UNIVERSIDADE Ð
COIMBRA

MANUEL GOMES CIPRIANO NABAIS CONDE

Simulation and Optimization of Stochastic Models in Physics with an Application to Economic Models

Thesis submitted to the
University of Coimbra for the degree of
Master in Physics Engineering

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Supervisors:
Prof. Dr. Orlando Olavo Aragão Aleixo e Neves de Oliveira (Universidade de Coimbra)
Dr. Bruno Gaminha

Coimbra, 2018

Acknowledgments

Being a master student has been a stimulating experience. Many people have contributed for this journey.

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Lastly, and most importantly, I would like to thank my family for their unending love and help. I wish to thank my parents, Augusta Cipriano and José Manuel Conde, for their unconditional support. My final words of gratitude go to my aunt Helena Cipriano for her encouragement and for letting me stay at her house whenever I needed to travel to Lisbon.

Abstract

The purpose of this thesis is to understand how the type of Ising model can be used to replicate human behaviour.

The output gap is understood as the variation of the Gross Domestic Product. The way people perceive its growth (expectations) influences the economy, which then influences the monetary policy of a central bank. A variation of the Ising model will be used to replicate the creation and diffusion of expectations of the output gap on a model society. The final objective is to understand how accurate this class of models can simulate the optimal monetary policy of a central bank.

List of Figures

List of Tables

Contents

1

Introduction

Even though human behaviour is highly non-linear and difficult to predict, if what we wish to model is simple and specific enough, it might be possible to explain its occurrence mathematically.

Wolfram's study of cellular automata[2] acts as an influence in this work, because it attempted to mathematically model highly complex systems. Somewhat closer to physics Gerard 't Hooft's[3] work tries to provide a cellular automata interpretation of quantum mechanics.

This thesis turns out as belonging more to the science of complex systems, than to either physics, engineering, mathematics or economics, as it tries to solve a problem in macroeconomics using mathematical methods from physics.

The Ising model provides an explanation for the occurrence of ferromagnetism, anti-ferromagnetism, and ferrimagnetism in materials. All these materials exhibit spontaneous magnetization in the absence of an external magnetic field. This model managed to explain the formation of magnetic domains within materials and the hysteresis curve.

In a lattice, each site has a corresponding magnetic dipole moment created by the atomic spin of each lattice site, which can either be $+1$ or -1. The way the individual magnetic dipole interacts with its nearest neighbours is the reason for the success of the Ising model. The domains are formed because the probability of a site having a spin $+1$ increases if its neighbours also have a $+1$ spin value.

The model was invented by physicist Wilhelm Lenz[4], who proposed it as a problem to his PhD student Ernst Ising. The one-dimensional case was solved by the latter in his PhD thesis[5]. The one-dimensional analytical solution fails to explain the phase transition from ferromagnetic to paramagnetic. Considering this outcome, Ising concluded - incorrectly - that this result would prevail for further dimensions. In 1940, Lars Onsager[6] solved the two-dimensional model which explains the phase transition.

Due to the complexity of the mathematics involved in multidimensional models, numerical simulations play an important role to estimate the physical properties associated to the Ising model. One can look at this problem as a Markov Chain, in which the probability of transitioning to a future state only depends on the current state. The most commonly used numerical method is the Metropolis-Hastings algorithm[7], which is a Markov chain Monte Carlo simulation.

Both the stochastic nature of physics and the formation of domains give the Ising model its distinctive feature, which has made this model applicable to various areas of knowledge outside Magnetism and even Physics.

The Ising model has also been used to explain the motion of atoms in a gas[8]. The "mirrored" variable of the atomic spin is if the lattice site is, or is not, occupied. Similar approaches are used to understand Biological systems with binding behaviours.

A model who drew inspiration from Wilhelm Lenz's work has also been used to statistically model the activity of neurons in a brain[9]. In this case, the equivalent variable to the atomic spin is if the neuron is, or is not, active.

Similar models inspired by the Ising model have been used to replicate the diffusion of expectations in a population. Even though it can be used in a variety of fields, this model was first used to replicate the stylized facts of a financial market[10][11]. Stylized facts are well-known empirical aspects of macroeconomic variables. This can either be correlations between different variables or the values of statistical moments of these variables. This will be further explored in Chapter 2. The model was called the Ant model because the herding behaviour of food collecting in an ant colony fairly resembles the implications word of mouth has in a group of people.

For any system with a binary variable, a finite number of agents and neighbours can use mathematics very similar to the Ising model. For the remainder of my master thesis, I will apply the Ant model to a Dynamic Stochastic General Equilibrium (DSGE) model as the one described in Paul de Grauwe's 2010 works[12][13]. DSGE models are a class of macroeconomic models, which are widely used by most central banks. In the case of this thesis, the binary variable is whether an Agent is optimistic or pessimistic towards the forecast of the central bank. Much like the Ising model, the probability of an agent being optimistic increases if its neighbours are also optimistic.

In the following chapters, I will not focus on the hysteresis curve of a ferromagnet, since it is not so important for the herding behaviour of the Ant-based model, which I wish to implement.

The Ising model

As it was first proposed by Wilhelm Lenz[4], the model is composed by a discrete number of sites ($\Lambda \in N$), atoms in this case, in a multidimensional lattice of dimension d ($d \in N$). Each individual site k ($k \in \Lambda$) has a discrete variable σ_k. This variable represents the site's spin whose values can be $\sigma_k \in -1, +1$. Let σ be a spin configuration of the lattice, which attributes a spin value to each site of the latice. The hamiltonian of a configuration is then:

$$H(\sigma) = -\sum_{<i,j>}^{N,\Lambda} J_{ij}\sigma_i\sigma_j - \mu\sum_{j}^{\Lambda} h_j\sigma_j \tag{1.1}$$

Each individual site has a discrete number of nearest neighbours N, which interact with the lattice site j through J_{ij}. Also, there can be an external magnetic field h_j applied to the lattice and μ symbolizes the magnetic moment of that interaction.

The first sum in Equation 1.1 is over the pairs of adjacents sites (first neighbours). Each pair $< i, j >$ is only counted once. The second sum is related to the effect of an external magnetic field h_j on a given site.

Given that: $\beta = (k_bT)^{-1}$, where k_b is the Boltmann constant and T is the temperature, the probability of a configuration is described by the Boltzmann distribution:

$$P_\beta = \frac{e^{-\beta H(\sigma)}}{Z_\beta} \tag{1.2}$$

Where the normalization constant Z_β is called the partition function and is given by:

$$Z_\beta = \sum_{\sigma} e^{-\beta H(\sigma))} \tag{1.3}$$

The coupling term J_{ij} is related to the interaction between nearest neighbours and characterizes the type of magnetic interaction:

- if $J_{ij} > 0$ the interaction is ferromagnetic
- if $J_{ij} < 0$ the interaction is anti-ferromagnetic
- if $J_{ij} = 0$ the interaction is non-existent

Usually, J_{ij} varies between $[-1, +1]$.

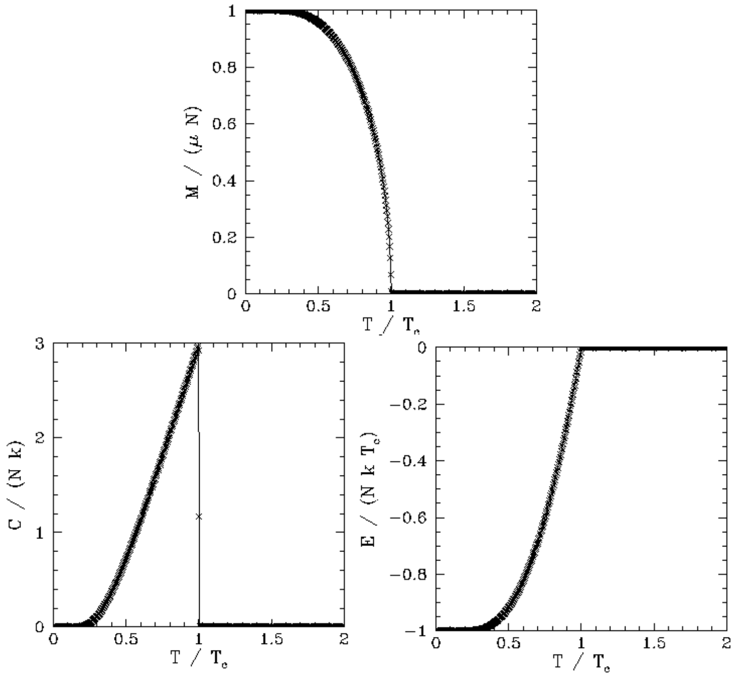

Figure 1.1: Expected results for Magnetization, Energy and Specific Heat for the Ising model in the absence of an external magnetic field. From: `http://farside.ph.utexas.edu/teaching/329/lectures/node110.html`

One Dimension Solution

The exact solution for the one-dimensional model[5] assumes that:

- J_{ij} is the same for all sites of the lattice
- h_j is the same for all sites of the lattice

As a first approach, let us assume the external magnetic field is zero ($h = 0$):

$$H(\sigma) = -J \sum_{<i,j>}^{L} \sigma_i \sigma_j \tag{1.4}$$

In a one dimension case with no boundary conditions each site, except for 1 and L, has two nearest neighbours. In such a case, the Hamiltonian of a spin configuration would be:

$$H(\sigma) = -J(\sigma_1\sigma_2 + ... + \sigma_{L-1}\sigma_L) \tag{1.5}$$

$$Z_\beta = \sum_{\sigma_1,...,\sigma_L} e^{\beta\sigma_1\sigma_2}...e^{\beta\sigma_{L-1}\sigma_L} \tag{1.6}$$

Changing the variables yelds:

$$\sigma'_j = \sigma_j\sigma_{j-1} \tag{1.7}$$

$$Z_\beta = 2\prod_{j=2}^{L}\sum_{\sigma'_j} e^{\beta J\sigma'_j} = 2(e^{\beta J} + e^{-\beta J})^{L-1} \tag{1.8}$$

Therefore the Helmhotz free energy is:

$$f(\beta,h=0) = -\lim_{L\to+\infty}\frac{1}{\beta L}ln(Z_\beta) = -\frac{1}{\beta}ln(e^{\beta J} + e^{-\beta J}) \tag{1.9}$$

Trying to solve for a non-zero external magnetic field ($h \neq 0$), using the same change of variables and this time with periodic boundary conditions:

$$Z_\beta = \sum_{\sigma_1...\sigma_L} e^{\beta\sigma_1}e^{\beta\sigma_1\sigma_2}...e^{\beta\sigma_{L-1}\sigma_L}e^{\beta\sigma_L} = \sum_{\sigma_1...\sigma_L} V_{\sigma_1\sigma_2}...V_{\sigma_L\sigma_1} \tag{1.10}$$

$$V_{\sigma\sigma'} = e^{\frac{\beta h}{2}\sigma}e^{\beta\sigma\sigma'}e^{\frac{\beta h}{2}\sigma'} \tag{1.11}$$

Defining the symmetric transfer matrix given by:

$$V = \begin{pmatrix} V(+,+) & V(+,-) \\ V(-,+) & V(-,-) \end{pmatrix} = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J+h)} \end{pmatrix} \tag{1.12}$$

The expression for the partition function can then be interpreted as:

$$Z_{L\beta} = \sum_{\sigma_1...\sigma_L} V_{\sigma_1\sigma_2}...V_{\sigma_L\sigma_1} = \sum_{i}^{L}(V^L)_{\sigma_i\sigma_{i+1}} = tr(V^L) \tag{1.13}$$

Diagonalizing V gives us the eigenvalues which are the solution to Equation 1.13.

$$\lambda_\pm = e^{\beta J}cosh(\beta h) \pm \sqrt{e^{2\beta J}sinh^2(\beta h) + e^{-2\beta J}} \tag{1.14}$$

Since $|\lambda_+| > |\lambda_-|$ and $Z_{L\beta} = \lambda_+^L + \lambda_-^L \sim \lambda_+^L$ as $L \to +\infty$. In this limit, the Helmotz free energy per site is:

$$f(h,T) = -\frac{1}{\beta}ln(\lambda_+) \tag{1.15}$$

From which follows that, the Magnetization is given by:

$$M(h,T) = -\left(\frac{\partial f(h,T)}{\partial h}\right)_T = \frac{sinh(\beta h)}{\sqrt{sinh^2(\beta h) + e^(-4\beta J)}} \tag{1.16}$$

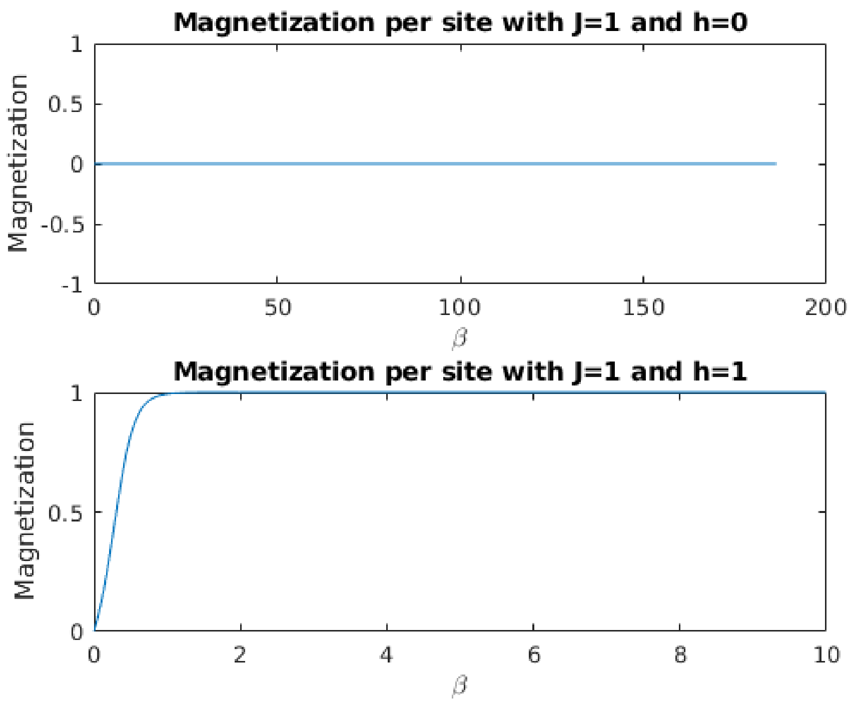

Figure 1.2: Magnetization curve of the teoretical 1D solution

As it can be seen in Figure 1.2, the magnetization without external magnetic field explodes when $\beta \rightarrow +\infty$, somewhere around $\beta \approx 180$, which is when $T \rightarrow 0$. So, at one dimension,this model is unable to explain spontaneous ferromagnetism.

Metropolis-Hastings Sampling

The Metropolis-Hastings algorithm is a Monte Carlo Markov Chain approach used to sample from a probability distribution from which it is difficult to sample directly. It is an algorithm that is usually used when the exact probability function is unknown or it is computationally hard to sample directly from it.

In the 1950's the model was first derived at Los Alamos by Nicholas Metropolis[7] at the Los Alamos National Laboratory. Later, in the 1970's, the algorithm was generalized to any distribution by W. K. Hastings[14].

This algorithm is used to produce a Markov chain with stationary density, π, from which we wish to sample from. The steps it follows are:

1. Assume that the current state is: $X_{k-1} = x$

2. Generate an educated guess for the next step state of the chain. This educated guess depends on the system to which this algorithm is being used. The proposed next step should follow a distribution, $g(x,y)$, and sometimes the Gaussian distribution is used for this purpose.

   $Y_k = y = g(x)$

3. Compute the acceptance probabilty for the proposed test:

   $\alpha(x,y') = min\left(\frac{\pi(y)g(y,x)}{\pi(x)g(x,y)},1\right)$

4. Generate a random number from a uniform random distribution:

   $r \sim U(0,1)$

5. If $r \leq \alpha$ accept the newly generated step: $X_k = Y_k$

6. If $r \geq \alpha$ reject the newly generated step: $X_k = X_{k-1}$

Ising model Simulation

It is important to say that all the simulations done in this chapter were written using the MATLAB programing language.

Now, we shall apply the Metropolis-Hastings algorithm to the Ising model[15]. It has the following steps:

1. Initiate the spin lattice
2. Choose a random site in the lattice and flip it. Therefore, we have two configurations σ and σ'
3. Calculate $E(\sigma)$ and $E(\sigma')$ assuming that each site has k neighbours:

   $E(\sigma) = -J\sum_{j}^{k} \sigma\sigma_j - h\sigma$
4. Calculate $\alpha = min\left(e^{-\beta(E(\sigma')-E(\sigma))},1\right)$
5. Generate a random number from a uniform random distribution:

   $r \sim U(0,1)$
6. If $r \leq \alpha$ accept the flipped spin
7. If $r \geq \alpha$ reject the flipped spin
8. Repeat steps from 2 to 7 N times

As I said previously if the chosen site has a spin value of -1 the probability α of becoming $+1$ increases if the neighbours of the site have a spin value of $+1$. The converse of this is also true.

For every iteration (steps from 2 to 7) there shall be a corresponding Energy and Magnetization. To calculate the Energy per site one shall use Equation 1.1 and to calculate the Magnetization you simply count over all the spins:

$$M = \sum_{i}^{L} \sigma_i \tag{1.17}$$

In order to get statistically relevant results, we must repeat this procedure N times.

For every temperature, there is an independent Markov Chain. For each temperature we must have a corresponding value for each one of the physical properties of the system. The value of these properties are given by averaging the physical variables over the course of the Markov Chain. So the physical properties will be calculated using:

$$< M >= \frac{1}{L} E[M] \tag{1.18}$$

$$< E >= \frac{1}{L} E[E] \tag{1.19}$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{L,V} \propto \frac{(\Delta E)^2}{k_B T^2} \tag{1.20}$$

$$X_M = \left(\frac{\partial M}{\partial H}\right)_{L,V} \propto \frac{(\Delta M)^2}{k_B T^2} \tag{1.21}$$

Because of the Third Law of Thermodynamics, at the absolute zero, the Entropy is zero, which means that our system must occupy a state of minimum Energy. So at the beginning of every Temperature iteration, our spin-lattice will be initialized with all spins equaling +1, $\sigma = +1$.

Since for every new temperature, a new lattice with $\sigma = +1$ is initialized, the initial values and states of the Markov Chain are not representative of the system in the new temperature. These first samples are not statistically relevant. This initialization only makes sense for temperatures near $0K$. In such a case, the values of the physical properties would be incorrect. Therefore, before the actual sampling Chain, it is necessary to prepare the system. To do so, a first Markov Chain is performed to ensure that the first state of the sampling Markov Chain is already an educated guess. So the code should look something like this:

```
for i=1:size(temperatures)
    %temperatures is a vector containing all the temperatures
    %J is the coupling constant
    $h is the external magnetic field
    %Mu is the magnetic moment
    spin=GridBuilding;
    %creates the spin lattice with only spin ups
    spin=metropolisEquilibrium(size(temperatures),spin, temperatures(i),J,h,Mu);
    %stabilization markov chain
    [spin,Energies,Magnetizations]=metropolisSampling(size(temperatures),spin,temperatures(i),J,h,Mu);
    %sampling markov chain
    Emean(tempI) = mean(Energies);
    CvVar(tempI) =cvIsing(Energies,temperatures(i));
    Mmean(tempI) = mean(Magnetizations);
    MgSuspVar(tempI) = suspIsing(Magnetizations,temperatures(i));
    %the above functions are the calculation of the physical properties
    %over the course of the markov chain
    %averaging over these vectors gives us the value of a thermodynamic
    %variable at a certain temperature
end
```

I have also added the measurement of Entropy for every Temperature value since it does not make much sense to talk about Physical Properties such as Energy in the

Ant-based model (even though Magnetization is still important in the Ant-based model).

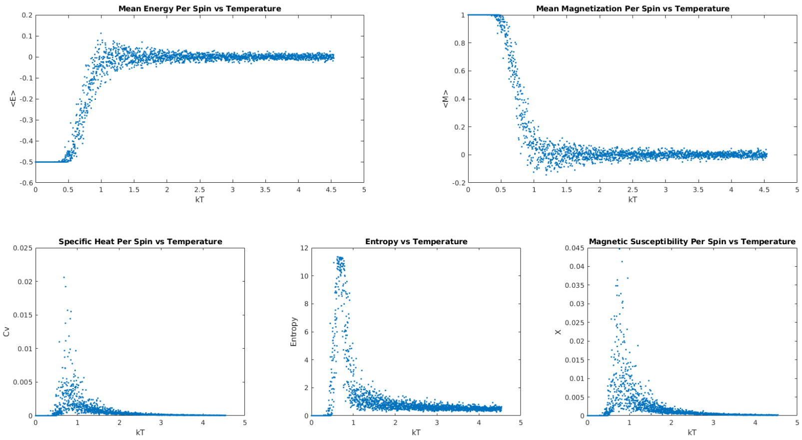

Figure 1.3: Simulation of the one dimension Ising model with $1 \times L = 500$, $J = 1$ and $h = 0$

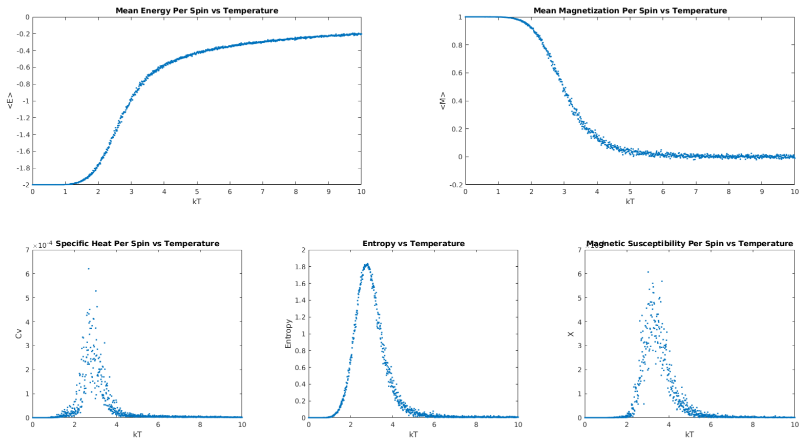

Figure 1.4: Simulation of the two dimension Ising model with $L \times L = 50 \times 50$, $J = 1$ and $h = 0$

The simulations for an one-dimensional and two-dimensional systems can be seen on Figure 1.3 and 1.4, respectively, with both simulations reporting a phase transition measure by $T = 2K$ and $T = 3K$, respectively. This temperature of the phase transition is called the Curie temperature (T_C). Moreover, the specific heat is always positive $C_V > 0$. In both cases, our simulation managed to replicate what was expected, as it can be seen comparing Figure 1.3 and 1.4 to Figure 1.1.

When an external magnetic field is applied ($h \neq 0$) to a ferromagnetic material, it will affect the spin arrangement. If a magnetic field is applied to a ferromagnetic material, the individual atomic magnetic dipoles will align with the external field orientation. So, an external magnetic field forces the spins to align, thus changing the order of the system. The existence of an external magnetic field changes the probability distribution and makes it more probable for the spin to align with the external field's direction.

The result of a 2D simulation of the Ising model with an external field applied is reported in Figure 1.5.

