"$+$"	not possible

We may conclude that the behaviour of the sign variation of $V_{s,*}^{(s-2)}$ is the same as for $V_{s,*}^{(s)}$. Once again, we may conclude that $V'_{s,*}$ has one of the following sign variations in $(0,+\infty)$: "$+,-$" or $-$".

sign variation of $V'_{s,*}$	"$+,-$"	"$-$"
monotonicity of $V_{s,*}$	"$\nearrow\searrow$"	"$\searrow$"
sign variation of $V_{s,*}$ in $(0,+\infty)$	"$+$"	not possible

Hence, $V_{s,*}(x) \geq 0$, for $x \geq 0$, which implies that $V_s(x) \geq 0$, for $x \geq 0$.

s odd: This corresponds to the behaviour of the derivative of order $s-1$ of $V_{s,*}$, when s is even, so the conclusion still holds.

Case 4: $1 < \lambda < a < 2a < \lambda + 1$. The sign of the coefficients in this case is "$+,+,-,+,-$", meaning that, at most, there are three real roots. This is exactly the same sign pattern that we found in **Case 1**. So, we can repeat the arguments to conclude that the sign variation of V_s, in $(0,+\infty)$, is at most "$-,+$".

Case 5: $1 < \lambda < a < \lambda + 1 < 2a$. This case is very simple to analyse. In fact, the sign pattern given by the coefficients of V_s is "$+,+,-,-,+$", so, taking into account Proposition 8, it has at most two real roots. Moreover, $\lim_{x\to-\infty} V_s(x) = +\infty$ and $\lim_{x\to-\infty} V_s(x) = 0^+$. But this behaviour is compatible to the following sign variations in $(0,+\infty)$: "$-,+$" or "$+$".

Case 6: $1 < \lambda < \lambda+1 < a < 2a$. The signs of the coefficients of V_s, in this case, are the same as in the previous case, so the result holds.

Therefore, we have shown that, in all the possible cases, the sign variation of V_s is, at most, "$-,+$" so, t_s is increasing and, consequently, $X \leq_{s-IFRA} Y$. ☐

Theorem 14. [2, Theorem 7.1] *Let X and Y be two random variables defined as in 4.1. For every $s \geq 1$, $X \leq_{s-IFR} Y$.*

Proof. In order to prove that $X \leq_{s-IFR} Y$, we will be applying similar arguments as the ones used in Proposition 16. However, we now want to prove that $c_s(x) = \overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x))$ is convex. Taking into account Theorem 11 and Theorem 13, it is enough to prove that $V_s(x) = \overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax+b)$ changes sign at most twice, in the order "$+,-,+$", for every $a>0$ and $b>0$, since we have already proved that $X \leq_{s-IFRA} Y$ (note that this is also the case where $b=0$, so we do not need to study it). Observe that, now, $V_s(0) = 1-\overline{T}_{X,s}(b) > 0$. Furthermore,

$$V_s(x) = \frac{1}{c(s,\lambda)}\left(e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-1}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-1}}\right) - \frac{2^s e^{-(ax+b)} - e^{-2(ax+b)}}{2^s-1},$$

$$V_s^{(k)}(x) = (-1)^k \left[\frac{1}{c(s,\lambda)}\left(e^{-x} + \frac{e^{-\lambda x}}{\lambda^{s-1-k}} - \frac{e^{-(\lambda+1)x}}{(\lambda+1)^{s-1-k}}\right) - \frac{2^s a^k e^{-(ax+b)} - 2^k a^k e^{-2(ax+b)}}{2^s-1}\right],$$

$$V_s^{(s)}(x) = (-1)^s \left[\frac{1}{c(s,\lambda)}\left(e^{-x} + \lambda e^{-\lambda x} - (\lambda+1)e^{-(\lambda+1)x}\right) - 2^s a^s \frac{e^{-(ax+b)} - e^{-2(ax+b)}}{2^s-1}\right].$$

We need to order decreasingly the exponential terms of these functions, with respect to the exponents, in order to apply Proposition 8. So, once again, we need to consider different cases according to the location of a with respect to the hazard rate λ. Note that $V_s^{(s)}(0) = \frac{(-1)^{s+1}2^s a^s e^{-b}(1-e^{-b})}{2^s-1}$, which has the same sign as $(-1)^{s+1}$, as $b>0$.

Case 1: $1 < a < 2a < \lambda < \lambda+1$. This sign pattern given by the coefficients of V_s is "$+,-,+,+,-$, so V_s has at most three real roots. Moreover, $\lim_{x\to-\infty} V_s(x) = -\infty$ and $\lim_{x\to+\infty} V_s(x) = 0^+$. Taking into account that $V_s(0) > 0$, it follows that the sign variation of V_s in $(0,+\infty)$, is either "$+,-,+$" or "$+$".

Case 2: $1 < a < \lambda < 2a < \lambda+1$. In this case, the signs of the coefficients of V_s are "$+,-,+,-,-$", implying that V_s has at most three roots, $\lim_{x\to-\infty} V_s(x) = -\infty$ and $\lim_{x\to+\infty} V_s(x) = 0^+$. So the behaviour of V_s is the same as the behaviour in the previous case. Therefore, the conclusion holds.

Case 3: $1 < a < \lambda < \lambda+1 < 2a$. The sign pattern of the coefficients of V_s is "$+,-,+,-,+$", so V_s may have, at most, four real roots. We also have that, $\lim_{x\to-\infty} V_s(x) = +\infty$ and $\lim_{x\to+\infty} V_s(x) = 0^+$. For $V_s^{(k)}$, when k is even the signs of the coefficients coincide with the coefficients' signs of V_s and, therefore, the limits when $x \to \pm\infty$, also coincide. On the other hand, when k is odd the sign pattern given by the coefficients of $V_s^{(k)}$ is "$-,+,-,+,-$", which implies that $\lim_{x\to-\infty} V_s^{(k)}(x) = -\infty$ and $\lim_{x\to+\infty} V_s^{(k)}(x) = 0^-$.

s even: In this case, the signs of the coefficients of $V_s^{(s)}$ are the same as the coefficients of V_s, so $V_s^{(s)}$ has, at most, four real roots. Moreover, $V_s^{(s)} < 0$. So the possible sign variations in $(0,+\infty)$ for $V_s^{(s)}$ are "$-,+,-,+$" or "$-,+$".

sign variation of $V_s^{(s)}$	"$-,+,-,+$"		"$-,+$"	
monotonicity of $V_s^{(s-1)}$	"$\searrow\nearrow\searrow\nearrow$"		"$\searrow\nearrow$"	
$V_s^{(s-1)}(0)$	positive	negative	positive	negative
sign variation of $V_s^{(s-1)}$ in $(0,+\infty)$	"$+,-,+,-$"	"$-,+,-$"	"$+,-$"	"$-$"

So the sign variation for $V_s^{(s-1)}$ is "$+,-,+,-$" or "$-,+,-$" or "$+,-$" or "$-$", in $(0,+\infty)$.

sign variation of $V_s^{(s-1)}$	"$+,-,+,-$"		"$-,+,-$"	
monotonicity of $V_s^{(s-2)}$	"$\nearrow\searrow\nearrow\searrow$"		"$\searrow\nearrow\searrow$"	
$V_s^{(s-2)}(0)$	positive	negative	positive	negative
sign variation of $V_s^{(s-2)}$ in $(0,+\infty)$	"$+,-,+$"	"$-,+,-,+$"	"$+,-,+$"	"$-,+$"

sign variation of $V_s^{(s-1)}$	"$+,-$"		"$-$"	
monotonicity of $V_s^{(s-2)}$	"$\nearrow\searrow$"		"$\searrow$"	
$V_s^{(s-2)}(0)$	positive	negative	positive	negative
sign variation of $V_s^{(s-2)}$ in $(0,+\infty)$	"$+$"	"$-,+$"	"$+$"	not possible

Hence, the most sign varying possibiblity, in $(0,+\infty)$, for $V_s^{(s-2)}$ is the same as for $V_s^{(s)}$, so we repeat the argument to obtain that the most sign varying possibility in $(0,+\infty)$ for V_s' is "$+,-,+,-$". Therefore, the monotonicity of V_s is "$\nearrow\searrow\nearrow\searrow$", which implies that the sign variation for V_s may be "$+$" or "$+,-,+$", since $V_s(0) > 0$.

s odd: Now we have that $V_s^{(s)}(0) > 0$ and the sign pattern of the coefficients of $V_s^{(s)}$ is "$-,+,-,+,-$". Taking into account that $\lim_{x\to-\infty} V_s^{(s)}(x) = -1\infty$ and $\lim_{x\to+\infty} V_s^{(s)}(x) = 0^-$, we conclude that the sign variation of this function in $(0,+\infty)$ may be "$+,-$" or "$+,-,+,-$". But this is what we found when analysing the derivative of order $s-1$ of V_s, when s was even, so the result holds.

Case 4: $1 < \lambda < a < 2a < \lambda+1$. This case coincides with **Case 1**, given that the signs of the coefficients of V_s "$+,+,-,+,-$", so the conclusion remains valid.

Case 5: $1 < \lambda < a < \lambda+1 < 2a$. The sign pattern of the coefficients of V_s, after ordering the exponentials decreasingly, with respect to the exponents, is "$+,+,-,-,+$", implying that there are, at most, two real roots, $\lim_{x\to-\infty} V_s(x) = +\infty$ and $\lim_{x\to+\infty} V_s(x) = 0^+$. As $V_s(0) > 0$, the only possibility for the sign variation of V_s, in $(0,+\infty)$, is "$+,-,+$" or "$+$".

Case 6: $1 < \lambda < \lambda+1 < a < 2a$. In this case we have that the signs of the coefficients of V_s are "$+,+,-,-,+$". But this coincides with the previous case.

Case 7: $0 < a < 1$. In this case, we have three different possibilities, according to the value of $2a$ being less or greater than 1: $a < 2a < 1 < \lambda < \lambda + 1$, $a < 1 < 2a < \lambda < \lambda + 1$ and $a < 1 < \lambda < 2a < \lambda + 1$. But, for all of these cases the sign pattern of the coefficients of V_s is "$-,+,+,+,-$". So, $\lim_{x \to -\infty} V_s(x) = -\infty$, $\lim_{x \to +\infty} V_s(x) = 0^-$ and V_s has, at most, two real roots. Therefore, as $V_s(0) > 0$, the only possible sign variation, in $(0, +\infty)$, is "$+,-$".

