New solutions of the *D*-dimensional Klein-Gordon equation via mapping onto the nonrelativistic one-dimensional Morse potential

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Abstract

New exact analytical bound-state solutions of the *D*-dimensional Klein-Gordon equation for a large set of couplings and potential functions are obtained via mapping onto the nonrelativistic bound-state solutions of the one-dimensional generalized Morse potential. The eigenfunctions are expressed in terms of generalized Laguerre polynomials, and the eigenenergies are expressed in terms of solutions of irrational equations at the worst. Several analytical results found in the literature, including the so-called Klein-Gordon oscillator, are obtained as particular cases of this unified approach.

1. Introduction

The generalized Morse potential $Ae^{-\alpha x} + Be^{-2\alpha x}$ [1–6], the singular harmonic oscillator (SHO) $Ax^2 + Bx^{-2}$ [3], [7–22] and the singular Coulomb potential (SCP) $Ax^{-1} + Bx^{-2}$ [3], [7–10], [19], [21], [23–30] have played an important role in atomic, molecular and solid-state physics. Bound states for systems modelled by those potentials are computed exactly in nonrelativistic quantum mechanics. In a recent paper [31], it was shown that the Schrödinger equation for all those exactly solvable problems mentioned above can be reduced to the confluent hypergeometric equation in such a way that it can be solved via Laplace transform method with closed-form eigenfunctions expressed in terms of generalized Laguerre polynomials. Connections between the Morse and those other potentials have also been reported. The Langer transformation [32] a nonunitary transformation consisting of a change of function plus a map from the full line of a one-dimensional problem to the half line appropriate to a radial *D*-dimensional problem, has been a key ingredient for those connections. The three-dimensional Coulomb potential has been mapped into the one-dimensional Morse potential and into the three-dimensional singular Coulomb potential [32]. The Morse potential with particular parameters has been mapped into the two-dimensional harmonic oscillator [33] and into the three-dimensional Coulomb potential [34]. Later, the generalized Morse potential was mapped into the three-dimensional harmonic oscillator and Coulomb potentials [35, 36]. Furthermore, a certain mapping between the Morse potential with particular parameters and a particular case of the three-dimensional SCP has been found with fulcrum on the algebra $so(2,1)$ and its representations [37]. Recently, the introduction of two additional parameters into the Langer transformation provided opportunity to bound-state solutions of the SHO and SCP in arbitrary dimensions be generated in a simple way from the bound states of the one-dimensional generalized Morse potential with straightforward determination of the critical attractive singular potential and the proper boundary condition on the radial eigenfunction at the origin [38]. More general connections for an extended Schrödinger equation in a position-dependent mass background have involved the three-dimensional nonsingular Coulomb potential [39], and the SHO and SCP in arbitrary dimensions [40].

Relativistic effects on systems described by SHO and SCP can be approached in the context of the Klein-Gordon (KG) theory. Actually, particular cases of SHO and SCP under the umbrella of the KG theory have been studied in

the literature for different couplings, space dimensions and methodologies. The free KG equation is expressed as $\left(p^\mu p_\mu - m^2 c^2 \right) \Psi = 0$ with the linear momentum operator given by $p^\mu = i\hbar \partial^\mu$. Vector and scalar interactions are considered by replacing p^μ by $p^\mu - V^\mu/c$ and m by $m + V_s/c^2$, respectively. From now on the time component of the vector potential will be denominated by V_v in this paper. The equal-magnitude mixing of vector and scalar couplings for arbitrary angular momentum has been considered for the harmonic oscillator [41–43], for S-waves of the SHO [44] and for arbitrary angular momentum for the pseudo-harmonic potential (a special particular case of the SHO) [45]. The Coulomb potential has been studied as a vector coupling [46–48], scalar coupling [47, 48], mixing of vector and scalar couplings with equal magnitudes [43], and an arbitrary mixing of vector and scalar couplings [49–51]. S-waves of the SCP with arbitrary mixing of vector and scalar potentials have been considered [52], and for arbitrary angular momentum with equal magnitudes of those potentials in the case of the Kratzer potential (a particular case of the SCP) [53]. In Refs. [51] and [54], the SCP for arbitrary angular momentum with a mixing of vector and scalar couplings with equal magnitudes has also been considered. Dubbed the KG oscillator, an unconventional form of interaction is achieved by replacing $\overrightarrow{p}^2$ by $\left(\overrightarrow{p} + im\omega r \widehat{r} \right) \cdot \left(\overrightarrow{p} - im\omega r \widehat{r} \right)$ in such a way that one obtains the Schrödinger equation with the harmonic oscillator potential in a nonrelativistic scheme [55]. This alternative form of interaction spurred a big and still growing literature [56–73]. It is interesting to remark that this bears a striking resemblance to the Dirac oscillator [74], where a harmonic oscillator potential is introduced via a radial linear tensor coupling in the Dirac equation. More recent literature treats the KG oscillator in two space dimensions under the influence of a scalar Coulomb potential [71], a scalar linear potential [72], and a mixing of scalar linear and vector Coulomb potentials [73], with solutions allowed only for certain parameters of the KG oscillator and eigenfunctions expressed in terms of Heun biconfluent functions. One caveat: in Ref. [62], the authors misidentified the one-dimensional KG equation with the space component of the linear vector potential (minimally coupled) as the KG oscillator.

