Solutions of the three-dimensional Dirac equation via mapping onto the nonrelativistic one-dimensional Morse potential

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Abstract

New exact analytical bound-state solutions of the 3+1 Dirac equation for sets of couplings and radial potential functions are obtained via mapping onto the nonrelativistic bound-state solutions of the one-dimensional generalized Morse potential. The eigenfunctions are expressed in terms of generalized Laguerre polynomials, and the eigenenergies are expressed in terms of solutions of equations that can be transformed into polynomial equations. Several analytical results found in the literature, including the Dirac oscillator, are obtained as particular cases of this unified approach.

Keywords: Klein-Gordon, DKP, Morse,

1. Introduction

In nonrelativistic quantum mechanics there are several potential with physical interest that allow for exact solutions, thus offering the possibility of extracting physical information in a way which is not possible otherwise. Among them is the the generalized Morse potential $Ae^{-\alpha x} + Be^{-2\alpha x}$ [1]-[6], the singular harmonic oscillator (SHO) $Ax^2 + Bx^{-2}$ [3], [7]-[22], and the singular Coulomb potential (SCP) $Ax^{-1} + Bx^{-2}$ [3], [7]-[10], [19], [21], [23]-[29], which have played an important role in atomic, molecular and solid-state physics.

In a recent paper [30], it was shown that nonrelativistic bound-state solutions of the well-known SHO and SCP in arbitrary dimensions can be systematically generated from the nonrelativistic bound states of the one-dimensional generalized Morse potential. The method amounts to a mapping via a Langer transformation [31]. In the present paper, that method is extended to the 3+1 Dirac equation with scalar, vector and tensor radial potentials. This extension of the method used in Ref. [30] is an interesting way of providing an unified treatment of many known relativistic problems via a mapping onto a unique well-known one-dimensional nonrelativistic problem, allowing to obtain some new exact analytical bound-state solutions for a large class of problems including new types of couplings and potential functions. We highlight vector-scalar SHO plus nonminimal vector Cornell potentials and nonminimal vector Coulomb (space component) and harmonic oscillator (time component) potentials, vector-scalar Coulomb plus nonminimal vector Cornell potentials

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and nonminimal vector shifted Coulomb potentials, vector-scalar SCP plus nonminimal vector Coulomb potentials, and also a pure nonminimal vector constant potential. Furthermore, we show that several exactly soluble bound states explored in the literature are obtained as particular cases of those cases. In all those circumstances the eigenfunctions are expressed in terms of the generalized Laguerre polynomials and the eigenenergies are expressed in terms of irrational equations.

The paper is organized as follows. In Sec. II we review, as a background, the generalized Morse potential in the Schrödinger equation. The Dirac with vector, scalar and tensor couplings and its connection with the generalized Morse potential and the proper form for the potential functions, are presented in Sec. III. In Sec. IV we draw some conclusions.

2. Nonrelativistic bound states in a one-dimensional generalized Morse potential

The time-independent Schrödinger equation is an eigenvalue equation for the characteristic pair (E, ψ) with $E \in \mathbb{R}$. For a particle of mass M embedded in the generalized Morse potential it reads

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2M}{\hbar^2} \left(E - V_1 e^{-\alpha x} - V_2 e^{-2\alpha x} \right) \psi(x) = 0, \tag{1}$$

where $\alpha > 0$. Bound-state solutions demand $\int_{-\infty}^{+\infty} dx \, |\psi|^2 = 1$ and occur only when the generalized Morse potential has a well structure ($V_1 < 0$ and $V_2 > 0$). The eigenenergies are given by

$$E_n = -\frac{V_1^2}{4V_2} \left[1 - \frac{\hbar \alpha \sqrt{2MV_2}}{M|V_1|} \left(n + \frac{1}{2} \right) \right]^2. \tag{2}$$

with

$$n = 0, 1, 2, \ldots < \frac{M|V_1|}{\hbar \alpha \sqrt{2MV_2}} - \frac{1}{2}. \tag{3}$$

