UNIVERSIDADE DE COIMBRA

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A study on fundamental concepts of lower semicontinuous and extended convex functions in Rn

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Abstract

In this dissertation we show the main ideas concerning on convex sets (in an understood way) and extended convex functions. Our aim is to deal, didactically, with the main topics about convexity, as well as the consequent we explore the the involved concepts in Convex Analysis. In this sense, we carried out a bibliographical review that contemplated theorems, lemmas, corollaries and relevant propositions in several Articles, and Books. We hope that this study may constitute an important research source either for students, teachers or researchers who wish to learn more about lower semicontinuous and extended convex function in $\mathbb{R}^n$.

Keywords: Lower semicontinuous function, extended convex function, convex envelope, convexity.

Resumo

Nesta dissertação, apresentamos as principais ideias concernentes aos conjuntos convexos (de forma subentendida) e às funções convexas estendidas. Nosso principal foco é tratar, de forma didática, os principais tópicos envolvidos na Análise Convexa, assim como a consequente exploração dos conceitos matemáticos envolvidos. Nesse sentido, realizamos uma revisão bibliográfica que contemplou teoremas, lemas, corolários e proposições relevantes em diversos Artigos, e Livros. Assim, esperamos que este material constitua uma importante fonte de pesquisa a estudantes, professores e pesquisadores que almejem estudar conteúdos relacionados as funções estendidas em $\mathbb{R}^n$ semicontínuas inferiormente e convexas.

Palavras chave: Funções semicontínuas inferiores, funções convexas extendidas, convexidade.

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Chapter 1

Introduction

The research for this thesis was motivated by the article [5], where the goal is to replace a nonconvex objective function by the best possible convex underestimator. This process is well-known by *Convexification* and several approaches to do it have been proposed in recent years, see [7] and [10]. Once a good convex underestimator is found, the underlying global optimization of the original objective function is an algorithm of the branch-and-bound type (ver um artigo de tawarmalani em global optimization).

In this thesis, we review basic topological concepts related to closed functions, to convex functions and both. The functions we are interested in are of the type $f\ colon\mathbb{R}^n \to [-\infty, \infty]$, so-called extended functions in $\mathbb{R}^n$. Most of the known basic results on convex analysis apply to functions that are proper (i.e., , never $-\infty$ and not $+\infty$ everywhere). Since the tools that we have in mind using [4] generate functions that are not proper, we thought have revisit basic results on convex and/or closed functions to understand where the theory might fail. A paragraph on [8, pp. 24] roughly explains what are the remedies when the results apply to extended functions, but it is not obvious or clear that it does work.

1.1 Notation

A quick review of the notation follows, which is fairly standard. For a set $K \subseteq \mathbb{R}^n$, $\mathrm{int}(K)$ denotes the interior of K, $\mathrm{cl}(K)$ the closure and $\mathrm{aff}(K)$ the affine hull. A set K is convex if it contains the line segment between any two points in K. The $\mathrm{conv}(K)$ represents the convex hull of K, the smallest convex set that contains K. The $\mathrm{ri}(K)$ denotes the relative interior of a convex set K, a slightly more complex definition. For $x \in \mathbb{R}^n$ and $r \geq 0$, we use $\mathscr{B}(x;r)$ to denote the ball of radius r centered at x.

Theoretical results on convex sets will in general be assumed - from the classical [8], or the more recent [6] with a focus on optimization. Some of most relevant to our work are, e.g. ,

Lemma 1.1.1 *If $K \subseteq \mathbb{R}^n$ is a convex set, $x \in \mathrm{ri}(K)$ and $y \in \mathrm{cl}(K)$ then, $x + t(y - x) \in \mathrm{ri}(K)$, for every $t \in [0, 1)$.*

Lemma 1.1.2 *If $A\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping and $K \subseteq \mathbb{R}^n$ is a convex set then, $A(K) \equiv \{Ax\colon x \in K\} \subseteq \mathbb{R}^m$ is also a convex set.*

Lemma 1.1.3 *If $A\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping and $K \subseteq \mathbb{R}^n$ is a convex set then, $A(\mathrm{ri}(K)) = \mathrm{ri}(A(K))$.*

Lemma 1.1.4 *If $C, K \in \mathbb{R}^n$ are convex sets such that* $\mathrm{ri}(C) \cap \mathrm{ri}(K) \neq \emptyset$ *then,* $\mathrm{ri}(C \cap K) = \mathrm{ri}(C) \cap \mathrm{ri}(K)$ *and* $\mathrm{cl}(C \cap K) = \mathrm{cl}(C) \cap \mathrm{cl}(K)$.

Let $A, B \subset \mathbb{R}^n$. The Minkwoski sum of two sets A and B, namely $A + B = \{a + b \colon a \in A, b \in B\}$. In particular if either A or B is the emptyset then $A + B$ is the emptyset. The multiplication of nonnegative scalar λ with a set A defined as $\lambda \cdot A = \{\lambda a \colon a \in A\}$. In particular, if A is the emptyset then $\lambda \cdot A$ is the emptyset.

Lemma 1.1.5 *If $C \subseteq \mathbb{R}^n$ is convex and contains the origin then λC is convex and increasing in $\lambda > 0$.*

Proof: The proof that λC is convex is straighforward. Let $0 < \lambda < \mu$, arbitrary and let $x = \lambda y \in \lambda C$, for some $y \in C$. Since both $0, \mu y \in \mu C$ and $x = \lambda y = (1 - \lambda/\mu)0 + (\lambda/\mu)(\mu y)$, where $\lambda/\mu \in (0, 1)$, we conclude that $x \in \mu C$. □

1.2 Structure of the thesis

This is a synthesis work on the concepts of convex analysis that seems to us to be fundamental to that objective, and trying to make the definitions as geometric as possible because they convey more intuition without forget that their algebraic version allow numerical calculation.