In the present paper, the method developed in [38] is extended to a modified D-dimensional KG equation featuring a vector interaction nonminimally coupled. Actually, this kind of generalized KG equation has the same form as that one for the physical component of the five-component spinor in a one-dimensional space [75] and also in a three-dimensional space [76] appearing in the scalar sector of the Duffin-Kemmer-Petiau theory. Specially noteworthy is the inclusion of the KG oscillator in that framework. This extension of the method used in Ref. [38] is an interesting way of providing a unified treatment of many known relativistic problems via a mapping onto a unique well-known one-dimensional nonrelativistic problem, allowing to obtain some new exact analytical bound-state solutions for a large class of problems including new types of couplings and potential functions. We highlight vector-scalar SHO plus nonminimal vector Cornell potentials and nonminimal vector Coulomb (space component) and harmonic oscillator (time component) potentials, vector-scalar Coulomb plus nonminimal vector Cornell potentials and nonminimal vector shifted Coulomb potentials, vector-scalar SCP plus nonminimal vector Coulomb potentials, and also the curious case of a pure nonminimal vector constant potential. Furthermore, we show that several exactly soluble bound states explored in the literature are obtained as particular cases of those cases. In all those circumstances the eigenfunctions are expressed in terms of the generalized Laguerre polynomials and the eigenenergies are expressed in terms of irrational equations.

The paper is organized as follows. In Sec. 2 we review, as a background, the generalized Morse potential in the Schrödinger equation. The D-dimensional KG equation with vector, scalar and nonminimal vector couplings, its connection with the generalized Morse potential and the proper form for the potential functions, are presented in Sec. 3. In Sec. 4 we present some concluding remarks.

2. Nonrelativistic bound states in a one-dimensional generalized Morse potential

The time-independent Schrödinger equation is an eigenvalue equation for the characteristic pair (E, ψ) with $E \in \mathbb{R}$. For a particle of mass M embedded in the generalized Morse potential it reads

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2M}{\hbar^2} \left(E - V_1 e^{-\alpha x} - V_2 e^{-2\alpha x} \right) \psi(x) = 0, \tag{1}$$

where $\alpha > 0$. Bound-state solutions demand $\int_{-\infty}^{+\infty} dx\, |\psi|^2 = 1$ and exist only when the generalized Morse potential has a well structure ($V_1 < 0$ and $V_2 > 0$). The eigenenergies are then given by (see, e.g., [31], [38])

$$E_n = -\frac{V_1^2}{4V_2}\left[1 - \frac{\hbar\alpha\,\sqrt{2MV_2}}{M|V_1|}\left(n + \frac{1}{2}\right)\right]^2,\tag{2}$$

with

$$n = 0, 1, 2, \ldots < \frac{M|V_1|}{\hbar\alpha\,\sqrt{2MV_2}} - \frac{1}{2}.\tag{3}$$

This restriction on n limits the number of allowed states and requires $M|V_1|/\left(\hbar\alpha\,\sqrt{2MV_2}\right) > 1/2$ to make possible the existence of bound states. On the other hand, with the substitutions

$$\hbar\alpha s_n = \sqrt{-2ME_n},\quad \hbar\alpha\xi = 2\sqrt{2MV_2}\,e^{-\alpha x},\tag{4}$$

the eigenfunctions are expressed in terms of the generalized Laguerre polynomials as

$$\psi_n(\xi) = N_n\,\xi^{s_n}e^{-\xi/2}L_n^{(2s_n)}(\xi),\tag{5}$$

where N_n are arbitrary constants.