This restriction on n limits the number of allowed states and requires $M|V_1| / \left(\hbar \alpha \sqrt{2MV_2} \right) > 1/2$ to make the existence of a bound state possible. On the other hand, on making the substitutions

$$\hbar \alpha s_n = \sqrt{-2ME_n}, \quad \hbar \alpha \xi = 2 \sqrt{2MV_2} \, e^{-\alpha x}, \tag{4}$$

the eigenfunctions are expressed in terms of the generalized Laguerre polynomials as

$$\psi_n(\xi) = N_n \xi^{s_n} e^{-\xi/2} L_n^{(2s_n)}(\xi), \tag{5}$$

where N_n are arbitrary constants.

3. The Dirac equation

The time-independent Dirac equation for a spin $1/2$ fermion with energy ε and with mass m, in the presence of a potential reads (with $\hbar = c = 1$)

$$\left(\vec{\alpha} \cdot \vec{p} + \beta m + \mathcal{V} \right) \Psi = \varepsilon \Psi, \tag{6}$$

where $\vec{p}$ is the momentum operator and $\vec{\alpha}$ and β are 4×4 matrices which, in the usual representation, take the form

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{7}$$

Here $\vec{\sigma}$ is a three-vector whose components are the Pauli matrices, and I_N stands for the $N \times N$ identity matrix. In the following, we consider

$$\boldsymbol{V}(r) = V_v(r) + \beta V_s(r) + i\beta\vec{\alpha} \cdot \hat{r}U(r).\tag{8}$$

In the last term, $\hat{r} = \vec{r}/r$, and the radial functions in Eq. (8) are named after the properties their respective terms have under Lorentz transformations: V_v corresponds to the time component of a vector potential, V_s is a scalar potential, and U is a tensor potential [32]. In spherical coordinates, Ψ is expressed in terms of spinor spherical harmonics

$$\Psi(\vec{r}) = \begin{pmatrix} i\dfrac{g_\kappa(r)}{r}\mathcal{Y}_{\kappa m_j}(\hat{r}) \\[2mm] -\dfrac{f_{\tilde{\kappa}}(r)}{r}\mathcal{Y}_{\tilde{\kappa} m_j}(\hat{r}) \end{pmatrix},\tag{9}$$

where $\kappa = \pm(j + 1/2) = -\tilde{\kappa}$ are eigenvalues of the spin-orbit operator $K = -\beta\left(2\vec{S} \cdot \vec{L} + I_4\right)$, j is the total angular momentum quantum number (m_j refers to its third component), and $\vec{S}$ and $\vec{L}$ are the spin and angular momentum operators, respectively. More explicitly, the spin-orbit coupling quantum number κ is related to the upper component orbital angular momentum quantum number l by

$$\kappa = \begin{cases} -(l+1) = -(j+1/2), & j = l + 1/2 \ (\kappa < 0) \\[2mm] l = +(j+1/2), & j = l - 1/2 \ (\kappa > 0). \end{cases}\tag{10}$$

The upper and lower radial functions obey the coupled first-order equations:

$$\left[\frac{d}{dr} + \frac{\kappa}{r} + U(r)\right]g_\kappa(r) = [m + \varepsilon - V_\Delta(r)]\,f_{\tilde{\kappa}}(r)$$

$$\left[\frac{d}{dr} - \frac{\kappa}{r} - U(r)\right]f_{\tilde{\kappa}}(r) = [m - \varepsilon + V_\Sigma(r)]\,g_\kappa(r),\tag{11}$$

where we have introduced the "sum" and the "difference" potentials defined by $V_\Sigma = V_v + V_s$ and $V_\Delta = V_v - V_s$.

It is instructive to note that the charge-conjugation operation is accomplished by the changes of sign of ε, V_v, U and κ. In turn, this means that V_Σ turns into $-V_\Delta$, V_Δ into $-V_\Sigma$, g into f and f into g. Therefore, to be invariant under charge conjugation, the Dirac equation must contain only a scalar potential. Furthermore, g and f should be square-integrable functions for bound states.

