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# Relativistic rotation — how does the energy vary with angular momentum?

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**Abstract.** A theorem in relativistic mechanics, establishing the relation between the variation of the rotational energy with the angular momentum variation, is derived. The theorem has a counterpart for relativistic translations. On the other hand, it also has a counterpart in non-relativistic rotational mechanics. Interestingly enough, to the best of our knowledge, that relation has never been presented in articles or textbooks. We discuss the relativistic rotation of a ring to illustrate how rotational dynamics works and to show the usefulness of the above mentioned theorem.

## 1. Introduction

Relativistic rotations are not usually addressed in conventional special relativity textbooks. However, the inertia and the moment of inertia of rotating bodies, whose parts experience velocities which are close to the speed of light, and the relativistic rotation equation itself, have merited the interest of a few authors [1, 2]. This attention is due to the qualitative differences in the comparison with the classical description of rotating bodies [3].

This work is about relativistic rotations around a principal axis of a rigid body (which remains rigid, even when it rotates at extremely high angular velocities). In the framework of the appropriate formalism applicable to the relativistic rotation, we prove a remarkable theorem which establishes the relationship between the variation of the rotational kinetic energy with the variation of the angular momentum. We study a rotating ring submitted to a constant torque, resulting from a binary of constant forces, to illustrate how the formalism works.

The rotational kinetic energy is the kinetic energy of a system as measured in the reference frame in which its linear momentum does vanish. Thus, it is part of the internal energy of the system. Let us denote by $E_0$ the energy of a zero momentum solid body in the absence of a rotation. This energy is internal energy too. Hence, when the rigid body rotates with angular velocity $\omega$, its internal energy in the pure rotational state, $E(\omega)$, is bound to be larger than $E_0$. Therefore, according to the Einstein's principle of the inertia of the energy [4], the inertia of a rotating body $\mathcal{M}(\omega) = E(\omega)\, c^{-2} > E_0\, c^{-2}$ depends on its angular velocity, $\omega$, and, surely, also depends on other magnitudes, such as its chemical composition, its temperature, etc.

By the same reasoning, the relativistic moment of inertia of a rotating rigid body, $I(\omega)$, is angular velocity dependent. This is not the case in classical mechanics, where the moment of inertia of a rigid body, $I_0$, is constant. The bottom line is that the body's moment of inertia and inertia depend on its angular velocity. For example, the inertia of a rotating object such as a compact star, a black-hole, a galaxy, etc., is higher than the inertia of the very same non-rotating object, this being a pure relativistic effect.

The present work on relativistic rotations around a principal axis resulted from a fruitful collaboration between physicists and mathematicians. Its main result was recently published as a letter [5].

## 2. The relativistic rotation

A problem most commonly pointed out in the context of rotations in special relativity is the non-existence of rigid bodies in rotation when some of its parts have linear velocities of the order of magnitude of the speed of light. However, here we assume that the rotating solid body is robust enough so that its distortion and strain response to stress, caused by the stationary revolving state, is negligible. Our study is a merely theoretical one and, so to say, we are not concerned with practical aspects of a real situation.

Therefore, let us consider a rigid solid body (a ring, a disc, a bar, etc.) whose volume we denote by $\Omega$. For the sake of simplicity, we assume that the body is homogeneous, i.e. its density (at rest), $\rho$, is constant. The body rotates around a principal axis with angular velocity $\omega$, and, differently from classical mechanics, now the inertia of the body is angular velocity dependent, $\mathcal{M}(\omega)$, and it is defined by [2]

$$\mathcal{M}(\omega) = \rho \int_\Omega \gamma\left(\omega\, r(x)\right) \mathrm{d}x\,. \tag{1}$$

Here, $r(x)$ is the distance from point $x$ to the rotation axis and

$$\gamma(v) = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{2}$$

is the usual relativistic factor, $c$ being the speed of light. Equation (1) clearly states a major different with respect to the classical situation.