3. The D-dimensional KG equation

Incorporating the KG oscillator as a particular case, a generalized Lorentz-covariant KG equation for a particle of mass m is written (with $\hbar = c = 1$) as

$$\left[(p^\mu - V^\mu - iU^\mu)\left(p_\mu - V_\mu + iU_\mu\right) - (m + V_s)^2\right]\Psi = 0.\tag{6}$$

In the absence of a scalar coupling, Eq. (6) has the same form as that one for the physical component of the five-component spinor in a one-dimensional space [75] and also in a three-dimensional space [76] appearing in the scalar sector of the Duffin-Kemmer-Petiau theory. A continuity equation of the form $\partial_\mu J^\mu = 0$ is satisfied with the current density J^μ proportional to $i\Psi^*\overleftrightarrow{\partial^\mu}\Psi - 2V^\mu|\Psi|^2$. In contrast to V^μ, the vector potential U^μ is not minimally coupled. Furthermore, invariance under the time-reversal transformation demands that V^μ and U^μ have opposite behaviours. Similarly to the scalar potential, the nonminimal vector potential does not couple to the charge because it does not change its sign under the charge-conjugation operation ($\Psi \rightarrow \Psi^*$, $V^\mu \rightarrow -V^\mu$). In other words, U^μ and V_s do not distinguish particles from antiparticles and so the system does not exhibit Klein's paradox in the absence of the minimal coupling. Note also that $J^\mu \rightarrow -J^\mu$ under charge conjugation, as should be expected for a charge current density.

At this stage, we make $\vec{V} = \vec{0}$ (the space component of the minimal coupling can be gauged away for spherically symmetric potentials) and consider only time-independent potentials in such that the factorization $\Psi(\vec{r}, t) = e^{-i\varepsilon t}\Phi(\vec{r})$ yields

$$\left\{\left(\vec{p} - i\vec{U}\right)\cdot\left(\vec{p} + i\vec{U}\right) - U_0^2 - \left[(\varepsilon - V_v)^2 - (m + V_s)^2\right]\right\}\Phi = 0,\tag{7}$$

or, equivalently,

$$\left[\nabla^2 - \left(\vec{\nabla}\cdot\vec{U}\right) - \vec{U}^2 + U_0^2 + (\varepsilon - V_v)^2 - (m + V_s)^2\right]\Phi = 0.\tag{8}$$

Eq. (8) is an eigenvalue equation for the characteristic pair (ε, Φ) with $\varepsilon \in \mathbb{R}$. From this equation, it is clear that the spectrum is distributed symmetrically about $\varepsilon = 0$ in the absence of the minimal coupling (charge conjugation changes ε by $-\varepsilon$). Because the charge density is proportional to $(\varepsilon - V_v)|\Phi|^2$, bound states demand $\Phi \rightarrow 0$ as $r \rightarrow \infty$. In spherical coordinates of a D-dimensional space, the position vector is $\vec{r} = (r, \Omega)$, where Ω denotes a set of $D - 1$ angular variables.

For spherically symmetric potentials, $V_v(\vec{r}) = V_v(r)$, $V_s(\vec{r}) = V_s(r)$, $U_0(\vec{r}) = V_0(r)$ and $\vec{U}(\vec{r}) = V_r(r)\hat{r}$, one can write

$$\Phi(\vec{r}) = \frac{u(r)}{r^k}Y(\Omega),\tag{9}$$

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where Y denotes the hyperspherical harmonics labelled by $2k$ quantum numbers (see, e.g. [18], [77]),

$$k = (D - 1)/2, \tag{10}$$

and u obeys the Schrödinger-like radial equation

$$\frac{d^2 u(r)}{dr^2} + 2M \left[\widetilde{\varepsilon} - V(r) - \frac{L(L+1)}{2Mr^2} \right] u(r) = 0, \tag{11}$$

with M denoting a positive parameter having dimension of mass. The effective energy $\widetilde{\varepsilon}$ and the effective potential V are expressed by

$$
\begin{aligned}
2M\widetilde{\varepsilon} &= \varepsilon^2 - m^2 \\
2MV &= V_s^2 - V_v^2 + 2(mV_s + \varepsilon V_v) + \frac{dV_r}{dr} + 2k\frac{V_r}{r} + V_r^2 - V_0^2,
\end{aligned} \tag{12}
$$

and L in the centrifugal barrier potential takes the values

$$L = l + k - 1 \quad \text{or} \quad L = -l - k, \tag{13}$$

in which $l = 0, 1, 2 \ldots$ is the orbital momentum quantum number, and $u \to 0$ as $r \to \infty$ for bound-state solutions.

Following Ref. [38], with effective potentials expressed by

$$V(r) = Ar^\delta + \frac{B}{r^2} + C, \quad \delta = +2 \text{ or } 0 \text{ or } -1 \tag{14}$$

the Langer transformation [32]

$$u(r) = \sqrt{r/r_0}\, \phi(x), \quad r/r_0 = e^{-\Lambda \alpha x}, \tag{15}$$

with $r_0 > 0$ and $\Lambda > 0$, transmutes the radial equation (11) into

$$\frac{d^2\phi(x)}{dx^2} + 2M \left\{ -\frac{(\Lambda\alpha S)^2}{2M} - (\Lambda\alpha r_0)^2 \left[Ar_0^\delta e^{-\Lambda\alpha(\delta+2)x} + (C - \widetilde{\varepsilon})\, e^{-2\Lambda\alpha x} \right] \right\} \phi(x) = 0, \tag{16}$$