Due to charge conjugation, solutions for $V_\Sigma = 0$ can be conveniently obtained from those ones for $V_\Delta = 0$, provided those solutions are analytical. These correspond, respectively, to so-called pseudospin and spin symmetry conditions of the Dirac equation (see [33] for a recent review). Therefore, we concentrate our attention to the case $V_\Delta = 0$. In this case, one obtains a second-order differential equation for g when $\varepsilon \neq -m$ and a first-order differential equation for g when $\varepsilon = -m$.

3.1. The Sturm-Liouville problem for $V_\Delta = 0$ ($\varepsilon \neq -m$)

For $V_\Delta = 0$ and $\varepsilon \neq -m$,

$$\frac{d^2 g_\kappa(r)}{dr^2} + 2M\left[\tilde{\varepsilon} - V(r) - \frac{\kappa(\kappa+1)}{2Mr^2}\right]g_\kappa(r) = 0,\tag{12}$$

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with M denoting a positive parameter having dimension of mass. The effective energy $\widetilde{\varepsilon}$ and the effective potential V are expressed by

$$
\begin{aligned}
2M\widetilde{\varepsilon} &= \varepsilon^2 - m^2 \\
2MV &= (\varepsilon + m)V_\Sigma - \frac{dU}{dr} + 2\kappa\frac{U}{r} + U^2,
\end{aligned}
\tag{13}
$$

$g_\kappa \to 0$ as $r \to \infty$ for bound-state solutions.

Following Ref. [30], with effective potentials expressed by

$$
V(r) = Ar^\delta + \frac{B}{r^2} + C, \quad \delta = +2, -1
\tag{14}
$$

the Langer transformation [31]

$$
g_\kappa(r) = \sqrt{r/r_0}\,\phi_\kappa(x), \quad r/r_0 = e^{-\Lambda\alpha x},
\tag{15}
$$

with $r_0 > 0$, $\Lambda > 0$ and α being as in eq. (1), transmutes the radial equation (12) into

$$
\frac{d^2\phi_\kappa(x)}{dx^2} + 2M\left\{-\frac{(\Lambda\alpha S)^2}{2M} - (\Lambda\alpha r_0)^2\left[Ar_0^\delta e^{-\Lambda\alpha(\delta+2)x} + (C - \widetilde{\varepsilon})\,e^{-2\Lambda\alpha x}\right]\right\}\phi_\kappa(x) = 0,
\tag{16}
$$

with

$$
S = \sqrt{(\kappa + 1/2)^2 + 2MB}.
\tag{17}
$$

A connection with the bound states of the generalized Morse potential of eq. (1) is obtained if the pair (δ, Λ) is equal either to $(2, 1/2)$ or $(-1, 1)$, and, as an immediate consequence of the reality of S, i.e., $S^2 > 0$, one must have

$$
2MB > -(\kappa + 1/2)^2.
\tag{18}
$$

Actually, if $2MB > -1/4$, the above condition will satisfied for all values of κ.

Furthermore, since the asymptotic behaviour of (16) implies that $\phi_\kappa(x) \underset{x\to+\infty}{\to} e^{-\Lambda\alpha S\,x}$ and therefore, from (15), one has

$$
g_\kappa(r) \underset{r\to 0}{\to} r^{1/2+S}.
\tag{19}
$$

Effective potentials with the general form (14) are achieved by choosing the potentials in the Dirac equation as follows

$$
\begin{aligned}
V_\Sigma(r) &= \alpha_\Sigma/r^2 + \beta_\Sigma/r + \gamma_\Sigma r^2, \tag{20}\\
U(r) &= \beta_u/r + \gamma_u r^{\delta_u}, \quad \delta_u = 0 \text{ or } 1 . \tag{21}
\end{aligned}
$$

In these last expressions, when $\delta = 2$ one must have $\beta_\Sigma = 0$, $\delta_u = 1$ and when $\delta = -1$ one has $\gamma_\Sigma = 0$, $\delta_u = 0$.