Similarly, the moment of inertia is also angular velocity dependent, $I(\omega)$, and it is given by [2]

$$I(\omega) = \rho \int_\Omega r^2(x)\gamma\left(\omega r(x)\right) \mathrm{d}x\,. \tag{3}$$

The classical inertia, $\mathcal{M}(0)$, and the moment of inertia, $I(0)$, are simply given by

$$\mathcal{M}(0) = \int_\Omega \rho \mathrm{d}x, \quad I(0) = \rho \int_\Omega r^2(x) \mathrm{d}x\,. \tag{4}$$

We shall assume that $\mathcal{M}(0)$ and $I(0)$ are fixed quantities.

### 2.1. *The theorem*

The variation of the magnitude $\mathcal{M}$ with the quantity $I\omega$, in the course of a relativistic rotation, is given by

$$\frac{\mathrm{d}\left[\mathcal{M}(\omega)c^2\right]}{\mathrm{d}[I(\omega)\omega]} = \omega\,. \tag{5}$$

Physically, this means that the variation of the energy, $E = \mathcal{M}(\omega)c^2$, with the angular momentum, defined as $J = I(\omega)\omega$, is equal to the angular velocity, $\omega$.

### *2.2. Proof*

Firstly, it is important to note that the relativistic factor, $\gamma(v)$, given by equation (2), as well as its derivative, $\gamma'(v) = \frac{\mathrm{d}\gamma(v)}{\mathrm{d}v}$, are strictly increasing functions. Therefore, the angular velocity dependent functions (1) and (3) (the latter firstly multiplied by $\omega$) are differentiable under the integral sign. The differentiations with respect to $\omega$ lead, respectively, to

$$\frac{\mathrm{d}\left[\mathcal{M}(\omega)c^2\right]}{\mathrm{d}\omega} = c^2 \int_\Omega \rho\gamma'\left(\omega r(x)\right) r(x)\mathrm{d}x, \tag{6}$$

and

$$\frac{\mathrm{d}\left[I(\omega)\omega\right]}{\mathrm{d}\omega} = \omega \int_\Omega \rho\gamma'\left(\omega r(x)\right) r^3(x)\mathrm{d}x + \int_\Omega \rho r^2(x)\gamma\left(\omega r(x)\right)\mathrm{d}x\,. \tag{7}$$

Now we use the following identity,

$$\gamma'(v) = \frac{v}{c^2 - v^2}\,\gamma(v), \quad (0 \leq v < c) \tag{8}$$

to obtain: from (6),

$$\frac{\mathrm{d}\left[\mathcal{M}(\omega)c^2\right]}{\mathrm{d}\omega} = \omega \int_\Omega \rho\gamma\left(\omega r(x)\right) \frac{c^2 r^2(x)}{c^2 - \omega^2 r^2(x)}\mathrm{d}x\,; \tag{9}$$

and, from (7),

$$\begin{aligned} \frac{\mathrm{d}\left[I(\omega)\omega\right]}{\mathrm{d}\omega} &= \int_\Omega \rho\gamma\left(\omega r(x)\right) \frac{\omega^2 r^4(x)}{c^2 - \omega^2 r^2(x)}\mathrm{d}x + \int_\Omega \rho r^2(x)\gamma\left(\omega r(x)\right)\mathrm{d}x \\ &= \int_\Omega \rho\gamma\left(\omega r(x)\right) \frac{c^2 r^2(x)}{c^2 - \omega^2 r^2(x)}\mathrm{d}x. \end{aligned} \tag{10}$$

Finally, after termwise dividing equations (9) and (10), one arrives at equation (5), which completes the proof. This mathematical proof, based on the identity (8), is general for $\mathcal{M}(\omega)$ and $I(\omega)$ as defined by (1) and (3), respectively, with fixed $\mathcal{M}(0)$ and $I(0)$ — see equation (4).

Equation (5) can also be expressed in a physically more meaningful way as the relation between the relativistic rotational energy variation and the relativistic angular momentum variation, in the course of a rotation

$$\frac{\mathrm{d}K_{\mathrm{rot}}}{\mathrm{d}J} = \omega\,, \tag{11}$$

just as in non-relativistic physics, where $K_{\mathrm{rot}} = E(\omega) - E_0$.