with

$$S = \sqrt{(l + k - 1/2)^2 + 2MB}. \tag{17}$$

At this point, it is instructive to note that not only $L(L+1)$ is insensible to the different choices of L as prescribed by (13) but also S. One can see from (16) that there is no bound-state solution if V is an inversely quadratic potential ($A = 0$, or $A \neq 0$ and $\delta = 0$ or $\delta = -2$). A connection with the bound states of the generalized Morse potential is reached only if the pair (δ, Λ) is equal to $(2, 1/2)$ or $(-1, 1)$, and, as can be seen from the identification of the first term in the curly brackets with the (negative) energy parameter E in (1) one has to have $S^2 > 0$. Thus,

$$2MB > -(2k - 1)^2/4. \tag{18}$$

Furthermore, from eq. (16), the asymptotic behaviour of $\phi(x)$ is $\phi(x) \underset{x \to +\infty}{\to} e^{-\Lambda\alpha S x}$ such that, from (15)

$$u(r) \underset{r \to 0}{\to} r^{1/2+S}. \tag{19}$$

Effective potentials of the general form (14) can be realized by the following particular choices for the radial potentials in the Klein-Gordon equation (8)

$$
\begin{aligned}
V_r &= \beta_r/r + \gamma_r r^{\delta_r}, \quad \delta_r = 0 \text{ or } 1 \\
V_0 &= \beta_0/r + \gamma_0 r^{\delta_0}, \quad \delta_0 = 0, 1 \text{ or } -1/2 \\
V_s &= \alpha_s/r^2 + \beta_s/r + \gamma_s r^2, \\
V_v &= \alpha_v/r^2 + \beta_v/r + \gamma_v r^2,
\end{aligned} \tag{20}
$$

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with $\alpha_v^2 = \alpha_s^2$ and $\alpha_v \beta_v = \alpha_s \beta_s$ to eliminate inversely quartic and cubic terms in V, respectively. This means $\alpha_v = \alpha_s = 0$, or $\alpha_v = \pm \alpha_s \neq 0$ with $\beta_v = \pm \beta_s$. On the other hand, because in (14) δ can take just one value, when $\gamma_{s,v} \neq 0$, $\beta_{s,v} = 0$ and vice-versa. For $\delta_0 = -1/2$ one has $\gamma_v = \gamma_s = \delta_r = \beta_0 = 0$.

The eigenvalue equation for the energy ε is still obtained from eq. (2), but with different roles for E, V_1 and V_2 in eq. (1). Indeed, in the present case all these parameters can depend on the energy ε in two essential ways: either explicitly through the term $(\Lambda \alpha r_0)^2 (C - \widetilde{\varepsilon})$ in (16), which can be equal to V_1 or to V_2, depending on whether the pair (δ, Λ) is equal to $(2, 1/2)$ or $(-1, 1)$, respectively, or through the parameters A and B of the potential $V(r)$ (14) whenever the potential V_v in (12) has the correspondent radial dependence.

3.1. The effective singular harmonic oscillator

With $(\delta, \Lambda) = (2, 1/2)$ plus the definition $A = M\omega^2/2$, the identification of the bound-state solutions of Eq. (11) with those ones from the generalized Morse potential is done by choosing $V_1 = -\alpha^2 r_0^2 (\widetilde{\varepsilon} - C)/4$ and $V_2 = \alpha^2 r_0^4 M\omega^2/8$, necessarily with $\widetilde{\varepsilon} > C$ and $\omega^2 > 0$. With $\omega > 0$ one can write, from eq. (4),

$$\xi = M\omega r^2. \tag{21}$$

Furthermore, (3) implies $\widetilde{\varepsilon} > C + \omega(2n+1)$. Using (2) and (17) one can write the complete solution of the problem as

$$\widetilde{\varepsilon} = C + \omega(2n + 1 + S) \tag{22}$$
$$u(r) = N r^{1/2+S} e^{-M\omega r^2/2} L_n^{(S)}(M\omega r^2), \tag{23}$$

where $\widetilde{\varepsilon}$ and ω depend on ε because of eqs. (12) and (14). The condition (3) means that

$$n \leq N = \left[\frac{M|V_1|}{\alpha \sqrt{2MV_2}} - \frac{1}{2} \right] = \left[\frac{\widetilde{\varepsilon} - C - \omega}{2\omega} \right] \tag{24}$$

where $[x]$ stands for the largest integer less or equal to x. This means that there is no limitation on the value of N, because it can be as large as the energy can, which in turns means that n in (22) has no upper bound.