So, for all the possible cases, we have that V_s changes sign at most twice, and if the change occurs it is in the order "$+,-,+$", as x traverses from 0 to $+\infty$, so, by Theorem 13, since we have proven that $X \leq_{s-IFRA} Y$, in Proposition 16, we may conclude that $X \leq_{s-IFR} Y$. □

Remark 6. *The previous theorem is only a partial extension of the result that Kochar and Xu [6] proved, referred at the beginning of this section, since we are working with parallel systems with two components, instead of an arbitrary number of components. This is due to the inherent difficulty of working with iterated distributions.*

4.2 Failure rate ordering of parallel systems with three exponentially distributed components

After having established the s-IFR order for parallel systems with two components, it is also of interest to study what happens for parallel systems with three components. In order to apply Theorem 13, we need first to prove that the s-IFRA order is satisfied, when comparing these two systems. In what follows, reproducing the approach used in the previous section, some progress with respect to the s-IFRA order, when comparing two exponentially distributed parallel systems with three components, are presented as well as some difficulties, which we were not able to solve. In the sequel we will consider

$$\begin{cases} X = \max(X_1, X_2, X_3), & \text{where } X_1, X_2 \text{ and } X_3 \text{ are independent random variables,} \\ & \text{exponentially distributed with hazard rate } 1 \\ Y = \max(Y_1, Y_2, Y_3), & \text{where } Y_1, Y_2 \text{ and } Y_3 \text{ are independent random variables,} \\ & \text{exponentially distributed with hazard rates } 1, 1 \text{ and } m < 1, \text{ respectively.} \end{cases} \tag{4.2}$$

We want to prove that $X \leq_{s-IFRA} Y$. For that, we will follow the arguments used for the analysis of parallel systems with two components.

Remark 7. *Assume that Y_1 and Y_2 are two random variables having an exponential distribution with hazard rate λ_1 and Y_3 is another random variable also exponentially distributed with hazard rate λ_2. Since λ_1 and λ_2 are scale parameters, we can normalize the variables by dividing all the parameters by λ_1 and take $m = \frac{\lambda_2}{\lambda_1}$, where m is assumed to be a constant smaller than* 1.

By (3.3) we may conclude that

$$\overline{T}_{X,s}(x) = \frac{1}{d(s)} \left(3e^{-x} - \frac{3e^{-2x}}{2^{s-1}} + \frac{e^{-3x}}{3^{s-1}} \right),$$

where $d(s) = 3 - \frac{3}{2^{s-1}} + \frac{1}{3^{s-1}}$, and

$$\overline{T}_{Y,s}(x) = \frac{1}{c(s,m)} \left(\frac{e^{-mx}}{m^{s-1}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1}} - \frac{e^{-2x}}{2^{s-1}} + \frac{e^{-(m+2)x}}{(m+2)^{s-1}} \right),$$

where $c(s,m) = 2 + \frac{1}{m^{s-1}} - \frac{2}{(m+1)^{s-1}} - \frac{1}{2^{s-1}} + \frac{1}{(m+2)^{s-1}}$. We will denote by $V_s(x)$ the difference $\overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax)$, $a > 0$. In order to prove that $X \leq_{s-IFRA} Y$ we need to show that V_s changes sign at most once and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$, for every $a > 0$. The study of the sign variation for V_s starts with the case where $a = 1$.

Proposition 17. *Let X and Y be two random variables defined as in (4.2). For $s \geq 1$, $x \geq 0$ we have that $V_s(x) = \overline{T}_{Y,s}(x) - \overline{T}_{X,s}(x)$ changes sign at most once and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$.*

Proof. In this case, V_s can be written as

$$V_s(x) = \frac{1}{c(s,m)} \left(\frac{e^{-mx}}{m^{s-1}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1}} - \frac{e^{-2x}}{2^{s-1}} + \frac{e^{-(m+2)x}}{(m+2)^{s-1}} \right) - \frac{1}{d(s)} \left(3e^{-x} - \frac{3e^{-2x}}{2^{s-1}} + \frac{e^{-3x}}{3^{s-1}} \right),$$

that is,

$$V_s(x) = e^{-x} \left(\frac{2}{c(s,m)} - \frac{3}{d(s)} \right) + \frac{e^{-2x}}{2^{s-1}} \left(\frac{3}{d(s)} - \frac{1}{c(s,m)} \right) - \frac{e^{-3x}}{3^{s-1} d(s)} + \frac{1}{c(s,m)} \left(\frac{e^{-mx}}{m^{s-1}} + \frac{e^{-(m+2)x}}{(m+2)^{s-1}} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1}} \right).$$

Note that since we are considering $m < 1$ the only case that we need to study is the case where $m < 1 < m+1 < 2 < m+2 < 3$. We will start by analysing the sign of the coefficients of e^{-x} and e^{-2x}. For e^{-x}, we have that the sign of its coefficient is the same as the sign of $A(s,m) = 2d(s) - 3c(s,m)$.

$$A(s,m) = 2\left(3 - \frac{3}{2^{s-1}} + \frac{1}{3^{s-1}}\right) - 3\left(2 + \frac{1}{m^{s-1}} + \frac{1}{(m+2)^{s-1}} - \frac{2}{(m+1)^{s-1}} - \frac{1}{2^{s-1}}\right)$$

$$= \left(\frac{2^s - 3^s}{6^{s-1}}\right) + 3\left(\frac{2}{(m+1)^{s-1}} - \frac{1}{(m+2)^{s-1}} - \frac{1}{m^{s-1}}\right) < 3\left(\frac{2}{(m+1)^{s-1}} - \frac{1}{(m+2)^{s-1}} - \frac{1}{m^{s-1}}\right).$$

Observe now that if we denote $h(m) = \frac{2}{(m+1)^{s-1}} - \frac{1}{(m+2)^{s-1}}$, we have, for $m \in (0,1)$, that h is decreasing and $h(1) \leq 1$. Hence, we may conclude that $h(m) < \frac{1}{m^{s-1}}$, for $m \in (0,1)$. Therefore, the coefficient of e^{-x} is negative. Analogously, we may conclude that the sign of the coefficient of e^{-2x} is the same as the sign of $B(s,m) = 3c(s,m) - d(s)$.

$$B(s,m) = 3\left(2 + \frac{1}{m^{s-1}} + \frac{1}{(m+2)^{s-1}} - \frac{2}{(m+1)^{s-1}} - \frac{1}{2^{s-1}}\right) - \left(3 - \frac{3}{2^{s-1}} + \frac{1}{3^{s-1}}\right)$$

$$= 3 - \frac{1}{3^{s-1}} - 3\left(\frac{2}{(m+1)^{s-1}} - \frac{1}{(m+2)^{s-1}} - \frac{1}{m^{s-1}}\right),$$

which, using the argument above, is positive. Thus, we may conclude that the signs of the coefficients of V_s are "$+,-,-,+,+,-$". So, V_s has at most three real roots and we have that $\lim_{x \to -\infty} V_s(x) = -\infty$

and $\lim_{x\to+\infty} V_s(x) = 0^+$. One root is easily located, since $V_s(0) = 0$. In order to locate the other two roots, we need to study the sign variation of the derivatives of orders $k < s$ and s of V_s. Differentiating V_s we have

$$V_s^{(k)}(x) = (-1)^k \left[e^{-x}\left(\frac{2}{c(s,m)} - \frac{3}{d(s)}\right) + \frac{e^{-2x}}{2^{s-1-k}}\left(\frac{3}{d(s)} - \frac{1}{c(s,m)}\right) - \frac{e^{-3x}}{3^{s-1-k}d(s)} \right.$$
$$\left. + \frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1-k}} + \frac{e^{-(m+2)x}}{(m+2)^{s-1-k}} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1-k}}\right)\right],$$

and

$$V_s^{(s)}(x) = (-1)^s \left[e^{-x}\left(\frac{2}{c(s,m)} - \frac{3}{d(s)}\right) + 2e^{-2x}\left(\frac{3}{d(s)} - \frac{1}{c(s,m)}\right) - \frac{3e^{-3x}}{d(s)} \right.$$
$$\left. + \frac{1}{c(s,m)}\left(me^{-mx} + (m+2)e^{-(m+2)x} - 2(m+1)e^{-(m+1)x}\right)\right].$$

Hence the signs of the coefficients of V_s alternate with each differentiation step and $V_s^{(s)}(0) = 0$. In order to analyse the behaviour of these functions we need to separate the analysis into two cases: when s is even and when s is odd.

s even: The sign pattern given by the coefficients of $V_s^{(s)}$ is "$+,-,-,+,+,-$". This implies that $\lim_{x\to-\infty} V_s^{(s)}(x) = -\infty$, $\lim_{x\to+\infty} V_s^{(s)}(x) = 0^+$ and that $V_s^{(s)}$ has at most three real roots. Since $V_s^{(s)}(0) = 0$, the sign variation of $V_s^{(s)}$, in $(0,+\infty)$, may be "$+,-,+$" or "$-,+$" or "$+$". Analysing $V_s^{(s-1)}$ we have that the signs of its coefficients are "$-,+,+,-,-,+$", implying that $\lim_{x\to-\infty} V_s^{(s-1)}(x) = +\infty$, $\lim_{x\to+\infty} V_s^{(s-1)}(x) = 0^-$ and that $V_s^{(s-1)}$ has at most three real roots. Now, observe that,

$$V_s^{(s-1)}(0) = -\frac{1}{c(s,m)} + \frac{1}{d(s)},$$

and, therefore, we cannot determine its sign. Hence, we need to analyse what happens to the sign variation of $V_s^{(s-1)}$ when $V_s^{(s-1)}(0)$ is positive or negative.