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3.1.1. The effective singular harmonic oscillator

With $(\delta, \Lambda) = (2, 1/2)$ plus the definition $A = M\omega^2/2$, the identification of the bound-state solutions of Eq. (12) with those ones from the generalized Morse potential is done by setting $V_1 = -\alpha^2 r_0^2 (\widetilde{\varepsilon} - C)/4$ and $V_2 = \alpha^2 r_0^4 M\omega^2/8$, with $\widetilde{\varepsilon} > C$ and $\omega^2 > 0$, since $V_1 < 0$ and $V_2 > 0$. With $\omega > 0$ one can write

$$\xi = M\omega r^2. \tag{22}$$

Furthermore, (3) implies $\widetilde{\varepsilon} > C + \omega(2n + 1)$. Using (2) and (17) one can write the complete solution of the problem as

$$\widetilde{\varepsilon} = C + \omega(2n + 1 + S) \tag{23}$$
$$g_\kappa(r) = Nr^{1/2+S}e^{-M\omega r^2/2}L_n^{(S)}\left(M\omega r^2\right), \tag{24}$$

The condition (3) means that

$$n \leq \left[\frac{\widetilde{\varepsilon} - C - \omega}{2\omega}\right] \tag{25}$$

where $[x]$ stands for the largest integer less or equal to x. Since $\widetilde{\varepsilon}$ depends quadratically on ε from eqs. (13) and ω may depend at most on $\sqrt{\varepsilon}$ (see eq. (27) below), the condition (25) means that there is no limitation on the value of n, because it can be as large as the energy can, which in turns means that n in (23) has no upper bound.

Examples of this class of solutions can be reached by choosing

$$V_\Sigma(r) = \frac{\alpha_\Sigma}{r^2} + \gamma_\Sigma r^2, \quad U(r) = \frac{\beta_u}{r} + \gamma_u r. \tag{26}$$

This includes solutions like the harmonic oscillator plus a tensor linear potential [34], the harmonic oscillator plus a tensor Cornell potential [35], [36], the SHO plus a tensor linearpotential [37], the SHO [39], the tensor Cornell potential [40] and the Dirac oscillator [41].

The complete identification with the generalized Morse potential is done with the equalities

$$\begin{aligned} M\omega &= \sqrt{\gamma_u^2 + \gamma_\Sigma(\varepsilon + m)} \\ 2MB &= (\beta_u + \kappa + 1/2)^2 - (\kappa + 1/2)^2 + \alpha_\Sigma(\varepsilon + m) \\ 2MC &= \gamma_u(2\beta_u + 2\kappa - 1), \end{aligned} \tag{27}$$

which lead, in general, to an irrational equation in ε:

$$(\varepsilon + m)(\varepsilon - m) - \gamma_u(2\beta_u + 2\kappa - 1) = 2(2n + 1 + S)\sqrt{\gamma_u^2 + \gamma_\Sigma(\varepsilon + m)}$$
$$= 2\left(2n + 1 + \sqrt{(\beta_u + \kappa + 1/2)^2 + \alpha_\Sigma(\varepsilon + m)}\right)\sqrt{\gamma_u^2 + \gamma_\Sigma(\varepsilon + m)} \tag{28}$$

We note that if $\alpha_\Sigma > 0$, $\gamma_\Sigma > 0$ and $\beta_u = 0$, one gets a harmonic oscillator type energy spectrum for positive energy states with $\varepsilon > m$, but there also states with negative energy, although there would a minimum value for that energy, because one must have $(\kappa + 1/2)^2 + \alpha_\Sigma(\varepsilon + m) \geq 0$. If in addition $\alpha_\Sigma = 0$, one has the (positive energy) generalized relativistic harmonic oscillator with $\gamma_\Sigma = 1/2\, m\omega_1^2$, $\gamma_u = m\omega_2$ where ω_1 and ω_2 are the frequencies defined in [34].