3.1.1. Vector-scalar SHO plus nonminimal vector Cornell potentials

An example of this class of solutions can be reached by choosing

$$V_r = \frac{\beta_r}{r} + \gamma_r r, \quad V_0 = \frac{\beta_0}{r} + \gamma_0 r, \quad V_s = \frac{\alpha_s}{r^2} + \gamma_s r^2, \quad V_v = \pm V_s. \tag{25}$$

This class represents a generalization of the cases found in [41] ($V_r = V_0 = 0$ and $V_v = V_s = \gamma_s r^2$ in three dimensions), [42] and [43] ($V_r = V_0 = 0$ and $V_v = V_s = \gamma_s r^2$ in two dimensions), [44] ($V_r = V_0 = 0$ and $V_v = V_s = \alpha_s/r^2 + \gamma_s r^2$ for S-waves in three dimensions), [45] ($V_r = V_0 = 0$ and the pseudo-harmonic potential in two dimensions) and [55] ($V_0 = V_v = V_s = 0$ and $\beta_r = 0$, as a matter of fact the space component of a nonminimal vector anisotropic linear potential in three dimensions).

The complete identification with the generalized Morse potential is done with the identifications

$$M\omega = \sqrt{\gamma_r^2 - \gamma_0^2 \pm 2\gamma_s(\varepsilon \pm m)}$$
$$2MB = \beta_r(\beta_r - 1 + 2k) - \beta_0^2 \pm 2\alpha_s(\varepsilon \pm m) \tag{26}$$
$$2MC = \gamma_r(2\beta_r + 1 + 2k) - 2\gamma_0 \beta_0,$$

which lead, in general, using eq. (22), to an irrational equation in ε

$$(\varepsilon + m)(\varepsilon - m) - \gamma_r(2\beta_r + 1 + 2k) + 2\gamma_0 \beta_0 = 2(2n + 1 + S)\sqrt{\gamma_r^2 - \gamma_0^2 \pm 2\gamma_s(\varepsilon \pm m)}, \tag{27}$$

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or, more explicitly, using eq. (17),

$$(\varepsilon + m)(\varepsilon - m) - \gamma_r(2\beta_r + 1 + 2k) + 2\gamma_0\beta_0$$
$$= 2\left(2n + 1 + \sqrt{(l + k - 1/2)^2 + \beta_r(\beta_r - 1 + 2k) - \beta_0^2 \pm 2\alpha_s(\varepsilon \pm m)}\right)$$
$$\times \sqrt{\gamma_r^2 - \gamma_0^2 \pm 2\gamma_s(\varepsilon \pm m)}. \tag{28}$$

One notices immediately that, if β_r, γ_r, and the product $\gamma_0\beta_0$ are equal or greater than zero, one has $(\varepsilon + m)(\varepsilon - m) > 0$, meaning that either $\varepsilon > m$ or $\varepsilon < -m$, a feature characteristic of harmonic oscillator states for positive energy (particle) or for negative energy (anti-particle) states. Furthermore, since the products $\pm\gamma_s(\varepsilon \pm m)$ and $\pm\alpha_s(\varepsilon \pm m)$ under the square roots must be positive for arbitrary high values of $|\varepsilon|$, both α_s and γ_s must be positive for $V_s = V_v$ and negative for $V_s = -V_v$ for positive energy states, that is, V_v is always positive for these states. The reverse is true for negative energy states.

Squaring Eq. (28) successively results into a nonequivalent polynomial equation of degree 8. Solutions of this algebraic equation that are not solutions of the original equation can be removed by backward substitution. A quartic algebraic equation is obtained when $\alpha_s = 0$ (the case of a nonminimal vector Cornell potential plus an equal-magnitude mixing of vector and scalar harmonic oscillators). For $\alpha_s = \gamma_r(2\beta_r + 1 + 2k) - 2\gamma_0\beta_0 = \gamma_r^2 - \gamma_0^2 = 0$ (which includes the case of a nonminimal vector Coulomb potential and the case of a space component of a nonminimal vector Coulomb potential plus a time component of a nonminimal vector linear potential, both plus an equal-magnitude mixing of vector and scalar harmonic oscillators) one obtains a cubic algebraic equation. However, (28) can be written as a quadratic algebraic equation rendering two branches of solutions symmetrical about $\varepsilon = 0$ in the case of a pure nonminimal vector Cornell potential ($\alpha_s = \gamma_s = 0$, $|\gamma_r| \neq |\gamma_0|$):

$$\varepsilon = \pm\sqrt{m^2 + \gamma_r(2\beta_r + 1 + 2k) - 2\beta_0\gamma_0 + 2\sqrt{\gamma_r^2 - \gamma_0^2}(2n + 1 + S)}, \tag{29}$$