sign variation of $V_s^{(s)}$	"$+,-,+$"		"$-,+$"		"$+$"	
monotonicity of $V_s^{(s-1)}$	"$\nearrow\searrow\nearrow$"		"$\searrow\nearrow$"		"$\nearrow$"	
$V_s^{(s-1)}(0)$ sign variation of $V_s^{(s-1)}$ in $(0,+\infty)$	positive "$+,-$"	negative "$-,+,-$"	positive "$+,-$"	negative "$-$"	positive not possible	negative "$-$"

Thus, $V_s^{(s-1)}$ has three possible sign variations, when $x \in (0,+\infty)$, which are "$-,+,-$" or "$+,-$ or "$-$". Proceeding to the analysis of $V_s^{(s-2)}$ we have that the signs of its coefficients are "$+,-,-,+,+.-$". Hence, $\lim_{x\to-\infty} V_s^{(s-2)}(x) = -\infty$, $\lim_{x\to+\infty} V_s^{(s-2)}(x) = 0^+$ and $V_s^{(s-2)}$ has at most three real roots. Since

$$V_s^{(s-2)}(0) = \frac{1}{c(s,m)}\left(\frac{1}{m}+2-\frac{2}{m+1}-\frac{1}{2}+\frac{1}{m+2}\right)-\frac{11}{6d(s)},$$

we cannot determine its sign. Once again we need to analyse what happens when $V_s^{(s-2)}(0)$ is positive and negative.

sign variation of $V_s^{(s-1)}$	"$-,+,-$"		"$+,-$"		"$-$"	
monotonicity of $V_s^{(s-2)}$	"$\searrow\nearrow\searrow$"		"$\nearrow\searrow$"		"$\searrow$"	
$V_s^{(s-2)}(0)$ sign variation of $V_s^{(s-2)}$ in $(0,+\infty)$	positive "$+,-,+$"	negative "$-,+$"	positive "$+$"	negative "$-,+$"	positive "$+$"	negative not possible

We may conclude that there are three possible sign variations for $V_s^{(s-2)}$ when $x \in (0,+\infty)$: "$+,-,+$" or "$-,+$" or "$+$". As it is possible to see, we have that the sign variation of $V_s^{(s-2)}$ has the same behaviour has the sign variation of $V_s^{(s)}$. So repeating this process, we are able to conclude that the sign variation of $V_s^{'}$ has the same behaviour as the sign variation of $V_s^{(s-1)}$. Remembering that $V_s(0)=0$ and that $\lim_{x\to+\infty} V_s(x) = 0^+$, we have the following table:

sign variation of $V_s^{'}$	"$-,+,-$"	"$+,-$"	"$-$"
monotonicity of V_s sign variation of V_s in $(0,+\infty)$	"$\searrow\nearrow\searrow$" "$-,+$"	"$\nearrow\searrow$" "$+$"	"$\searrow$" not possible

Hence, we may conclude that V_s changes sign at most once, and if the change occurs it is in the order "$-,+$", when x traverses from 0 to $+\infty$.

s odd: In this case, the sign pattern given by the coefficients of $V_s^{(s)}$ is "$-,+,+,-,-,+$". This implies that $V_s^{(s)}$ has at most three real roots and that $\lim_{x\to-\infty} V_s^{(s)}(x) = +\infty$ and $\lim_{x\to+\infty} V_s^{(s)}(x) = 0^-$. Therefore, $V_s^{(s)}$ can have the following sign variations: "$-,+,-$" or "$+,.$" or "$-$". But this is the same behaviour that we find for the derivative of order $s-1$, when s is even. Applying similar arguments, the previous conclusion holds. □

In order to prove that $X \leq_{s-IFRA} Y$, we also need to analyse the sign variation of V_s for $a>1$ and $a<1$.

Proposition 18. *Let X and Y be two random variables defined as in (4.2). For $s \geq 1$, $x \geq 0$ and $a>1$ we have that $V_s(x) = \overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax)$ changes sign at most once and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$.*

Proof. In this case we have that

$$V_s(x) = \frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1}}+2e^{-x}-\frac{2e^{-(m+1)x}}{(m+1)^{s-1}}-\frac{e^{-2x}}{2^{s-1}}+\frac{e^{-(m+2)x}}{(m+2)^{s-1}}\right)-\frac{1}{d(s)}\left(3e^{-ax}-\frac{3e^{-2ax}}{2^{s-1}}+\frac{e^{-3ax}}{3^{s-1}}\right).$$

To see how many roots this function has at most, we need to order decreasingly with respect to the exponents of the exponential terms in V_s, which means we need to separate our analysis into different cases, depending on the location of a with respect to $m+1$ or $m+2$. Recall that for $m<1$.

Case 1: $m<1<m+1<2<m+2<a<2a<3a$. The sign pattern given by the coefficients of V_s is "$+,+,-,-,+,-,+,-$", hence there could exist up to 5 roots. One of the roots is easily located, since $V_s(0)=0$. Due to the number of possible roots a direct usage of those arguments does not allow us to conclude anything about a sign variation compatible with the one we want. Note that, in this case, $a>m+2$, so, for every fixed $x\geq 0$, $\overline{T}_{X,s}(ax)<\overline{T}_{X,s}((m+2)x)$, since $\overline{T}_{X,s}$ is a decreasing function. Therefore, $V_s(x)>\overline{T}_{Y,s}(x)-\overline{T}_{X,s}((m+2)x)=V_{s,*}(x)$. Our aim is to prove that $V_{s,*}$ is positive. The function $V_{s,*}$ can be written as

$$V_{s,*}(x)=\frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1}}+2e^{-x}-\frac{2e^{-(m+1)x}}{(m+1)^{s-1}}+\frac{e^{-(m+2)x}}{(m+2)^{s-1}}\right)-$$

$$-\frac{1}{d(s)}\left(3e^{-(m+2)x}-\frac{3e^{-2(m+2)x}}{2^{s-1}}+\frac{e^{-3(m+2)x}}{3^{s-1}}\right).$$

Now, we need to determine the sign of the coefficient of $e^{-(m+2)x}$. Note, that its sign is the same as the sign of $A(s,m)=d(s)-3c(s,m)(m+2)^{s-1}$. But,

$$A(s,m)=3-\frac{3}{2^{s-1}}+\frac{1}{3^{s-1}}-3(m+2)^{s-1}\left(2+\frac{1}{m^{s-1}}+\frac{1}{(m+2)^{s-1}}-\frac{2}{(m+1)^{s-1}}-\frac{1}{2^{s-1}}\right)=$$

$$=-\frac{3}{2^{s-1}}+\frac{1}{3^{s-1}}-6(m+2)^{s-1}+\frac{6(m+2)^{s-1}}{(m+1)^{s-1}}-\frac{3(m+2)^{s-1}}{m^{s-1}}+\frac{3(m+2)^{s-1}}{2^{s-1}}<0.$$

Hence, the coefficient of $e^{-(m+2)x}$ is negative and the signs of the coefficients of $V_{s,*}$ are "$+,+,-,-,-,+,-$". This implies that $V_{s,*}$ has at most three real roots, $\lim_{x\to-\infty}V_{s,*}(x)=-\infty$ and $\lim_{x\to+\infty}V_{s,*}(x)=0^+$. Moreover, we have that $V_{s,*}(0)=0$, so one root is already located. As we have done before, we will look at the derivatives of orders $k<s$ and s, in order to find the remaining roots. Differentiating we have

$$V_{s,*}^{(k)}(x)=(-1)^k\left[\frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1-k}}+2e^{-x}-\frac{2e^{-(m+1)x}}{(m+1)^{s-1-k}}-\frac{e^{-2x}}{2^{s-1-k}}+\frac{e^{-(m+2)x}}{(m+2)^{s-1-k}}\right)\right.$$

$$\left.-\frac{1}{d(s)}\left(3(m+2)^ke^{-(m+2)x}-\frac{3(m+2)^ke^{-2(m+2)x}}{2^{s-1-k}}+\frac{(m+2)^ke^{-3(m+2)x}}{3^{s-1-k}}\right)\right].$$

$$V_{s,*}^{(s)}(x)=(-1)^s\left[\frac{1}{c(s,m)}\left(me^{-mx}+2e^{-x}-2(m+1)e^{-(m+1)x}-2e^{-2x}+(m+2)e^{-(m+2)x}\right)\right.$$
$$\left.-\frac{1}{d(s)}\left(3(m+2)^se^{-(m+2)x}-6(m+2)^se^{-2(m+2)x}+3(m+2)^se^{-3(m+2)x}\right)\right].$$

Hence the signs of the coefficients of $V_{s,*}$ alternate with each differentiation and $V_{s,*}^{(s)}(0)=0$. As before we need to analyse separately the cases depending on the oddity of s.

s even: In this case, the signs of the coefficients of $V_{s,*}^{(s)}$ are "$+,+,-,-,-,+,-$, implying that $V_{s,*}^{(s)}$ has at most three real roots and that $\lim_{x\to-\infty}V_{s,*}^{(s)}(x)=-\infty$ and $\lim_{x\to+\infty}V_{s,*}^{(s)}(x)=$

0^+. Therefore, $V_{s,*}^{(s)}$ has three possible sign variations, when $x \in (0,+\infty)$: "$+,-,+$" or "$-,+$" or "$+$". Analysing $V_{s,*}^{(s-1)}$, we have that the sign pattern given by its coefficients is "$-,-,+,+,+,-,+$. This means that $\lim_{x\to-\infty} V_{s,*}^{(s-1)}(x) = +\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(s-1)}(x) = 0^-$. Moreover, the sign of $V_{s,*}^{(s-1)}(0)$ is the same as the sign of $B(s,m) = -d(s)+c(m+2)^{s-1}$. Note that we only need to study the sign of $B(s,m)$ for $s \geq 2$, given that for $s=1$ we have $V_{s,*}^{(s-1)}(0) = 0$. Therefore,

$$
\begin{aligned}
B(s,m) &= -3+\frac{3}{2^{s-1}}-\frac{1}{3^{s-1}}+(m+2)^{s-1}\left(2+\frac{1}{m^{s-1}}+\frac{1}{(m+2)^{s-1}}-\frac{2}{(m+1)^{s-1}}-\frac{1}{2^{s-1}}\right) \\
&> -2+2(m+2)^{s-1}-\frac{(m+2)^{s-1}}{2^{s-1}}+(m+2)^{s-1}\left(\frac{1}{m^{s-1}}-\frac{2}{(m+1)^{s-1}}\right) \\
&= (m+2)^{s-1}\left(-\frac{2}{(m+2)^{s-1}}+2-\frac{1}{2^{s-1}}\right)+(m+2)^{s-1}\left(\frac{1}{m^{s-1}}-\frac{2}{(m+1)^{s-1}}\right) \\
&> (m+2)^{s-1}\left(\frac{1}{m^{s-1}}-\frac{2}{(m+1)^{s-1}}\right) = \frac{(m+2)^{s-1}}{m^{s-1}}\left(1-\frac{2m^{s-1}}{(m+1)^{s-1}}\right).
\end{aligned}
$$