Therefore, in chapter 2 we define the epigraphic set and study its properties on this context, and their operations. In Chapter 3, we define function lower semicontinuous, or closed, and study its properties, where, we explain why it is relevant in the context of optimization to handle functions that are lower semicontinuous. In chapter 4 we define convex function and study its most elementary properties, but excluding differentiability, trying to highlight the relevance of being or not own, be closed or not. In Chapter 5, we study the recession function, starting from the concept of recession cone, in order to characterize the perspective function that is a function that preserves the convexity. In Chapter 6 we apply the disjunctive programming tool to finding the convex envelope of x/y on a (positive) rectangle. Though the results are not fully new, see [9] and [10], we believe that the arguments may lead to novel ways of finding the convex envelope of other nonconvex functions. And finally the conclusion.

Chapter 2

Basic concepts with functions

In this chapter we present the main theoretical foundations that we use in the development of this thesis and we describe the settings we wish to consider and introduce some terminology which will be used throughout. We also provide some general results on the construction the epigraphical hull of a certain function over a general domain, some propositions are posed in a more specific way, attending to our needs in the development of the thesis. We state some definitions and results that are derived from the convex analysis, see , [6], [7] and [8]. In our development we use the function in extended value, i.e., , if f is real function $f\colon \mathbb{R}^n \to [-\infty, +\infty]$. The domain is always $\mathbb{R}^n$.

2.1 The epigraph

In this section we start with the definitions of the epigraph set, which is loosely speaking all the points that are above the graph including the graph line itself. Distinguish weather the epigraph whose functions value $-\infty$ or not or simply formed with real numbers.

Definition 2.1.1 *Given a function f*

$$\begin{array}{llll} \mathrm{Epi}(f) & \stackrel{\text{def}}{=} & \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \colon f(x) \le t\} & \textit{is the epigraph of } f; \\ \mathrm{epi}(f) & \stackrel{\text{def}}{=} & \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \colon f(x) \le t, f(x) \in \mathbb{R}\} & \textit{is the finite epigraph of } f. \end{array}$$

The definition between the two concepts is our decision. The reason will become throughout the text.

Not only the function values determine the epigraph, the epigraph also determines the function values through

$$f(x) = \inf\{t \colon t \in \mathrm{Epi}(f)|_x\}, \tag{2.1}$$

where, $\mathrm{Epi}(f)|_x = \{t \colon f(x) \le t, \text{ for some } x \in \mathrm{Epi}(f)\}$, as convention, we assume that the infimum over the empty set is $+\infty$.

Definition 2.1.2 *We say that $E \subseteq \mathbb{R}^n \times \mathbb{R}$ is an epigraph set if, for every $x \in \mathbb{R}^n$, the set $E|_x$ is either $\emptyset, \mathbb{R}$ or a closed interval $[\bar{t}, +\infty)$.*

The set $\mathrm{Epi}(f)$ is always an epigraph set. If E is an epigraph set then, for some f, $E = \mathrm{Epi}(f)$. The effective domain of f is the set of objects where the function value is either a real number or $-\infty$.

The next definitions is derived from epigraph definitions as follows:

Definition 2.1.3 *Given a function f*

$$\begin{array}{lll} \mathrm{Dom}(f) & \stackrel{\text{def}}{=} & \{x \in \mathbb{R}^n : \ \mathrm{Epi}(f)|_x \neq \emptyset\} \quad \textit{is the effective domain of } f; \\ \mathrm{dom}(f) & \stackrel{\text{def}}{=} & \{x \in \mathbb{R}^n : \ \mathrm{Epi}(f)|_x \neq \emptyset, \mathbb{R}\} \quad \textit{is the finite effective domain of } f. \end{array}$$

The operation of taking linear combinations of functions will be prevalent throughout the text. Some simple technical difficulties are avoided if we restrict the analysis to proper functions.

Definition 2.1.4 *We say that a function f is proper if $f(x) > -\infty$, for all $x \in \mathbb{R}$, and $f \not\equiv +\infty$.*

If f is proper then $\mathrm{Epi}(f) = \mathrm{epi}(f)$ and $\mathrm{Dom}(f) = \mathrm{dom}(f)$. A function that is not proper is called *improper*. Improper functions do arise from proper ones in many natural situations. Moreover, simple operations with proper functions, e.g. the sum are not assured to be well defined.

Certain properties of the function f can be cast into properties of the epigraph. For example, a function is proper if and only if its epigraph is nonempty and it does not contain a vertical line. The following result is another example.

Proposition 2.1.5 *The set* $\mathrm{Epi}(f)$ *is a cone if and only if f is positively homogeneous (i.e., , $f(\lambda x) = \lambda f(x)$,for every $x \in \mathbb{R}^n$ and every $\lambda > 0$).*

Proof: Assume f is positively homogeneous. Then, for every $\lambda > 0$,

$$(x,t) \in \mathrm{Epi}(f) \ \Rightarrow \ f(x) \leq t \ \Rightarrow \ \lambda f(x) = f(\lambda x) \leq \lambda t \ \Rightarrow \ (\lambda x, \lambda t) = \lambda(x,t) \in \mathrm{Epi}(f).$$

Reciprocally, assume $\mathrm{Epi}(f)$ is a cone. Then, for every $\lambda > 0$,

$$\begin{aligned} f(\lambda x) &= \inf\{t : \ (\lambda x, t) \in \mathrm{Epi}(f)\} \\ &= \lambda \inf\{t/\lambda : \ \lambda(x, t/\lambda) \in \mathrm{Epi}\, f\} \\ &= \lambda f(x). \end{aligned}$$

□

2.2 The epigraphical hull

Given a nonconvex epigraph, a natural idea is to take a convex hull of its epigraph.