where $S = \sqrt{(l + k - 1/2)^2 + \beta_r(\beta_r - 1 + 2k) - \beta_0^2}$. One may also note that the (positive energy) D-dimensional relativistic harmonic oscillator is obtained when $\gamma_r = \gamma_0 = \beta_r = \beta_0 = \alpha_s = 0$ and $\gamma_s = \pm(1/2)m\Omega^2$ for $V_v = \pm V_s$, where Ω is the harmonic oscillator frequency. In this case $B = C = 0$ and one obtains $M\omega = \sqrt{m\Omega^2(\varepsilon \pm m)}$ and $\varepsilon^2 - m^2 = 2(2n + 1 + |l + k - \frac{1}{2}|)\sqrt{m\Omega^2(\varepsilon \pm m)}$. The Klein-Gordon 3$D$ oscillator is obtained of course as a particular case with $k = 1$. At this point, it is worth remarking that, when there are only scalar and vector potentials, the conditions $V_v = \pm V_s$ correspond to having spin(pseudospin) symmetry conditions in the Dirac equation with $D = 3$ (a recent review of this subject is given in [78]) and that, under those conditions, the energy spectrum of the Dirac and the Klein-Gordon equations is the same [79].

3.1.2. Vector-scalar SHO plus nonminimal vector Coulomb (space component) and harmonic oscillator (time component) potentials

Another example is given by

$$V_r = \frac{\beta_r}{r}, \quad V_0 = \gamma_0 r^2, \quad V_s = \frac{\alpha_s}{r^2} + \gamma_s r^2, \quad V_v = \pm\frac{\alpha_s}{r^2} + \gamma_v r^2, \tag{30}$$

with $\gamma_s^2 = \gamma_v^2 + \gamma_0^2$ and

$$\begin{aligned}
M\omega &= \sqrt{2(\varepsilon\gamma_v + m\gamma_s)} \\
2MB &= \beta_r(\beta_r - 1 + 2k) \pm 2\alpha_s(\varepsilon \pm m) \\
2MC &= 2\alpha_s(\gamma_s \mp \gamma_v).
\end{aligned} \tag{31}$$

In this case, the quantization condition

$$(\varepsilon + m)(\varepsilon - m) - 2\alpha_s(\gamma_s \mp \gamma_v) = 2(2n + 1 + S)\sqrt{2(\varepsilon\gamma_v + m\gamma_s)} \tag{32}$$

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where $S = \sqrt{(l + k - 1/2)^2 + \beta_r (\beta_r - 1 + 2k) \pm 2\alpha_s (\varepsilon \pm m)}$, is again convertible to an algebraic equation of degree 8. A quartic algebraic equation is obtained when $\alpha_s = 0$, and a quadratic algebraic equation when $\alpha_s = \gamma_v = 0$:

$$\varepsilon = \pm \sqrt{m^2 + 2 \sqrt{2m\gamma_s} (2n + 1 + S)}, \tag{33}$$

with $m \neq 0$ and $\gamma_s > 0$. In this case, $|\gamma_0| = \gamma_s$ and the spectrum is indifferent to the sign of γ_0.

3.2. The effective singular Coulomb potential

Comparison of the bound states of Eq. (11) with those ones from the generalized Morse potential with the pair $(\delta, \Lambda) = (-1, 1)$ is done by choosing $V_1 = \alpha^2 r_0 A$ and $V_2 = -\alpha^2 r_0^2 (\widetilde{\varepsilon} - C)$, with $A < 0$ and $\widetilde{\varepsilon} < C$. Now,

$$\xi = 2 \sqrt{2M (C - \widetilde{\varepsilon})} \, r \tag{34}$$

and (3) implies $\widetilde{\varepsilon} > C - MA^2/[2 (n + 1/2)^2]$. Using (2) and (17) one can write

$$\widetilde{\varepsilon} = C - \frac{MA^2}{2\zeta^2}$$

$$\tag{35}$$

$$u(r) = N r^{1/2+S} e^{-M|A|r/\zeta} L_n^{(2S)} \left(\frac{2M|A|r}{\zeta} \right),$$

with

$$\zeta = n + 1/2 + S. \tag{36}$$

Again, we see that there is no upper limit on the value of n. The spectrum is bound from below and above, since, when $n \to \infty$, $\widetilde{\varepsilon} \to C$, corresponding to energies $\varepsilon = \pm \sqrt{m^2 + 2MC}$, in which case there are no bound states, and of course one must have $2MC > -m^2$. One has in general

$$2MC - \left(\frac{2MA}{2S + 1} \right)^2 + m^2 \leq \varepsilon^2 < 2MC + m^2 . \tag{37}$$

This is the characteristic behaviour of Coulomb-like spectra.