The first inequality is obtained by noting that $\frac{1}{2^{s-1}} > \frac{1}{3^{s-1}}$. Furthermore, since $\frac{1}{(m+2)^{s-1}} < \frac{1}{2^{s-1}}$, and $\frac{3}{2^{s-1}} < \frac{3}{2}$, the second inequality follows. Let $h(m) = \left(1-\frac{2m^{s-1}}{(m+1)^{s-1}}\right)$, $m \in (0,1)$. It is easy to see that h is a decreasing function and that $h(0) = 1$ and $0 \leq h(1) = 1-\frac{2}{2^{s-1}} < 1$, since $s > 1$. Hence, $h(m) \geq 0$ for $m \in (0,1)$, which implies that $B(s,m) > 0$. So, $V_{s,*}^{(s-1)}(0) > 0$. Thus, we have the following possibilities for the sign variation of $V_{s,*}^{(s-1)}$,

sign variation of $V_{s,*}^{(s)}$	"$+,-,+$"	"$-,+$"	"$+$"
monotonicity of $V_{s,*}^{(s-1)}$	"$\nearrow\searrow\nearrow$"	"$\searrow\nearrow$"	"$\nearrow$"
sign variation of $V_{s,*}^{(s-1)}$ in $(0,+\infty)$	"$+,-$"	"$+,-$"	not possible

Therefore, $V_{s,*}$ has one possible sign variation when $x > 0$: "$+,-$". Observe that, when k is even, the sign pattern given by the coefficients of $V_{s,*}^{(k)}$ is "$+,+,-,-,-,+,-$", implying that $\lim_{x\to-\infty} V_{s,*}^{(k)}(x) = -\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(k)}(x) = 0^+$. On the other hand, when k is odd the signs of the coefficients of the derivative of order k are "$-,-,+,+,+,-,+$". which implies that $\lim_{x\to-\infty} V_{s,*}^{(k)}(x) = +\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(k)}(x) = 0^-$. However, the sign of the derivatives of order $k < s-1$ of the function that we are studying at 0 cannot be determined, hence, we need to analyse what happens when that value is negative and positive. We will start by analysing the sign variation of $V_{s,*}^{(s-2)}$.

sign variation of $V_{s,*}^{(s-1)}$	"$+,-$"	
monotonicity of $V_{s,*}^{(s-2)}$	"$\nearrow\searrow$"	
$V_{s,*}^{(s-2)}(0)$	positive	negative
sign variation of $V_{s,*}^{(s-2)}$ in $(0,+\infty)$	"$+$"	"$-,+$"

Thus, $V_{s,*}^{(s-2)}$, when $x \in (0,+\infty)$, has two possible sign variations: "$-,+$" or "$+$". Taking this into account, for $V_{s,*}^{(s-3)}$ we have the following table:

sign variation of $V_{s,*}^{(s-2)}$	“$-,+$”		“$+$”	
monotonicity of $V_{s,*}^{(s-3)}$	“$\searrow\nearrow$”		“$\nearrow$”	
$V_{s,*}^{(s-3)}(0)$	positive	negative	positive	negative
sign variation of $V_{s,*}^{(s-3)}$ in $(0,+\infty)$	“$+,-$”	“$-$”	not possible	“$-$”

So, $V_{s,*}^{(s-3)}$ has two possible sign variations as x traverses from 0 to $+\infty$: “$+,-$” or “$-$”. Finally, we analyse the sign variation of $V_{s,*}^{(s-4)}$, when $x > 0$.

sign variation of $V_{s,*}^{(s-3)}$	“$+,-$”		“$-$”	
monotonicity of $V_{s,*}^{(s-4)}$	“$\nearrow\searrow$”		“$\searrow$”	
$V_{s,*}^{(s-4)}(0)$	positive	negative	positive	negative
sign variation of $V_{s,*}^{(s-4)}$ in $(0,+\infty)$	“$+$”	“$-,+$”	“$+$”	not possible

So, $V_{s,*}^{(s-4)}$ has two possible sign variations, when $x \in (0,+\infty)$: “$-,+$” or “$+$”. We may conclude that the sign variation of $V_{s,*}^{(s-4)}$ has the same behaviour has the sign variation of $V_{s,*}^{(s-2)}$. Hence, repeating this process we find that the behaviour of the sign variation of $V_{s,*}^{(s-3)}$ is the same as for the sign variation of $V_{s,*}^{'}$. Taking into account that $V_{s,*}(0) = 0$ and $\lim_{x\to+\infty} V_{s,*}(x) = 0^+$, we have that $V_{s,*}(x) \geq 0$, for $x \geq 0$. This implies that $V_s(x) \geq 0$, for $x \geq 0$.

s odd: The signs of the coefficients of $V_{s,*}^{(s)}$ are “$-,-,+,+,+,-,+$”. This implies that $\lim_{x\to-\infty} V_{s,*}^{(s)}(x) = +\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(s)}(x) = 0^-$. Hence, $V_{s,*}^{(s)}$ has three possible sign variations when $x > 0$: “$-,+,-$” or “$+,-$” or “$-$”. Since, for s even $V_{s,*}^{(s-1)}(0) > 0$, for s odd $V_{s,*}^{(s-1)}(0) < 0$. Moreover, when k is even, the sign pattern given by the coefficients of $V_{s,*}^{(k)}$ is “$+,+,-,-,-,+,-$”, implying that $\lim_{x\to-\infty} V_{s,*}^{(k)}(x) = -\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(k)}(x) = 0^+$. On the other hand, when k is odd the signs of the coefficients of the derivative of order k are “$-,-,+,+,+,-,+$”. which implies that $\lim_{x\to-\infty} V_{s,*}^{(k)}(x) = +\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(k)}(x) = 0^-$. The table bellow summarizes the sign variation of the derivative of order $s-1$.

sign variation of $V_{s,*}^{(s)}$	“$-,+,-$”	“$+,-$”	“$-$”
monotonicity of $V_{s,*}^{(s-1)}$	“$\searrow\nearrow\searrow$”	“$\nearrow\searrow$”	“$\searrow$”
sign variation of $V_{s,*}^{(s-1)}$ in $(0,+\infty)$	“$-,+$”	“$-,+$”	not possible

We conclude that, when $x > 0$, $V_{s,*}^{(s-1)}$ changes sign exactly once and the change occurs in the order “$-,+$”. The possible sign variations for $V_{s,*}^{(s-2)}$ are described in the following table.

sign variation of $V_{s,*}^{(s-1)}$	“$-,+$”	
monotonicity of $V_{s,*}^{(s-2)}$	“$\searrow\nearrow$”	
$V_{s,*}^{(s-2)}(0)$	positive	negative
sign variation of $V_{s,*}^{(s-2)}$ in $(0,+\infty)$	“$+,-$”	“$-$”

So, $V_{s,*}^{(s-2)}$ has to possible sign variations, “$+,-$” or “$-$”, when $x \in (0,\infty)$. But this coincides with the behaviour of the sign variation of $V_{s,*}^{(s-3)}$, when s is even. Applying the previous arguments, we may conclude that $V_{s,*} \geq 0$, for $x \geq 0$.

Case 2: $m < 1 < a < m+1 < 2 < 2a < m+2 < 3a$. The signs of the coefficients of V_s are "$+,+,-,-,-,+,+,-$". Hence, we have that $\lim_{x\to-\infty} V_s(x) = -\infty$, $\lim_{x\to+\infty} V_s(x) = 0^+$ and V_s has at most three real roots. So, we may repeat the arguments used in the course of Proposition 17 to conclude that V_s changes sign at most once and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$.

Case 3: $m < 1 < a < m+1 < 2 < m+2 < 2a < 3a$. This case coincides with the behaviour observed in the previous case, so the conclusion holds.

Case 4: $m < 1 < m+1 < a < 2 < 2a < m+2 < 3a$. We can observe that the sign pattern given by the coefficients of V_s is "$+,+,-,-,-,+,+,-$". Hence, the analysis to be done in this case is the same as the analysis done in **Case 2** above. Therefore, the conclusion holds.

Case 5: $m < 1 < m+1 < a < 2 < m+2 < 2a < 3a$. The behaviour in this case for the sign pattern given by the coefficients of V_s is the same as the one observed in the previous case. Once again, we may conclude that V_s changes sign at most once and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$.

Case 6: This case coincides with the behaviour observed in the previous case, so the conclusion holds.

We have proved that, for $a > 1$ and for all the cases, depending on the location of a with respect to $m+1$ and $m+2$, V_s changes sign at most once and if the change occurs it is in the order "$-,+$", as x traverses from 0 to $+\infty$. □

The only case left to study, now, is when $a < 1$. However, this case is not simple, because of the large number of possible cases.

Proposition 19. *Let X and Y be two random variables defined as in (4.2). Then, for each of the following cases,*

Case 1: $a < m < 1 < 2a < m+1 < 2 < 3a < m+2$,
Case 2: $a < m < 1 < 2a < m+1 < 3a < 2 < m+2$,
Case 3: $a < m < 1 < 2a < 3a < m+1 < 2 < m+2$,
Case 4: $a < m < 2a < 1 < 3a < m+1 < 2 < m+2$,
Case 5: $a < m < 2a < 3a < 1 < m+1 < 2 < m+2$,

$V_s(x) = \overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax) \leq 0$, *for* $s \geq 1$, $x \geq 0$.