Proposition 2.2.1 *The intersection of a family of epigraphs is an epigraph set.*

Proof: Denote by $\mathscr{F}$ a family of epigraph sets $F \subseteq \mathbb{R}^n \times \mathbb{R}$ and let $E = \cap_{F \in \mathscr{F}} F$. For a fixed $x \in \mathbb{R}^n$,

$$E|_x = \left\{ t : \ (x,t) \in \bigcap_{F \in \mathscr{F}} F \right\} = \bigcap_{F \in \mathscr{F}} (F|_x).$$

If $F|_x = \emptyset$, for some $F \in \mathscr{F}$, then $E|_x = \emptyset$; if $F|_x = \mathbb{R}$, for every $F \in \mathscr{F}$, then $E|_x = \mathbb{R}$; Otherwise,

$$E|_x = \cap_{F \in \mathscr{F}'} [\bar{t}_F, +\infty),$$

for some nonempty $\mathscr{F}' \subseteq \mathscr{F}$, where $\bar{t}_F \in \mathbb{R}$, for each $F \in \mathscr{F}'$. Let

$$\bar{t} = \sup \left\{ t_F : F \in \bigcap F' \right\}.$$

If $\bar{t} = +\infty$ then $E|_x = \emptyset$, while if $\bar{t} \in \mathbb{R}$ then $E|_x = [\bar{t}, +\infty)$. In any case, the desired result follows. □

Definition 2.2.2 *Let F be a subset of $\mathbb{R}^n \times \mathbb{R}$. We say that $E = \mathrm{Epi.hull}(F)$ is the epigraphical hull of F if E is the intersection of all epigraphs that contain F.*

From Proposition 2.2.1, the epigraphical hull is always an epigraph set. Identifying the epigraphical hull of a given set F is straightforward. For each $x \in \mathbb{R}^n$, $\mathrm{Epi.hull}(F)|_x$ is either: $\emptyset$, if $F|_x$ is empty; $\mathbb{R}$, if $F|_x$ is unbounded below; or, $[\bar{t}, +\infty)$, for $\bar{t}$ equal to the largest lower bound on $F|_x$.

2.3 Some epigraph calculus

In this section, we show that the epigraphical hull may be used as tool a to assure that certain operations with functions are always well defined.

2.3.1 The supremum function

The supremum function $\sup_{h \in \mathscr{H}} h$ where $\mathscr{H}$ denotes a family of functions, is always well defined.

Proposition 2.3.1 *Let $\mathscr{H}$ be a family of functions. Then,*

$$\mathrm{Epi}\left(\sup_{h \in \mathscr{H}} h \right) = \bigcap_{h \in \mathscr{H}} \mathrm{Epi}(h).$$

Proof:

$$(x,t) \in \mathrm{Epi}\left(\sup_{h \in \mathscr{H}} h \right) \leftrightarrow \sup_{h \in \mathscr{H}} h(x) \leq t \leftrightarrow h(x) \leq t,\ h \in \mathscr{H} \leftrightarrow (x,t) \in \bigcap_{h \in \mathscr{H}} \mathrm{Epi}(h).$$

□

Note that the supremum function may be improper even if all the functions $h, h \in \mathscr{H}$, are proper.

2.3.2 The sum function

The sum of two functions may not always well defined. At points x where $|g(x)| = |h(x)| = \infty$ and $g(x) = -h(x)$ the function $g + h$ is undefined.

Proposition 2.3.2 *Let $g \boxplus h \colon \mathbb{R}^n \to [-\infty, +\infty]$ defined by*

$$(g \boxplus h)(x) \equiv \begin{cases} +\infty & \text{if } |g(x)| = |h(x)| = \infty \text{ and have opposite signs}, \\ g(x) + h(x) & \text{otherwise.} \end{cases}$$

Then, $\mathrm{Epi}(g \boxplus h)|_x \equiv \mathrm{Epi}(g)|_x + \mathrm{Epi}(h)|_x$.

Proof: Let $x \in \mathbb{R}^n$. If either $\mathrm{Epi}(g)|_x$ or $\mathrm{Epi}(h)|_x$ is the empty set then $(g \boxplus h)(x) = +\infty$ so that

$$\mathrm{Epi}(g \boxplus h)|_x = \emptyset = \mathrm{Epi}(g)|_x + \mathrm{Epi}(h)|_x.$$

So, now, we assume that both $\mathrm{Epi}(g)|_x$ and $\mathrm{Epi}(h)|_x$ are nonempty. Then,

$$\begin{aligned} t \in \mathrm{Epi}(g \boxplus h)|_x \quad &\Longleftrightarrow \quad g(x) + h(x) \le t \\ &\Longleftrightarrow \quad s_1 + s_2 = t, \text{ for some } s_1 \in \mathrm{Epi}(g)|_x, s_2 \in \mathrm{Epi}(h)|_x, \\ &\Longleftrightarrow \quad t \in \mathrm{Epi}(g)|_x + \mathrm{Epi}(h)|_x. \end{aligned}$$

□

The function $g \boxplus h$ may be improper even if both g, h are proper, in which case $g + h = g \boxplus h$. In simple terms, $g \boxplus h$ coincides with $g + h$ except at those points where $\infty - \infty$ occurs. At such points, $g \boxplus h$ is set to $+\infty$.

2.3.3 The scalar multiplication

The multiplication of a positive scalar and a function always be well defined. When λ is zero, λg is undefined when $|g(x)| = \infty$.

Proposition 2.3.3 *Let* $\lambda \boxdot g \colon \mathbb{R}^n \to [-\infty, +\infty]$*, where* $\lambda \ge 0$*, be defined by* $(\lambda \boxdot g)(x) \equiv \lambda g(x)$ *if* $\lambda > 0$*, and*

$$(0 \boxdot g)(x) \equiv \begin{cases} 0 & \text{if } g(x) < +\infty \\ +\infty & \text{if } g(x) = +\infty. \end{cases}$$

Then, $\mathrm{Epi}(\lambda \boxdot g)|_x = \lambda \cdot \mathrm{Epi}(g)|_x$ *if* $\lambda > 0$*, and*

$$\mathrm{Epi}(0 \boxdot g)|_x \equiv \left\{ \begin{array}{ll} [0, +\infty) & \text{if } g(x) < +\infty \\ \emptyset & \text{if } g(x) = +\infty. \end{array} \right\} \equiv \mathrm{Epi.hull}(0 \cdot \mathrm{Epi}(g))|_x.$$