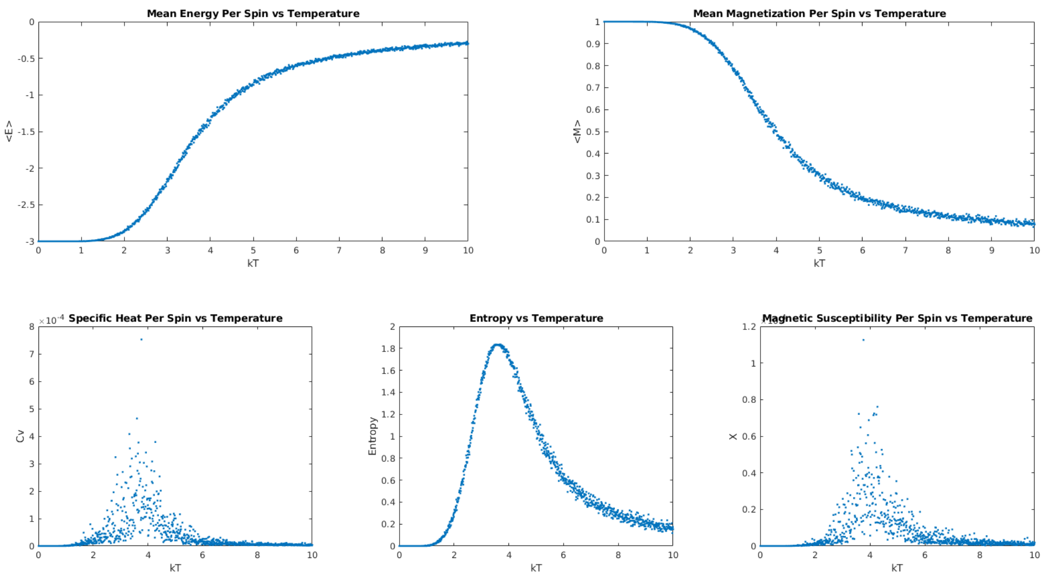

Figure 1.5: Simulation of the two dimension Ising model with $L \times L = 50 \times 50$, $J = 1$ and $h = 1$

Ant-based model and simulation

The Ant model to be studied is similar to the one presented by Chia-Ling Chang and Shu-Heng Chen[16].

In a crystal lattice, the neighbours are defined as a site's first neighbours. Since crowds are not crystal lattices, one has to define a topology, which sets the neighbours of each agent. The most basic topologies are exposed in Figure 1.6.

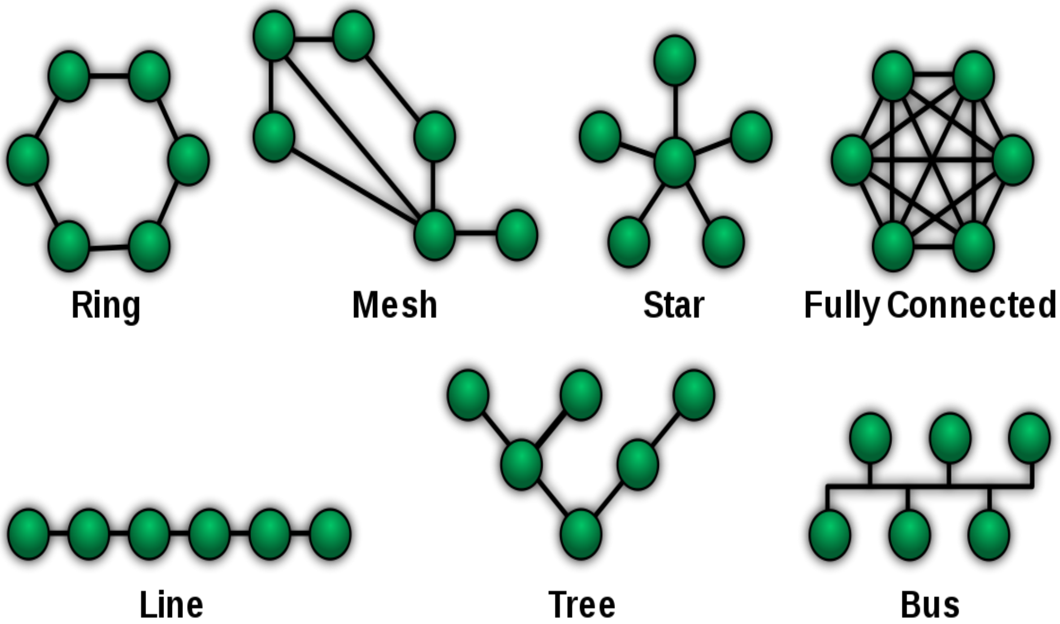

Figure 1.6: Typical topologies. From: `https://simple.wikipedia.org/wiki/Network_topology`

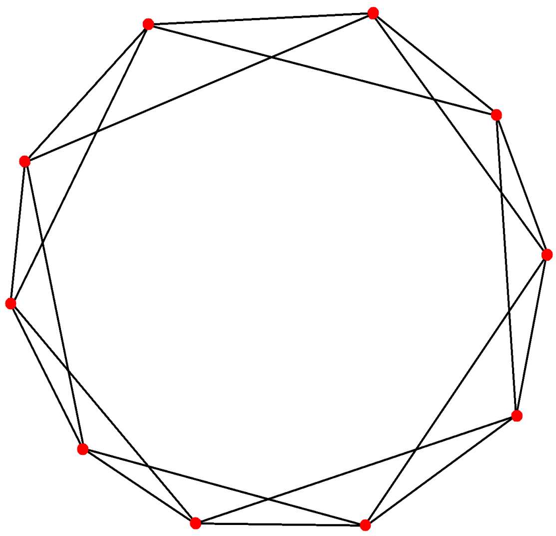

Figure 1.7: Second order ring/circular topology. Each agent has 4 neighbours.

This model is used in macroeconomics to simulate the diffusion of expectations related to the interest rate proposed by the central bank. In other words, whether agents in an economy rely on the forecast of the central bank, optimist, or do not, pessimist.

There are many physical properties and variables which make sense when we are dealing with actual physics, but become useless outside the world of the physics.

What we call temperature in the Ising model is related to the intensity of choice, which means of how closely related are two agents in our network. In our case, it is called intensity of choice and it measures the strength of the social interaction[16].

$$\lambda = \frac{1}{k_B T} \tag{1.22}$$

If an agent i is pessimistic the probabilty of becoming optimistic is given by:

$$\alpha = min\left(\frac{1}{1+e^{-2\lambda M_i}},1\right) \tag{1.23}$$

If an agent i is optimistic the probabilty of becoming pessimistic is given by:

$$\alpha = min\left(1 - \frac{1}{1+e^{-2\lambda M_i}},1\right) \tag{1.24}$$

M_i is the equivalent of the spin sum in the Ising model and therefore accounts for the agents' interaction.

$$M_i = s_i + \sum_{N}^{j=1} w_{ij} Z_o(i,j) \tag{1.25}$$

$$w_{ij} = \frac{1}{j \in \vartheta_i} \tag{1.26}$$

$$Z_o(i,j) = \begin{cases} +1 & \text{, if the neighbour is optimistic} \\ -1 & \text{, if the neighbour is pessimistic} \end{cases} \tag{1.27}$$

In the above Equations, s_i is the self-conversion rate, the following sum is the sum of the neighbours and w_{ij} is the normalization factor over the neighbours' sum. This sum is identical to the Ising Model.

The simulation algorithm will be pretty similar to the one used in Section 1.4, except for initial condition. In physics, there are laws and empirical evidence which help to understand what happens in the limits of temperature ($T \to 0$ and $T \to +\infty$).

Even though the Ant-based model is inspired by the Ising model, there are some differences. There is not any empirical evidence that suggests how our agents must be initialized. For this reason, one must initiate the lattice with the same number of pessimistic and optimistic agents. The simulation will be done in a one-dimensional lattice in which each site has $2k$ neighbours, k to its right and k to its left.

The code developped in this Section is based on the code of Section 1.4. In Figures 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14 and 1.15 λ^{-1} is the inverse of the intensity of choice. The scale of temperature/ inverse of intensity of choice may look like it does not make any sense in this application. In De Grauwe's Work[18], λ varies between 0 and 2. For the multi-agent simulations performed in Chapter 3, λ varies between 25 and 100. To satisfy both cases, I have choosen this scale, from 0 to 1000, for the inverse of intensity of choice.

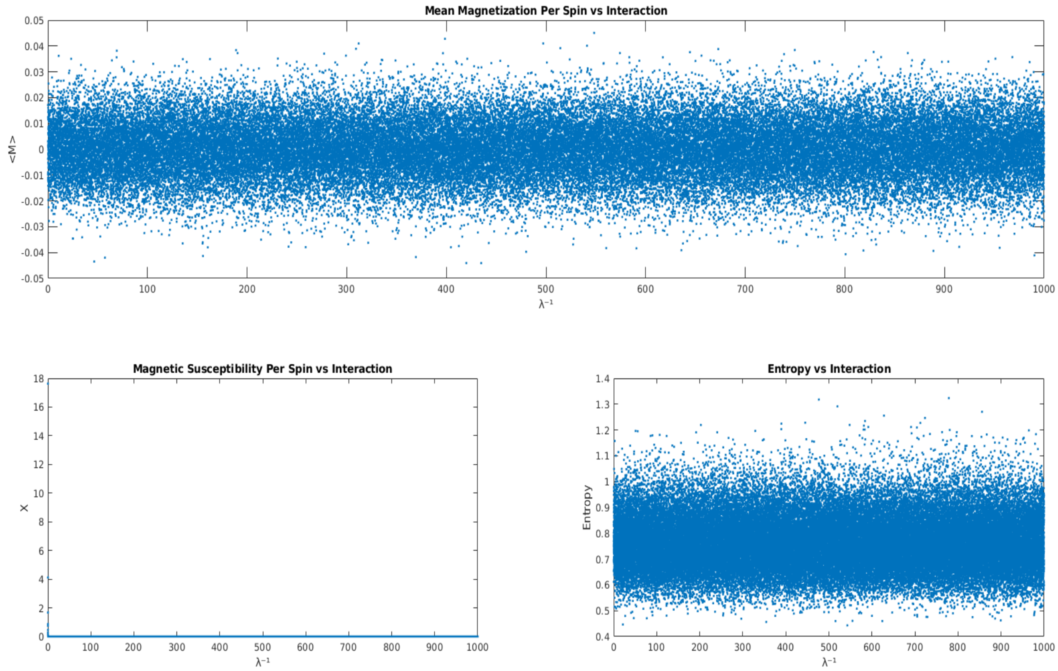

Figure 1.8: Simulation Ant-Based model with $1 \times L = 250$, $Neigbours = 2$ and $s_i = 0$

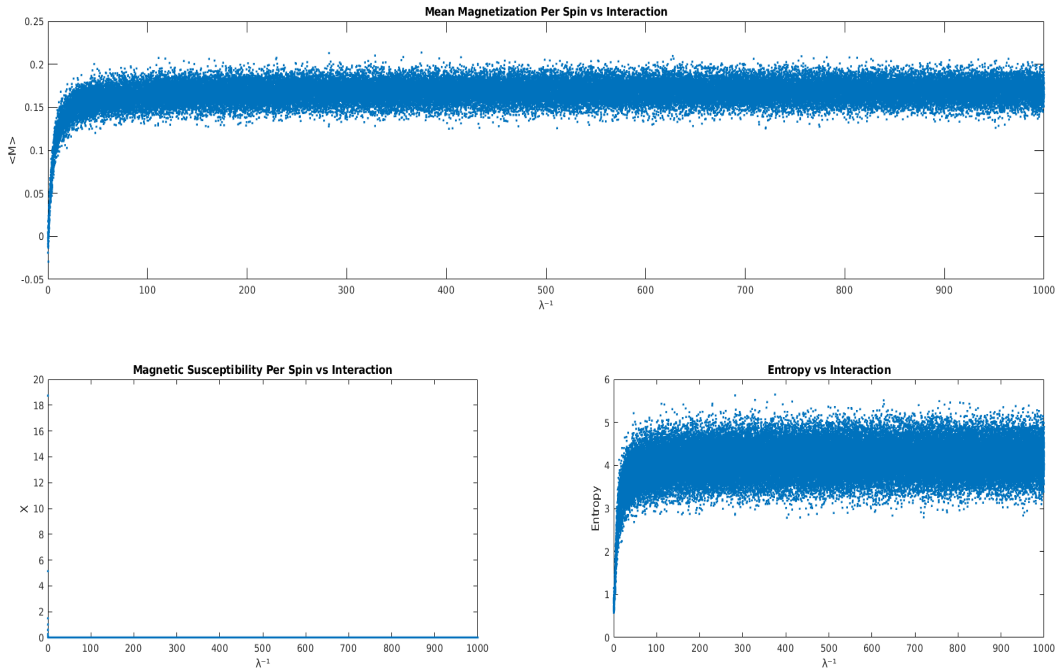

Figure 1.9: Simulation Ant-Based model with $1 \times L = 250$, $Neigbours = 2$ and $s_i = 1$

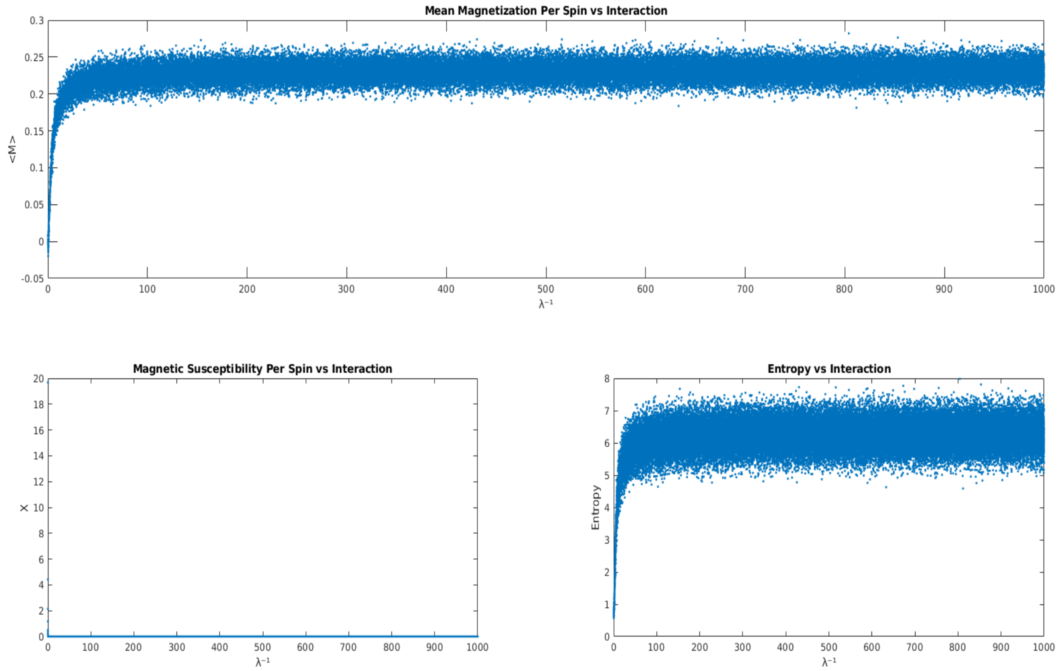

Figure 1.10: Simulation Ant-Based model with $1 \times L = 250$, $Neigbours = 4$ and $s_i = 1$

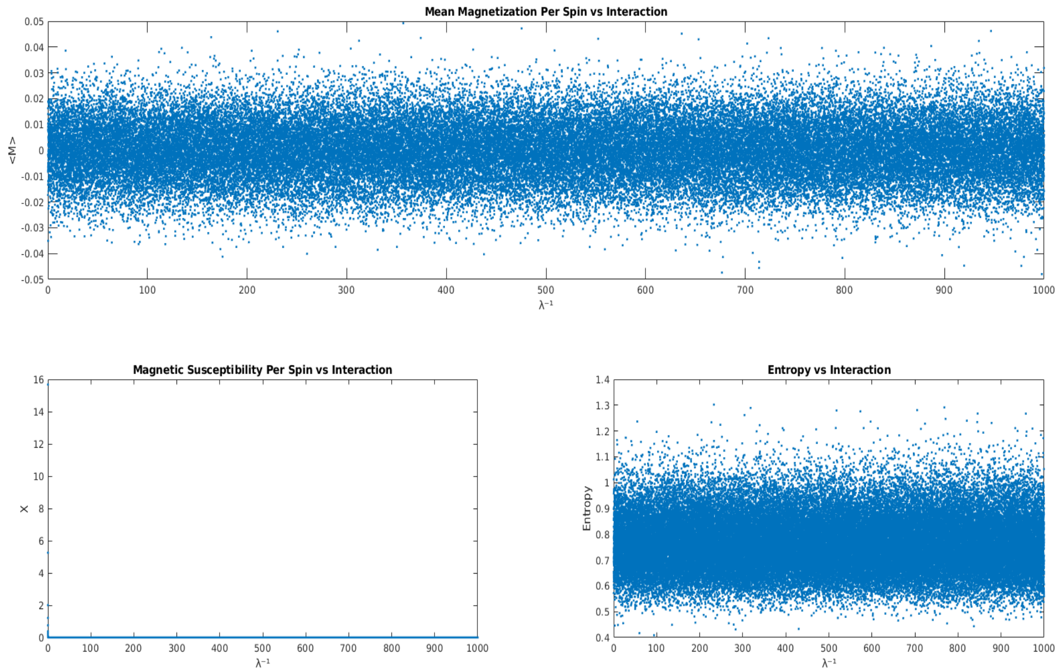

Figure 1.11: Simulation Ant-Based model with $1 \times L = 250$, $Neigbours = 6$ and $s_i = 0$

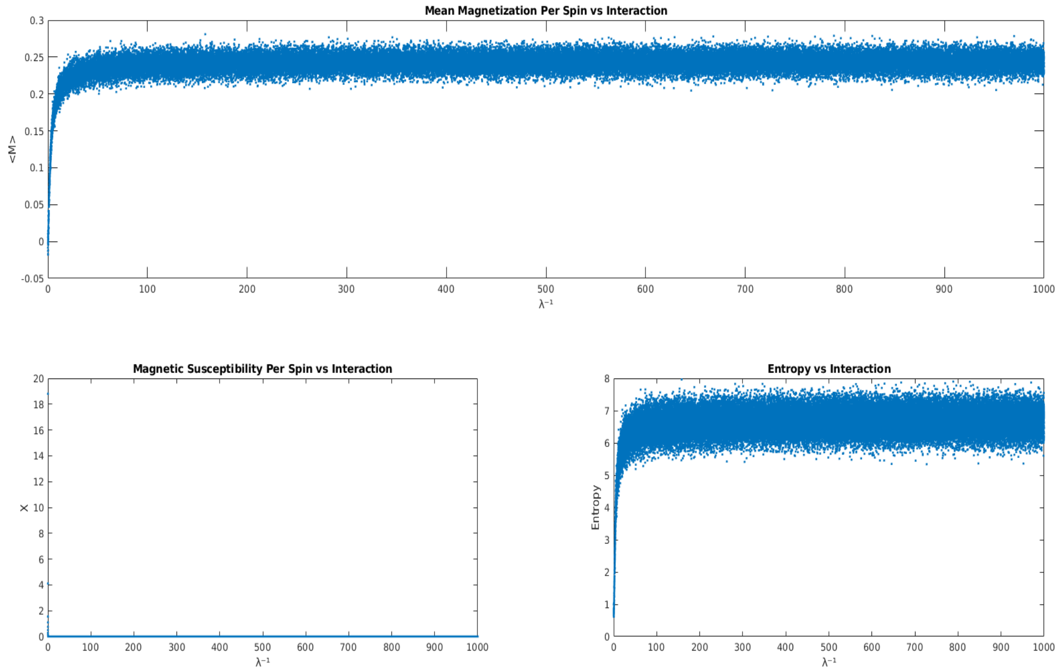

Figure 1.12: Simulation Ant-Based model with $1 \times L = 250$, $Neigbours = 6$ and $s_i = 1$

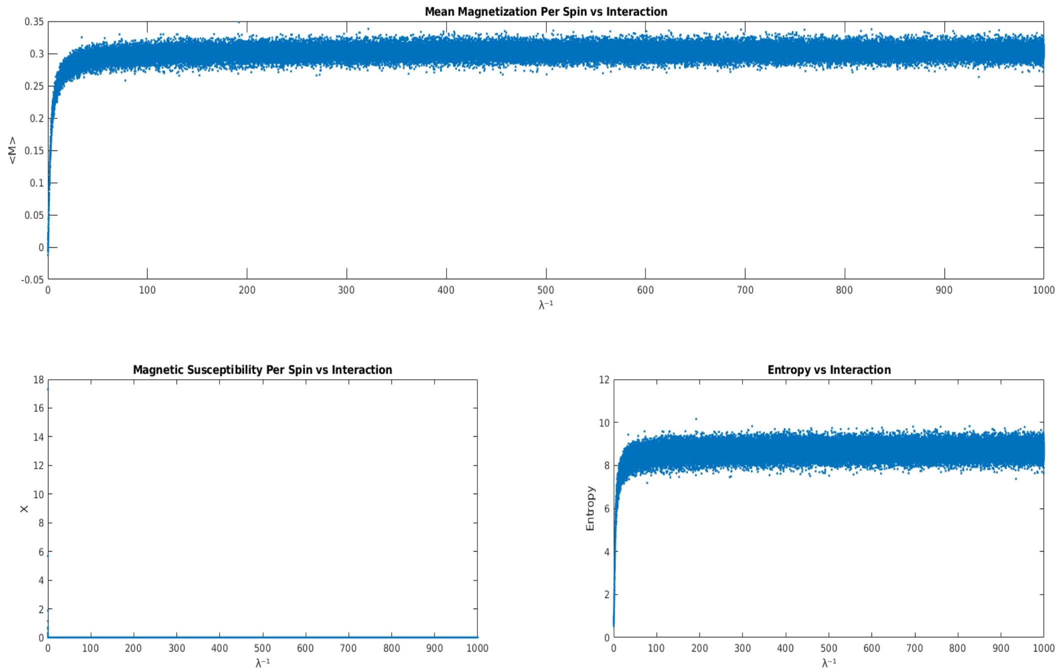

Figure 1.13: Simulation Ant-Based model with $1 \times L = 250$, $Neigbours = 20$ and $s_i = 1$

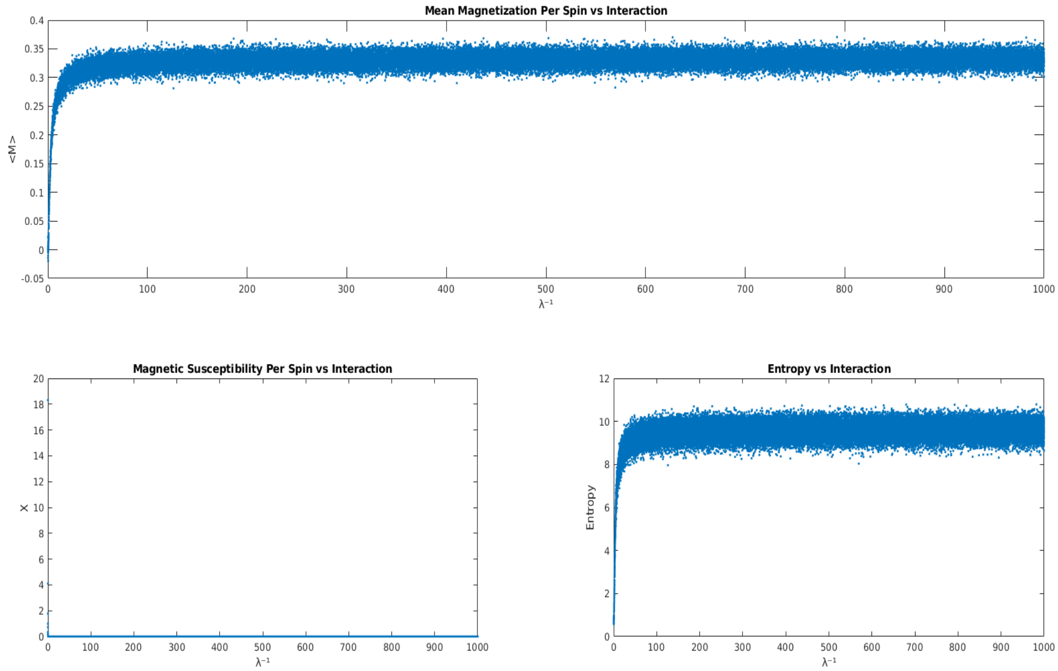

Figure 1.14: Simulation Ant-Based model with $1 \times L = 250$, $Neigbours = 200$ and $s_i = 1$

First of all, when $s_i \neq 0$ it is difficult for a lattice to sustain an opinion. In this case, there is no diffusion of opinions in the lattice.