3.2.1. Vector-scalar Coulomb plus nonminimal vector Cornell potentials

An example of this class of solutions can be reached by choosing

$$V_r = \frac{\beta_r}{r} + \gamma_r r, \quad V_0 = \frac{\beta_0}{r} \pm \gamma_r r, \quad V_s = \frac{\beta_s}{r}, \quad V_v = \frac{\beta_v}{r}. \tag{38}$$

This class generalizes the results found in [43] ($V_r = V_0 = 0$ and $V_v = V_s = \beta_s/r$ in two dimensions), [49–51] ($V_r = V_0 = 0$ in D dimensions).

There results

$$\begin{aligned} 2MA &= 2 (m\beta_s + \varepsilon\beta_v) \\ 2MB &= \beta_s^2 - \beta_v^2 + \beta_r (\beta_r - 1 + 2k) - \beta_0^2 \\ 2MC &= \gamma_r (2\beta_r + 1 + 2k \mp 2\beta_0), \end{aligned} \tag{39}$$

so that

$$\varepsilon = \frac{-m\beta_v\beta_s \pm \zeta \sqrt{(\beta_v^2 + \zeta^2) [m^2 + \gamma_r (2\beta_r + 1 + 2k \mp 2\beta_0)] - m^2\beta_s^2}}{\beta_v^2 + \zeta^2}, \tag{40}$$

necessarily with $\varepsilon^2 < m^2 + \gamma_r (2\beta_r + 1 + 2k \mp 2\beta_0) .$

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3.2.2. Vector-scalar Coulomb plus nonminimal vector shifted Coulomb potentials

Another example, generalizing the results found in [43] ($V_r = V_0 = 0$ and $V_v = V_s = \beta_s/r$ in two dimensions), [49–51] ($V_r = V_0 = 0$ in D dimensions), is

$$V_r = \frac{\beta_r}{r} + \gamma_r, \quad V_0 = \frac{\beta_0}{r} + \gamma_0, \quad V_s = \frac{\beta_s}{r}, \quad V_v = \frac{\beta_v}{r}, \tag{41}$$

resulting in

$$
\begin{aligned}
2MA &= 2\left[m\beta_s + \varepsilon\beta_v + \gamma_r(\beta_r + k) - \gamma_0\beta_0\right] \\
2MB &= \beta_r(\beta_r - 1 + 2k) - \beta_0^2 + \beta_s^2 - \beta_v^2 \\
2MC &= \gamma_r^2 - \gamma_0^2,
\end{aligned}
\tag{42}
$$

so that, defining

$$\tau = m\beta_s + \gamma_r(\beta_r + k) - \gamma_0\beta_0 \tag{43}$$

one finds

$$\varepsilon = \frac{-\beta_v\tau \pm \zeta\sqrt{(\beta_v^2 + \zeta^2)(m^2 + \gamma_r^2 - \gamma_0^2) - \tau^2}}{\beta_v^2 + \zeta^2}, \tag{44}$$

with $\varepsilon^2 < m^2 + \gamma_r^2 - \gamma_0^2$. The pure nonminimal vector shifted Coulomb potential holds the spectrum

$$\varepsilon = \pm\sqrt{m^2 + \gamma_r^2 - \gamma_0^2 - \left[\frac{\gamma_r(\beta_r + k) - \gamma_0\beta_0}{\zeta}\right]^2}. \tag{45}$$

Curiously, there can be solutions even if $\beta_r = \beta_0 = 0$ with $\gamma_r < 0$, i.e., one has bound solutions only with constant nonminimal vector potentials. Note that in this example, "±" denotes two different solutions regarding the same potentials parameters.

3.2.3. Vector-scalar SCP plus nonminimal vector Coulomb potentials

Now consider the case of a nonminimal vector Coulomb potential plus an equal-magnitude mixing of vector and scalar SCP:

$$V_r = \frac{\beta_r}{r}, \quad V_0 = \frac{\beta_0}{r}, \quad V_s = \frac{\alpha_s}{r^2} + \frac{\beta_s}{r}, \quad V_v = \pm V_s. \tag{46}$$

Chiefly due to the presence of the nonminimal vector Coulomb potential, this last example generalizes those ones found in Refs. [51] (SCP in D dimensions), [52] (Kratzer potential for S-waves in three dimensions, as a matter of fact the SCP), [53] (Kratzer potential with $V_v = V_s$ in three dimensions), and [54] (SCP with $V_v = V_s$ in three dimensions). One finds

$$
\begin{aligned}
2MA &= \pm 2\beta_s(\varepsilon \pm m) \\
2MB &= \beta_r(\beta_r - 1 + 2k) - \beta_0^2 \pm 2\alpha_s(\varepsilon \pm m) \\
2MC &= 0,
\end{aligned}
\tag{47}
$$

in such a way that for $m \neq 0$ one finds the irrational equation in ε

$$
\begin{aligned}
(\varepsilon + m)(\varepsilon - m) &= -\left[\frac{\beta_s(\varepsilon \pm m)}{\zeta}\right]^2 \\
&= -\left[\frac{\beta_s(\varepsilon \pm m)}{n + 1/2 + \sqrt{(l + k - 1/2)^2 + \beta_r(\beta_r - 1 + 2k) - \beta_0^2 \pm 2\alpha_s(\varepsilon \pm m)}}\right]^2,
\end{aligned}
\tag{48}
$$

necessarily with $|\varepsilon| < m$ and $\beta_s < 0$.