Proof: Let $x \in \mathbb{R}^n$. If $g(x) = +\infty$ then $(\lambda \boxdot g)(x) = +\infty$ so that

$$\mathrm{Epi}(\lambda \boxdot g)|_x = \emptyset = \lambda \cdot \mathrm{Epi}(g)|_x.$$

So, now, we assume that both $\mathrm{Epi}(g)|_x$ is nonempty. Assuming $\lambda > 0$, then,

$$\begin{aligned} t \in \mathrm{Epi}(\lambda \boxdot g)|_x \quad &\Longleftrightarrow \quad \lambda g(x) \le t \\ &\Longleftrightarrow \quad \lambda s = t, \text{ for some } s \in \mathrm{Epi}(g)|_x \\ &\Longleftrightarrow \quad t \in \lambda \cdot \mathrm{Epi}(g)|_x. \end{aligned}$$

Assuming $\lambda = 0$, then $t \in \mathrm{Epi}(\lambda \boxdot g)|_x$ if and only if $t \ge 0$, which concludes the proof. □

If g is proper then $\lambda \boxdot g$ is proper. Moreover, $\lambda g \equiv g$. In simple terms, $\lambda \boxdot g$ coincides with λg except at those points where $0 \times (+\infty)$ or $0 \times (-\infty)$ occurs. At such points, $\lambda \boxdot g$ is set to $+\infty$ or 0, respectively.

2.3.4 The infimal convolution

The sum of two epigraphs may not be an epigraph set. For example, if $g(x) \equiv 0$, $h(x) \equiv \begin{cases} 1/x & \text{if } x > 0, \\ +\infty & \text{if } x \leq 0. \end{cases}$ then

$$\mathrm{Epi}(g) + \mathrm{Epi}(h) = \{(x,y) + (z,w) \colon y \geq 0,\ z > 0, w \geq 1/z\} = \{(\bar{x},\bar{y}) \colon \bar{y} > 0\},$$

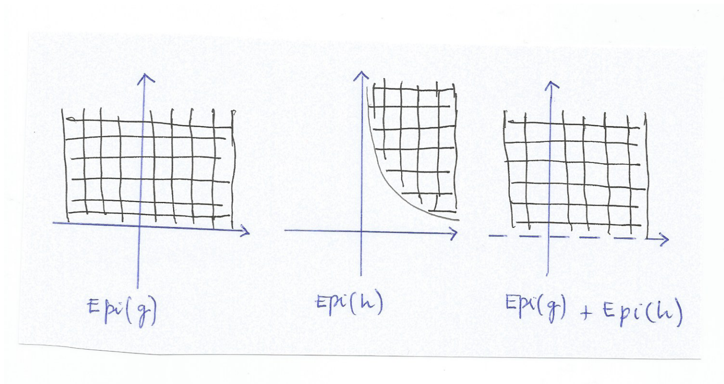


Fig. 2.1

the open positive half space plan, which is not an epigraph. One inclusion is trivial. For the other, set $x = \bar{x} - 1/\bar{y}$, $z = \bar{x} - x$ and $y = \bar{y} - 1/z$, $w = 1/z$, see [1, pp. 250].

It is interest to understand what is the function whose epigraph is the epigraphical hull of the sum of two epigraphs. The answer, justified by our next result, is the *infimal convolution* or *epi-sum*.

Proposition 2.3.4 *Let* $g\Box h \colon \mathbb{R}^n \to [-\infty, +\infty]$ *be defined by*

$$(g\Box h)(x) \equiv \begin{cases} +\infty & \text{if } \Omega(x) = \emptyset \\ \inf\limits_{y \in \Omega(x)} g(y) + h(x-y) & \text{if } \Omega(x) \neq \emptyset. \end{cases}$$

where $\Omega(x) = \mathbb{R}^n \setminus \{y \in \mathbb{R}^n \colon |g(y)| = |h(x-y)| = \infty$ *and have opposite signs*$\}$. *Then,* $\mathrm{Epi}(g\Box h) = \mathrm{Epi.\,hull}\,(\mathrm{Epi}(g) + \mathrm{Epi}(h))$.

Proof: Let $x \in \mathbb{R}^n$, arbitrary. We will show that

$$\mathrm{Epi}(g\Box h)|_x = \mathrm{Epi.\,hull}\,(\mathrm{Epi}(g) + \mathrm{Epi}(h))\,|_x. \tag{2.2}$$

If $\Omega(x) = \emptyset$ then $(g\Box h)(x) = +\infty$, so that $\mathrm{Epi}(g\Box h)|_x = \emptyset$. Now, we show, by contradiction, that $\mathrm{Epi}(g) + \mathrm{Epi}(h)|_x = \emptyset$ as well. Suppose there is $(x,t) \in \mathrm{Epi}(g) + \mathrm{Epi}(h)$. Hence, there are $(y, s_1) \in \mathrm{Epi}(g), (x-y, s_2) \in \mathrm{Epi}(h)$ such that $t = s_1 + s_2$, a contradiction because, then, $y \in \Omega(x)$. Hence, $\mathrm{Epi}(g) + \mathrm{Epi}(h)|_x = \emptyset$ which implies $\mathrm{Epi.\,hull}\,(\mathrm{Epi}(g) + \mathrm{Epi}(h))\,|_x = \emptyset$. If $\Omega(x) \neq \emptyset$ then

$$(x,t) \in \mathrm{Epi}(g\Box h) \tag{2.3}$$

if and only if

$$\inf_{y\in\Omega(x)} g(y)+h(x-y)\leq t$$

which, in turn, is equivalent to

$$(y,s_1)+(x-y,s_2)=(x,t+\varepsilon),\ \text{for some } (y,s_1)\in \operatorname{Epi}(g),(x-y,s_2)\in \operatorname{Epi}(h),\ \text{for every } \varepsilon>0,$$

or, in other words,

$$(x,t+\varepsilon)\in \operatorname{Epi}(g)+\operatorname{Epi}(h),\ \text{for every } \varepsilon>0,$$

or, equivalently,

$$(x,t)\in \operatorname{Epi.hull}\left(\operatorname{Epi}(g)+\operatorname{Epi}(h)\right).$$

□

If both g,h are proper then $\Omega(x)$ is the emptyset everwhere. Thus $g\Box h$ is simply given by the well-known infimum expression. It may occur that $g\Box h$ is improper. In simple terms, $\operatorname{Epi}(g\Box h)$ coincides with $\operatorname{Epi}(g)+\operatorname{Epi}(h)$ except at those points where $(\infty-\infty)$ occurs.