Furthermore, the asymptotic value of mean magnetization, as $\lambda \to 0^+$, does not change that much when the number of neighbours is increased beyond 2. So, the dissipation of expectations saturates rapidly with the number of neighbours. Just like the Ising model, there is also a phase transition, but in this case, it appears when $\lambda = +\infty$.

Much like in the original Ising model, there can also be an equivalent to the external magnetic field. This can be understood as a way to steer the crowd's opinion due to some kind of external influence. Such a simulation is presented in Figure 1.15.

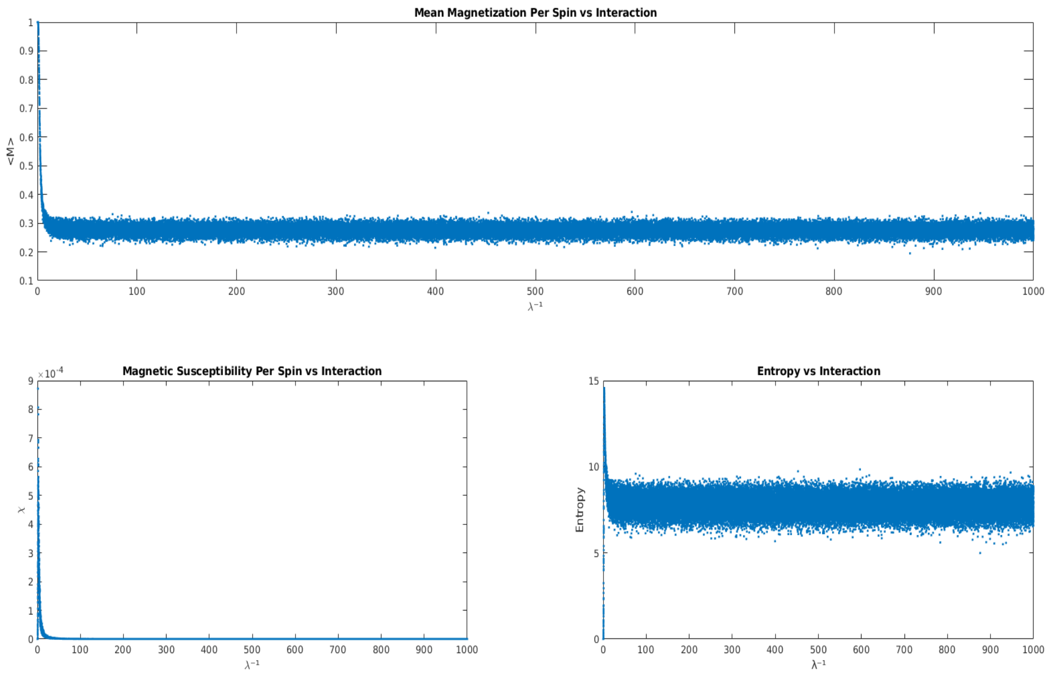

Figure 1.15: Simulation Ant-Based model with $1 \times L = 250$, $Neigbours = 4$, $s_i = 0$ and $h = 1$

Comparing Figure 1.15 with the previous Ant-based simulations, specially Figure 1.10, the way the magnetization varies with temperature is very different. In Figure 1.10 when $\lambda^{-1} \sim 0$, then $< M >= 0$. In Figure 1.10 when $\lambda^{-1} \sim 0$, then $< M >> 0$. This means that our system responds to an external trigger. If the the "magnetic" field is positive, the magnetization will also be positive. This is very similar to what happens in the Ising model.

So we have a way to externally influence a crowd's opinion if we choose the values of intensity of choice wisely.

2

Methods

It is important to clarify that this piece of work is a physics engineering master thesis, but since I will be applying physical models to macroeconomics, I will do a brief introduction to macroeconomic mathematical models and to the role of the central bank in the economy.

First of all, I will define some concepts which are fundamental to economics and understanding the following work.

Gross domestic product (GDP) is a measure of all the goods and services existent in a given economy[1]. It is measured in US dollars ($). It is usually used as a metric of the power of a given market.

Inflation, on the other hand, is related to the sustained growth over a period of time of the prices of goods and services[1]. There is some empirical evidence sustaining a negative correlation between inflation and unemployment. If unemployment is high, the number of people looking for jobs should be higher than the number of jobs available. So, the demand for labour is lower than its supply. The converse of this is also true. However, this trade-off is not always true and is a subject of debate in modern economics. This debate will be further developed in Section 2.2.

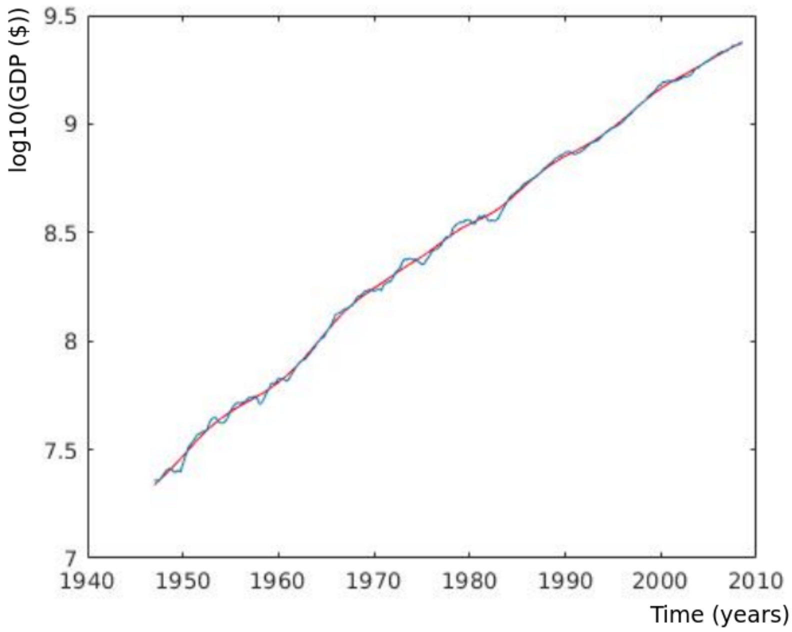

Figure 2.1: Time evolution of the natural logarithm of USA's GDP. From: `https://fred.stlouisfed.org/`

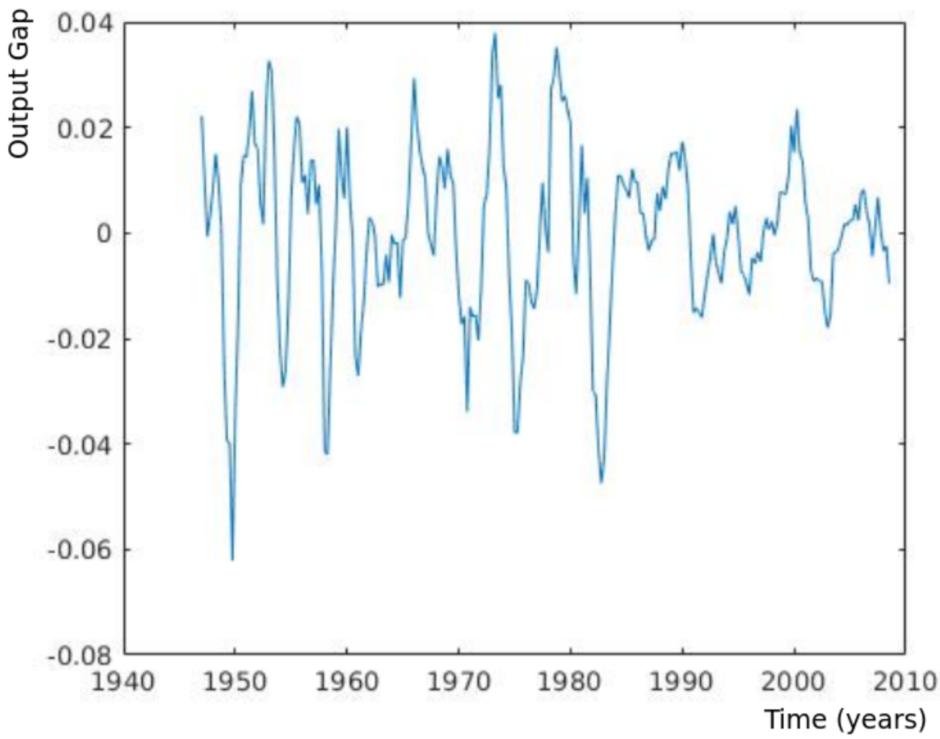

Figure 2.2: Time variation of USA's Output Gap. From: `https://fred.stlouisfed.org/`

In Figure 2.1, one can see the evolution of the American GDP after the end of World War 2. The blue line represents the real GDP. The red line on Figure 2.1 represents the Hodrick-Prescott filter[17] applied to the GDP time series. This acts as a low-pass filter. The economic interpretation of the filtered data is the natural state of the economy, which is the output the economy should be achieving at the top of its capacity. The difference between the blue and red line is called the output gap (Figure 2.2). A positive output gap will eventually lead to inflationary pressures. On the other hand, a negative output gap should lead to a decrease in inflation. When the output gap is positive, it is usually referred to as a “boom”. When the output gap is negative, it is called a recession.

Another important concept is price stickiness[1]. Price stickiness is the reluctance of a set of prices to change, even when the economic paradigm suggests a different price. This means that prices set for a certain good may resist the urge to change despite variations in cost or demand. Variations in the money supply affect the economy, which leads to changes in investment, unemployment, GDP and internal consumption. The inefficiency of a market to adjust immediately in the economic conditions creates a market disequilibrium. This concept also applies to wages.

Monetary Policy

A Central Bank (CB) is an independent government agency, whose goal is to stabilize and regulate the economy[1]. Usually, its board is composed of all lending banks. A neglecting Central Bank may not only cause economic turmoil but also cost thousands of people to lose their jobs. The regulation of an economy involves three things.

First of all, they should audit the activities of banks, in order to protect the depositors’ funds[1]. Secondly, a Central Bank can also provide financial services to banks, by lending them money if needed[1]. Lastly, they can determine the monetary policy, thus determining the amount of money circulating in the economy. They have three tools to achieve this last objective[1].

The first tool is setting a reserve requirement for each bank, which is the minimum amount of cash each bank must have every night[1]. By doing this, the Central Bank can control how much money a bank can lend.

Secondly, they can also use open market operations, by selling and buying securities from banks[1]. This affects the amount of cash on hand without having to change the reserve requirement.

The third tool is by setting the target interest that banks can lend[1]. By raising interest rates, growth is slowed down, lowering the output gap and preventing inflation. This is called a contractionary monetary policy. On the other hand, by lowering interest rates, growth is promoted, raising the output gap and shortening, or even preventing a recession. This is called expansionary monetary policy. This tool takes between 6 and 18 months to have visible effects on an economy. The effect of this tool on macroeconomic models is going to be the focus of this thesis.

It is also important to clarify that the sustained growth of the GDP is not a responsibility of the Central Bank: the government is responsible for providing a fruitful economic environment for companies and economic agents to thrive[1], thus generating a growth in the GDP.

One of the aims of macroeconomic models is to provide better tools to the Central Bank to better predict the outcome of a certain monetary policy in an economy.

A good macroeconomic model gives us a good representation of reality. There is always room for improvements in these models and this piece of work is an attempt at enhancing an already established macroeconomic model.

Stylized facts is an expression which is used to express the known dynamics and characteristics of macroeconomic variables. For instance, it is a known fact that inflation is contracyclical to the output gap. These empirical facts are summarized in Figure 2.3 and are shown in Figures 2.4 and 2.5.

Furthermore, our model should be statistically coherent with the reality. This means that the statistical moments (usually first, second and fourth statistical moments) of the simulated data must be in the same order of magnitude as the real data. If this is not achieved, the relative magnitude of the statistical moments of different macroeconomic variables must be coherent with empirical data.

	Cyclicality	*Lead/Lag*	*Variability Relative to GDP*
Consumption	Procyclical	Coincident	Smaller
Investment	Procyclical	Coincident	Larger
Price Level	Countercyclical	Coincident	Smaller
Money Supply	Procyclical	Leading	Smaller
Employment	Procyclical	Lagging	Smaller
Real Wage	Procyclical	?	?
Average Labor Productivity	Procyclical	Coincident	Smaller

Figure 2.3: Summary of the relations of many Macroeconomic variables with GDP. From Stephen Williamson, *Macroeconomics*, Addison-Wesley, 2005[1].

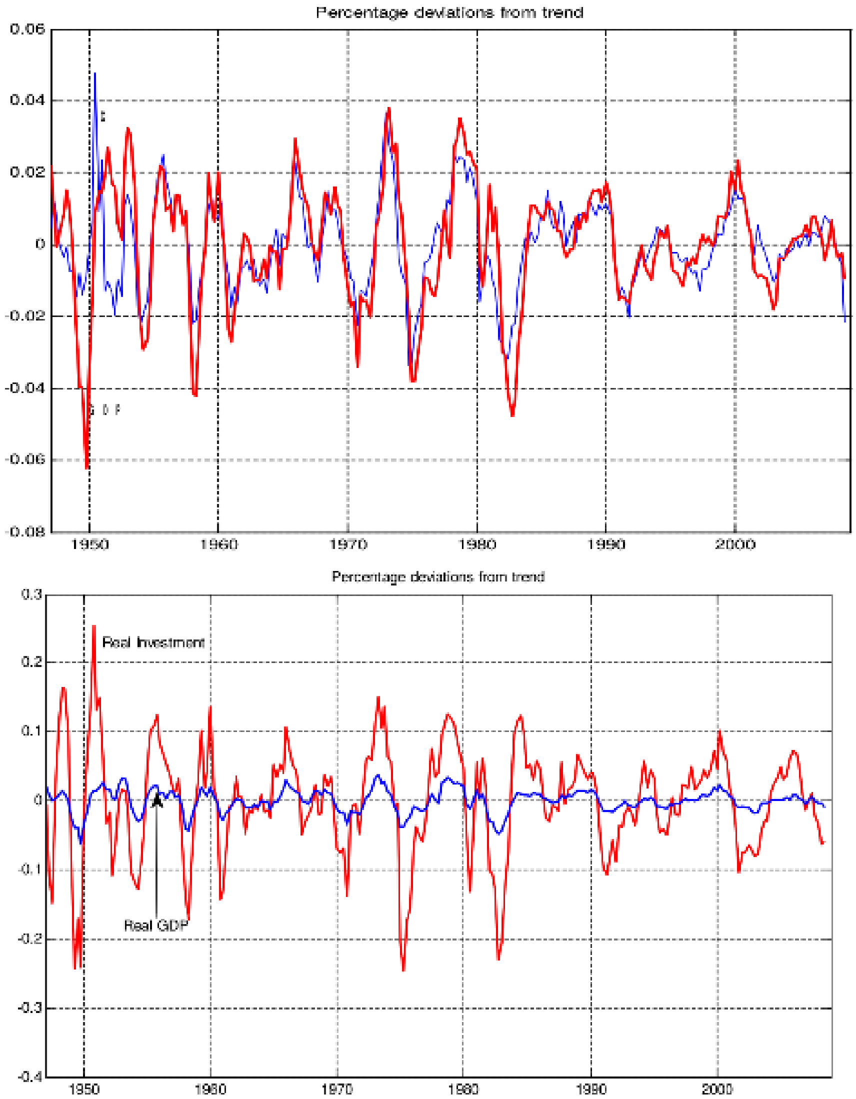

Figure 2.4: Time series of the fluctuations of Consumption vs Real GDP and Real Investment vs Real GDP. Data collected from from: `https://fred.stlouisfed.org/`

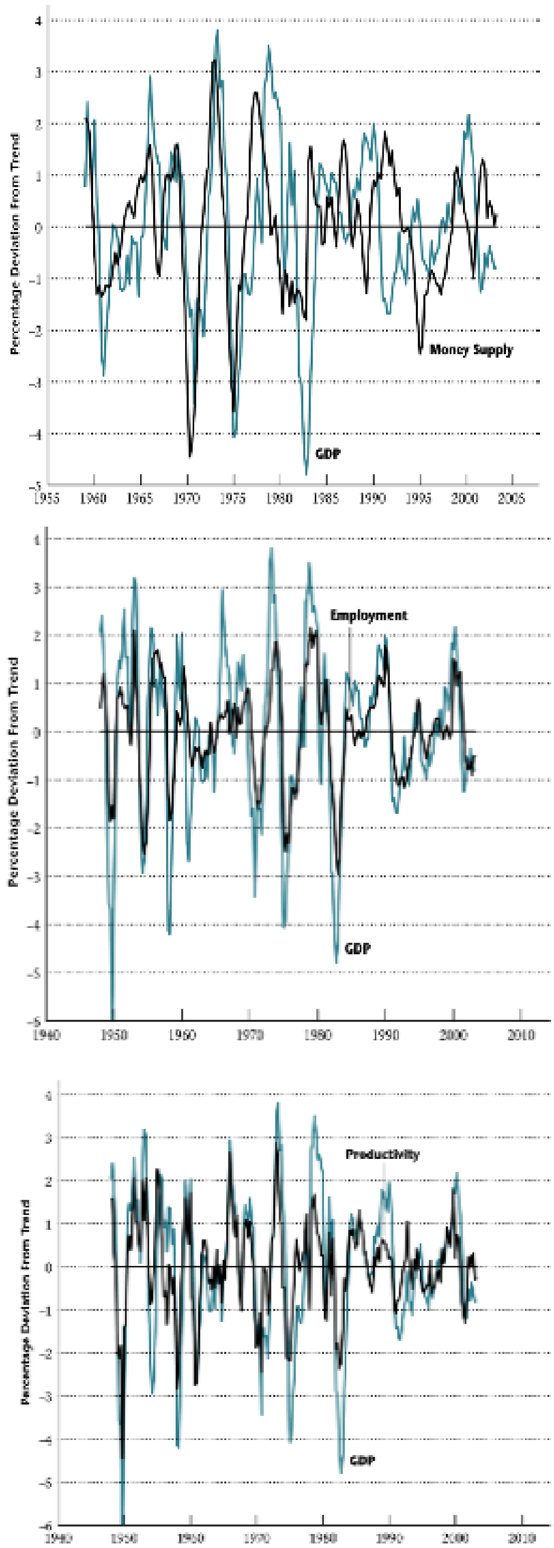

Figure 2.5: Time series of the fluctuations of Money Supply vs Real GDP, Employment vs Real GDP and Productivity vs Real GDP. From Stephen Williamson, *Macroeconomics*, Addison-Wesley, 2005[1].

Typically, from the mid 90's up until the sub-prime crisis in 2007, most Central Banks were adopting inflation targeting as its monetary policy and until then it was quite successful[1]. This was seen as a victory of macroeconomics and it was believed we would be living in a secure macroeconomic called the "Great Moderation".

It is important to explain some terms mathematically because it helps to bridge the gap between macroeconomics and the simulations that will be done in the next Chapter.

In economic terms, strict inflation targeting means that the Central Bank acts in order to keep inflation close to its target[1]. Mathematically speaking, this means that over a set of simulations with the same initial conditions, we need to find the minimum of the average of standard deviations of inflation.

This does not mean, however, that in a world of strict inflation targeting there is no role for output gap stabilization[18]. Price stickiness means that strict inflation targeting might not be the best option for a monetary policy, which paves the way for output gap stabilization combined with inflation targeting. Therefore, a Central Bank should be flexible in its inflation targeting. When the inflation is sufficiently different from its target, the Central Bank should adopt a gradual return of inflation to its target. If it adopted a more abrupt policy to return the inflation to its target, high-interest rates would be needed to lower the output gap.

However, until 2007 output gap stabilization was still limited[18]. The rationale was that the output gap was only needed because of price rigidities. The circumscribed extent of output gap stabilization is the basis of a model whose main characteristic is a stable equilibrium. The fluctuations around this equilibrium in the output gap only had one source, price rigidities.

If another source of fluctuations is known, the way we look at monetary policy and flexible inflation targeting would drastically change. In such a scenario, one would have to construct a trade-off between inflation and output gap targeting.

Typical Macroeconomic models

Throughout history, there have been two schools of thought who have dominated the way we think about economics[1]. These two lines have had a profound effect on the way one thinks about fiscal policy and how the central bank should act. Subsequently, these also affect the underlying mathematical models.

Classical School

The birth of modern economics is generally attributed to Adam Smith's work of 1776 "The Wealth of Nations"[19]. Not only is he considered the father of modern economics, but he is also considered the father of the classical line of thought. Many other intellectuals contributed to the development of these ideas, such as David Ricardo[20].

The classical economic theory is rooted in the idea of the invisible hand[19]. According to it, individuals act for their own benefit and self-interest economic-wise which assures that economic assets are allocated according to the market's best interest. This idea has obvious repercussions on how the state should operate, as far as fiscal and monetary policies are concerned. A *laissez-faire* type of policy requires little to no government regulation. The classical view of monetary policy is based on the notion that money itself doesn't affect the overall wealth of the economy, but simply the price level of services and goods. According to classical economists, an economy is always operating at, or near, the natural value of the GDP.

This means that for the most time, the Central Bank does not act. The Central Bank only intervenes if the economy is clearly not operating at its natural level. In such a situation, the Central Bank adopts very aggressive monetary policies, thus launching the economy into a long-term equilibrium.

A lot of critics of this school of thought argue that the implications of this line of thought in the monetary policy lead to a "Darwinistic" view of the economy. The survival of the "fittest" companies is done at the expense of many jobs and people's lives. According to its critics, from a macroeconomy point of view, a monetary policy that might only be effective in the long term is, arguably, not efficient at all.

A more modern formulation of a classic is the Real Business Cycle (RBC)[1]. Even though it managed to reproduce some stylized facts correctly, it failed to replicate the dynamics of the output gap.

One of the major differences from previous classical models was the adoption of a bottom-up approach. This was done by trying to model mathematically how individuals function in an economy and how it affects the bigger picture. Thus, the RBC provided a microeconomic foundation to a macroeconomic model. This reason is the main argument in favour of the RBC. Since most intellectuals which follow this school are located in universities by Northern American freshwater lakes, the New Classical economists are also called Freshwater Economists.

Although it was a flawed model, it introduced the need for sound microeconomic foundations as a pillar of any macroeconomic model. It was a flawed model since it failed to reproduce business cycles accurately.