It is worth noting that there is a counterpart equation for (5) in the context of a pure translation [6]. That counterpart equation is $\mathrm{d}[\gamma(v)c^2] = \vec{v} \cdot \mathrm{d}[\gamma(v)\vec{v}]$, relating the energy variation with and the variation of the linear momentum of a system with constant inertia. Here $\vec{v} = (v_x, v_y, v_z)$ denotes the velocity of the system moving as a whole in a certain reference frame. In other words, this is the velocity of the reference frame in which the linear momentum of the system vanishes. This moving inertial reference frame is the relativistic equivalent of the centre-of-mass reference frame in classical mechanics.

### *2.3. Classical limit*

Let us verify whether our result properly fulfils the classical limit.

In the limit $c \to \infty$, the relativistic rotational kinetic energy, $K_{\mathrm{rot}}$, gives rise to the corresponding classical rotational kinetic energy. In the limit

$$\lim_{c\to\infty} K_{\mathrm{rot}} = \lim_{c\to\infty} \frac{1}{(1/c^2)}\left[\int_\Omega \rho\gamma\left(\omega r(x)\right)\mathrm{d}x - \int_\Omega \rho\mathrm{d}x\right], \tag{12}$$

the L'Hôpital rule can be applied and, after differentiation with respect to $c$, one obtains

$$\lim_{c\to\infty} K_{\rm rot} = \lim_{c\to\infty} \frac{1}{2}\int_\Omega \frac{\omega^2 r^2(x)\rho \mathrm{d}x}{\sqrt{[1-\omega^2 r^2(x)/c^2]^3}}\,. \tag{13}$$

The denominator reduces to 1 and, using the second definition in (4), one obtains $\lim_{c\to\infty} K_{\rm rot} = \frac{1}{2}I(0)\omega^2$ , which is indeed the correct classical limit. Classically, equation (11) is expressed by $\mathrm{d}K_{\rm rot}/\mathrm{d}(I\omega) = \omega$, where $I = I(0)$ is the classical (constant) moment of inertia.

## 3. The equations for the rotational process

The total energy of a body, moving as a whole with linear velocity $v$ and simultaneously rotating with angular velocity $\omega$, is [4]

$$E(\omega, v) = \gamma(v)\mathcal{M}(\omega)c^2\,. \tag{14}$$

However, for the case of a body with zero total linear momentum, $p = \gamma(v)\mathcal{M}(\omega)v = 0$, its total energy, i.e., its rest energy, reduces to $E(\omega) = \mathcal{M}(\omega)c^2$. We stress that this energy does include the internal rotational kinetic energy. Therefore, it is $\omega$ dependent.

Let us consider a rigid body that rotates submitted to a set of external conservative forces $\{\vec{F}_i^{\rm ext}\}$, of zero resultant, each of which with arm $r_i$. The relativistic Hamiltonian for the body is given by

$$H(J,\theta) = [\mathcal{M}(\omega) - \mathcal{M}(0)]\,c^2 + \sum_i V_i(\theta)\,, \tag{15}$$

with $\omega = \omega(J)$, i.e. the angular velocity is a function of the angular momentum, $J$; in (15) each potential energy is such that ($\mathrm{d}x_i = r_i\mathrm{d}\theta$)

$$-\frac{\partial V_i(\theta)}{\partial\theta} = -\frac{\partial V_i(\theta)}{\partial x_i}\frac{\partial x_i}{\partial\theta} = F_i^{\rm ext} r_i = \tau_i^{\rm ext}\,. \tag{16}$$

The external forces, whose resultant must be zero, as mentioned above, produce a vanishing linear impulse on the body. However, in general, they may perform work as well as angular impulse.