The infimum sum operation between two extended functions is associative.

We believe that the extend the infimal convolution of two functions to more than two in the following way:

$$(\Box_{i=1}^m g_i)(x)=\begin{cases} +\infty & \text{if } \Omega(x)=\emptyset \\ \left\{\begin{array}{ll} \inf & \sum_{i=1}^m g_i(y^i) \\ \text{s.t.} & (y^1,y^2,\ldots,y^m)\in\Omega(x) \end{array}\right\} & \text{if } \Omega(x)\neq\emptyset. \end{cases}$$

where $\Omega(x)$ is the set of those $(y^1,y^2,\ldots,y^m)\in\mathbb{R}^n\times\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ such that

$$x=y^1+y^2+\cdots+y^m$$

and there is no pair $(y^i,y^j), i\neq j$, where $|g_i(y^i)|=|g_j(y^j)|=\infty$, and $g_i(y^i)=-g_j(y^j)$. Then,

$$\operatorname{Epi}(\Box_{i=1}^m g_i)=\operatorname{Epi.hull}\left(\sum_{i=1}^m \operatorname{Epi}(g_i)\right).$$

2.3.5 The epi-multiplication

The multiplication of a positive scalar and an epigraph is always an epigraph. When $\lambda=0$, the set $\lambda\operatorname{Epi}(g)$ may not be an epigraph.

Our next result applies to a function operation known as *epi-multiplication*, which is closely related to the perspective function concept to be introduced later.

Proposition 2.3.5 *Let* $g \boxdot \lambda : \mathbb{R}^n \to [-\infty, +\infty]$, *where* $\lambda \geq 0$, *defined by* $(g \boxdot \lambda)(x) \equiv \lambda g(x/\lambda)$, *when* $\lambda > 0$, *and*

$$(g \boxdot 0)(x) \equiv \begin{cases} +\infty & \textit{if } g \equiv +\infty, \\ 0 & \textit{if } x = 0,\ g \not\equiv +\infty, \\ +\infty & \textit{if } x \neq 0,\ g \not\equiv +\infty. \end{cases}$$

Then, $\mathrm{Epi}(g \boxdot \lambda) = \lambda \cdot \mathrm{Epi}(g)$, *when* $\lambda > 0$, *and*

$$\mathrm{Epi}(g \boxdot 0) \equiv \left\{ \begin{array}{ll} \{0\} \times [0, +\infty) & \textit{if } g \not\equiv \infty \\ \emptyset & \textit{if } g \equiv +\infty. \end{array} \right\} \equiv \mathrm{Epi.\,hull}(0 \cdot \mathrm{Epi}(g)).$$

Proof: Assume $\lambda > 0$ and let $x \in \mathbb{R}^n$. Then,

$$\begin{aligned} (x,t) \in \mathrm{Epi}(g \boxdot \lambda) \quad &\Longleftrightarrow \quad \lambda g(x/\lambda) \leq t, \\ &\Longleftrightarrow \quad g(x/\lambda) \leq t/\lambda \\ &\Longleftrightarrow \quad (x,t)/\lambda \in \mathrm{Epi}(g) \\ &\Longleftrightarrow \quad (x,t) \in \lambda \cdot \mathrm{Epi}(g). \end{aligned}$$

Therefore, when $\lambda > 0$,

$$\mathrm{Epi}(g \boxdot \lambda) = \lambda \cdot \mathrm{Epi}(g) = \mathrm{Epi.\,hull}\,(\lambda \cdot \mathrm{Epi}(g))\,.$$

Now, assume $\lambda = 0$ and consider separately the cases: $g \equiv +\infty$ and $g \not\equiv +\infty$. If $g \equiv +\infty$ then

$$\mathrm{Epi}(g \boxdot \lambda) = \emptyset = \lambda \cdot \mathrm{Epi}(g) = \mathrm{Epi.\,hull}\,(\lambda \cdot \mathrm{Epi}(g))\,.$$

If $g \not\equiv +\infty$ then

$$\mathrm{Epi}(g \boxdot \lambda)| = \{(0,t) \colon t \geq 0\} = \mathrm{Epi.\,hull}\{(0,0)\}.$$

On the other hand,

$$\lambda \cdot \mathrm{Epi}(g) = \{(0,0)\},$$

The desired result follows. $\square$

If g is proper then $g \boxdot \lambda$ is always proper, for every $\lambda \geq 0$. In simple terms, $\lambda > 0$, $\mathrm{Epi}(g \boxdot \lambda)$ is an enlargement or reduction (in size) of $\mathrm{Epi}(f)$ through multiplication by λ.

Chapter 3

Closedness and Lower semi-continuity

In this chapter, we will study the characterization of functions whose epigraph are closed sets, the lower semicontinuous function. The infimum function correspond a minimum, see [3], [6] and [8].

3.1 Lower semicontinuity

Definition 3.1.1 *We say that f is closed if its epigraph is a closed set.*

Proposition 3.1.2 *f is closed if and only if all sublevel sets $L_\alpha(f) \equiv \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ are closed.*

Proof: $(\Rightarrow)$

Let $\{x_k\} \subseteq L_\alpha(f)$ be a convergent sequence to $x \in \mathbb{R}^n$, arbitrary. Then, $\{(x_k, \alpha)\} \subseteq \text{Epi}(f)$ is a convergent sequence to (x, α), which belongs to $\text{Epi}(f)$ because $\text{Epi}(f)$ is closed. Thus, $x \in L_\alpha(f)$.