In this specific case, the microeconomic foundations were rational expectations. This means that economic agents were able to understand the underlying mathematical models of the economy, have all the relevant information and have the cognitive ability to process it[1][18].

Keynesian School

Due to failing to change the economic paradigm after the Great-Depression a new school of economic thought arose, which was heavily influenced by the work of John Maynard Keynes[21], from whom it draws its name. The Keynesian economic theory relies on spending and aggregate demand to define the economic marketplace. Keynesian economists believe that the aggregate demand is influenced by both public (government agencies and municipalities) and private decisions (individuals and businesses in the economic marketplace). The Keynesian economic theory relies heavily on the fact that a nation's monetary and fiscal policy can affect a business' economy. Therefore, the state is a major economic agent. This economic theory also rejects the notion that an economy is always operating near the natural level of the GDP.

The 1970's oil crisis drove the first wave of Keynesian models obsolete, which lead to the resurgence of classical models, such as the RBC.

More modern takes on Keynesian models have been developed since the 1980's[1][22]. These models and its derivatives are now known as the New-Keynesian models (NKM). As opposed to Freshwater economics, most relevant schools are in cities near the sea. For this reason, this school of thought is often called Saltwater Economics.

These new models still had some similarities with the "Old" Keynesian models:

- Prices are sticky, which leads to rigid nominal wages
- Staggered Calvo contracts[23], which is the ability of a company to reset a price which is independent of its previous price. This phenomenon occurs with constant probability

On top of this, they also made some changes to the original model, in order to revert the changes that drove these models obsolete in the 1970's:

- Forward-looking rational expectations instead of adaptive expectations, which was an attempt to build the model upon some microeconomic foundations
- No permanent trade-off between inflation and unemployment, in order to avoid a stagflation scenario (high unemployment and economic stagnation).

These "new" models were then called Dynamic stochastic general equilibrium (DSGE) models. The most basic New Keynesian Model has three Equations: the IS function (Equation 2.1)[1], for the demand side of the economy; the AS function (Equation 2.2)[1], for the supply side of the economy and the Taylor rule (Equation 2.3)[1], for the monetary policy (interest rate).

In 1970, William Poole[24] demonstrated a fundamental result: the interest rate is a better instrument than monetary base if the variance of money demand shocks is larger than the variance of aggregate demand shocks. This result has been confirmed empirically. In his analysis, the Central Bank's objective was to minimize fluctuations in the output or deviations from the natural level of output

Since 1989, the Fed has used fed funds rates as instruments to hit its intermediate target of short-term interest rates.

The Taylor rule[25] was a purely empirical Equation which modelled how the interest rate should act given a certain inflation rate and the output gap.

In Equations 2.1, 2.2 and 2.3, the subscript t indicates the time index.

$$\pi_{t+1} = b_1 \widetilde{E}_t \pi_{t+1} + (1 - b_1)\pi_t + b_2 y_t + \eta_t \tag{2.1}$$

Equation 2.1 explains how inflation π changes.The first term is the expected value of inflation at $t + 1$. The second term is a backwards-looking inflation variable. The third term gives us how much does the price changes when the supply changes. Therefore, b_2 is a measure of the price stickiness. When $b_2 = 0$ there is a total price stickiness. The last term is a white noise retrieved from a normally distributed random number generator. It represents the action of exogenous shocks in an economy.

$$y_{t+1} = a_1 \widetilde{E}_t y_{t+1} + (1 - a_1) y_t + a_2 (r_t - \widetilde{E}_t \pi_{t+1}) + \epsilon_t \tag{2.2}$$

The logic behind Equation 2.2 is very similar to Equation 2.1, but this last Equation is related to the output gap y. The first term is the expected value of the output gap at $t + 1$. The second term is a backwards-looking output gap variable. The value

a_1 is related to how the future values of output gap react to its previous values. The third term gives us the expected change in supply, with the difference between the interest rate r_t and the expected inflation $\widetilde{E}_t\pi_{t+1}$. This difference is called the effective interest rate. The value a_2 is related to the expected changes in supply. The last term is also a white noise retrieved from a normally distributed random number generator, acting as an exogenous shock.

$$r_t = c_1(\pi_t - \pi^*) + c_2 y_t + c_3 r_t + u_t \tag{2.3}$$

Equation 2.3 gives us the interest rate that should exist given a certain inflation rate π and output gap y. The first term is the difference between the current inflation and the inflation target of the Central Bank, π^*. The constant c_1 is related to how aggressively does the interest rate react to a change in inflation. The second term is related to the current output gap value, y. Much like in the previous term, c_2 is a measure of how vigorous is the reaction of the interest rate to a change in the output gap. The third term (c_3) is a backwards-looking interest rate variable. It is important in order to smoothen the interest rate curve. The last term is a normally distributed random number, which acts as an exogenous shock.

The values of the constants in Equations 2.1, 2.2 and 2.3 are stated for the regular New Keynesian Model.

It is also important to add that in its simplest form this is not a multi-agent model. This means that there is a single economic agent whose action is an average of how an economic agent would act.

This specific model uses rational expectations: our agent fully understands the underlying mechanics of the economy[1]. In other words, he considers the targets of the Central Bank to be fully credible. In other words:

$$\widetilde{E}_t y_{t+1} = y^*, \tag{2.4}$$

$$\widetilde{E}_t \pi_{t+1} = \pi^*, \tag{2.5}$$

where y^* and π^* are the Central Bank's targets for the output gap and inflation, respectively.

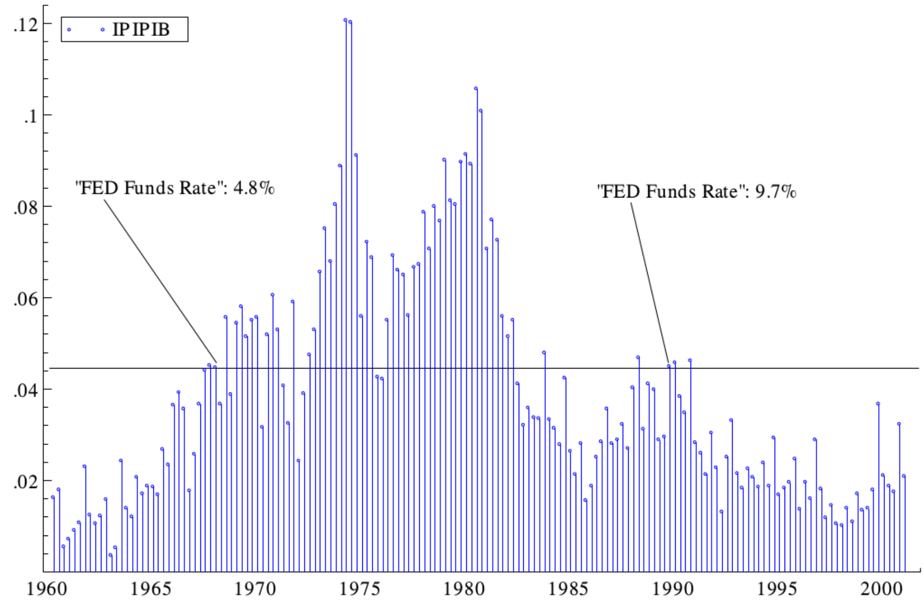

Figure 2.6: Time evolution of the inflation in the USA. From: `https://fred.stlouisfed.org/`

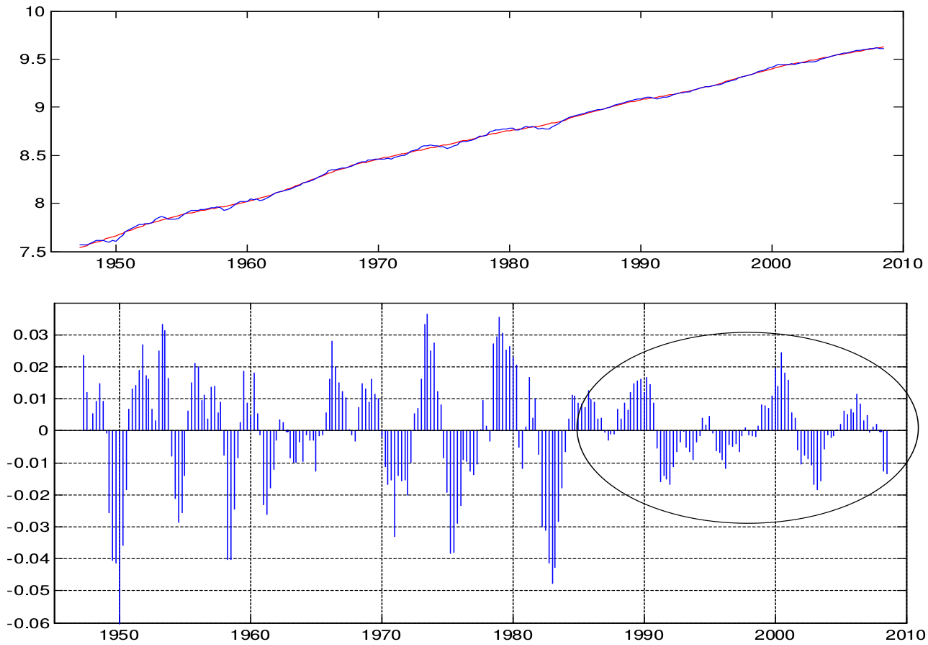

Figure 2.7: Time evolution of GDP and output gap in the USA. From: `https://fred.stlouisfed.org/`

As one can see in Figures 2.6 and 2.7, when the implementation of New Keynesian began to become the norm (the 1980's), the Central Bank acted much more aggressively and fastly to combat the increasing inflation. The new paradigm made it mandatory to change the behaviour of the Central Bank. On one hand, "the old view" determined that the Central Bank controls the money supply and the market determines the interest rate. On the other hand, "the new view" stated that the Central Bank sets the interest rate and then the market determines the level of money flowing around.

Summed up, this model has 4 distinctive characteristics:

- The instrument of monetary policy ought to be the short-term interest rate
- Policy should be focused on the control of inflation
- Inflation can be reduced by aggressively increasing short-term interest
- The Central Bank should conduct monetary policy adopting a strategy of commitment in a forward-looking environment, instead of discretion

These models had tremendous results and managed quite well to stabilize the economy. The empirical data supports this (Figures 2.6 and 2.7). The introduction of the interest rate as a tool managed to decrease the variations of both output gap and inflation.

“One of the most striking features of the economic landscape over the past twenty years or so has been a substantial decline in macroeconomic volatility. In a recent article, Olivier Blanchard and John Simon (2001)[26] documented that the variability of quarterly growth in real output (as measured by its standard deviation) has declined by half since the mid-1980s, while the variability of quarterly inflation has declined by about two thirds. Several writers on the topic have dubbed this remarkable decline in the variability of both output and inflation "the Great Moderation." Similar declines in the volatility of output and inflation occurred at about the same time in other major industrial countries, with the recent exception of Japan, a country that has faced a distinctive set of economic problems in the past decade.” Remarks by Governor Ben S. Bernanke, At the meetings of the Eastern Economic Association, Washington DC, 20 February 2004.

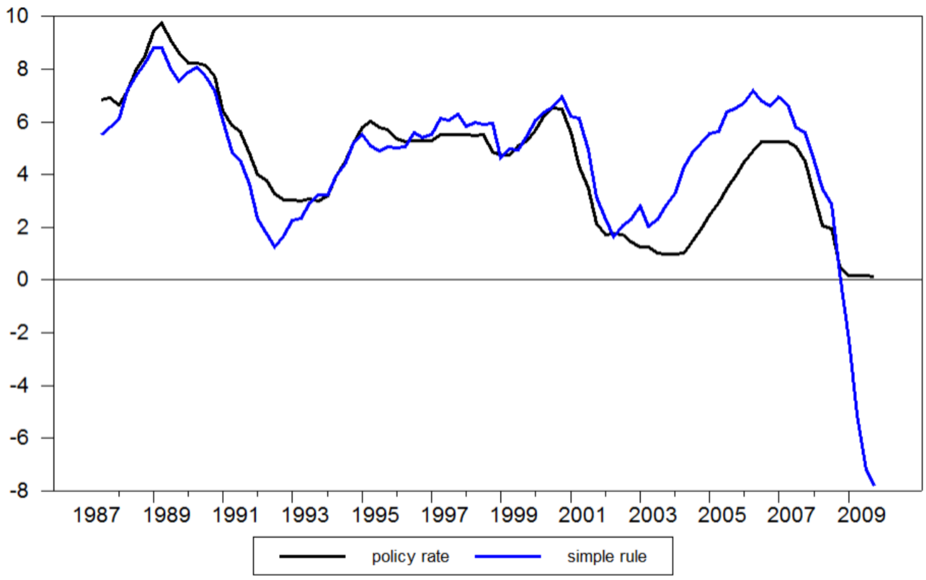

Figure 2.8: Time series of the interest rate given by the taylor rule vs the actural interest rate. Data collected from: `https://fred.stlouisfed.org/`

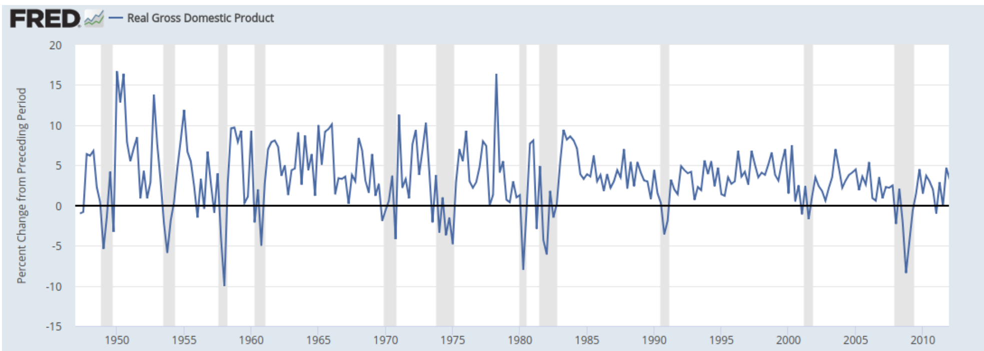

Figure 2.9: Time series of the percentual change of the GDP relatively to the previous year. From: `https://fred.stlouisfed.org/`

So, as it can be observed in Figures 2.8 e 2.9, up until 2008 the Taylor rule managed to produce quite satisfactory results, but something went wrong. I am referring to the subprime crisis of 2007. After 2008, the New Keynesian Models were subject to a lot of criticism.

One of the biggest problems with New Keynesian models was the dependence of exogenous shocks in order to accurately reproduce economic cycles.

From this criticism, a lot of questions arose. Some of these questions were about the role of the Central Bank. Other questions were related to the theoretical and mathematical foundations of New Keynesian Models.

Rational expectations assume that our agents know the underlying mathematical model of the economy, have all the relevant information and have the cognitive ability to process it[1][18]. The assumption of the *homo economicus* seems very odd. Assuming that human beings, even from an economic point of view, are fully rational and always make the best decision might have caused the failure of these models. Other criticisms and questions arose, but they will not be the focus of this thesis. Works in the area of neural sciences and behavioural psychology have corroborated with the idea of men having limited information and knowledge of how the world works, but eager to learn from its mistakes[27][28].

Basic Behavioral Model

In this section, I will describe a simple behavioural model, which will serve as a starting point. This model, with basic heuristics and an almost rule of thumb approach, is described by Paul De Grauwe (2010)[29]. This model starts as a classical New Keynesian Model, but with a twist.

This models uses the same Equations as the New Keynesian model (Section 2.2.2) but refuses the idea of rational expectations.

$$\widetilde{E}_t \pi_{t+1} = \pi^* \tag{2.6}$$

As it can be seen in Equation 2.6, we will assume a fully credible inflation targeting. The reason for this is: in an imperfect inflation targeting scenario the volatility of both output gap and inflation become too high to have significant conclusions of whether a certain monetary policy given by this model is, or is not, optimal[18]. Since the results obtained from such simulations are in accordance with real data, this is an accurate assumption.

$$\widetilde{E}_t y_{t+1} = \alpha_{f,t} \widetilde{E}^f_{t+1} y_{t+1} + \alpha_{e,t} \widetilde{E}^e_{t+1} y_{t+1} \tag{2.7}$$

$$\widetilde{E}^f_{t+1} y_{t+1} = y^* = 0 \tag{2.8}$$

$$\widetilde{E}^e_{t+1} y_{t+1} = y_t \tag{2.9}$$

So this model has two rules for the expectation of the output gap: either is a fundamentalist rule (Equation 2.8) or an extrapolative rule (Equation 2.9). The fundamentalist rule is the fully credible agent, which believes that the Central Bank's forecast is true. The extrapolative rule picks up the last value of the output gap instead of the Central Bank's prediction. The operator $\widetilde{E}$ is the expectative operator, which is the expected future value of a certain rule. $\widetilde{E}_t y_{t+1}$ is the expected value of the output gap at $t+1$. $\widetilde{E}$

$$\alpha_{f,t} + \alpha_{e,t} = 1 \tag{2.10}$$

So, since this is a single agent simulation, the $\alpha`s$ in Equation 2.7 act as a way to hide this simplification. Since $\alpha`s$ are a probability, they exist to know how many agents would choose any of the rules. Therefore, $\alpha_{e,t}$ is the proportion of agents in a society which chooses the extrapolative rule.

$$U_{f,t} = -\sum_{k=1}^{\infty} \omega_k (y_{t-k-1} - \widetilde{E}_{f,t-k-2} y_{t-k-1})^2 \tag{2.11}$$

$$U_{e,t} = -\sum_{k=1}^{\infty} \omega_k (y_{t-k-1} - \widetilde{E}_{e,t-k-2} y_{t-k-1})^2 \tag{2.12}$$

$$\alpha_{f,t} = \frac{exp(\gamma U_{f,t})}{exp(\gamma U_{f,t}) + exp(\gamma U_{e,t})} \tag{2.13}$$

$$\alpha_{e,t} = \frac{exp(\gamma U_{e,t})}{exp(\gamma U_{e,t}) + exp(\gamma U_{f,t})} = 1 - \alpha_{f,t} \tag{2.14}$$

The parameter $\omega_k`s$ in Equations 2.12 and 2.11 acts as a memory to each one of the rules, it records whether a choice was or was not successful, which will then affect the probability α and therefore the output gap expectation. This is seen as the memory of the agent. $U_{f,t}$ is related to the difference of the current output gap value and the fundamentalist expectation (the Central Bank objective). $U_{e,t}$ is related to the difference between the current value of the output gap and the extrapolative expectation (in this case the previous value of the output gap). Both of them are dimensionless variables.

Much like the New Keynesian model, this model, in its simplest form, is not a multi-agent model. This means that there is a single economic agent whose action is an average of how an economic agent would act.

The variable γ is called the intensity of choice and is used to quantify how related are the agents. When γ is zero, the switching is purely random. As γ increases, agents are more willing to learn from their past mistakes. γ is dimensionless.

These type of models also introduced a new variable called "animal spirits" (AS)[29], which is related to the evolution of the probabilities of choosing each of the rules. Mathematically speaking, it is the temporal evolution of $\alpha_{e,t}$ over the course of a simulation. It is a way of knowing how agents feel about the economy and if they believe, or do not believe, in the forecast of the Central Bank. It is related to waves of pessimism and optimism in the economy.

Animal spirits vary between -1+1, in order to be in accordance with the Magnetization variable of both Ant-Based and Ising models described in Chapter 1.

$$AS = \frac{1}{2}\alpha_y + \frac{1}{2} \tag{2.15}$$

It is also important to say that α_y is always calculated in relation to the extrapolative rule, which means that when $AS = +1$ our agents do not believe at all in the central bank's forecast, being therefore pessimistic.

These Equations and its meaning are more thoroughly explained in the works of Brock and Hommes (1997)[30], Branch and Evans(2006)[31], De Grauwe and Grimaldi(2006)[32].

Agent-based Behavioral Model

Starting from the single agent model described previously (Section 2.3), I will now describe a similar multi-agent model. This is similar to the one described by Paul de Grauwe (2010)[18].

First of all, it is necessary to define a topology. The topology sets the interactions in a network. Each agent has a corresponding number. For every agent there is a list with the corresponding number of the neighbours, setting whom does the agent interact with. Each agent is initialized with a starting expectation, retrieved from a random distributed uniform distribution:

$$y_{t,i} \sim U(-0{,}3, +0{,}3) \tag{2.16}$$

Each agent has:

$$U_{f,t,i} = \rho.U_{f,t-1,i} - (1-\rho)(y_t - y^*)^2 \tag{2.17}$$

$$U_{e,t,i} = \rho.U_{f,t-1,i} - (1-\rho)(y_t - \widetilde{E}_{e,t,i}y_{t,i})^2 \tag{2.18}$$

$$\alpha_{e,t,i} = \rho_{BH}\alpha_{e,t-1,i} + (1-\rho_{BH})\frac{exp(\gamma U_{e,t,i})}{exp(\gamma U_{e,t,i}) + exp(\gamma U_{f,t,i})} \tag{2.19}$$

The index i in Equations 2.17, 2.18 and 2.19 indicates the agent number in the grid. It is important to say that Equations 2.17, 2.18 and 2.19 are more simple versions of Equations 2.11, 2.12 and 2.14 in a multi-agent scenario. Also, ρ is the same as ω_k.

Agent i chooses the neighbour, or itself, with the highest $\alpha_{e,t,i}$ and takes the expectation and $\alpha_{e,t,i}$ from its "best" neighbour.

There is also a low chance that the agent is able to innovate, which in practice means that a new agent is formed with a new expectation and a new value of $\alpha_{e,t,i}$. If there is innovation, the newly created agent will be compared as if it was a regular agent.

After the assessment if the chosen $\alpha_{e,t,i} < 0{,}5$, then the agent follows the fundamentalistic rule, which means that $y_{t,i} = y^*$ and $\alpha_{e,t,i} = 0$. The final expectation of the whole system is the average of all expectations after all these assessments.

It is important to say that the property called λ (intensity of choice) in Chapter 1 will be named γ from now on.

In this case, animal spirits are the proportion of agents in the system which follows each of the rules.

This is done in a synchronous network, which means that agents hold their old values for the same instant of time when assessing a possible change. All the Agent-Based simulations will be using a synchronous network.

Ant-Based Behavioral Model

The model described in this Section is based in Kirman's Ant-Based Model[10] and Shu-Heng Chen's Boltzmann-Gibbs Machine[16] but with some subtleties. The main difference is how the external influence on the crowd is generated.

Much like in previous models, there is a topology, which sets the interactions in the network.

This model is very similar to the single agent behavioural model, however, the value of $\alpha_{e,t}$ will be transformed into a kind of external magnetic field, which will act upon the network of agents.

$$h_t = 2 * \alpha_{e,t} - 1 \tag{2.20}$$

Much like in the previous model (Section 2.3), all agents are numbered, to facilitate the indexing of neighbours. Each agent is given a value, either +1 or -1. This binary value is called s_i.