Squaring Eq. (28) successively results into a nonequivalent algebraic equation of degree 8. Solutions of this algebraic equation that are not solutions of the original equation can be removed by backward substitution. A quartic algebraic equation is obtained when $\alpha_\Sigma = 0$. For $\alpha_\Sigma = \gamma_u = 0$ one obtains a cubic

algebraic equation. However, (28) can be written as a quadratic algebraic equation rendering two branches of solutions symmetrical about $\varepsilon = 0$ in the case of a pure tensor Cornell potential ($\gamma_u \neq 0$):

$$
\begin{aligned}
\varepsilon &= \pm \sqrt{m^2 + \gamma_u (2\beta_u + 2\kappa - 1) + 2|\gamma_u| (2n + 1 + S)} \\
&= \pm \sqrt{m^2 + \gamma_u (2\beta_u + 2\kappa - 1) + 2|\gamma_u| (2n + 1 + |\beta_u + \kappa + 1/2|)}.
\end{aligned}
\tag{29}
$$

3.1.2. The effective singular Coulomb potential

To get the bound states of eq. (12) from those of the generalized Morse potential equation eq. (16) with the pair $(\delta, \Lambda) = (-1, 1)$ on must choose $V_1 = \alpha^2 r_0 A$ and $V_2 = \alpha^2 r_0^2 (C - \widetilde{\varepsilon})$, with $A < 0$ and $\widetilde{\varepsilon} < C$. Now,

$$
\xi = 2\sqrt{2M(C - \widetilde{\varepsilon})}\, r
\tag{30}
$$

and (3) implies $\widetilde{\varepsilon} > C - MA^2/[2(n + 1/2)^2]$. Using (2) and (17) one can write

$$
\widetilde{\varepsilon} = C - \frac{MA^2}{2\zeta^2},
\tag{31}
$$

$$
g_\kappa(r) = N r^{1/2+S} e^{-M|A|r/\zeta} L_n^{(2S)}\left(\frac{2M|A|r}{\zeta}\right).
$$

with

$$
\zeta = n + 1/2 + S = n + 1/2 + \sqrt{(\kappa + 1/2)^2 + 2MB}.
\tag{32}
$$

This class of solutions can be obtained by choosing

$$
V_\Sigma(r) = \frac{\alpha_\Sigma}{r^2} + \frac{\beta_\Sigma}{r}, \qquad U(r) = \frac{\beta_u}{r} + \gamma_u.
\tag{33}
$$

There results

$$
\begin{aligned}
2MA &= \beta_\Sigma (\varepsilon + m) + 2\gamma_u (\beta_u + \kappa) \\
2MB &= (\beta_u + \kappa + 1/2)^2 - (\kappa + 1/2)^2 + \alpha_\Sigma (\varepsilon + m) \\
2MC &= \gamma_u^2,
\end{aligned}
\tag{34}
$$

in such a way that one finds the irrational equation in ε

$$
(\varepsilon + m)(\varepsilon - m) = \gamma_u^2 - \left[\frac{2\gamma_u (\beta_u + \kappa) + \beta_\Sigma (\varepsilon + m)}{2\left(n + 1/2 + \sqrt{(\beta_u + \kappa + 1/2)^2 + \alpha_\Sigma (\varepsilon + m)}\right)}\right]^2,
\tag{35}
$$

One example of solutions for these type of radial potentials in the Dirac equation is the Coulomb potential plus a tensor Coulomb potential [42], and the SCP plus a tensor Coulomb potential [43], [44].

The very special case $\alpha_\Sigma = \gamma_u = 0$, necessarily with $\beta_\Sigma < 0$, holds a spectrum given by

$$
\varepsilon = m \frac{1 - [\beta_\Sigma/(2\zeta)]^2}{1 + [\beta_\Sigma/(2\zeta)]^2}.
\tag{36}
$$

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with $\zeta = n + 1/2 + |\beta_u + \kappa + 1/2|$. It is interesting that the dependance on the tensor potential parameter β_u is done only through ζ, which contains the quantity $2MB$. Therefore, the spectrum is formally similar to the solution of pure ($\beta_u = 0$) Coulomb scalar and vector potentials in spin symmetry conditions [45]. It amounts to have an effective value of κ, given by $\bar{\kappa} = \kappa + \beta_u$.