$(\Leftarrow)$

Let $\{(x_k, t_k)\} \subseteq \text{Epi}(f)$ be a convergent sequence to (x, t), arbitrary. Fix a finite $\alpha > t$, arbitrary. Then, for all k large enough, $f(x_k) \leq t_k < \alpha$. Thus, for all k large enough, $x_k \in L_\alpha(f)$, which implies that $x \in L_\alpha(f)$ because $L_\alpha(f)$ is closed. Thus, $(x, \alpha) \in \text{Epi}(f)$, for all $\alpha > t$, which implies that $(x, t) \in \text{Epi}(f)$. □

The proposition above justifies the importance of closed functions in the context of optimization. Namely, when the feasible region of an optimization problem is a subset of $\mathbb{R}^n$ defined as the intersection of a finite number of sublevel sets then closedness of all the functions involved implies closedness of the feasible region. Moreover, when the feasible region is compact, closedness of the objective function implies the existence of a global minimizer.

Lower semi-continuity of f is, as we shall see, a necessary and sufficient condition for closedness.

Definition 3.1.3 *We say that f is lower semicontinuous (Lsc) at x if*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } f(y) + \varepsilon \geq f(x),\ \forall y \in \mathscr{B}(x, \delta). \tag{3.1}$$

The definition above stems from the definition of continuity: we say that f is continuous at x, and mathematically say $\lim_{y\to x} f(y) = f(x)$, if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \underbrace{-\varepsilon \le f(y) - f(x)}_{\text{Lsc}} \le \varepsilon,\ \forall y \in \mathscr{B}(x,\delta). \tag{3.2}$$

Note that the definition of Lsc allows for the arithmetic with ∞ values, while the definition of continuity does not.

Lemma 3.1.4 *If f is closed function, then f is Lsc at x if and only if*

$$\liminf_{y\to x} f(y) \equiv \sup_{\delta>0} \left(F(\delta) \equiv \inf_{y\in B(x,\delta)} f(y) \right) \ge f(x). \tag{3.3}$$

Proof: Assume (3.1) holds and, by contradiction, assume (3.3) does not hold. Since F is nonincreasing, there exists $\varepsilon > 0$ such that $F(\delta) + \varepsilon < f(x)$, for every $\delta > 0$, which contradicts (3.1). Reciprocally, assume (3.3) holds. Fix $\varepsilon > 0$, arbitrary. Then, there exists $\delta > 0$ such that $F(\delta) + \varepsilon \ge f(x)$. Thus, (3.1) holds. □

When f is Lower semicontinuous at every $x \in \mathbb{R}^n$ we will simply say that f is Lower semicontinuous, or Lsc for short.

Proposition 3.1.5 *f is closed if and only if f is Lsc.*

Proof: ($\Rightarrow$)

Assume that $\mathrm{Epi}(f)$ is closed. By contradiction, suppose f is not Lsc at some $x \in \mathbb{R}^n$. Clearly, $f(x) > -\infty$. Moreover, from Lemma 3.1.4, there exists $\varepsilon > 0$ and $L \in \mathbb{R}$ such that

$$F(\delta) + \varepsilon < L \le f(x), \qquad \text{for every } \delta > 0. \tag{3.4}$$

Now, consider two sequences $\{\delta_k \equiv 1/k\}$ and $\{z_k\}$ where $z_k \in B(x,\delta_k)$ and

$$f(z_k) - \frac{\varepsilon}{2} \le \inf_{y\in B(x,\delta_k)} f(y)\ \ (F(\delta_k)).$$

Since $\{\delta_k\}$ is a positive sequence converging to zero, then $\{z_k\}$ is a sequence converging to x. Moreover, since $f(z_k) \le F(\delta_k) + \varepsilon/2 < L - \varepsilon/2$, for every k, the sequence $\{(z_k, L-\varepsilon/2)\} \subseteq \mathrm{Epi}(f)$ is convergent to $(x, L-\varepsilon/2) \in \mathrm{Epi}(f)$, because $\mathrm{Epi}(f)$ is closed. Thus, $f(x) \le L - \varepsilon/2 \le f(x) - \varepsilon/2$, a contradiction.

($\Leftarrow$)

Assume that f is Lsc. Let $\{(x_k, t_k)\} \subseteq \mathrm{Epi}(f)$ be a convergent sequence to (x,t), arbitrary. Since f is Lsc at x,

$$\begin{aligned} f(x) &\le \sup_{\delta>0}\left(\inf_{y\in B(x,\delta)} f(y)\right) \le \sup_{\delta>0}\left(\inf_{x_k:\, x_k\in B(x,\delta)} f(x_k)\right) \le \sup_{\delta>0}\left(\inf_{x_k:\, x_k\in B(x,\delta)} t_k\right) \\ &= \lim_k t_k = t. \end{aligned}$$

□

3.2 Operations with closed functions (checking closedness)

In this section, we consider some of the operations functions studied in section (2.3).

Proposition 3.2.1 *Let $\mathscr{H}$ be a family of closed functions $h\colon \mathbb{R}^n \to [-\infty, +\infty]$. Then, the function $\sup_{h\in\mathscr{H}} h$ is closed.*

Proof: Let $\mathscr{H}$ be a family of Lsc functions h and denote $f \equiv \sup_{h\in\mathscr{H}} h$. Fix $x \in \mathbb{R}^n$, arbitrary. Then, for every $h \in \mathscr{H}$,

$$\sup_{\delta>0}\left(\inf_{y\in B(x,\delta)} f(y)\right) \geq \sup_{\delta>0}\left(\inf_{y\in B(x,\delta)} h(y)\right) \geq h(x),$$

because h is Lsc. Thus,

$$\sup_{\delta>0}\left(\inf_{y\in B(x,\delta)} f(y)\right) \geq \sup_{h\in\mathscr{H}} h(x) = f(x).$$

Hence, f is Lsc. □

In general, the sum of two closed functions is not closed, as our next example shows

$$g(x) \equiv \begin{cases} 1/x & \text{if } x > 0 \\ +\infty & \text{if } x \leq 0 \end{cases}, \qquad h(x) \equiv \begin{cases} -1/x & \text{if } x > 0 \\ -\infty & \text{if } x \leq 0 \end{cases}, \qquad (g \boxplus h)(x) \equiv \begin{cases} 0 & \text{if } x > 0 \\ +\infty & \text{if } x \leq 0 \end{cases}$$

Both g and h are closed but $g \boxplus h$ is not closed.