The network of agents is initialized with the same number of agents being +1 or −1. In other words:

$$s_i =\sim U\{-1,+1\}\ \{0\} \tag{2.21}$$

The rest of the algorithm looks something like this:

$$M_i = s_i(\lambda + \epsilon.h_t) + \beta \sum_{j=1}^{N_i} w_{ij}s_j \tag{2.22}$$

$$w_{ij} = \frac{1}{j \in N_i} \tag{2.23}$$

if $s_i = 1$:

$$\alpha_i = min(1, \frac{1}{1 + e^{-2\gamma M_i}}) \tag{2.24}$$

else:

$$\alpha_i = min(1, 1 - \frac{1}{1 + e^{-2\gamma M_i}}) \tag{2.25}$$

if $y \sim U(0{,}1) < \alpha_i$:

$$s_i = -s_i \tag{2.26}$$

- M_i is the value of Magnetization
- λ is the self-conversion rate, which is the importance the agent gives its current value
- β is the imitation rate, which is the importance the agent gives to its neighbours' values
- ϵ is the importance the magnetic field has on the Magnetization
- γ is again the intensity of choice

It is also easy to implement innovation in this algorithm. If innovation exists, one has simply to add a random value $s_i \sim U\{-1,+1\}\ \{0\}$ to the sum in Equation 2.22.

Much like in the models described previously (Section 2.3 and 2.4), the animal spirits are related to the proportion of agents which are optimistic or pessimistic towards the forecast of the Central Bank. The final $\alpha_{e,t}$ is calculated averaging the total magnetization of the network. Calculating the average magnetization ($< M_{TOTAL} >$) is done simply by averaging all the spins in the network.

$$\alpha_{e,t} = \frac{1}{2} < M_{TOTAL} > + \frac{1}{2} \tag{2.27}$$

The most interesting part of this model is that by tweaking the values (λ, β, ϵ and γ) of the Ising/Ant-Based interactions, it is possible to model $\alpha_{e,t}$, and consequently animal spirits, to one's will.

One can validate a model by finding the optimal monetary policy and check if the stylized facts and macroeconomic cycles produced by it make any sense when compared to empirical data.

In this specific case and in the case of the model described in Section 2.4, the optimal monetary policy is given by choosing the values of c_1 and c_2 in Equation 2.3. If the Central Bank is focused only on inflation targeting, then $c_2 = 0$. c_3 is not as important since it only exists to smoothen the interest rate curve. It is setted with the same value for all simulations ($c_3 = 0{,}1$). For a given pair of values of c_1 and c_2 one has to run the simulations n times and calculate the average of the standard deviations of the output gap(σ_y) and inflation(σ_π). Even though this is a stochastic phenomenon, these simulations will generate two paraboloids with a vast minimum, which I will call a “valley”. Since this is statistical phenomena, one can not use numerical methods to find this minimum, but it is known from empirical data that c_1 and c_2 do not exceed the value 3. The optimal monetary policy is given by the “best” overlap of the valleys of both paraboloids (standard deviation and inflation)[18]. All the simulations presented in the next chapter were done using python3. By finding this ”overlap”, it means that the optimal monetary policy focuses both on output gap and inflation stabilization, even though it is not the actual minimum of the standard deviation of both variables. Therefore, this will be done visually inspecting both plots.

These type of simulations are very demanding for a machine, so I had to take advantage of having a multicore computer at my disposal and parallelize this process. For this purpose, I used python’s 3 multiprocessing library[33].

In the next sections, I will compare the results, with the same topology, of the Agent-based Behavioral model and the Ant-Based Behavioral model. I will then tweak with the values inside the Ising/Ant-Based interaction and different topologies. I will then check in which way does it affect the optimal monetary policy, the stylized facts and macroeconomic cycles.

3

Results

In this chapter, I will show the results of the simulations. For each interaction, I will test with two topologies: a normal circular topology (Figure 3.1) and an "American" topology (Figure 3.2).

The first topology is a fairly simple ring topology. Each agent has two neighbours. It will be similar to the topology in Figure 3.1 but with 650 agents. I chose 650 agents, so it has a similar number of agents as the "American" topology.

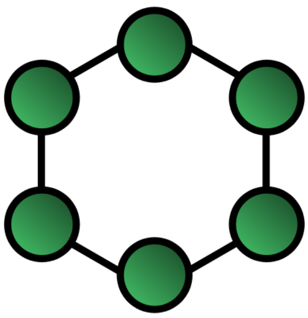

Figure 3.1: Standard ring/circular topology. From: `https://networktopologytutorials101.wordpress.com/2015/09/07/physical-topology-ring-topology/`

The American topology is an attempt to mimic the United States of America. Each state will have a circular topology, in which an agent is equivalent to half a million people. In each state, there is a privileged agent, which talks to a central agent, which will act as the Federal Reserve. It is a mixture of a circular topology and a ring topology. In total there are 648 agents in this topology. It is similar to the network described in Figure 3.2.

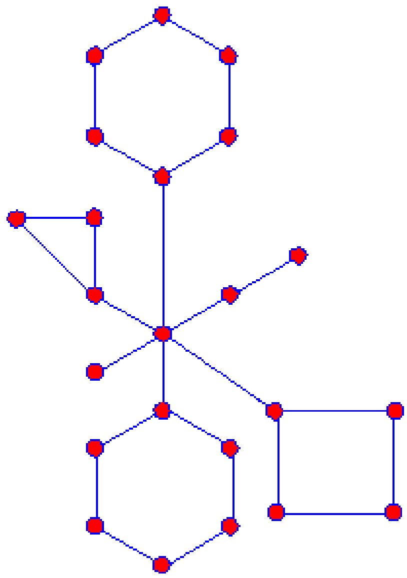

Figure 3.2: Simplified version of the american topology.

The circular network is highly regular since the agent with index i only "speaks" with the agents with the agents of index $i-1$ and $i+1$. This means that expectations and innovations have a clear path to spread themselves. The "American" network has not this type of regularity. Each state acts as a cluster where there is a clear path for expectations and innovations. For an innovation to spread from a non-privileged agent in a state to another non-privileged agent in another state, the path is not as clear, thus delaying the spreading of information. The goal of this is to understand if these two different topologies affect the economic cycles and the optimal monetary policy of a central bank. This can be achieved if the values of c_1 and c_2 of the optimal monetary policy differ for these two different topologies.

Some of the simulation conditions have already been studied so its values are known. These values are exposed in Table 3.1.

After finding the optimal monetary policy, a simulation will be performed using the values of c_1 and c_2 given by it. The time and frequency plots of the macroeconomic variables (output gap, inflation, interest rate and animal spirits) of the simulated data using the optimal monetary policy will be presented for each topology and type of interaction.

	Value	Description
p_i	0,05	Probability of innovation of a single agent
t	100	Number of time iterations per simulation
π^*	0	Inflation objective by the Central Bank
y^*	0	Output Gap objective by the Central Bank
a_1	0,5	From the AS Function
a_2	−0,2	From the AS Function
ϵ	0,5	Gaussian shocks from the AS Function
b_1	0,5	From the IS Function
b_2	0,05	From the IS Function
η	0,5	Gaussian shocks from the IS Function
u	0,5	Gaussian shocks in the Taylor rule
c_3	0,1	From the Taylor rule

Table 3.1: Common simulation conditions.

For each interaction, I will first try to find the optimal monetary policy. This will be done by trying to find the optimal values of c_1 and c_2, which produce the best outcome in the standard deviation of both the inflation and the output gap. Using the optimal monetary policy I will then simulate the economic cycles and calculate the statistical moment of the generated macroeconomic variables.

Though it is not specified in the literature, a single time iteration corresponds to a time period of weeks/months.

Agent-based behavioral model

Since we are working in a fast changing environment, with a lot of agents, there will always be a considerable amount of agents that are innovating. In such a scenario, agents have a short memory, so there is no problem to assume that they have no memory. This assumption is widely accepted in these type of models. The specific simulation conditions are exposed in Table 3.2

	Value
$\rho_{BH,i}$	0
ρ_i	0
γ	100

Table 3.2: Agent-based behavioral model specific simulation conditions.

Circular Topology

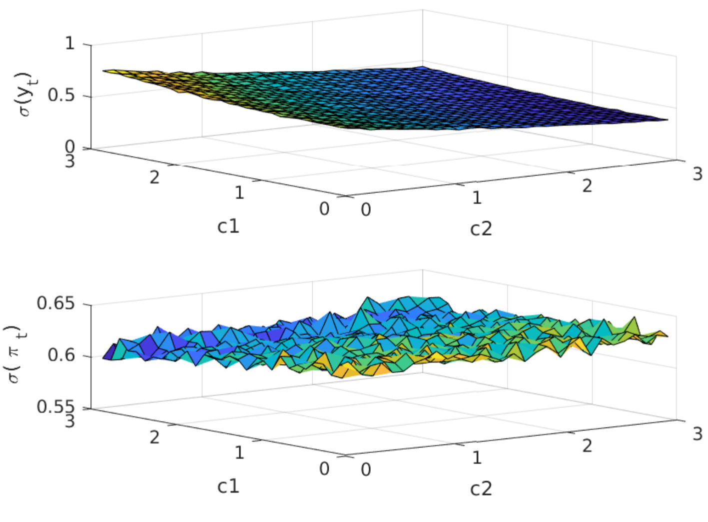

Figure 3.3: Finding the optimal monetary policy for the circular topology with an Agent-based behavioral model.

By analyzing Figure 3.3, one can say that a good guess for the optimal monetary policy will be: $c_1 = 2$ and $c_2 = 2$. As $c_1 \to 0$ one can see that there is an increase in the standard deviation of the inflation. The pair $c_{1,2} = 2$ produces the best outcome for both variables.

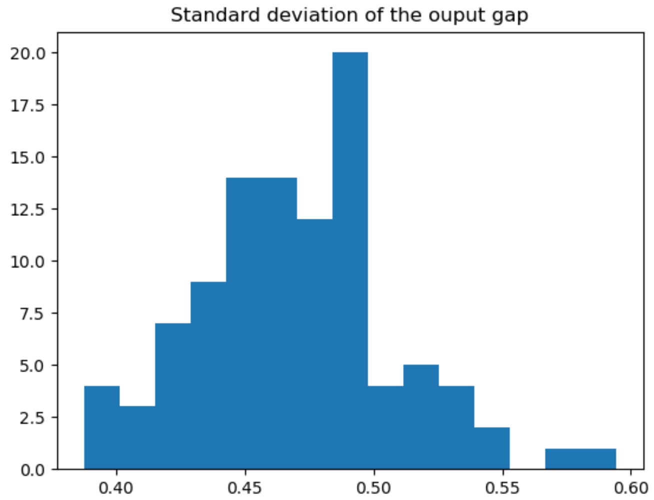

Figure 3.4: Distribution of the standard deviation of the ouput gap using the optimal monetary policy over 100 simulations.

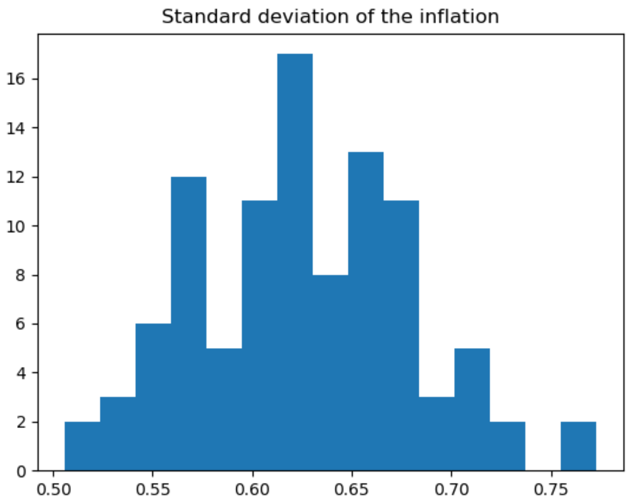

Figure 3.5: Distribution of the standard deviation of the inflation using the optimal monetary policy over 100 simulations.

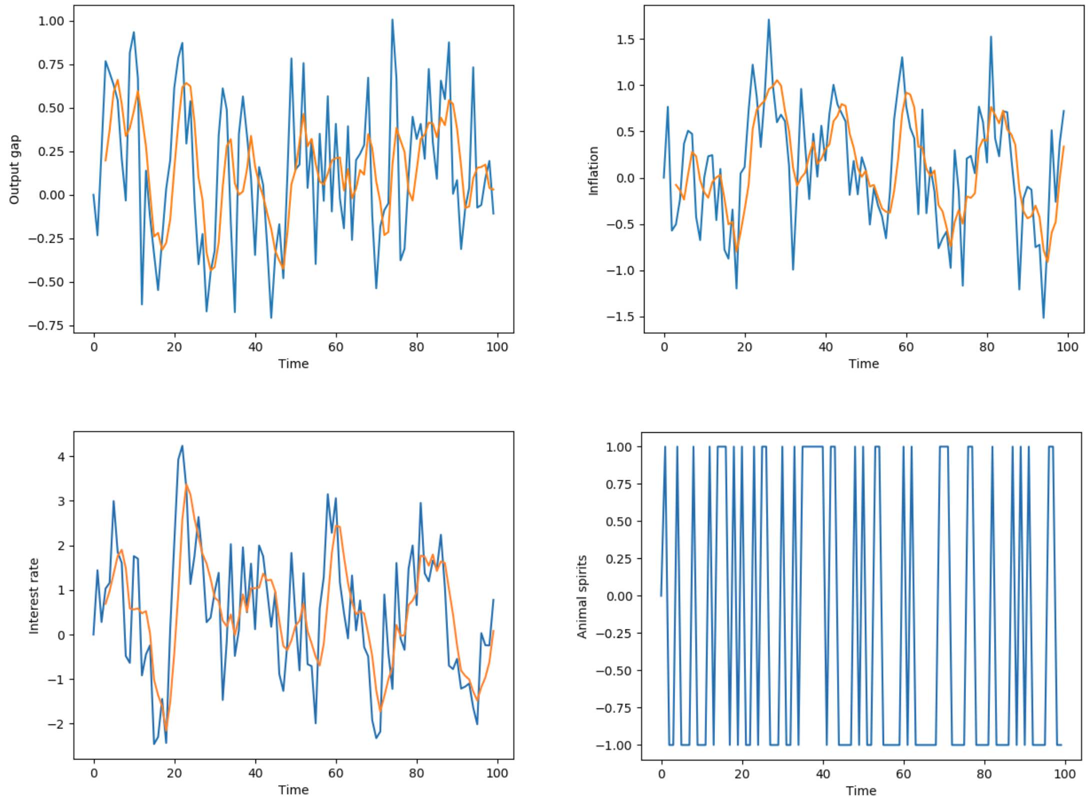

Figure 3.6: Time evolution of the macroeconomic values. The orange line is a moving mean.

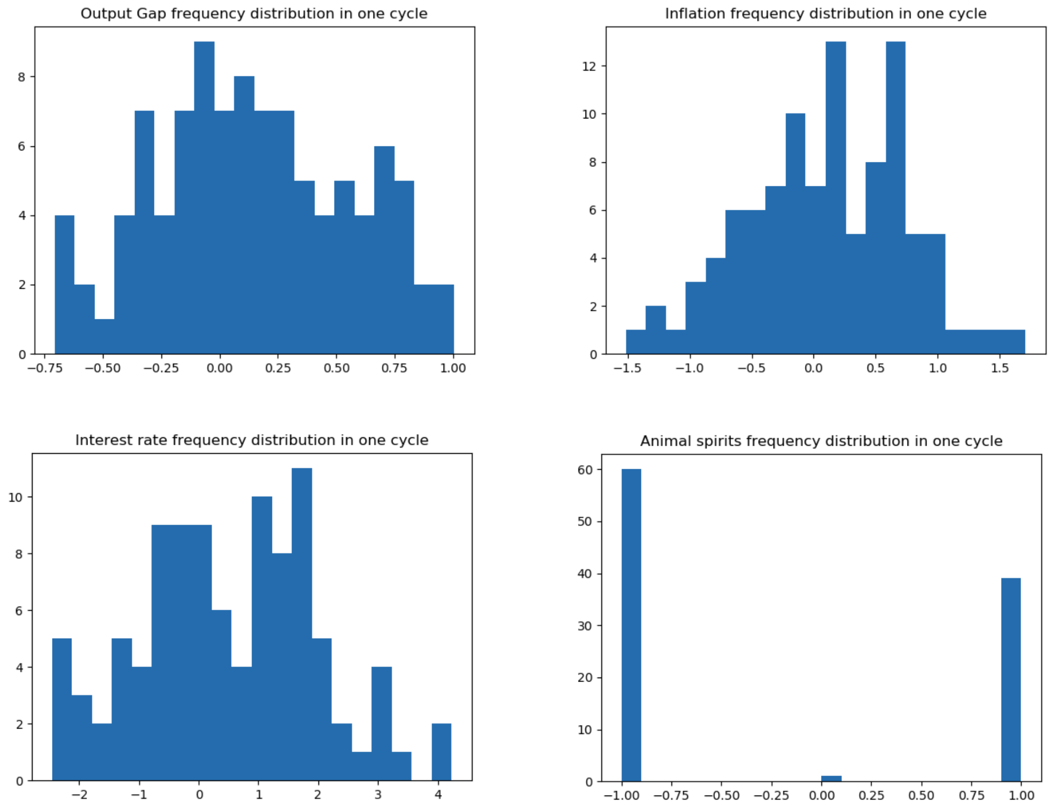

Figure 3.7: Frequency distribution of the macroeconomic values.

	Value
Standard deviation of the ouput gap	0,470
Standard deviation of the inflation	0,626
Standard deviation of animal spirits	0,989
Autocorrelation coeficient of the output gap	0,405
Autocorrelation coeficients of the output gap	0,460
Autocorrelation coeficient of the inflation	-13,804
Mean of the animal spirits	0,010
Mean of the output gap	0,009
Mean of the inflation	0,004
Kurtosis of the ouput gap	-0,039
Kurtosis of the inflation	-0,074
Standard deviation of the interest rate	1,337
Mean of interest rate	-0,003
Kurtosis of the interest rate	-0,087

Table 3.3: Averages of the statistical moments ($N = 100$) using the optimal monetary policy.

American Topology

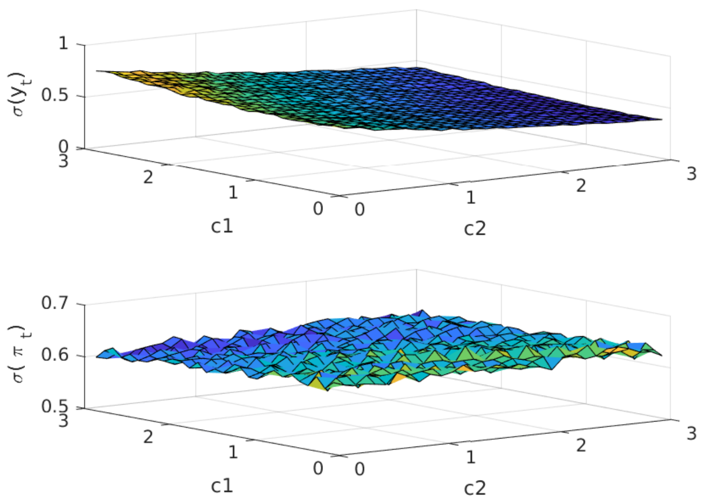

Figure 3.8: Finding the optimal monetary policy for the "american" topology with an Agent-based behavioral model.

Much like in the first case, by analyzing Figure 3.8, a good guess for the optimal monetary policy is: $c_1 = 2$ and $c_2 = 2$.

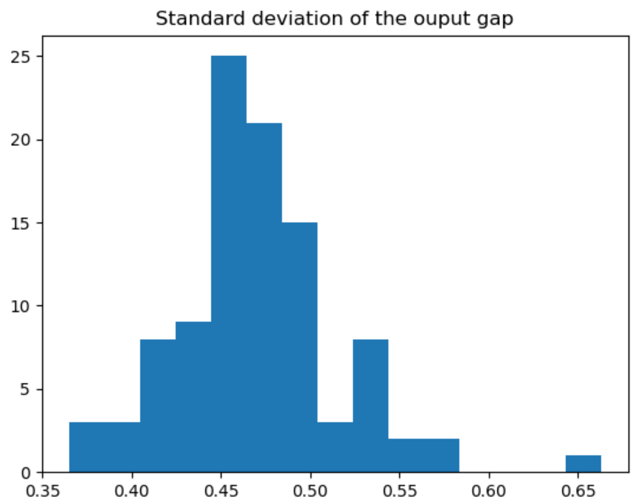

Figure 3.9: Distribution of the standard deviation of the ouput gap using the optimal monetary policy over 100 simulations.

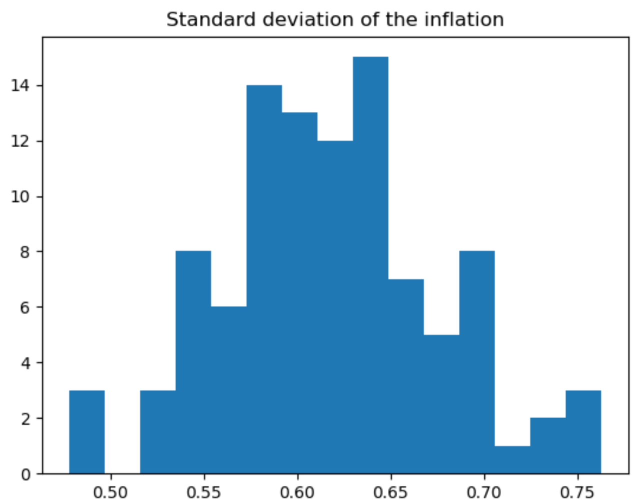

Figure 3.10: Distribution of the standard deviation of the inflation using the optimal monetary policy over 100 simulations.

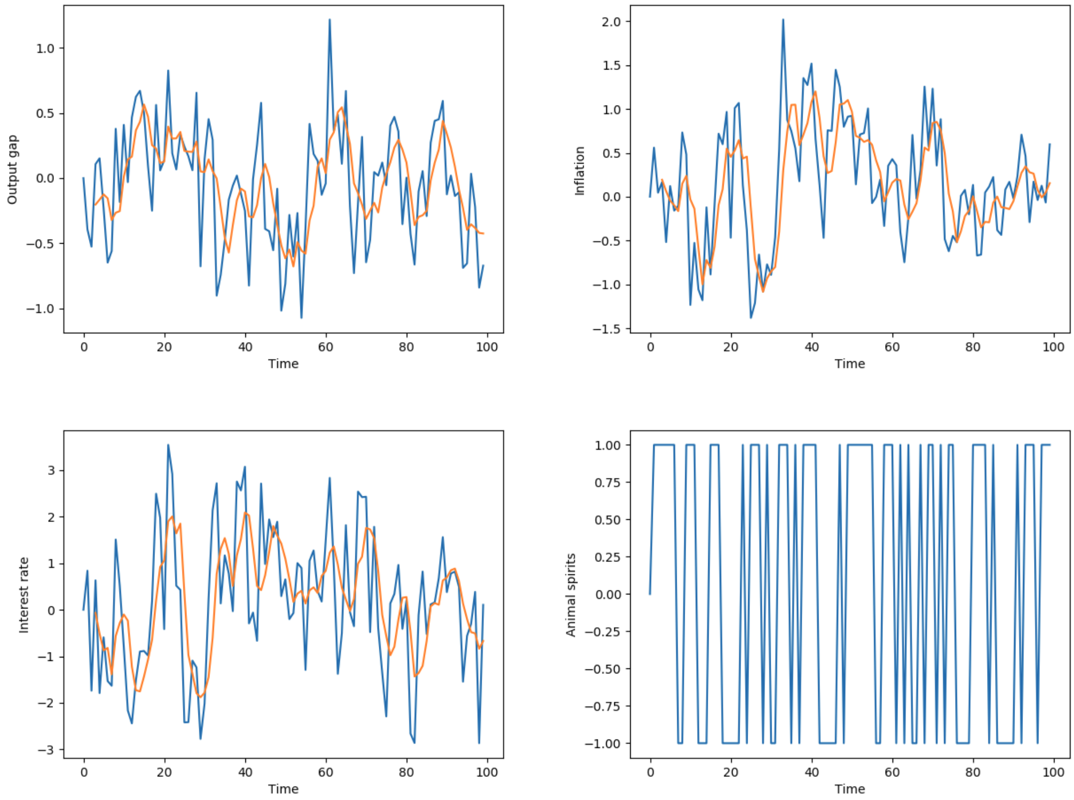

Figure 3.11: Time evolution of the macroeconomic values. The orange line is a moving mean.