Proof. Recall that,

$$V_s(x) = \frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1}} - \frac{e^{-2x}}{2^{s-1}} + \frac{e^{-(m+2)x}}{(m+2)^{s-1}}\right) - \frac{1}{d(s)}\left(3e^{-ax} - \frac{3e^{-2ax}}{2^{s-1}} + \frac{e^{-3ax}}{3^{s-1}}\right).$$

Therefore, the derivatives of V_s of orders $s < k$ and s, have the following expressions

$$V_s^{(k)}(x) = (-1)^k\left[\frac{1}{c(s,m)}\left(\frac{e^{mx}}{m^{s-1-k}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1-k}} - \frac{e^{-2x}}{2^{s-1-k}} + \frac{e^{-(m+2)x}}{(m+2)^{s-1-k}}\right)\right.$$

$$\left. - \frac{1}{d(s)}\left(3a^k e^{-ax} - 3a^k\frac{e^{-2ax}}{2^{s-1-k}} + a^k\frac{e^{-3ax}}{3^{s-1-k}}\right)\right],$$

$$V_s^{(s)}(x) = (-1)^s \left[\frac{1}{c(s,m)} \left(me^{-mx} + 2e^{-x} - 2(m+1)e^{-(m+1)x} - 2e^{-2x} + (m+2)e^{-(m+2)x} \right) - \frac{1}{d(s)} \left(3a^s e^{-ax} - 6a^s e^{-2ax} + 3a^s e^{-3ax} \right) \right].$$

Case 1: $a < m < 1 < 2a < m+1 < 2 < 3a < m+2$. The sign pattern of the coefficients of V_s is "$-,+,+,+,-,-,-,+$". Therefore, $\lim_{x\to-\infty} V_s(x) = +\infty$, $\lim_{x\to+\infty} V_s(x) = 0^-$ and V_s has at most three real roots. One of the roots is easily located, since $V_s(0) = 0$. Observe that for k even, the signs of the coefficients are the same as for V_s, so the limits at $x \to \pm\infty$ will also be the same, while for k odd, the sign pattern given by the coefficients of the derivative of order k is "$+,-,-,-,+,+,+,-$", implying that $\lim_{x\to-\infty} V_s^{(k)}(x) = -\infty$ and $\lim_{x\to+\infty} V_s^{(k)}(x) = 0^+$. So, in order to locate the other roots of V_s we need to study what happens at the derivative of order s, when s is even and odd.

s even: The signs of the coefficients of $V_s^{(s)}$ are "$-,+,+,+,-,-,-,+$. Hence, $V_s^{(s)}$ has, at most, three real roots, $\lim_{x\to-\infty} V_s^{(s)}(x) = +\infty$ and $\lim_{x\to+\infty} V_s^{(s)}(x) = 0^-$. Since $V_s^{(s)}(0) = 0$, we may conclude that there are three possible sign variations for this function, in $(0,+\infty)$: "$-,+,-$" or "$+,-$" or "$-$". Now, we need to analyse the sign variation of $V_s^{(s-1)}$, but for that it is necessary to see if we can determine the sign of $V_s^{(s-1)}$. In fact, given that $a < m$, we have that

$$\begin{aligned} V_s^{(s-1)}(0) &= -d(s) + a^{s-1}c(s,m) < -d(s) + m^{s-1}c(s,m) \\ &= -2 + \frac{3}{2^{s-1}} - \frac{1}{3^{s-1}} + 2m^{s-1} + \frac{m^{s-1}}{(m+2)^{s-1}} - \frac{2m^{s-1}}{(m+1)^{s-1}} - \frac{m^{s-1}}{2^{s-1}}. \end{aligned}$$

Let $h(m) = -2 + \frac{3}{2^{s-1}} - \frac{1}{3^{s-1}} + 2m^{s-1} + \frac{m^{s-1}}{(m+2)^{s-1}} - \frac{2m^{s-1}}{(m+1)^{s-1}} - \frac{m^{s-1}}{2^{s-1}}$, for $m \in (0,1)$. Differentiating, we obtain

$$h'(m) = 2(s-1)m^{s-2} \left(1 + \frac{1}{(m+2)^s} - \frac{1}{(m+1)^s} - \frac{1}{2^s} \right) = 2(s-1)m^{s-2}N(m).$$

Differentiating N, we have that $N'(m) = -\frac{s}{(m+2)^{s+1}} + \frac{s}{(m+1)^{s-1}} > 0$. So, N is increasing and $N(0) = 0$, implying that $N(m) > 0$, for $m \in (0,1)$. But this implies that $h'(m) > 0$, for $m \in (0,1)$, i.e, that h is increasing. Moreover, $h(0) = -2 + \frac{3}{2^{s-1}} - \frac{1}{3^{s-1}} \le 0$ and $h(1) = 0$. Therefore, $h(m) < 0$, for $m \in (0,1)$ and, consequently, $V_s^{(s-1)}(0) < 0$. The next table shows the possible sign variations for $V_s^{(s-1)}$, in $(0,+\infty)$, according to the possible sign variations of $V_s^{(s)}$.

sign variation of $V_s^{(s)}$	"$-,+,-$"	"$+,-$"	"$-$"
monotonicity of $V_s^{(s-1)}$	"$\searrow\nearrow\searrow$"	"$\nearrow\searrow$"	"$\searrow$"
sign variation of $V_s^{(s-1)}$ in $(0,+\infty)$	"$-,+$"	"$-,+$"	not possible

Hence, $V_s^{(s-1)}$ has one possible sign variation "$-,+$", in $(0,+\infty)$. Since, the sign of the derivatives of orders $k < s-1$ at zero cannot be determined, we need to analyse what happens in each possible case. For $V_s^{(s-2)}$ we have that,

sign variation of $V_s^{(s-1)}$	"$-,+$"	
monotonicity of $V_s^{(s-2)}$	"$\searrow\nearrow$"	
$V_s^{(s-2)}(0)$	positive	negative
sign variation of $V_s^{(s-2)}$ in $(0,+\infty)$	"$+,-$"	"$-$"

So, the sign variation of $V_s^{(s-2)}$ is, at most, "$+,-$", as x traverses from 0 to $+\infty$.

sign variation of $V_s^{(s-2)}$	"$+,-$"		"$-$"	
monotonicity of $V_s^{(s-3)}$	"$\nearrow\searrow$"		"$\searrow$"	
$V_s^{(s-3)}(0)$	positive	negative	positive	negative
sign variation of $V_s^{(s-3)}$ in $(0,+\infty)$	"$+$"	"$-,+$"	"$+$"	not possible

Therefore, the possible sign variations for the derivative of V_s of order $s-3$ are: "$-,+$" or "$+$", for $x \in (0,+\infty)$. Finally, we study the sign variation of $V_s^{(s-4)}$.

sign variation of $V_s^{(s-3)}$	"$-,+$"		"$+$"	
monotonicity of $V_s^{(s-4)}$	"$\searrow\nearrow$"		"$\nearrow$"	
$V_s^{(s-4)}(0)$	positive	negative	positive	negative
sign variation of $V_s^{(s-4)}$ in $(0,+\infty)$	"$+,-$"	"$-$"	not possible	"$-$"

We obtain the the most sign varying possibility for $V_s^{(s-4)}$ is the same as for $V_s^{(s-2)}$. Thus, applying the same arguments we may conclude that V_s' has sign variations "$-,+$" or "$+$". But, given that $V_s(0)=0$, $\lim_{x\to+\infty} V_s(x) = 0^-$, we conclude that $V_s(x) \leq 0$, for $x \leq 0$.

s odd: For this case, the sign pattern of the coefficients of $V_s^{(s)}$ is "$+,-,-,-,+,+,+,-$, implying that $\lim_{x\to-\infty} V_s^{(s)}(X) = -\infty$ and $\lim_{x\to+\infty} V_s^{(s)}(x) = 0^+$. Furthermore, $V_s^{(s)}$ has at most three real roots, so the possible sign variations in $(0,+\infty)$, are "$+,-,+$" or "$-,+$" or "$+$". Here, we have that $V_s^{(s-1)} > 0$. So, $V_s^{(s-1)}$ has only one possible sign variation, as x traverses from 0 to $+\infty$: "$+,-$". The following table summarizes the possible sign variations for $V_s^{(s-2)}$.

sign variation of $V_s^{(s-1)}$	"$+,-$"	
monotonicity of $V_s^{(s-2)}$	"$\nearrow\searrow$"	
$V_s^{(s-2)}(0)$	positive	negative
sign variation of $V_s^{(s-2)}$ in $(0,+\infty)$	"$+$"	"$-,+$"

Hence, $V_s^{(s-2)}$ changes sign at most once, in the order "$-,+$", as x traverses from 0 to $+\infty$. We find for $V_s^{(s-2)}$ the same behaviour as for the derivative of order $s-3$, when s even, so the previous conclusion holds.

Case 2: $a < m < 1 < 2a < m+1 < 3a < 2 < m+2$. The sign of the coefficients of V_s are "$-,+,+,+,-,-,-,+$". But this coincides with the previous case, so the result still holds.

Case 3: $a < m < 1 < 2a < 3a < m+1 < 2 < m+2$. This case is similar to **Case 1**, given that the sign pattern of the coefficients of V_s is the same.

Case 4: $a < m < 2a < 1 < 3a < m+1 < 2 < m+2$. The sign pattern given by the coefficients of V_s is "$-,+,+,+,-,-,-,+$" so, once again, the conclusion holds.