The infimal convolution of two closed functions is not closed, in general. This is essentially a consequence that the sum of two unbounded closed (convex) sets need not be closed[1]. For example, the following subsets of $\mathbb{R}^2$,

$$S = \{(x_1,x_2)\colon x_2 = 0\}, \quad T = \{(x_1,x_2)\colon x_1 > 0,\ x_2 \geq 1/x_1\}$$

are both closed (and convex). However, $S+T = \{(x_1,x_2)\colon x_2 > 0\}$, which is not closed (though convex). Now, consider the following functions

$$g(x) \equiv \mathbb{1}_S(x) \equiv \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S \end{cases}, \qquad h(x) \equiv \mathbb{1}_T(x) \equiv \begin{cases} 0 & \text{if } x \in T \\ +\infty & \text{if } x \notin T \end{cases},$$

which are both closed because

$$\mathrm{Epi}(g) = S \times [0,\infty), \qquad \mathrm{Epi}(h) = T \times [0,\infty).$$

One may verify that $(g \square h)(x) \equiv \mathbb{1}_{S+T}(x)$ so that

$$\mathrm{Epi}(g \square h) = (S+T) \times [0,\infty),$$

which is not closed. Note that any choice of closed sets S,T such that $S+T$ is not closed would serve the same purpose of defining a counter example.

Proposition 3.2.2 *If g is a closed function and $\lambda \geq 0$ then, $\lambda \boxdot g$ is a closed function.*

[1] Obviously the sum of two bounded convex sets are always closed.

Proof: Assume $\lambda > 0$. Then,

$$\liminf_{y\to x} g(y) \geq g(x) \Rightarrow \liminf_{y\to x} \lambda g(y) \geq \lambda g(x).$$

Thus, in this case, $\lambda \boxdot g$ is closed. Now, assume $\lambda = 0$ and consider two cases: $g(x) < +\infty$ or $g(x) = +\infty$.

If $g(x) < +\infty$ then $(0 \boxdot g)(x) = 0$ and $(0 \boxdot g)(y) \geq 0$, for all y. Thus, in this case

$$\liminf_{y\to x} (0 \boxdot g)(y) \geq 0 = (0 \boxdot g)(x).$$

If $g(x) = +\infty$ then, since g is closed, there exists $\delta > 0$ such that $g(y) = +\infty$, for all $y \in B(x,\delta)$. Thus, $(0 \boxdot g)(y) = +\infty$, for all $y \in B(x,\delta)$, which implies

$$\liminf_{y\to x} (0 \boxdot g)(y) = +\infty = (0 \boxdot g)(x).$$

$\square$

Proposition 3.2.3 *If g is a closed function and $\lambda \geq 0$ then, $g \boxdot \lambda$ is a closed function.*

Proof: Assume $\lambda > 0$. Then, $\mathrm{Epi}(g \boxdot \lambda) = \lambda\,\mathrm{Epi}(g)$. Since $\mathrm{Epi}(g)$ is closed, $\mathrm{Epi}(g \boxdot \lambda)$ is also closed. Now, assume $\lambda = 0$ and consider two cases: $g \equiv +\infty$ or $g \not\equiv +\infty$. If $g \equiv +\infty$ then $\mathrm{Epi}(g \boxdot 0) = \emptyset$, which is closed; If $g \not\equiv +\infty$ then $\mathrm{Epi}(g \boxdot 0) = \{0\} \times [0,+\infty)$ (cartesian product), which is closed as well. $\square$

3.3 Closure of f

When f is not closed we may consider the problem of finding the “closest” closed function.

Lemma 3.3.1 *If E is an epigraph set then $\mathrm{cl}\,E$ is also an epigraph set.*

Proof: This is a direct consequence of Theorem 3.3.3 below. Here we present a direct proof. Let $(x,t) \in \mathrm{cl}(E)$. Then, there exists a sequence $\{(x_k,t_k)\} \subset E$ convergent to (x,t). Let $s > t$, arbitrary. Then, for all k large enough, $f(x_k) \leq t_k < s$. Thus, there exists a sequence $\{(x_k,s)\} \subset E$ convergent to (x,s). Hence, $(x,s) \in \mathrm{cl}(E)$, for all $s > t$. Now, for any $\bar{x} \in \mathbb{R}^n$,

$$\{(\bar{x},t) \in \mathbb{R}^n \times \mathbb{R} \colon (\bar{x},t) \in \mathrm{cl}(E)\} = \mathrm{cl}\,E \cap \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \colon x = \bar{x}\}, \tag{3.5}$$

which is closed because it is the intersection of two closed sets. The set $\{t \in \mathbb{R} \colon (\bar{x},t) \in \mathrm{cl}(E)\}$ is the linear projection of the set (3.5) onto $\mathbb{R}^n$. Thus, it is closed as well. $\square$

Definition 3.3.2 *The closure of f, or lower semicontinuous regularization of f, is the function $(\mathrm{cl}\,f)\colon \mathbb{R}^n \to [-\infty,+\infty]$ whose epigraph is $\mathrm{cl}\,\mathrm{Epi}(f)$.*

Theorem 3.3.3 *If f is a closed function, the function $g\colon \mathbb{R}^n \to [-\infty, +\infty]$ defined by*

$$g(x) \equiv \liminf_{y \to x} f(y)$$

is Lsc. Moreover, $\mathrm{Epi}(g) = \mathrm{cl}\,\mathrm{Epi}(f)$.