3.2.4. Vector-scalar SCP plus nonminimal shifted Coulomb and inverse square root potentials

An additional example containing SCP potentials is given by

$$V_r = \frac{\beta_r}{r} + \gamma_r, \ V_0 = \frac{\gamma_0}{\sqrt{r}}, \ V_s = \frac{\alpha_s}{r^2} + \frac{\beta_s}{r}, \ V_v = \pm V_s, \tag{49}$$

resulting in

$$
\begin{aligned}
2MA &= 2\gamma_r (\beta_r + k) - \gamma_0^2 \pm 2\beta_s (\varepsilon \pm m) \\
2MB &= \beta_r (\beta_r - 1 + 2k) \pm 2\alpha_s (\varepsilon \pm m) \\
2MC &= \gamma_r^2,
\end{aligned}
\tag{50}
$$

and giving, after properly squaring the original equation, an algebraic equation of degree 6 in ε. For $\alpha_s = 0$, though, one obtains a quadratic algebraic equation with the solutions

$$\varepsilon = \frac{-\beta_v \tau \pm \sqrt{\beta_v^2 \tau^2 + (\beta_v^2 + \zeta^2) \left[4\zeta^2 (m^2 + \gamma_r^2) - \tau^2\right]}}{2 (\beta_v^2 + \zeta^2)}, \tag{51}$$

where

$$\tau = 2\left[\gamma_r (\beta_r + k) - \gamma_0^2/2 + \beta_s m\right] \tag{52}$$

In the case of a pure nonminimal vector potential one finds

$$\varepsilon = \pm \sqrt{m^2 + \gamma_r^2 - \left[\tau/(2\zeta)\right]^2}. \tag{53}$$

3.2.5. A special case
The very special case

$$V_r = \frac{\beta_r}{r}, \quad V_0 = \frac{\beta_0}{r}, \quad V_s = \frac{\beta_s}{r}, \quad V_v = \pm V_s \tag{54}$$

consists in particular cases of the first three preceding cases, yielding a spectrum for $m \neq 0$ given by

$$\varepsilon = \pm m \frac{1 - (\beta_s/\zeta)^2}{1 + (\beta_s/\zeta)^2}. \tag{55}$$

This spectrum is formally identical to the relativistic Coulomb potential for Klein-Gordon and Dirac equations in the condition of spin or pseudospin symmetry mentioned before ($V_v = \pm V_s$). The reason is that the nonminimal Coulomb potentials enter only in the expression of ζ, which is given by

$$\zeta = n + 1/2 + \sqrt{(l + k - 1/2)^2 + \beta_r (\beta_r - 1 + 2k) - \beta_0^2}. \tag{56}$$

When there are only scalar and vector potentials $\zeta = n + 1/2 + |l + k - 1/2| = N$ and one recovers the expressions for the energy of a D-dimensional relativistic Coulomb spin-0 system in spin or pseudospin symmetry conditions (if D is odd, N is an integer, and for $D = 3$ it is the principal quantum number of the non-relativistic hydrogen atom). One may note that, contrary to general case of some examples discussed before, here the choice $V_s = V_v$ (spin condition) implies that there is only one (positive) energy bound solution and the other choice $V_s = -V_v$ (pseudospin condition) means that there is only a negative energy bound solution. As might be expected, this is exactly what happens in the corresponding (3-dimensional) Dirac equation [80].

4. Concluding remarks

Based on Ref. [38], we have described a straightforward and efficient procedure for finding a large class of new solutions of the D-dimensional Klein-Gordon equation with radial scalar, vector, and nonminimal coupling potentials, whose wave functions are all expressed in terms of generalized Laguerre polynomials and whose energy eigenvalues obey analytical equations, either polynomial or irrational which can be cast as polynomial. These include harmonic oscillator-type and Coulomb-type potentials and their extensions. Although the solutions for those systems could be found by standard methods, this procedure, based on the mapping from the one-dimensional generalized Morse potential via a Langer transformation to the D-dimensional radial Klein-Gordon equation, provide an easier and powerful way to find the solutions of a very general class of potentials which otherwise one might not know that would have analytical solutions in the first place. We were able to reproduce well-known particular cases of relativistic harmonic oscillator and Coulomb spin-0 systems, especially when the scalar and vector potentials have the same magnitude, but there are a wealth of other particular cases with physical interest that are left for further study.

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