É também interessante ver o caso de um potencial tensorial puro ($\alpha_\Sigma = \beta_\Sigma = 0$),

$$U(r) = \frac{\beta_u}{r} + \gamma_u, \tag{37}$$

com o espectro dado por

$$\varepsilon = \pm \sqrt{m^2 + \gamma_u^2 \left[1 - \left(\frac{\beta_u + \kappa}{\zeta} \right)^2 \right]}. \tag{38}$$

Como $A < 0$ e usando a equação (32) encontramos que $\gamma_u (\beta_u + \kappa) < 0$. Dessa inequação e da equação de autovalor percebe-se que não há estados ligados para $\gamma_u = 0$. Se $\beta_u = 0$ então $\gamma_u \kappa < 0$ e teremos estados ligados para partículas com spin alinhado ou desalinhado dependendo do sinal de γ_u. Nesta situação, no entanto, o espectro é o mesmo em ambos os casos, pois não depende do sinal de $\gamma_u \kappa$. Para que haja estados ligados para $\beta_u \neq 0$ e $\gamma_u < 0$, a parte coulombiana do potencial tensorial tem que ter um limite inferior dado por $\beta_u > \mp |\kappa|$ para spin alinhado e desalinhado respectivamente. Então, pode-se concluir que nesta situação, só é possível ter estados ligados de partículas com spin *desalinhado* para um potencial coulombiano repulsivo, e para spin *alinhados*, pode-se ter estados ligados com um potencial atrativo ou repulsivo. Por outro lado, para que haja estados ligados para $\beta_u \neq 0$ e $\gamma_u > 0$ devemos ter um limite superior para a parte coulombiana do potencial tensorial dado por $\beta_u < \mp |\kappa|$, para spin alinhado e desalinhado respectivamente, donde se conclui que nesta situação, só é possível ter estados ligados de partículas de spin *alinhado* para um potencial atrativo, e ara spin *desalinhado*, pode-se ter estados ligados com um potencial atrativo ou repulsivo. Desta vez, o espectro não será necessariamente o mesmo para spin alinhado e desalinhado. É engraçado que β_u sozinho não consiga gerar estados ligados, ou seja, para o caso do potencial puramente tensorial, a parte constante do potencial é mais "forte" do que a parte coulombiana. Como já tínhamos visto, um potencial tensorial constante é capaz de gerar estados ligados.

3.2. Isolated solutions for $V_\Delta = 0$ ($\varepsilon = -m$)

We shall now deal with possible solutions for potentials that can not be expressed by means of the Sturm-Liouville problem. For $V_\Delta = 0$ and $\varepsilon = -m$, one can write

$$g_\kappa(r) = N_g e^{-v(r)}, \quad f_\kappa(r) = \left[N_f + N_g I(r) \right] e^{+v(r)}, \tag{39}$$

where N_g and N_f are constants, and

$$v(r) = \int^r dy \left[\frac{\kappa}{y} + U(y) \right], \quad I(r) = \int^r dy \left[2m + V_\Sigma(y) \right] e^{-2v(y)}. \tag{40}$$

It is worthwhile to note that this sort of isolated solution can not describe scattering states.

Setting

$$(\delta_u + 1) \lambda = 2\gamma_u, \quad (\delta_u + 1) \tau = -2 (\beta_u + \kappa), \tag{41}$$

one finds

$$2v(r) = -(\delta_u + 1) \tau \ln r + \lambda r^{\delta_u + 1} \tag{42}$$

7

and

$$
\begin{aligned}
(\delta_u + 1)\lambda^\tau I(r) \;=\; & 2m\lambda^{-1/(\delta_u+1)}\gamma\left(\frac{1}{\delta_u + 1} + \tau,\, \lambda r^{\delta_u+1}\right) \\[2mm]
& +\alpha_\Sigma \lambda^{1/(\delta_u+1)}\gamma\left(-\frac{1}{\delta_u + 1} + \tau,\, \lambda r^{\delta_u+1}\right) \\[2mm]
& +\beta_\Sigma \gamma\left(\tau,\, \lambda r^{\delta_u+1}\right) \\[2mm]
& +\gamma_\Sigma \lambda^{-3/(\delta_u+1)}\gamma\left(\frac{3}{\delta_u + 1} + \tau,\, \lambda r^{\delta_u+1}\right)
\end{aligned}
\tag{43}
$$

where $\gamma(a, z)$ is the incomplete gamma function [46]

$$
\gamma(a, z) = \int_0^z dt\, e^{-t} t^{a-1},\ \ \mathrm{Re}\ a > 0.
\tag{44}
$$