From the Hamilton equation

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\frac{\partial H}{\partial\theta}\,, \tag{17}$$

one obtains the relativistic Euler equation [3] for the body's rotation, namely

$$\mathrm{d}[I(\omega)\omega] = \left(\sum_i F_i^{\rm ext} r_i\right)\mathrm{d}t\,. \tag{18}$$

On the other hand, from equation (18) and equation (5) one derives the pseudo-work–kinematic rotational energy variation [7], namely

$$\mathrm{d}[\mathcal{M}(\omega)c^2] = \left(\sum_i F_i^{\rm ext} r_i\right)\mathrm{d}\theta\,. \tag{19}$$

Note that, it is equation (5) that makes it possible, starting from the equation of motion, to obtain the pseudo-work – rotational kinetic energy variation equation (19).

From the Hamiltonian (15), and because equation (19) can also be expressed as $\mathrm{d}[\mathcal{M}(\omega)c^2] + \sum_i \frac{\partial V_i}{\partial\theta}\mathrm{d}\theta = 0$, one concludes that $\mathrm{d}H = 0$ and one has

$$\frac{\mathrm{d}}{\mathrm{d}t}H(J,\theta) = 0\,. \tag{20}$$

As expected, mechanical energy is conserved for a process when only conservative forces act on the system.

### *3.1. Characteristic functions*

For a given body, it turns out to be convenient to define two characteristic functions: $\zeta(\omega)$, for the inertia, and $\chi(\omega)$, for the moment of inertia. These dimensionless functions are defined by

$$\zeta(\omega) = \frac{\mathcal{M}(\omega)}{\mathcal{M}(0)} \qquad \text{and} \qquad \chi(\omega) = \frac{I(\omega)}{I(0)} \,. \tag{21}$$

According to the previous Hamiltonian approach, one has the Euler equation (18) for relativistic rotation dynamics written as [8]

$$\mathrm{d}\left[I(0)\chi(\omega)\omega\right] = \Gamma^{\mathrm{ext}}\mathrm{d}t \,, \tag{22}$$

where $\Gamma^{\mathrm{ext}} = \sum_j \tau_j^{\mathrm{ext}}$ is the total resulting torque for the external forces (conservative or non-conservative). After integration in a time interval $[0, t]$, assuming constant torque, one gets the finite equation

$$I(0)\Big[\chi(\omega_{\mathrm{f}})\omega_{\mathrm{f}} - \chi(\omega_{\mathrm{i}})\omega_{\mathrm{i}}\Big] = \Gamma^{\mathrm{ext}}\, t \,. \tag{23}$$

This equation is the relativistic counterpart of the corresponding non-relativistic equation $I\left(\omega_{\mathrm{f}} - \omega_{\mathrm{i}}\right) = \Gamma^{\mathrm{ext}}\, t$.

Multiplying both sides of equation (22) by $\omega = \mathrm{d}\theta/\mathrm{d}t$, and taking into account equation (5), one obtains the equation for the relativistic pseudo-work – rotational kinetic energy variation:

$$\mathrm{d}\Big[\mathcal{M}(0)\zeta(\omega)c^2\Big] = \Gamma^{\mathrm{ext}}\mathrm{d}\theta \,. \tag{24}$$

Again, the right hand side includes the contribution of both conservative or non-conservative forces. After integration, between an initial and a final state, and for a constant torque, one gets the corresponding finite equation

$$\mathcal{M}(0)\Big[\zeta(\omega_{\mathrm{f}})\, c^2 - \zeta(\omega_{\mathrm{i}})\, c^2\Big] = \Gamma^{\mathrm{ext}}\left(\theta_{\mathrm{f}} - \theta_{\mathrm{i}}\right) \,. \tag{25}$$

This equation is the relativistic counterpart of the non-relativistic equation $\frac{1}{2}I\left(\omega_{\mathrm{f}}^2 - \omega_{\mathrm{i}}^2\right) = \Gamma^{\mathrm{ext}}\left(\theta_{\mathrm{f}} - \theta_{\mathrm{i}}\right)$, where the left hand side is the rotational kinetic energy variation and the right hand side is, in general, pseudo-work [7].