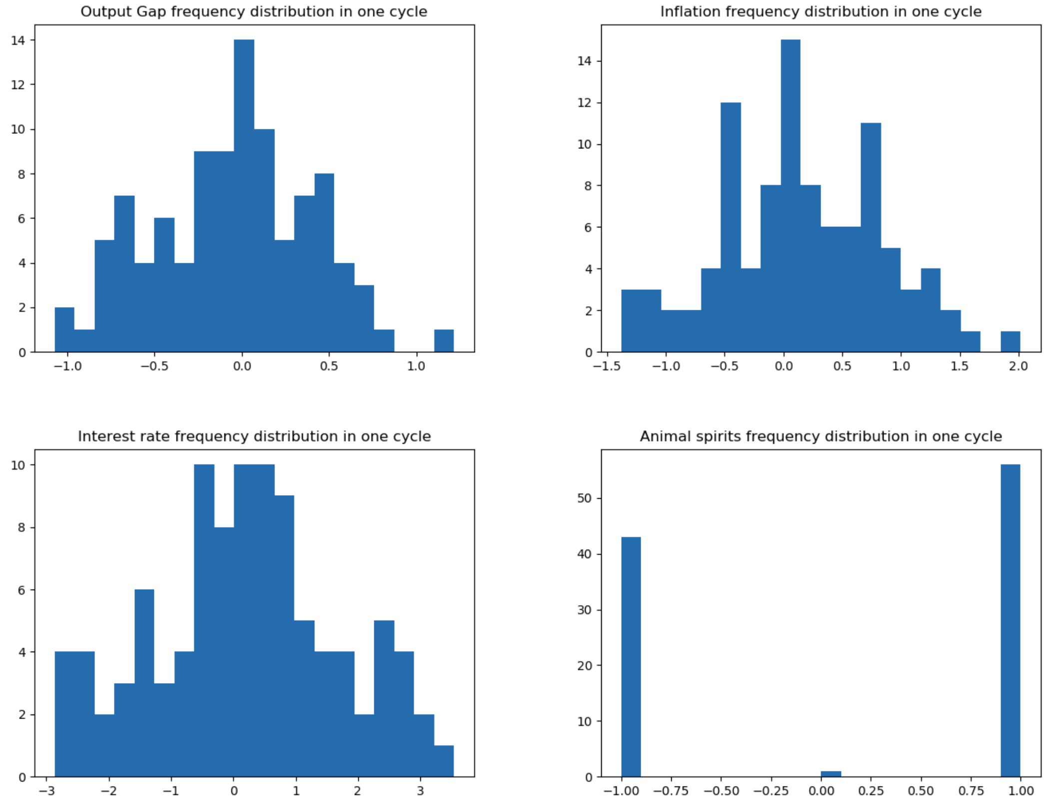

Figure 3.12: Frequency distribution of the macroeconomic values.

	Value
Standard deviation of the ouput gap	0,470
Standard deviation of the inflation	0,617
Standard deviation of animal spirits	0,989
Autocorrelation coeficient of the output gap	0,377
Autocorrelation coeficients of the output gap	0,467
Autocorrelation coeficient of the inflation	-13,992
Mean of the animal spirits	0,011
Mean of the output gap	-0,007
Mean of the inflation	0,014
Kurtosis of the ouput gap	-0,088
Kurtosis of the inflation	-0,086
Standard deviation of the interest rate	1,336
Mean of interest rate	0,005
Kurtosis of the interest rate	-0,126

Table 3.4: Averages of the statistical moments ($N = 100$) using the optimal monetary policy.

Ant-based behavioral model

As it was already said previously, we will be using three different types of interactions inside the Ising model to generate expectations. Much like in the previous section and due to the same reasons, the mechanism which generates the external magnetic field will have no memory (Table 3.5).

	Value
ρ_{BH}	0
ρ	0

Table 3.5: Ant-based model specific simulation conditions.

First interaction

The objective of this interaction is to provide the agent with some ability to remain neutral. If the Animal spirits are equal to zero, then the agent will have some reluctance to change. The specifications of this interaction are exposed in Table 3.6.

	Value	**Meaning**
β	0,6	Imitation rate
α	0,4	Self-interaction rate
ϵ	0,3	Magnetic/Out-conversion rate
γ	25	Temperature/Intensity of choice

Table 3.6: First ant-based model simulation conditions of the Ising interaction.

Circular Topology

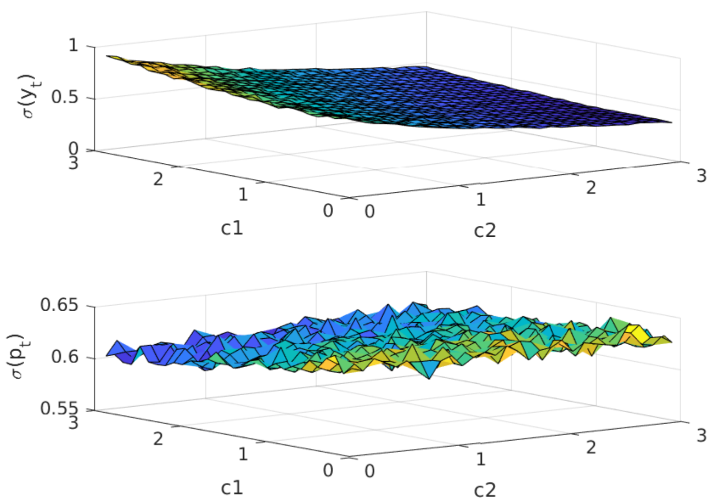

Figure 3.13: Finding the optimal monetary policy for the circular topology with an Ant-based behavioral model.

Similarly to the first case and due to the same reasons, one can conclude that the optimal monetary policy is $c_1 = 2$ and $c_2 = 2$, by analyzing Figure 3.13.

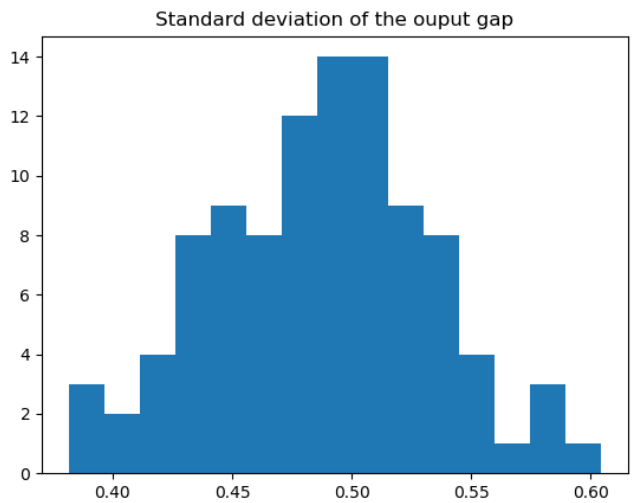

Figure 3.14: Distribution of the standard deviation of the ouput gap using the optimal monetary policy over 100 simulations.

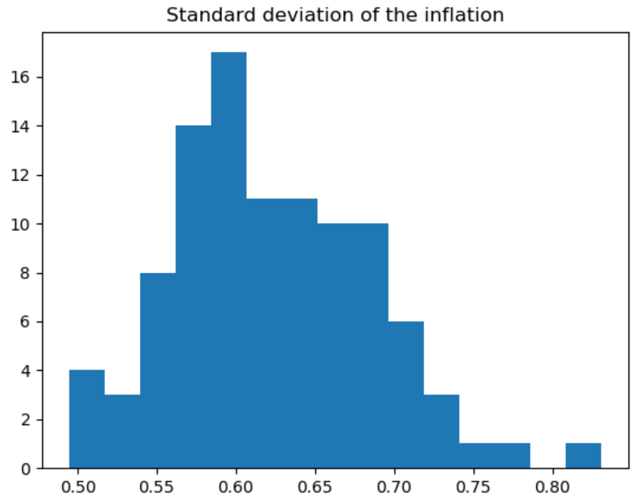

Figure 3.15: Distribution of the standard deviation of the inflation using the optimal monetary policy over 100 simulations.

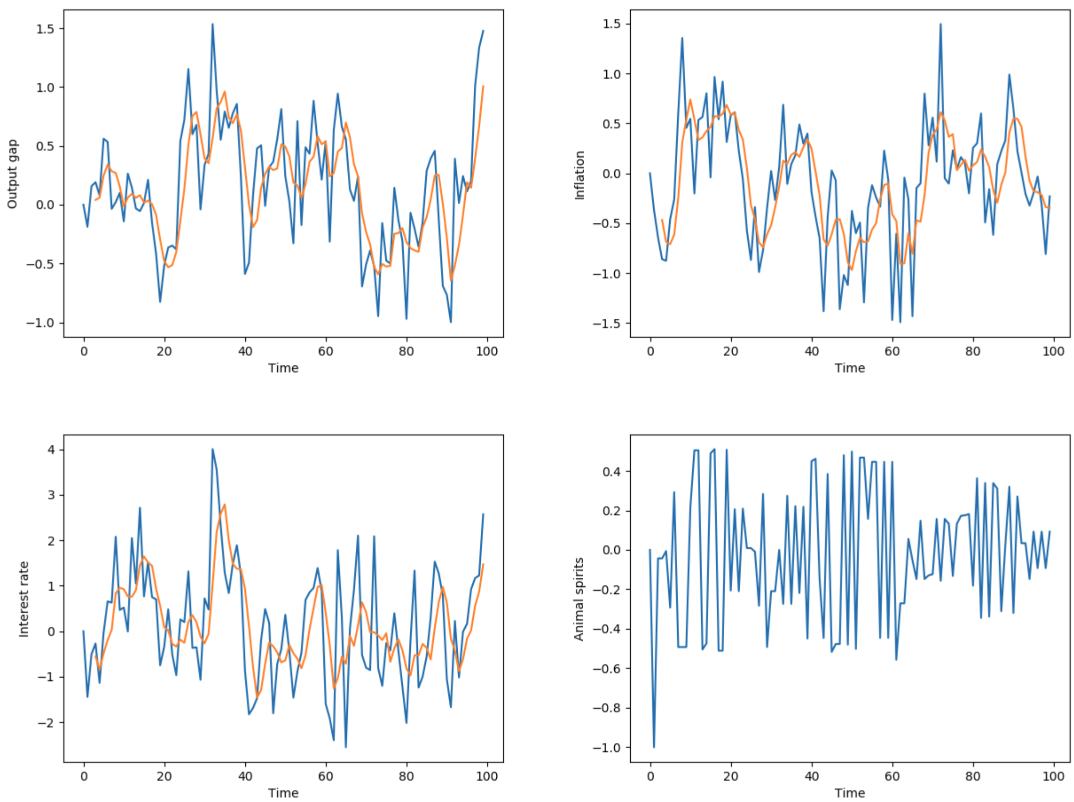

Figure 3.16: Time evolution of the macroeconomic values. The orange line is a moving mean.

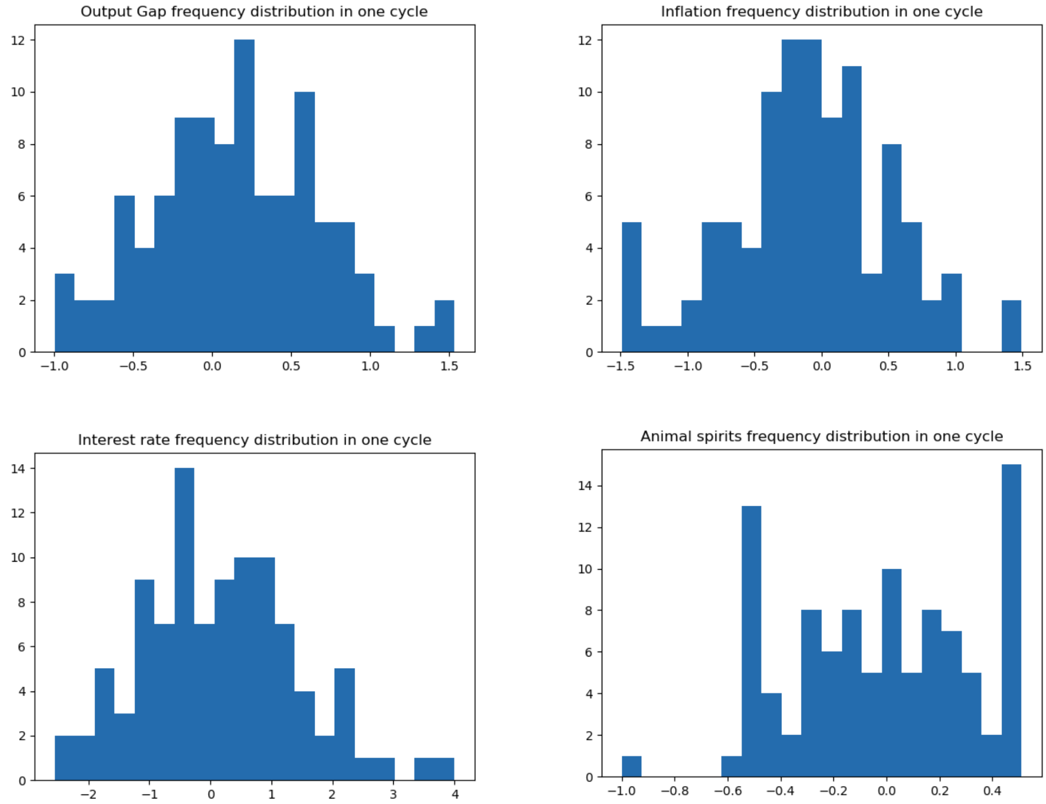

Figure 3.17: Frequency distribution of the macroeconomic values.

	Value
Standard deviation of the ouput gap	0,488
Standard deviation of the inflation	0,622
Standard deviation of animal spirits	0,300
Autocorrelation coeficient of the output gap	0,546
Autocorrelation coeficients of the output gap	0,465
Autocorrelation coeficient of the inflation	0,128
Mean of the animal spirits	-0,010
Mean of the output gap	-0,011
Mean of the inflation	0,005
Kurtosis of the ouput gap	-0,144
Kurtosis of the inflation	-0,082
Standard deviation of the interest rate	1,356
Mean of interest rate	0,048
Kurtosis of the interest rate	-0,064

Table 3.7: Averages of the statistical moments ($N = 100$) using the optimal monetary policy.

American Topology

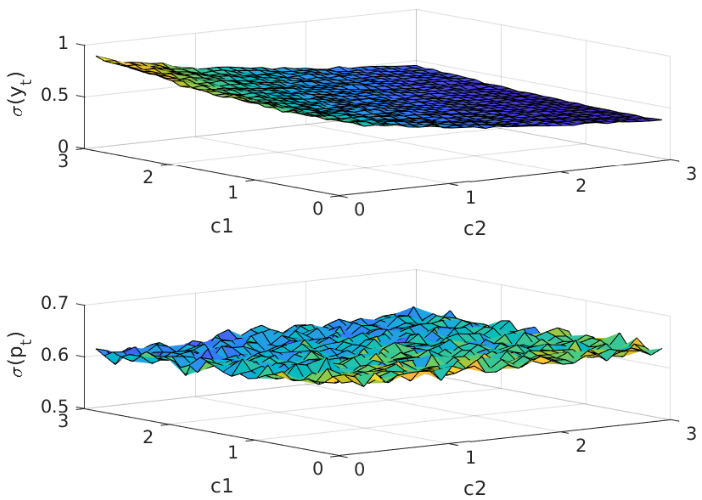

Figure 3.18: Finding the optimal monetary policy for the "american" topology with an Ant-based behavioral model.

For the same reasons stated for the previous cases, by examining Figure 3.18, one can say that a good guess for the optimal monetary policy will be: $c_1 = 2$ and $c_2 = 2$.

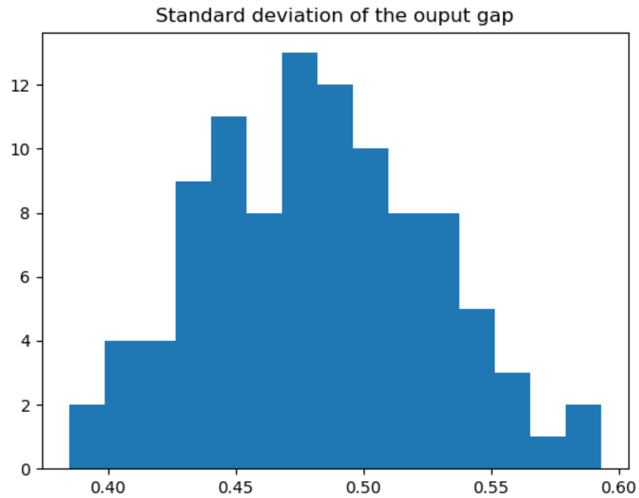

Figure 3.19: Distribution of the standard deviation of the ouput gap using the optimal monetary policy over 100 simulations.

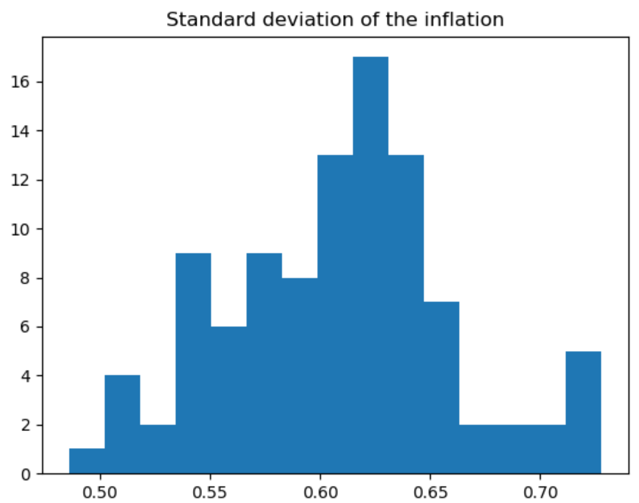

Figure 3.20: Distribution of the standard deviation of the inflation using the optimal monetary policy over 100 simulations.

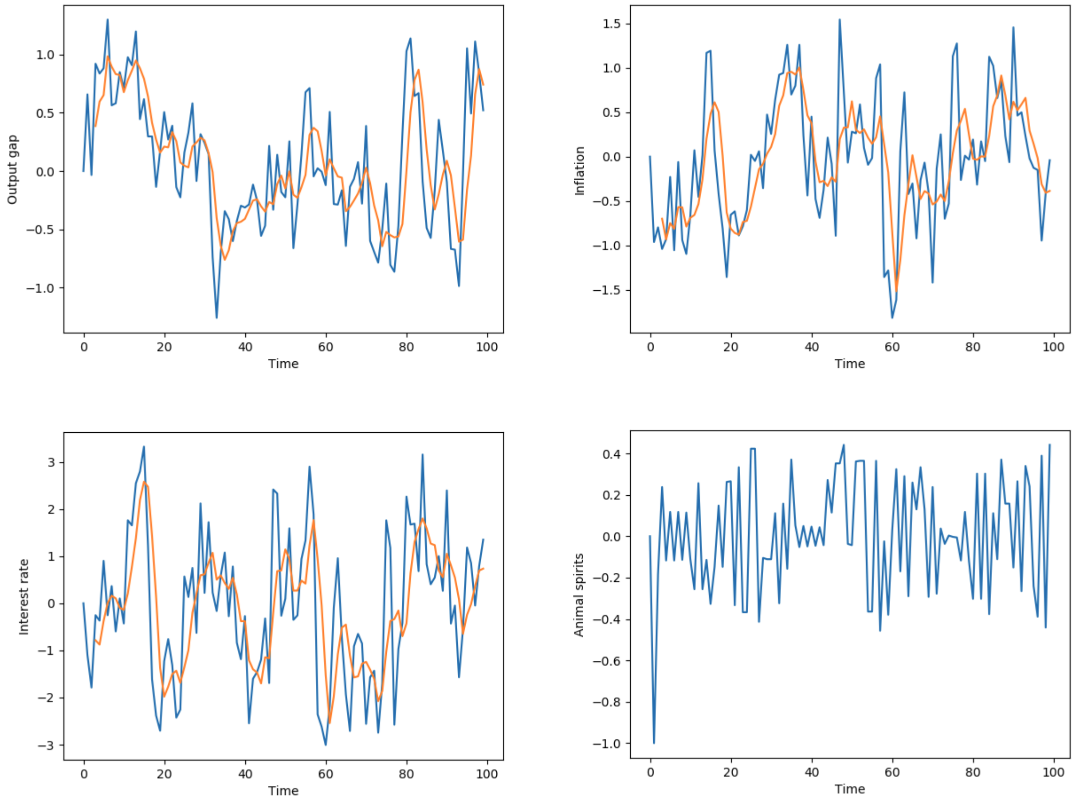

Figure 3.21: Time evolution of the macroeconomic values. The orange line is a moving mean.

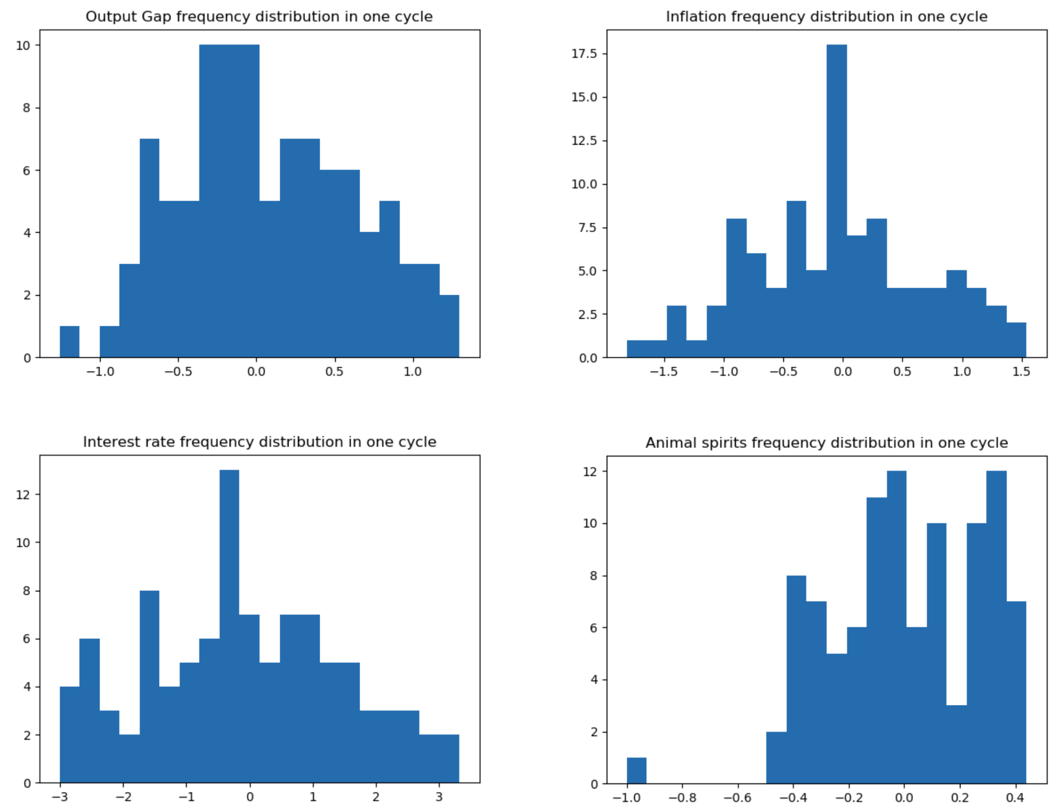

Figure 3.22: Frequency distribution of the macroeconomic values.