Hence,

$$
\begin{aligned}
g_\kappa(r) \;=\;& N_g\, r^{+(\delta_u+1)\tau/2} e^{-\lambda r^{\delta_u+1}/2} \\[2mm]
f_\kappa(r) \;=\;& \left[N_f + N_g I(r)\right] r^{-(\delta_u+1)\tau/2} e^{+\lambda r^{\delta_u+1}/2}
\end{aligned}
\tag{45}
$$

A proper asymptotic behaviour for $\lambda < 0$ ($\gamma_u < 0$) requires $N_g = 0$. Thus,

$$
\begin{aligned}
g_\kappa(r) \;=\;& 0 \\[2mm]
f_\kappa(r) \;=\;& N_f\, r^{-(\delta_u+1)\tau/2} e^{+\lambda r^{\delta_u+1}/2}
\end{aligned}
\tag{46}
$$

regardless of $\alpha_\Sigma, \beta_\Sigma$ and m. Nevertheless, a good behaviour of f_κ near the origin, in the sense of normalization, forces one to the choice $(\delta_u + 1)\tau < 2$, i.e. $\beta_u + \kappa > -1$.

As for $\lambda > 0$ ($\gamma_u > 0$), a good behaviour at infinity requires $N_f = 0$. Hence,

$$
\begin{aligned}
g_\kappa(r) \;=\;& N_g\, r^{+(\delta_u+1)\tau/2} e^{-\lambda r^{\delta_u+1}/2} \\[2mm]
f_\kappa(r) \;=\;& N_g I(r)\, r^{-(\delta_u+1)\tau/2} e^{+\lambda r^{\delta_u+1}/2}
\end{aligned}
\tag{47}
$$

We must, however, pay attention to the behaviour of $I(r)$ at infinity. As z increases $\gamma(a, z)$ approaches the limiting value $\Gamma(a)$ so that f_κ is not a square-integrable function. An exception, though, occurs when $m = \alpha_\Sigma = \beta_\Sigma = 0$ just for the reason that f_κ vanishes identically. Therefore,

$$
\begin{aligned}
g_\kappa(r) \;=\;& N_g\, r^{+(\delta_u+1)\tau/2} e^{-\lambda r^{\delta_u+1}/2} \\[2mm]
f_\kappa(r) \;=\;& 0
\end{aligned}
\tag{48}
$$

with $(\delta_u + 1)\tau > -2$, i.e. $\beta_u + \kappa < 1$.

4. Concluding remarks

Based on Ref. [30], we have described a straightforward and efficient procedure for finding a large class of new solutions of the 3+1 Dirac equation with radial scalar V_s vector V_v and tensor U radial potentials, when has $V_s = \pm V_v$. Their wave functions are all expressed in terms of generalized Laguerre polynomials and their energy eigenvalues obey analytical equations, either polynomial or irrational which can be cast as polynomial. These include harmonic oscillator-type and Coulomb-type potentials and their extensions. Although the solutions for those systems could be found by standard methods, this procedure, based on the mapping from the one-dimensional generalized Morse potential via a Langer transformation to the $3 + 1$ radial Dirac equation, provide an easier and powerful way to find the solutions of a very general class of potentials which otherwise one might not know that would have analytical solutions in the first place. We were able to reproduce well-known particular cases of relativistic harmonic oscillator and Coulomb spin-1/2 systems, when the scalar and vector potentials have the same magnitude, but there are a wealth of other particular cases with physical interest that are left for further study.

Acknowledgement

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