## 4. Example: a rotating ring

For a homogeneous ring of radius $R$ and moment of inertia $I_{\mathrm{ring}}(0) = \mathcal{M}(0)R^2$, rotating with angular velocity $\omega$ around its axis, the characteristic functions (21) are

$$\zeta_{\mathrm{ring}}(\omega) = \gamma\left(\omega R\right) \ , \qquad \chi_{\mathrm{ring}}(\omega) = \gamma\left(\omega R\right) \,. \tag{26}$$

The angular momentum of the rotating ring (if initially at rest, $\omega_{\mathrm{i}} = 0$) is $J = I_{\mathrm{ring}}(0)\gamma(\omega R)\omega$, and the angular velocity can be written explicitly as

$$\omega(J) = J\left[I_{\mathrm{ring}}^2(0) + J^2R^2/c^2\right]^{-1/2} \,. \tag{27}$$

The Hamiltonian (15) can be written in term of the canonical variables as

$$H(J,\theta) = \left[I_{\mathrm{ring}}^2(0) + \frac{J^2R^2}{c^2}\right]^{1/2}\frac{c^2}{R^2} - \mathcal{M}(0)c^2 + V(\theta) \,. \tag{28}$$

We can check that, for $c \to +\infty$, the classical Hamiltonian is recovered:

$$H(J,\theta) = \frac{J^2}{2I_{\mathrm{ring}}(0)} + V(\theta)\,. \tag{29}$$

We study the process depicted in fig. 1, representing a rotating ring of radius $R$ mounted in a rigid massless structure. Two conservative forces, $\vec{F}_1 = (F,0,0)$ and $\vec{F}_2 = (-F,0,0)$, are applied to the system, with arms $r_1$ and $r_2$, respectively ($r_1 = r_2 = r$). Since the resultant force vanishes, $F^{\mathrm{ext}} = 0$, the ring does not move as a whole in the reference frame where its initial total linear momentum is zero [6].

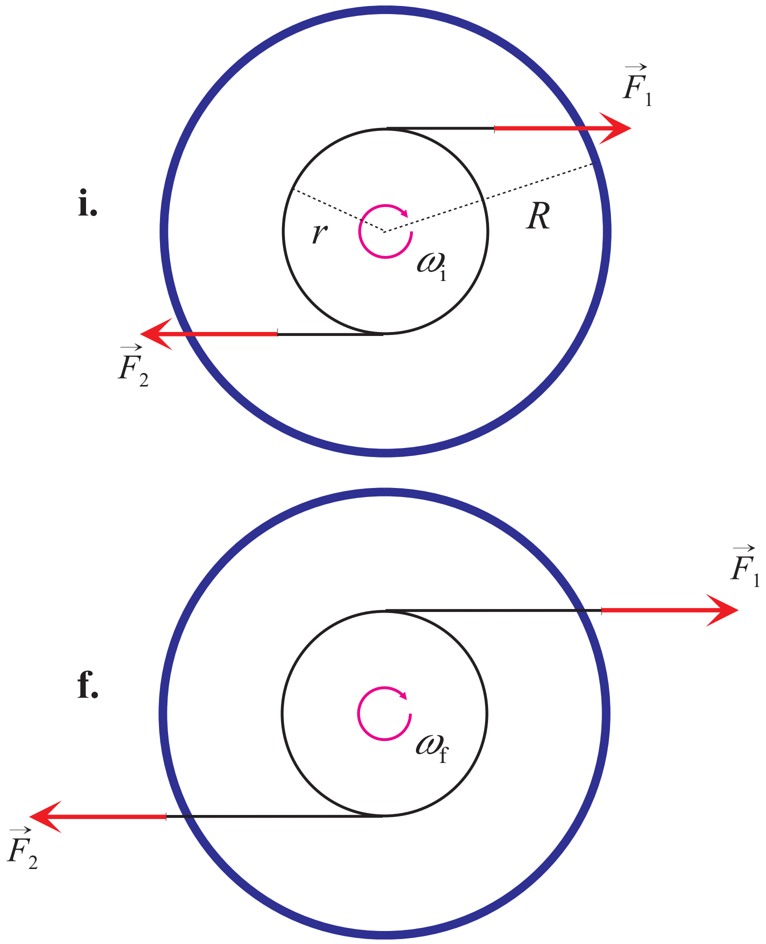


**Figure 1.** Rotating ring acted upon only by constant forces: (i) initial state and (f) final state, after a time interval $[0,t]$.