	Value
Standard deviation of the ouput gap	0,483
Standard deviation of the inflation	0,608
Standard deviation of animal spirits	0,295
Autocorrelation coeficient of the output gap	0,533
Autocorrelation coeficients of the output gap	0,455
Autocorrelation coeficient of the inflation	0,060
Mean of the animal spirits	-0,007
Mean of the output gap	-0,004
Mean of the inflation	0,013
Kurtosis of the ouput gap	-0,074
Kurtosis of the inflation	-0,009
Standard deviation of the interest rate	1,321
Mean of interest rate	0,012
Kurtosis of the interest rate	-0,103

Table 3.8: Averages of the statistical moments ($N = 100$) using the optimal monetary policy.

Second interaction

The objective of this interaction is to provide the agent with some ability to remain neutral. If the Animal spirits are equal to zero, then the agent will have some reluctance to change. The specifications of this interaction are exposed in Table 3.9.

	Value	Meaning
β	0,6	Imitation rate
α	0,15	Self-interaction rate
ϵ	1	Magnetic/Out-conversion rate
γ	75	Temperature/Intensity of choice

Table 3.9: Second ant-based model simulation conditions of the Ising interaction.

Circular Topology

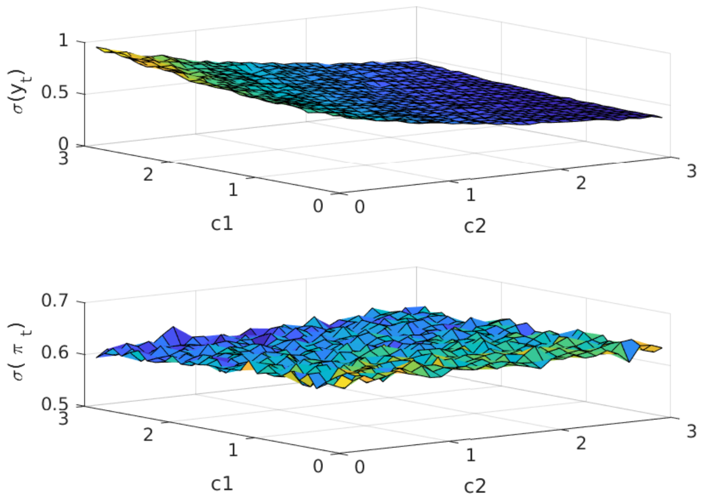

Figure 3.23: Finding the optimal monetary policy for the circular topology with an Ant-based behavioral model.

In similar fashion to previous cases, according to Figure 3.23, $c_1 = 2$ and $c_2 = 2$ are good guesses for the optimal monetary policy.

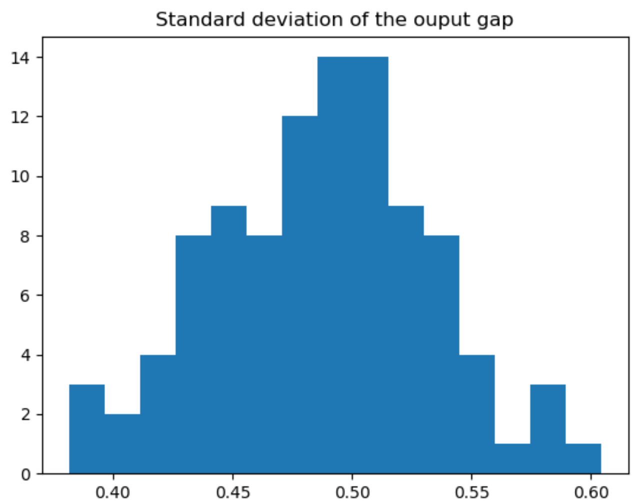

Figure 3.24: Distribution of the standard deviation of the ouput gap using the optimal monetary policy over 100 simulations.

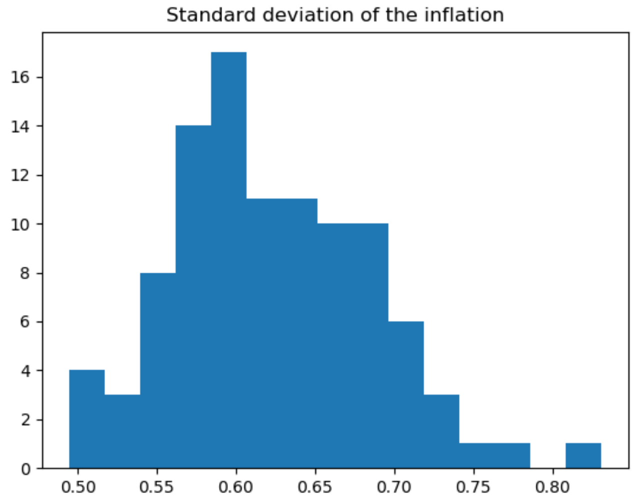

Figure 3.25: Distribution of the standard deviation of the inflation using the optimal monetary policy over 100 simulations.

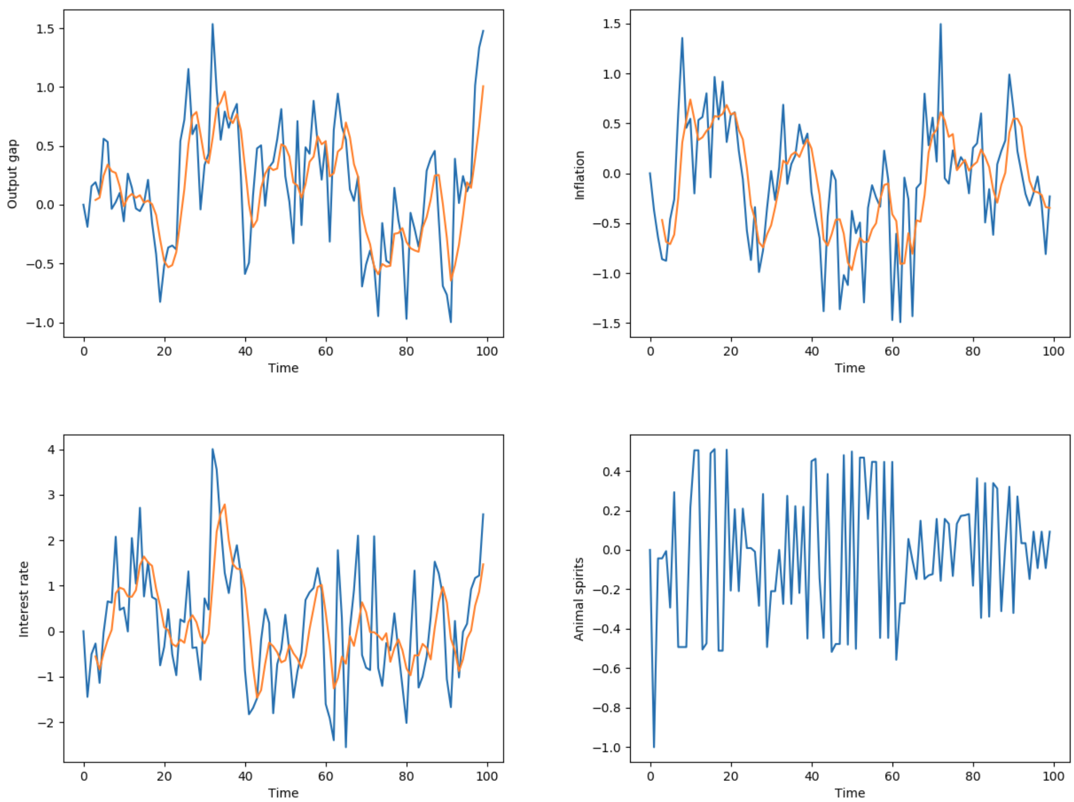

Figure 3.26: Time evolution of the macroeconomic values. The orange line is a moving mean.

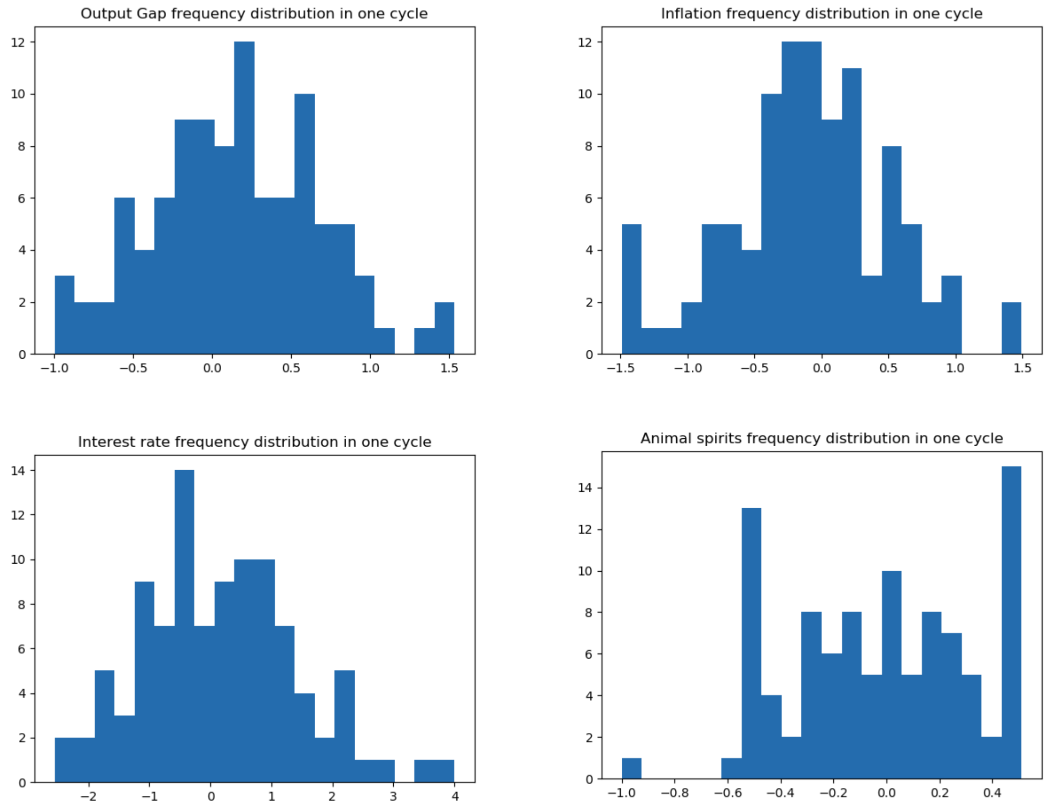

Figure 3.27: Frequency distribution of the macroeconomic values.

	Value
Standard deviation of the ouput gap	0,481
Standard deviation of the inflation	0,611
Standard deviation of animal spirits	0,947
Autocorrelation coeficient of the output gap	0,470
Autocorrelation coeficients of the output gap	0,464
Autocorrelation coeficient of the inflation	1,328
Mean of the animal spirits	0,002
Mean of the output gap	-0,007
Mean of the inflation	0,002
Kurtosis of the ouput gap	-0,042
Kurtosis of the inflation	0,024
Standard deviation of the interest rate	1,350
Mean of interest rate	0,031
Kurtosis of the interest rate	-0,004

Table 3.10: Averages of the statistical moments ($N = 100$) using the optimal monetary policy.

American Topology

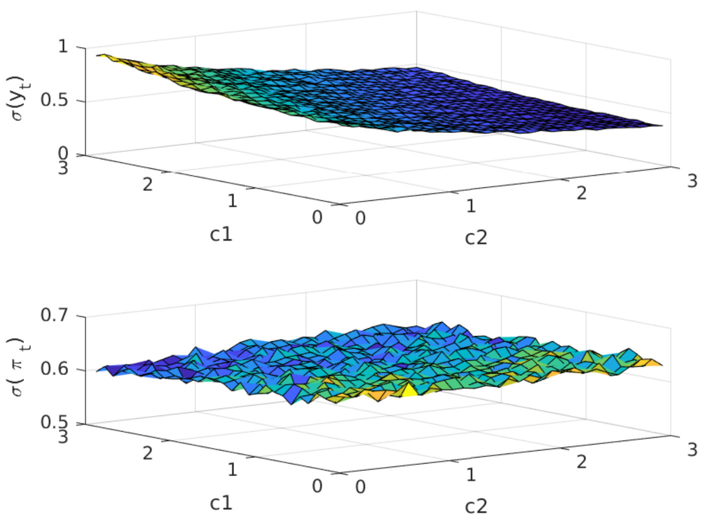

Figure 3.28: Finding the optimal monetary policy for the "american" topology with an Ant-based behavioral model.

Much like in the previous cases, according to Figure 3.28, $c_1 = 2$ and $c_2 = 2$ are good guesses for the optimal monetary policy.

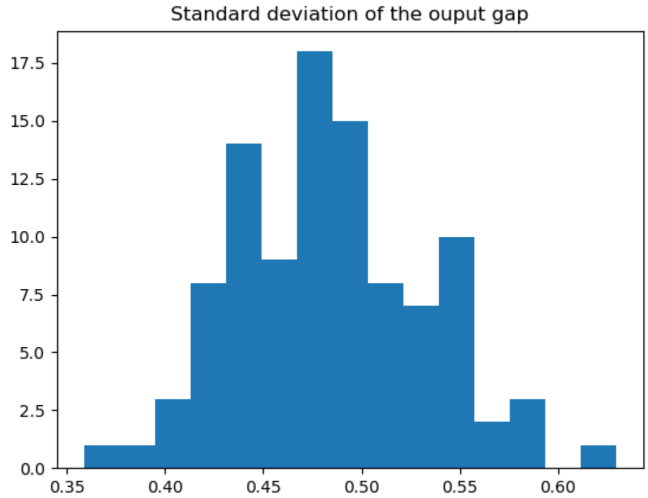

Figure 3.29: Distribution of the standard deviation of the ouput gap using the optimal monetary policy over 100 simulations.

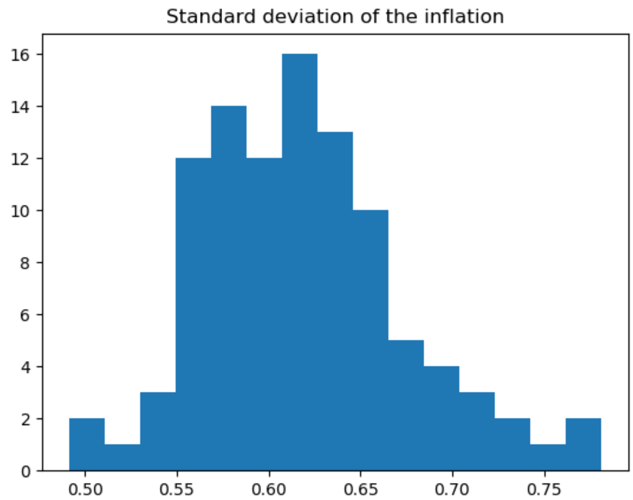

Figure 3.30: Distribution of the standard deviation of the inflation using the optimal monetary policy over 100 simulations.

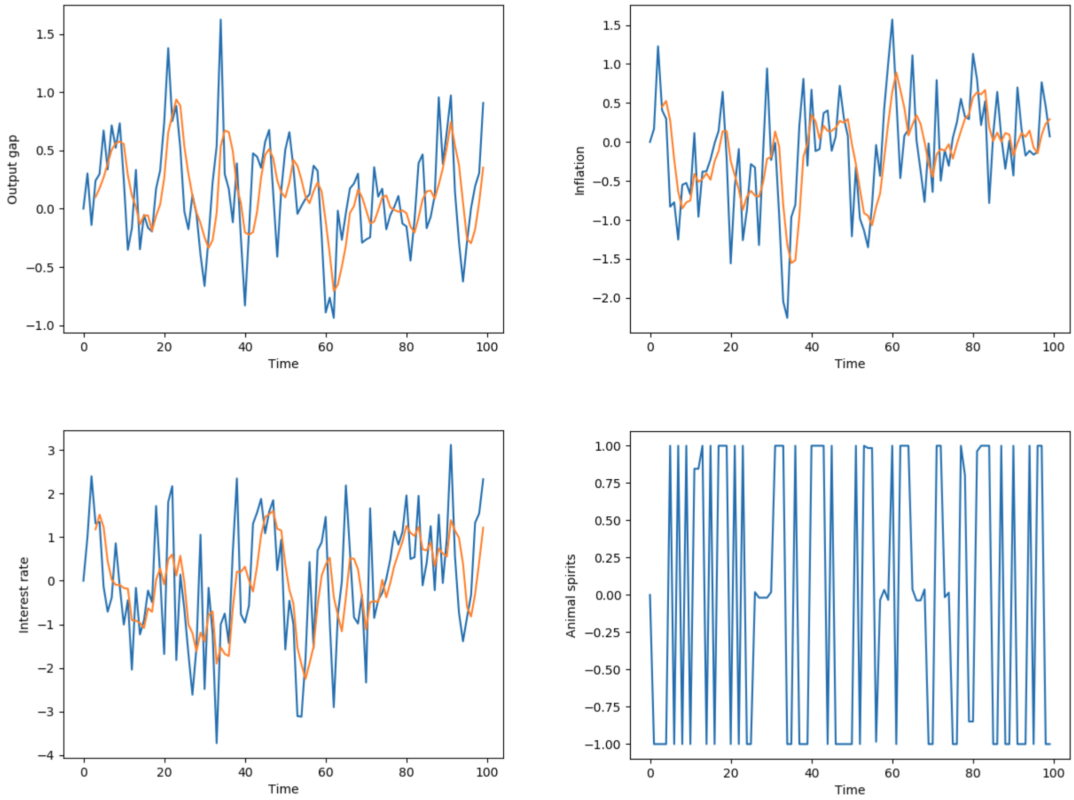

Figure 3.31: Time evolution of the macroeconomic values. The orange line is a moving mean.

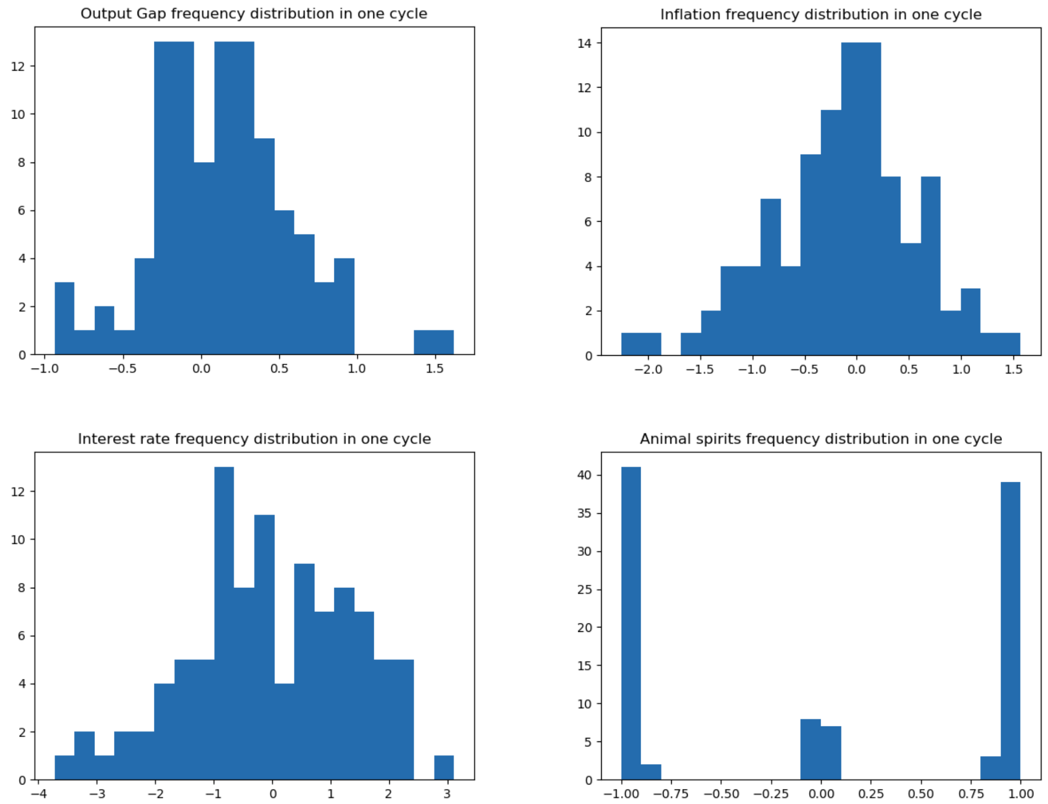

Figure 3.32: Frequency distribution of the macroeconomic values.

	Value
Standard deviation of the ouput gap	0,485
Standard deviation of the inflation	0,619
Standard deviation of animal spirits	0,935
Autocorrelation coeficient of the output gap	0,493
Autocorrelation coeficients of the output gap	0,464
Autocorrelation coeficient of the inflation	0,546
Mean of the animal spirits	-0,001
Mean of the output gap	0,000
Mean of the inflation	-0,020
Kurtosis of the ouput gap	-0,060
Kurtosis of the inflation	0,048
Standard deviation of the interest rate	1,333
Mean of interest rate	-0,006
Kurtosis of the interest rate	-0,032

Table 3.11: Averages of the statistical moments ($N = 100$) using the optimal monetary policy.

Third interaction

This last interaction disallows the agents to maintain null animal spirits but does not allow them to change them to either +1 or -1. It is similar to the previous interaction but on a smaller scale. The specifications of this model are exposed in Table 3.12.

	Value	Meaning
β	0,6	Imitation rate
α	0,15	Self-interaction rate
ϵ	0,1	Magnetic/Out-conversion rate
γ	10	Temperature/Intensity of choice

Table 3.12: Third ant-based model simulation conditions of the Ising interaction.

Circular Topology

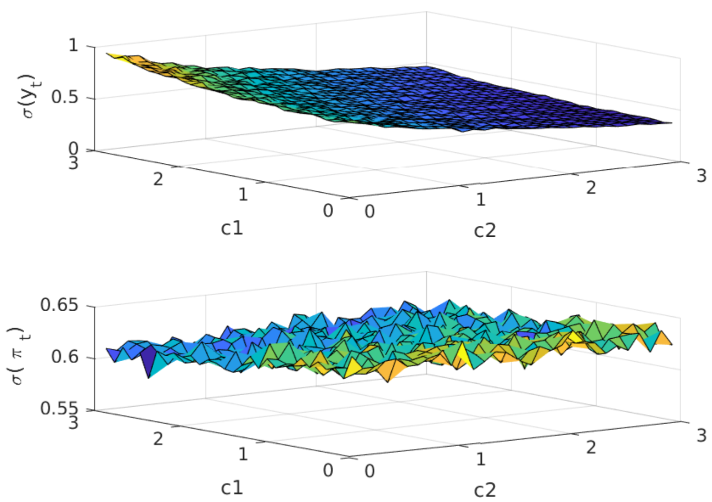

Figure 3.33: Finding the optimal monetary policy for the circular topology with an Ant-based behavioral model.