Case 5: $a < 2a < 3a < m < 1 < m+1 < 2 < m+2$. In this case, the sign pattern given by the coefficients of V_s is "$-,+,-,+,+,-,-,+$". By Proposition 8, V_s has at most five real roots. Therefore, the previous arguments will not allow us to conclude what we want. However, note that $3a < m$ and $2 < m+2$, implying that

$$V_s(x) < V_{s,*}(x) = \frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)s-1} - \frac{e^{-2x}}{2^{s-1}} + \frac{e^{-2x}}{(m+2)^{s-1}}\right) - \frac{1}{d(s)}\left(3e^{-ax} - \frac{3e^{-2ax}}{2^{s-1}} + \frac{e^{-mx}}{3^{s-1}}\right),$$

for every fixed $x \geq 0$. Therefore, if we prove that $V_{s,*}(x) \leq 0$, for $x \geq 0$, we have that $V_s(x) \leq 0$, for $x \geq 0$. Since $2 < m+2$, we have that the coefficient of e^{-2x} is clearly negative. The sign of e^{-mx} is given by the sign of $h(m) = 3^{s-1}d(s) - m^{s-1}c(s,m)$.

$$h(m) = 3^s - \frac{3^s}{2^{s-1}} - 2m^{s-1} - \frac{m^{s-1}}{(m+2)^{s-1}} + \frac{2m^{s-1}}{(m+1)^{s-1}} + \frac{m^{s-1}}{2^{s-1}}$$

If we look at $A(m,s)$ has only a function of m, $h(m)$, for $m \in (0,1)$, and if we differentiate, we have that

$$h'(m) = -2(s-1)m^{s-2}\left(-1 - \frac{1}{(m+2)^s} + \frac{1}{(m+1)^s} + \frac{1}{2^s}\right) = 2(s-1)N(m).$$

But we have already seen that this function is positive, which implies that the coefficient of e^{-mx} is positive. We may now conclude that the signs of the coefficients of $V_{s,*}$ are "$-,+,+,+,-,-$". But this implies that $V_{s,*}$ has at most two real roots, $\lim_{x\to-\infty} V_{s,*}(x) = -\infty$ and $\lim_{x\to+\infty} V_{s,*}(x) = 0^-$. We also have that $V_{s,*}(0) = 0$. In order to locate the remaining root we need to analyse the derivatives of orders $k < s$ and s of $V_{s,*}$.

$$V_{s,*}^{(k)}(x) = (-1)^k\left[\frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1-k}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1-k}} - \frac{e^{-2x}}{2^{s-1-k}} + \frac{2^k e^{-2x}}{(m+2)^{s-1}}\right)\right.$$
$$\left. - \frac{1}{d(s)}\left(3a^k e^{-ax} - \frac{3a^k e^{-2ax}}{2^{s-1-k}} + \frac{m^k e^{-mx}}{3^{s-1}}\right)\right],$$

$$V_{s,*}^{(s)}(x) = (-1)^s\left[\frac{1}{c(s,m)}\left(me^{-mx} + 2e^{-x} - 2(m+1)e^{-(m+1)x} - 2e^{-2x} + \frac{2^s e^{-2x}}{(m+2)^{s-1}}\right)\right.$$
$$\left. - \frac{1}{d(s)}\left(3a^s e^{-ax} - 6a^s e^{-2ax} + \frac{m^s e^{-mx}}{3^{s-1}}\right)\right],$$

Observe that for k even, the signs of the coefficients of $V_{s,*}^{(k)}$ are "$-,+,+,+,-,-$". Hence, $\lim_{x\to-\infty} V_{s,*}^{(k)}(x) = -\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(k)}(x) = 0^-$. When k odd, both the signs of the coefficients and of the limits reverse. Now, we need to study what happens for s even and odd.

s even: In this case, the sign pattern given by the coefficients of $V_{s,*}^{(s)}$ is the same as for V_s. Thus the limits when $x \to \pm\infty$ of $V_{s,*}^{(s)}$ coincide with the limits when $x \to \pm\infty$ of V_s.

$$V_{s,*}^{(s)}(0) = \frac{1}{c(s,m)}\left(-m-2-\frac{2^s}{(m+2)^{s-1}}\right) - \frac{1}{d(s)}\left(-3a^s+\frac{m^s}{3^{s-1}}\right),$$

hence, we cannot determine its sign. Therefore, $V_{s,*}^{(s)}$ has three possible sign variations in $(0,+\infty)$: "$-,+,-$" or "$+,-$" or "$-$". We need to study, now, the sign variation of $V_{s,*}^{(s-1)}$. For that, it is important to see if we can determine its sign at zero.

$$V_{s,*}^{(s-1)}(0) = -3^s2^{s-1}+3^s-2^{s-1}+2m^{s-1}(m+2)^{s-1}+(m+2)^{s-1}+m^{s-1}$$
$$-\frac{2m^{s-1}(m+2)^{s-1}}{(m+1)^{s-1}}-\frac{m^{s-1}(m+2)^{s-1}}{2^{s-1}}.$$

Let $h(m) = -3^s2^{s-1}+3^s-2^{s-1}+2m^{s-1}(m+2)^{s-1}+(m+2)^{s-1}+m^{s-1}$, for $m \in (0,1)$. Differentiating h we obtain $h'(m) = 2(s-1)m^{s-2}(m+2)^{s-1}+2m^{s-1}(m+2)^{s-2}+(s-1)(m+2)^{s-2}+(s-1)m^{s-2} > 0$. Thus, h is increasing. Moreover, $h(0) = -3^s2^{s-1}+3^s-2^{s-1} \leq 0$ ahd $h(1) = -3^s2^{s-1}+3^s2-2^{s-1}+1 \leq 0$. Therefore, $h(m) < 0$, for $m \in (0,1)$, so $V_{s,*}^{(s-1)}(0)$ is negative.

sign variation of $V_{s,*}^{(s)}$	"$-,+,-$"	"$+,-$"	"$-$"
monotonicity of $V_{s,*}^{(s-1)}$	"$\searrow\nearrow\searrow$"	"$\nearrow\searrow$"	"$\searrow$"
sign variation of $V_{s,*}^{(s-1)}$ in $(0,+\infty)$	"$-,+$"	"$-,+$"	not possible

So, $V_{s,*}^{(s-1)}$ has only one possible sign variation on $(0,+\infty)$ "$-,+$". For $V_{s,*}^{(s-2)}$ we have the following sign variation.

sign variation of $V_{s,*}^{(s-1)}$	"$-,+$"	
monotonicity of $V_{s,*}^{(s-2)}$	"$\searrow\nearrow$"	
$V_{s,*}^{(s-2)}(0)$	positive	negative
sign variation of $V_{s,*}^{(s-2)}$ in $(0,+\infty)$	"$+,-$"	"$-$"

Therefore, the possible sign variations for $V_{s,*}^{(s-2)}$, in $(0,+\infty)$ are "$+,-$" or "$-$".

sign variation of $V_{s,*}^{(s-2)}$	"$+,-$"		"$-$"	
monotonicity of $V_{s,*}^{(s-3)}$	"$\nearrow\searrow$"		"$\searrow$"	
$V_{s,*}^{(s-3)}(0)$	positive	negative	positive	negative
sign variation of $V_{s,*}^{(s-3)}$ in $(0,+\infty)$	"$+$"	"$-,+$"	"$+$"	not possible

So, in $(0,+\infty)$, the most sign varying possibility for the derivative of $V_{s,*}$ of order $s-3$ is "$-,+$". The table bellow summarizes the sign variation of $V_{s,*}^{(s-4)}$, according to the possible sign variations of $V_{s,*}^{(s-3)}$.

sign variation of $V_{s,*}^{(s-3)}$	"$-,+$"		"$+$"	
monotonicity of $V_{s,*}^{(s-4)}$	"$\searrow\nearrow$"		"$\nearrow$"	
$V_{s,*}^{(s-4)}(0)$	positive	negative	positive	negative
sign variation of $V_{s,*}^{(s-4)}$ in $(0,+\infty)$	"$+,-$"	"$-$"	not possible	"$-$"

Thus, the possible sign variations for $V_{s,*}^{(s-4)}$, in $(0,+\infty)$, are "$+,-$" or "$-$". Repeating iteratively this process, we may conclude that the sign variation of $V'_{s,*}$ is "$-,+$" or "$+$", as x traverses from 0 to $+\infty$. Remembering that $V_{s,*}(0)=0$, and that $\lim_{x\to+\infty} V_{s,*}(x)=0^-$, we obtain that $V_{s,*}(x)\leq 0$, for $x\geq 0$, implying that $V_s(x)\leq 0$, for $x\geq 0$.

s odd: In this case the signs of the coefficients of $V_{s,*}^{(s)}$ are "$+,-,-,-,+,+$". So, $V_{s,*}^{(s)}$ has at most two real roots, $\lim_{x\to-\infty} V_{s,*}^{(s)}(x)=+\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(s)}(x)=0^+$. Remembering that we cannot determine the sign of $V_{s,*}^{(s)}(0)$, we may conclude that the possible sign variations for $V_{s,*}^{(s)}$, in $(0,+\infty)$, are "$+,-,+$" or "$-,+$" or "$+$". Since s is odd, $s-1$ is even and, therefore, $V_{s,*}^{(s-1)}(0)>0$.

sign variation of $V_{s,*}^{(s)}$	"$+,-,+$"	"$-,+$"	"$+$"
monotonicity of $V_{s,*}^{(s-1)}$ sign variation of $V_{s,*}^{(s-1)}$ in $(0,+\infty)$	"$\nearrow\searrow\nearrow$" "$+,-$"	"$\searrow\nearrow$" "$+,-$"	"$\nearrow$" not possible

So, once again, $V_{s,*}^{(s-1)}$ changes sign exactly once, but in this case in the order "$+,-$", in $(0,+\infty)$.

sign variation of $V_{s,*}^{(s-1)}$	"$+,-$"	
monotonicity of $V_{s,*}^{(s-2)}$	"$\nearrow\searrow$"	
$V_{s,*}^{(s-2)}(0)$ sign variation of $V_{s,*}^{(s-2)}$ in $(0,+\infty)$	positive "$+$"	negative "$-,+$"

The behaviour for the sign variation of $V_{s,*}^{(s-2)}$ is the same as for $V_{s,*}^{(s-3)}$, when s even, in $(0,+\infty)$. So we can apply the same arguments used previously to conclude that $V_s(x)\leq 0$, for $x\geq 0$. □