For the description of the rotating ring by using equation (23) and equation (26) we get

$$I_{\mathrm{ring}}(0)\,[\gamma(\omega_{\mathrm{f}}R)\omega_{\mathrm{f}} - \gamma(\omega_{\mathrm{i}}R)\omega_{\mathrm{i}}] = \Gamma^{\mathrm{ext}}\,t\,, \tag{30}$$

where the external net torque is $\Gamma^{\mathrm{ext}} = F(r_1 + r_2) = 2Fr$.

Assuming, in (30), that the initial angular velocity is zero, the angular velocity, simply denoted by $\omega$, as a function of time, is obtained, as well as the angular dependence $\theta(t)$, yielding [9]

$$\omega(t) = \frac{\alpha t}{\sqrt{1 + (\alpha R t)^2/c^2}}\,, \tag{31}$$

and

$$\theta(t) = \frac{c^2}{\alpha R^2}\left(\sqrt{1+\frac{(\alpha R t)^2}{c^2}} - 1\right), \tag{32}$$

where $\alpha = \Gamma^{\rm ext}/I_{\rm ring}(0)$. It is straightforward to check the following limits:

$$\lim_{t\to 0}\frac{\omega(t)}{t} = \alpha \quad , \quad \lim_{t\to 0}\frac{\theta(t)}{t^2} = \frac{1}{2}\alpha\,; \tag{33}$$

$$\lim_{t\to +\infty}\omega(t) = \omega_{\rm L} = \frac{c}{R}\,, \quad \lim_{t\to +\infty}\frac{\theta(t)}{t} = \omega_{\rm L}\,, \tag{34}$$

where $\omega_{\rm L}$ is the limiting angular velocity for the ring and $t \to 0$ corresponds to the classical limit. The maximum angular velocity may also be derived from the requirement that the ring characteristic functions (26) are real. As the above limit indicates, that angular velocity is attained only for $t \to +\infty$ and the energy transferred to the system is $\lim_{\omega\to\omega_{\rm L}} \mathcal{M}(0)[\gamma(\omega R) - 1]\, c^2 = +\infty$.

Inserting the characteristic mass function for a ring, equation (26), into equation (25), for zero initial angle and zero initial angular velocity, one obtains the explicit dependence of the angular velocity with the rotation angle (we are setting $\omega_{\rm f} = \omega$ and $\theta_{\rm f} = \theta$):

$$\omega(\theta) = \frac{c}{R}\left\{1 - \left[1 + \left(\frac{a\theta}{c^2}\right)\right]^{-2}\right\}^{1/2}, \tag{35}$$

where $a = \Gamma^{\rm ext}/\mathcal{M}(0) = \alpha R^2$. We can also obtain the limits of the angular velocity for large and small angles:

$$\lim_{\theta\to +\infty}\omega(\theta) = \omega_{\rm L}\,, \quad \lim_{\theta\to 0}\frac{\omega(\theta)}{\theta^{1/2}} = \sqrt{2\alpha}\,. \tag{36}$$

## 5. Conclusions

Based on a relativistic Hamiltonian approach we addressed the relativistic rotation by developing an appropriate formalism.

We proved a remarkable theorem, established by the relation (5), which bridges the vector description of the relativistic rotation process, involving the angular impulse and the variation of the angular momentum, and its scalar description that relies on the concepts of energy and work.

To illustrate the usefulness of that relativistic equation (5), a process has been analysed in which only conservative forces (related to a work reservoir) intervene, a process that evolves with conservation of mechanical energy. Processes involving dissipation of mechanical energy (and heat production) or processes with production of rotational kinetic energy, due to forces that come from chemical reactions have also been analysed elsewhere [5].

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