Looking at Figure 3.33 and with the arguments used previously, $c_1 = 2$ and $c_2 = 2$ are good guesses for the optimal monetary policy.

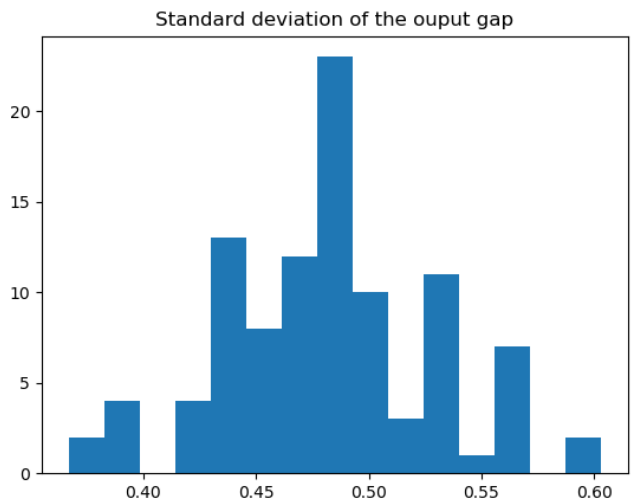

Figure 3.34: Distribution of the standard deviation of the ouput gap using the optimal monetary policy over 100 simulations.

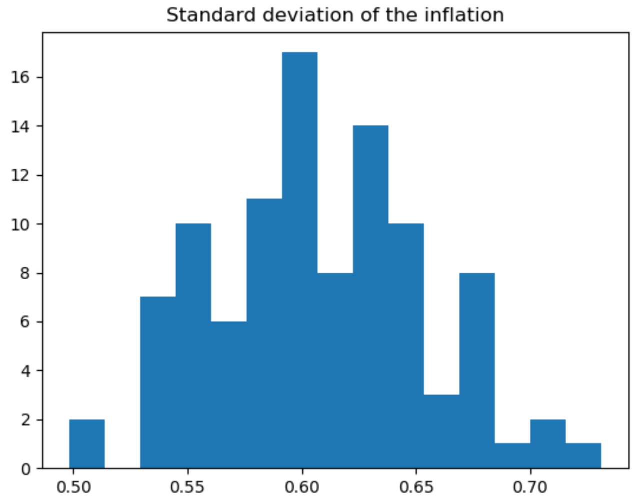

Figure 3.35: Distribution of the standard deviation of the inflation using the optimal monetary policy over 100 simulations.

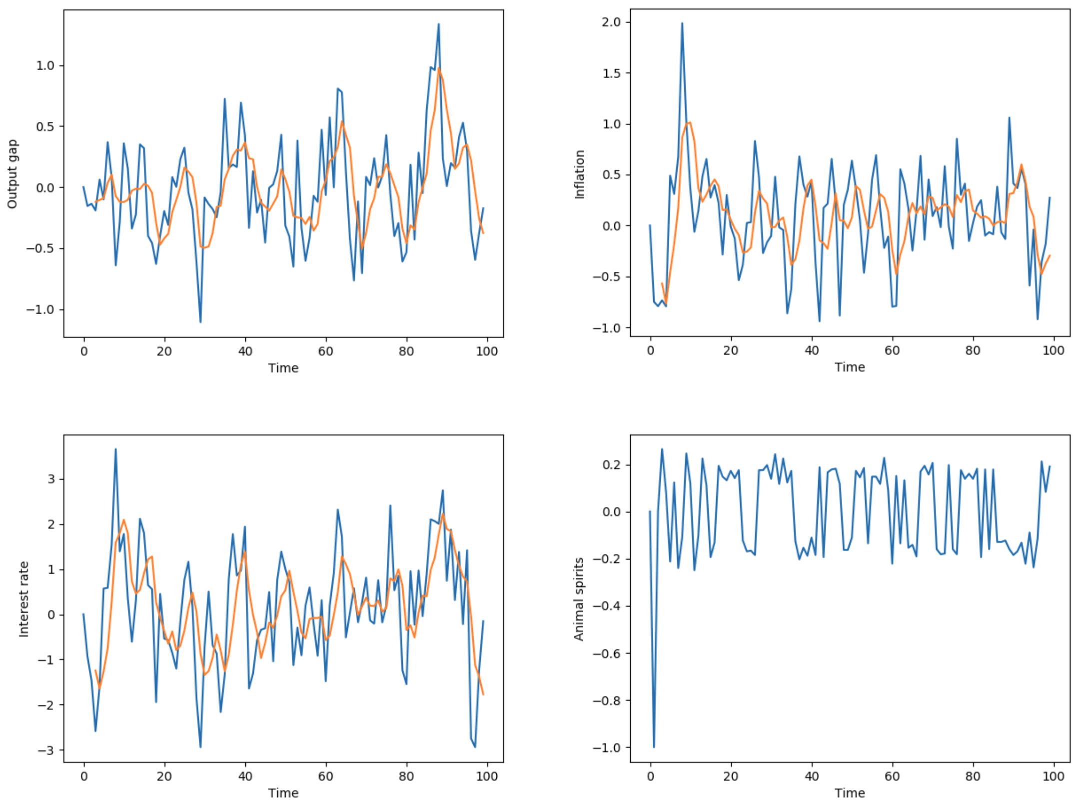

Figure 3.36: Time evolution of the macroeconomic values. The orange line is a moving mean.

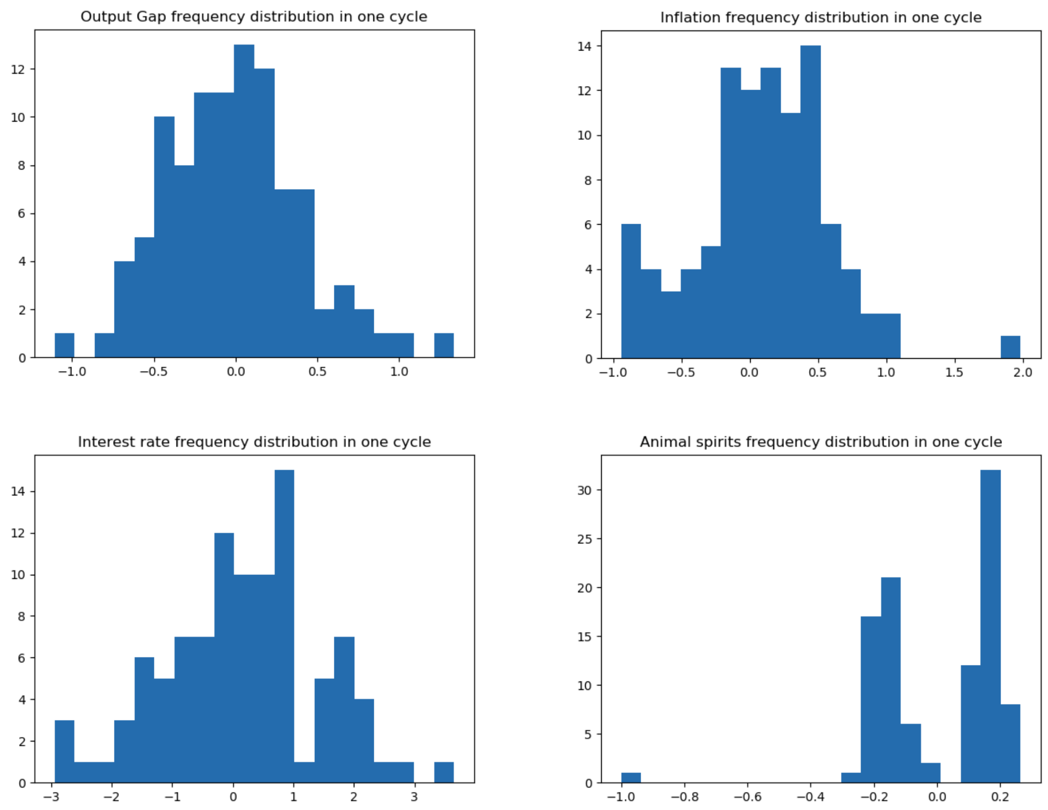

Figure 3.37: Frequency distribution of the macroeconomic values.

	Value
Standard deviation of the ouput gap	0,482
Standard deviation of the inflation	0,607
Standard deviation of animal spirits	0,193
Autocorrelation coeficient of the output gap	0,527
Autocorrelation coeficients of the output gap	0,444
Autocorrelation coeficient of the inflation	-3,489
Mean of the animal spirits	-0,012
Mean of the output gap	0,008
Mean of the inflation	-0,004
Kurtosis of the ouput gap	-0,099
Kurtosis of the inflation	-0,028
Standard deviation of the interest rate	1,359
Mean of interest rate	-0,001
Kurtosis of the interest rate	-0,105

Table 3.13: Averages of the statistical moments ($N = 100$) using the optimal monetary policy.

American Topology

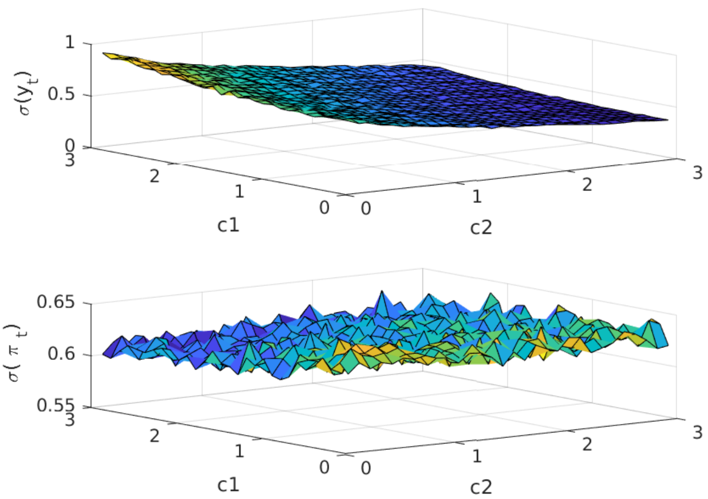

Figure 3.38: Finding the optimal monetary policy for the "american" topology with an Ant-based behavioral model.

Finally, like in all the above cases, the optimal monetary policy is $c_1 = 2$ and $c_2 = 2$.

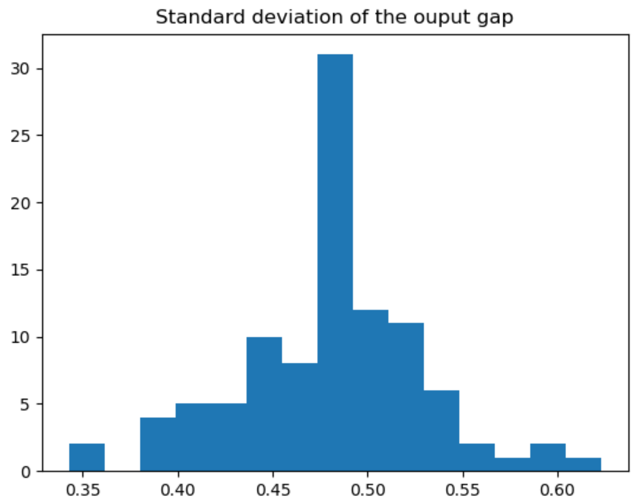

Figure 3.39: Distribution of the standard deviation of the ouput gap using the optimal monetary policy over 100 simulations.

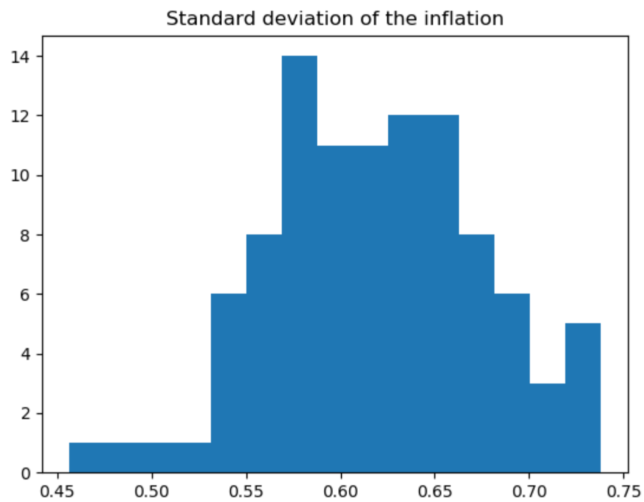

Figure 3.40: Distribution of the standard deviation of the inflation using the optimal monetary policy over 100 simulations.

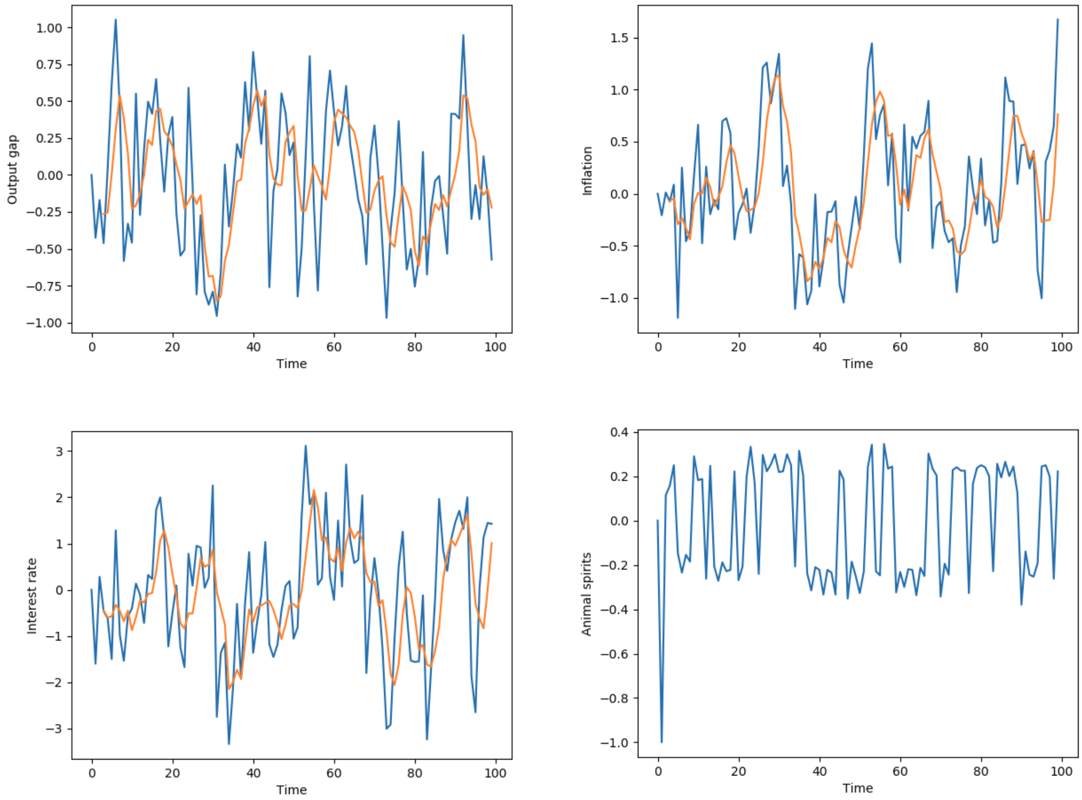

Figure 3.41: Time evolution of the macroeconomic values. The orange line is a moving mean.

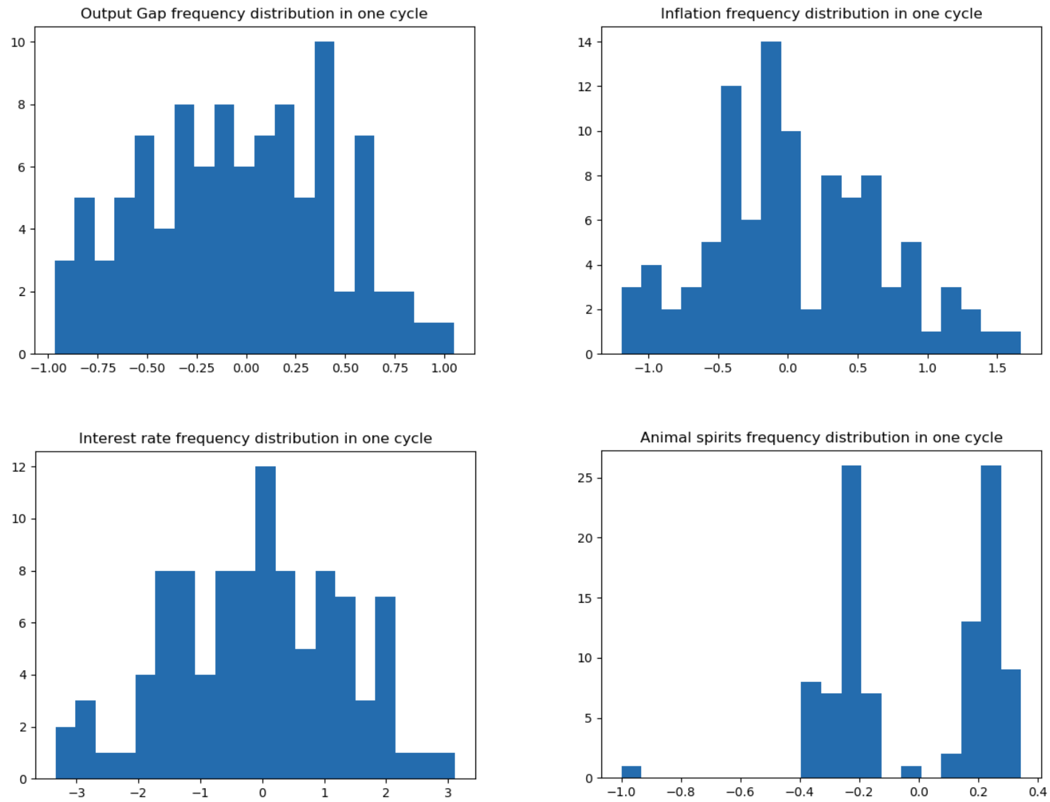

Figure 3.42: Frequency distribution of the macroeconomic values.

	Value
Standard deviation of the ouput gap	0,480
Standard deviation of the inflation	0,619
Standard deviation of animal spirits	0,261
Autocorrelation coeficient of the output gap	0,519
Autocorrelation coeficients of the output gap	0,457
Autocorrelation coeficient of the inflation	-4,992
Mean of the animal spirits	-0,013
Mean of the output gap	0,005
Mean of the inflation	-0,014
Kurtosis of the ouput gap	-0,095
Kurtosis of the inflation	-0,091
Standard deviation of the interest rate	1,331
Mean of interest rate	0,003
Kurtosis of the interest rate	-0,104

Table 3.14: Averages of the statistical moments ($N = 100$) using the optimal monetary policy.

4

Conclusions

Due to the nature of this thesis, one can not make the mistake of making solely quantitative or qualitative conclusions. They are also important, but the most relevant information comes from comparing the different cases of the model within the simulations.

Firstly, when comparing Figures 2.4 and 2.5 with Figures 3.6, 3.11, 3.16, 3.21, 3.26, 3.31, 3.36 and 3.41, the simulated data is a good approximation of the dynamics of real business and economic cycles. This is not quantitative at all, but it is important to identify these patterns when building a macroeconomic model.

σ_y	σ_π
1,909	1,055

Table 4.1: Standard deviation of American ouput gap (y) and inflation (π) between 1990-2018. Data collected from: `https://fred.stlouisfed.org/`

$\sigma_{y,DG,Circular}$	0,47
$\sigma_{\pi,DG,Circular}$	0,626
$\sigma_{y,1,Circular}$	0,488
$\sigma_{\pi,1,Circular}$	0,622
$\sigma_{y,2,Circular}$	0,481
$\sigma_{\pi,2,Circular}$	0,611
$\sigma_{y,3,Circular}$	0,482
$\sigma_{\pi,3,Circular}$	0,607
$\sigma_{y,DG,America}$	0,483
$\sigma_{\pi,DG,America}$	0,617
$\sigma_{y,1,America}$	0,483
$\sigma_{\pi,1,America}$	0,608
$\sigma_{y,2,America}$	0,486
$\sigma_{\pi,2,America}$	0,619
$\sigma_{y,3,America}$	0,48
$\sigma_{\pi,3,America}$	0,619

Table 4.2: Standard deviation of output gap and inflation. The first index relates to output gap (y) and inflation (π). The second index is related to the type of interaction (DG - Agent-based simulation; 1 - Ising1, etc). The third index is the topology used in the simulation.

Secondly, comparing the values from Table 4.1 with the values from Table 4.2, it is possible to visualize that there is a significant difference in the values of standard deviation. However, all of them are still in the same order of magnitude. By increasing the values of the external shocks in Equations 2.1, 2.2 and 2.3, it would be possible to replicate the statistical moments of real data. It would also be interesting to verify if by increasing the probability of innovation in the models described in Sections 2.4 and 2.5 this would also happen.

Thirdly, in a fully credible inflation targeting scenario, one can achieve better results for the stabilization of output gap than the actual inflation. Also, the optimal monetary policy found in the simulations is never strict inflation targeting, since c_2 on the Taylor Rule (Equation 2.3) is always $\neq 0$.

Furthermore, by analyzing Table 4.2 one can see that there is a slight increase in both statistical moments when simulating the American topology. This makes sense since there is not a clear path for expectations and innovations to diffuse. Nonetheless, the difference is not very significant. With this result, one can deduce that our system is not very susceptible to a change in topology.

Also by analyzing Table 4.2, one can conclude that our system is more sensitive to changes in topology than to changes in interactions. I was expecting the opposite when I initiated this work.

Moreover, one of the most important characteristics of the Ant-Based Behavioral model (the model described in Section 2.5), besides its simplicity, is its computational speed, especially when compared to more conventional agent-based macroeconomic models (the model described in Section 2.4).

Lastly, and what in my opinion is the most important conclusion, whatever the topology or the interaction used in a simulation is, the central bank is still flexible enough when it is trying to find an optimal monetary policy. This means that it is possible to find a monetary policy which suits almost-perfectly any case.

For future works, I have listed some interesting possibilities which occurred to me during this last year but ended up not being developed for the purpose of this thesis.

It would be very interesting to test with a totally random network, as it would make it very difficult for expectations to spread. This would be a more extreme scenario of the American topology.

I would also like to know what would happen if we were not working in a strict inflation targeting scenario. Not that I find strict inflation targeting a farfetched idea, but I would be interested to know how would it affect the optimal monetary policy when using different topologies and interactions. It would also interesting to use different types of interactions for the expectation of output gap and inflation.

Lastly, in Equation 2.20 I used the $\alpha_{e,t}$ as an external magnetic field, but solely for mathematical reasons. It solved my problem and made sense in the model, but I did not fully understand the mechanism. For that reason, I opted for a memoryless system. Therefore, I would also be interested to know if by adding memory to this mechanism the results obtained would be different.

I really enjoyed working on this thesis because it allowed me to study and learn about an area of knowledge totally different from what I am used to.

Most importantly, the implementation and results proved to be quite successful, using a mixed approach to physics and macroeconomics.

5

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Appendices

A

First appendix

All the codes developped and used in this piece of work are stored in `https://github.com/mconde94/Codigos-Tese`.