4.3 A difficult case

In order to show the main difficulties when analysing the sign variation of V_s and explain why the analysis for $a<1$ has not been completed, we will study what happens for the case $a<m<2a<3a<1<m+1<2<m+2$. In this case, the sign pattern given by the coefficients of V_s is "$-,+,+,-,+,-,-,+$". By Proposition 8, this implies that V_s has at most five real roots. Moreover, $\lim_{x\to-\infty} V_s(x)=+\infty$ and $\lim_{x\to+\infty} V_s(x)=0^-$. Due to the large number of possible roots, the direct study of the derivatives of V_s will not lead us to the desired result. However, taking into account the limit of V_s when $x\to+\infty$ and noting that $a<m$, we have that $V_s(x)<V_{s,*}(x)=\overline{T}_{Y,s}(x)-\overline{T}_{X,s}(mx)$, for every fixed $x\geq 0$. So,

$$V_{s,*}(x)=\frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1}}+2e^{-x}-\frac{2e^{-(m+1)x}}{(m+1)^{s-1}}-\frac{e^{-2x}}{2^{s-1}}+\frac{e^{-(m+2)x}}{(m+2)^{s-1}}\right)-\frac{1}{d(s)}\left(3e^{-mx}-\frac{3e^{-2mx}}{2^{s-1}}+\frac{e^{-3mx}}{3^{s-1}}\right).$$

If we manage to prove that $V_{s,*}(x) \leq 0$, for ≥ 0, we would have found a sign variation compatible with the s-IFRA ordering. To do that we need to analyse the derivatives of orders $k < s$ and s.

$$V_{s,*}^{(k)}(x) = (-1)^k \left[\frac{1}{c(s,m)} \left(\frac{e^{-mx}}{m^{s-1-k}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1-k}} - \frac{e^{-2x}}{2^{s-1-k}} + \frac{e^{-(m+2)x}}{(m+2)^{s-1-k}} \right) \right.$$

$$\left. - \frac{1}{d(s)} \left(3m^k e^{-mx} - 3m^k \frac{e^{-2mx}}{2^{s-1}} + m^k \frac{e^{-3mx}}{3^{s-1-k}} \right) \right],$$

$$V_{s,*}^{(s)}(x) = (-1)^s \left[\frac{1}{c(s,m)} \left(me^{-mx} + 2e^{-x} - 2(m+1)e^{-(m+1)x} - e^{-2x} + (m+2)e^{-(m+2)x} \right) \right.$$

$$\left. - \frac{1}{d(s)} \left(3m^s e^{-mx} - 6m^s e^{-2mx} + 3m^s e^{-3mx} \right) \right].$$

In order to apply Proposition 8 to $V_{s,*}$, we need to order decreasingly the exponentials terms, with respect to their exponents. This gives rise to different cases, according to the location of $2m$ and $3m$ with respect to $m+1$ and $m+2$. Before analysing the cases that arise, let us study the sign of the coefficient of e^{-mx}. Note that its sign is the same as the sign of $h(m) = d(s) - 3m^{s-1}c(s,m)$.

$$h(m) = -6m^{s-1} - \frac{3}{2^{s-1}} + \frac{1}{3^{s-1}} - \frac{3m^{s-1}}{(m+2)^{s-1}} + \frac{6m^{s-1}}{(m+1)^{s-1}} + \frac{3m^{s-1}}{2^{s-1}}.$$

Differentiating, we obtain

$$h'(m) = 6(s-1)m^{s-2} \left(-1 - \frac{1}{(m+2)^s} + \frac{1}{(m+1)^s} + \frac{1}{2^s} \right).$$

Define $N(m) = -1 - \frac{1}{(m+2)^s} + \frac{1}{(m+1)^s} + \frac{1}{2^s}$, for $m \in (0,1)$. Differentiating, we have $N'(m) = s\left(\frac{1}{(m+2)^{s+1}} - \frac{1}{(m+1)^{s+1}} \right) < 0$. Hence, N is decreasing. Since, $N(0) = 0$, we may conclude that $N(m) < 0$, for $m \in (0,1)$ and, consequently, that h is decreasing. Moreover, $h(0) = -\frac{3}{2^{s-1}} + \frac{1}{3^{s-1}} < 0$, implying that $h(m) < 0$, for $m \in (0,1)$. Therefore, the coefficient of e^{-mx} is negative. Let us now study the sign variation of $V_{s,*}$ for the possible different cases, according the location of $2m$ and $3m$.

Case 1: $m < 1 < 2m < m+1 < 2 < 3m < m+2$. In this case, the signs of the coefficients are "$-,+,+,-,-,-,+$", which implies that $\lim_{x\to-\infty} V_{s,*}(x) = +\infty$, $\lim_{x\to+\infty} V_{s,*}(x) = 0^-$ and $V_{s,*}$ has at most three real roots. Furthermore, $V_{s,*}(0) = 0$, so we only need to locate the other two possible roots. For that, we analyse the sign variation of the derivatives of $V_{s,*}$. Observe that, for this case, when k is even the signs of the coefficients of $V_{s,*}^{(k)}$ is the same as for $V_{s,*}$, so the limits when $x \to \pm\infty$ are the same, as well as the possible number of real roots. When k is odd, the signs of the coefficients are reversed, as well as the signs of the limits. Now, we need to analyse what happens for s even and odd.

s even: The sign pattern of the coefficients of $V_{s,*}^{(s)}$ is "$-,+,+,-,-,-,+$". Hence, $V_{s,*}^{(s)}$ has at most three real roots, $\lim_{x\to-\infty} V_{s,*}^{(s)}(x) = +\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(s)}(x) = 0^-$. Taking into account that $V_{s,*}(0) = 0$, we have that the possible sign variations for this function, in

$(0,+\infty)$, are "$-,+,-$" or "$+,-$" or "$-$". Note that, the sign of $V_{s,*}^{(s-1)}(0)$ is the same as the sign of $B(s,m) = -d(s) + c(s,m)m^{s-1}$. If we now fix $s \geq 1$, we have that

$$h(m) = -2 + \frac{3}{2^{s-1}} - \frac{1}{3^{s-1}} + 2m^{s-1} + \frac{m^{s-1}}{(m+2)^{s-1}} - \frac{2m^{s-1}}{(m+1)^{s-1}} - \frac{m^{s-1}}{2^{s-1}},$$

for $m \in (0,1)$. Differentiating, we obtain,

$$h'(m) = 2(s-1)m^{s-2}\left(1 + \frac{1}{(m+2)^s} - \frac{1}{(m+1)^s} - \frac{1}{2^s}\right) = 2(s-1)m^{s-2}N(m).$$

But, we have already seen that $-N(m) < 0$, for $m \in (0,1)$, so h is increasing. Given that $h(0) = -2 + \frac{3}{2^{s-1}} - \frac{1}{3^{s-1}} \leq 0$ and $h(1) = 0$, we may conclude that $h(m) < 0$, for $m \in (0,1)$, that is, $V_{s,*}^{(s-1)}(0) < 0$.

sign variation of $V_{s,*}^{(s)}$	"$-,+,-$"	"$+,-$"	"$-$"
monotonicity of $V_{s,*}^{(s-1)}$ sign variation of $V_{s,*}^{(s-1)}$ in $(0,+\infty)$	"$\searrow\nearrow\searrow$" "$-,+$"	"$\nearrow\searrow$" "$-,+$"	"$\searrow$" not possible

The only possible sign variation for $V_{s,*}^{(s-1)}$, in $(0,+\infty)$, is "$-,+$". Although we could determine the sign of $V_{s,*}^{(s-1)}$ at zero, we cannot do that for $V_{s,*}^{(s-2)}$, do we need to analyse what happens for $V_{s,*}^{(s-2)}(0)$ negative and positive.

sign variation of $V_{s,*}^{(s-1)}$	"$-,+$"	
monotonicity of $V_{s,*}^{(s-2)}$	"$\searrow\nearrow$"	
$V_{s,*}^{(s-2)}(0)$ sign variation of $V_{s,*}^{(s-2)}$ in $(0,+\infty)$	positive "$+,-$"	negative "$-$"

Thus, $V_{s,*}^{(s-2)}$, when $x \in (0,+\infty)$, has two possible sign variations: "$+,-$" or "$-$". Taking this into account, for $V_{s,*}^{(s-3)}$ we have the following table:

sign variation of $V_{s,*}^{(s-2)}$	"$+,-$"		"$-$"	
monotonicity of $V_{s,*}^{(s-3)}$	"$\nearrow\searrow$"		"$\searrow$"	
$V_{s,*}^{(s-3)}(0)$ sign variation of $V_{s,*}^{(s-3)}$ in $(0,+\infty)$	positive "$+$"	negative "$-,+$"	positive "$+$"	negative not possible

So, $V_{s,*}^{(s-3)}$ has two possible sign variations as x traverses from 0 to $+\infty$: "$-,+$" or "$+$". Finally, we analyse the sign variation of $V_{s,*}^{(s-4)}$, when $x > 0$.

sign variation of $V_{s,*}^{(s-3)}$	"$-,+$"		"$+$"	
monotonicity of $V_{s,*}^{(s-4)}$	"$\searrow\nearrow$"		"$\nearrow$"	
$V_{s,*}^{(s-4)}(0)$ sign variation of $V_{s,*}^{(s-4)}$ in $(0,+\infty)$	positive "$+,-$"	negative "$-$"	positive not possible	negative "$-$"

So, $V_{s,*}^{(s-4)}$ has two possible sign variations, when $x \in (0,+\infty)$: "$+,-$" or "$-$", which are the same possibilites as for $V_{s,*}^{(s-2)}$. Hence, repeating this process we find that the behaviour of the sign variation of $V_{s,*}^{(s-3)}$ is the same as for the sign variation of $V'_{s,*}$.

Taking into account that $V_{s,*}(0)=0$ and $\lim_{x\to+\infty} V_{s,*}(x)=0^-$, we have that $V_{s,*}(x)\leq 0$, for $x\geq 0$

s odd: The signs of the coefficients of $V_{s,*}^{(s)}$ are "$+,-,-,+,+,+,-$". This implies that $\lim_{x\to-\infty} V_{s,*}^{(s)}(x)=-\infty$ and $\lim_{x\to+\infty} V_{s,*}^{(s)}(x)=0^+$. Hence, $V_{s,*}^{(s)}$ has three possible sign variations when $x>0$: "$+,-,+$" or "$-,+$" or "$+$". Since, for s even $V_{s,*}^{(s-1)}(0)<0$, for s odd $V_{s,*}^{(s-1)}(0)>0$. The table bellow summarizes the sign variation of the derivative of order $s-1$.

sign variation of $V_{s,*}^{(s)}$	"$+,-,+$"	"$-,+$"	"$+$"
monotonicity of $V_{s,*}^{(s-1)}$	"$\nearrow\searrow\nearrow$"	"$\searrow\nearrow$"	"$\nearrow$"
sign variation of $V_{s,*}^{(s-1)}$ in $(0,+\infty)$	"$+,-$"	"$+,-$"	not possible

We conclude that, when $x>0$, $V_{s,*}^{(s-1)}$ changes sign exactly once and the change occurs in the order "$+,-$". The possible sign variations for $V_{s,*}^{(s-2)}$ are described in the following table.

sign variation of $V_{s,*}^{(s-1)}$	"$+,-$"	
monotonicity of $V_{s,*}^{(s-2)}$	"$\nearrow\searrow$"	
$V_{s,*}^{(s-2)}(0)$	positive	negative
sign variation of $V_{s,*}^{(s-2)}$ in $(0,+\infty)$	"$+$"	"$-,+$"

So, $V_{s,*}^{(s-2)}$ has to possible sign variations, "$-,+$" or "$+$", when $x\in(0,\infty)$. But this coincides with the behaviour of the sign variation of $V_{s,*}^{(s-3)}$, when s is even. Applying the previous arguments, we may conclude that $V_{s,*}\leq 0$, for $x\geq 0$.

Case 2: $m<1<2m<m+1<3m<2<m+2$. The sign pattern of the coefficients of $V_{s,*}$ coincides with the previous case, so the conclusion still holds.

Case 3: $m<2m<1<3m<m+1<2<m+2$. The sign of the coefficients of $V_{s,*}$ are "$-,+,+,-,-,-,+$". So, this is the same as **Case 1**.

Case 4: $m<2m<3m<1<m+1<2<m+2$. The sign pattern given by the coefficients of $V_{s,*}$ is "$-,+,-,+,-,-,+$". By Proposition 8, we may conclude that $V_{s,*}$ has at most five real roots. Once again, a direct usage of the arguments as in the previous cases with $V_{s,*}$ will not allow us to conclude a sign variation compatible with the s-IFRA order. Nevertheless, note that $m<\frac{1}{3}$, so we can define $U_s(x)>V_{s,*}(x)$, for every fixed $x\geq 0$, such that,

$$U_s(x)=\frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1}}+2e^{-mx}-\frac{2e^{-(m+1)x}}{(m+1)^{s-1}}-\frac{e^{-2x}}{2^{s-1}}+\frac{e^{-(m+2)x}}{(m+2)^{s-1}}\right)$$
$$-\frac{1}{d(s)}\left(3e^{-\frac{1}{3}x}-3\frac{e^{-\frac{2}{3}x}}{2^{s-1}}+\frac{e^{-x}}{3^{s-1}}\right).$$

Regardless of the sign of the coefficient of e^{-x}, we may conclude that U_s has at most three real roots, when $\frac{1}{3}<m<\frac{2}{3}<1<m+1<2<m+2$ and $\frac{1}{3}<\frac{2}{3}<m<1<m+1<2<m+2$. Moreover, $\lim_{x\to-\infty} U_s(x)=+\infty$ and $\lim_{x\to+\infty} U_s(x)=0^-$. The derivatives of U_s of orders $k<s$

and s can be written as

$$U_s^{(k)}(x) = (-1)^k \left[\frac{1}{c(s,m)} \left(\frac{e^{-mx}}{m^{s-1-k}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1-k}} - \frac{e^{-2x}}{2^{s-1-k}} + \frac{e^{-(m+2)x}}{(m+2)^{s-1-k}} \right) \right.$$

$$\left. - \frac{1}{d(s)} \left(\frac{3}{3^k} e^{-\frac{1}{3}x} - 3 \frac{e^{-\frac{2}{3}x}}{3^k 2^{s-1-k}} + \frac{e^{-x}}{3^{s-1}} \right) \right],$$

$$U_s^{(s)}(x) = (-1)^s \left[\frac{1}{c(s,m)} \left(me^{-mx} + 2e^{-x} - 2(m+1)e^{-(m+1)x} - 2e^{-2x} + (m+2)e^{-(m+2)x} \right) \right.$$

$$\left. - \frac{1}{d(s)} \left(\frac{3}{3^s} e^{-\frac{1}{3}x} - \frac{6e^{-\frac{2}{3}x}}{3^s} + \frac{e^{-x}}{3^{s-1}} \right) \right].$$

Consider s even. For this case, we may conclude that $U_s^{(s)}$ has at most three real roots, $\lim_{x\to-\infty} U_s^{(s)}(x) = +\infty$ and $\lim_{x\to+\infty} U_s^{(s)}(x) = 0^-$. Taking into account that $U_s^{(s)}(0) = 0$, the possible sign variations are "$-,+,-$" or "$+,-$" or "$-$", in $(0,+\infty)$. Observing now that, for k even $\lim_{x\to-\infty} U_s^{(k)}(x) = +\infty$ and $\lim_{x\to+\infty} U_s^{(k)}(x) = 0^-$ and for k odd $\lim_{x\to-\infty} U_s^{(k)}(x) = -\infty$ and $\lim_{x\to+\infty} U_s^{(k)}(x) = 0^+$, and that we cannot determine the sign of $U_s^{(s-1)}(0) = -\frac{1}{c(s,m)} + \frac{1}{3^{s-1}d(s)^{s-1}}$, we obtain the following table for the derivative of order $s-1$.

sign variation of $U_s^{(s)}$	"$-,+,-$"		"$+,-$"		"$-$"	
monotonicity of $U_s^{(s-1)}$	"$\searrow\nearrow\searrow$"		"$\nearrow\searrow$"		"$\searrow$"	
$U_s^{(s-1)}(0)$ sign variation of $U_s^{(s-1)}$ in $(0,+\infty)$	positive "$+,-,+$"	negative "$-,+$"	positive "$+$"	negative "$-,+$"	positive "$+$"	negative not possible

Thus, the possible sign variations for $U_s^{(s-1)}$, in $(0,+\infty)$ are "$+,-,+$" or "$-,+$" or "$+$".

sign variation of $U_s^{(s-1)}$	"$+,-,+$"		"$-,+$"		"$+$"	
monotonicity of $U_s^{(s-2)}$	"$\nearrow\searrow\nearrow$"		"$\searrow\nearrow$"		"$\nearrow$"	
$U_s^{(s-2)}(0)$ sign variation of $U_s^{(s-2)}$ in $(0,+\infty)$	positive "$+,-$"	negative "$-,+,-$"	positive "$+,-$"	negative "$-$"	positive not possible	negative "$-$"

Therefore, the possible sign variations for $U_s^{(s-2)}$, in $(0,+\infty)$, are "$-,+,-$" or "$+,-$" or "$-$", which are the same as for $U_S^{(s)}$. Consequently, we may conclude that U_s' has sign variation "$+,-,+$" or "$-,+$" or "$+$". Remembering that $U_s(0) = 0$,

sign variation of U_s'	"$+,-,+$"	"$-,+$"	"$+$"
monotonicity of U_s sign variation of U_s in $(0,+\infty)$	"$\nearrow\searrow\nearrow$" "$+,-$"	"$\searrow\nearrow$" "$-$"	"$\nearrow$" not possible

Hence, U_s has sign variation at most "$+,-$". But this does not allows to conclude that $V_{s,*}(x) \leq 0$ for $x \geq 0$.

As we can see, it is clear that the choice of $V_{s,*}$ was not the correct one. Note that, since, initially, we had that $a < m$, $3a < 1$ and $2 < m+2$, we could have chosen $V_{s,*}$ such that, for every fixed $x \geq 0$

$$V_s(x) < V_{s,*}(x) = \frac{1}{c(s,m)}\left(\frac{e^{-mx}}{m^{s-1}} + 2e^{-x} - \frac{2e^{-(m+1)x}}{(m+1)^{s-1}} - \frac{e^{-2x}}{2^{s-1}} + \frac{e^{-2x}}{(m+2)^{s-1}}\right) - \frac{1}{d(s)}\left(3e^{-mx} - \frac{3e^{-2ax}}{2^{s-1}} + \frac{e^{-x}}{3^{s-1}}\right).$$

However, a similar study would allow us to conclude that, once more, this was not the correct function to choose.

Remark 8. *Although we were not able to prove the required result for this case, it does not mean that the problem cannot be solved. A number of plots were made, that provide evidence that the result is true, however there was not enough time for further experiments with different choices of functions.*

Remark 9. *If we were considering, for example, the case* $m < a < 1 < m+1 < 2a < 2 < m+2 < 3a$, *the signs of the coefficients of* V_s *would be "*$+,-,+,-,+,-,+,-$*", implying that* V_s *would have at most seven real roots. Since,* $\lim_{x\to+\infty} V_s(x) = 0^+$, *for this case, instead of choosing a inferior function, we should choose a superior function to* V_s, *in order to prove that* $V_s(x) \geq 0$, *for* $x \geq 0$, *which is compatible with a sign variation for the s-IFRA order. However, the difficulty in choosing this function remains.*

4.4 Future work

As referred in Remark 8, due to time constraints it was not possible to finish the problem that we were studying. Therefore, for future work, we still need to prove, for the remaining cases, that V_s changes sign at most once, in the order "$-,+$", as x traverses from 0 to $+\infty$. However, the arguments used in this work may not lead us to a sign variation compatible with the s-IFRA, so it becomes important to search for new approaches that allows to establish the desired result. This may be an interesting topic for a PhD thesis.

After proving that $X \leq_{s-IFRA} Y$, the next step is to prove that $X \leq_{s-IFR} Y$. If we manage to prove this, it would be interesting to study the same problem for a more general case: instead of having Y as defined in (4.2), we would have $Y = \max(Y_1, Y_2, Y_3)$, where Y_1, Y_2 and Y_3 are independent random variables, with exponential distributions with hazard rates 1, λ and m, respectively. Of course, this problem is, at least with the present approach, much more difficult due to the large number of possible cases, making the location of the roots even more complicated.

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