Proof: Let $x \in \mathbb{R}^n$, arbitrary. Fix finite $\varepsilon > 0$, arbitrary. From the definition of g, there exists $\delta > 0$ such that

$$\inf_{y \in B(x,\delta)} f(y) \geq g(x) - \frac{\varepsilon}{2}. \tag{3.6}$$

On the other hand, for every $y \in \mathbb{R}^n$, there exists $\delta_y > 0$, such that

$$\inf_{z \in B(y,\delta_y)} f(z) \leq g(y) + \frac{\varepsilon}{2}. \tag{3.7}$$

Without loss of generality, we may assume that

$$B(y, \delta_y) \subseteq B(x, \delta), \text{ for every } y \in B(x, \delta).$$

Thus,

$$B(x,\delta) = \bigcup_{y \in B(x,\delta)} B(y, \delta_y),$$

which implies that

$$\inf_{y \in B(x,\delta)} f(y) = \inf_{y \in B(x,\delta)} \left(\inf_{z \in B(y,\delta_y)} f(z) \right) \leq \inf_{y \in B(x,\delta)} \left(g(y) + \frac{\varepsilon}{2} \right) = \frac{\varepsilon}{2} + \inf_{y \in B(x,\delta)} g(y)$$

Thus, from (3.6),

$$\inf_{y \in B(x,\delta)} g(y) + \varepsilon \geq g(x).$$

Thus, g is Lsc. It remains to be shown that $\mathrm{Epi}(g) = \mathrm{cl}\,\mathrm{Epi}(g)$. Let (x,t) be an interior point to $\mathrm{Epi}(f)$. Then, there is $\varepsilon > 0$ such that

$$f(y) \leq s, \qquad \text{for all } (y,s) \in B((x,t), \varepsilon).$$

In particular, $f(y) \leq t$, for all $y \in B(x, \varepsilon)$, which implies that $g(x) \leq t$. Thus, $(x,t) \in \mathrm{Epi}(g)$. We have thus proved that $\mathrm{int}\,\mathrm{Epi}(f) \subseteq \mathrm{Epi}(g)$. Now, from Proposition 3.1.5, the set $\mathrm{Epi}(g)$ is closed because, as we have proved g is Lsc. Thus,

$$\mathrm{cl}\,\mathrm{Epi}(f) = \mathrm{cl}\,(\mathrm{int}\,\mathrm{Epi}(f)) \subseteq \mathrm{cl}\,(\mathrm{Epi}(g)) = \mathrm{Epi}(g),$$

It remains to be shown that $\mathrm{Epi}(g) \subseteq \mathrm{cl}\,\mathrm{Epi}(f)$. Let $(x,t) \in \mathrm{Epi}(g)$, arbitrary. Fix a finite $s > t$, arbitrary. In every open ball $B(x, \delta)$, there exists y such that $f(y) \leq s$. Thus,

$$B((x,s),\delta) \cap \mathrm{Epi}(f) \neq \emptyset, \text{ for every } \delta > 0.$$

Thus, $(x,s) \in \operatorname{cl}\operatorname{Epi}(f)$, for every $s > t$. Since $\operatorname{cl}\operatorname{Epi}(f)$ is closed, $(x,t) \in \operatorname{cl}\operatorname{Epi}(f)$. □

Theorem 3.3.3 provides an characterization of the function values of $\operatorname{cl}(f)$, namely,

$$\operatorname{cl} f(x) = \liminf_{y \to x} f(y). \tag{3.8}$$

Chapter 4

Convex functions

Convex functions of real variables in $\mathbb{R}^n$ form an important class of functions in the context real analysis. They are widely used in optimization and also in many areas of applied mathematics. The definitions and properties of the following chapter can be found in classical book [8], or [2], [3] and [6].

4.1 Basic definition

Definition 4.1.1 *We say that f is convex if* $\mathrm{Epi}(f)$ *is convex.*

We recognize convex function with Jensen's Inequality. But extended function is a special because can take $\infty - \infty$.

Theorem 4.1.2 (Jensen's inequality) *f is convex if and only if* $\mathrm{Dom}(f)$ *is convex and*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{4.1}$$

for every $x, y \in \mathrm{Dom}(f)$ and $\lambda \in (0,1)$.

Proof: $(\Rightarrow)$

Let $x, y \in \mathrm{Dom}(f)$. Thus, there are $r, s \in \mathbb{R}$ such that $(x,r), (y,s) \in \mathrm{Epi}(f)$. Since and $\mathrm{Epi}(f)$ is convex, for every $\lambda \in (0,1)$,

$$\lambda(x,r) + (1-\lambda)(y,s) = (\lambda x + (1-\lambda)y, \lambda r + (1-\lambda)s) \in \mathrm{Epi}(f).$$

Therefore, $\lambda x + (1-\lambda)y \in \mathrm{Dom}(f)$, which implies that $\mathrm{Dom}(f)$ is convex.

Let $x, y \in \mathrm{Dom}(f)$ and $\lambda \in (0,1)$, arbitrary. Two separate cases need to be considered: either (i) both $x, y \in \mathrm{dom}(f)$, (ii) $x \notin \mathrm{dom}(f)$. The third case, $y \notin \mathrm{dom}(f)$ is not necessary to be verified.

If both $x, y \in \mathrm{dom}(f)$ then, $(x, f(x)), (y, f(y)) \in \mathrm{Epi}(f)$ and, since $\mathrm{Epi}(f)$ is convex,

$$\lambda(x, f(x)) + (1-\lambda)(y, f(y)) = (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda) f(y)) \in \mathrm{Epi}(f),$$

from where (4.1) follows. If $x \notin \mathrm{dom}(f)$ then, $(x,s) \in \mathrm{Epi}(f)$, for all $s \in \mathbb{R}$ and $(y,t) \in \mathrm{Epi}(f)$, for some $t \in \mathbb{R}$. Since $\mathrm{Epi}(f)$ is convex,

$$\lambda(x,s)+(1-\lambda)(y,t)=(\lambda x+(1-\lambda)y, \lambda s+(1-\lambda)t) \in \mathrm{Epi}(f),$$

for any $s \in \mathbb{R}$. Thus, $f(\lambda x+(1-\lambda)y)=-\infty$ so that (4.1) follows.

($\Leftarrow$)

Let $(x,r),(y,s) \in \mathrm{Epi}(f)$ and $\lambda \in (0,1)$, arbitrary. If $x,y \in \mathrm{dom}(f)$ then, from (4.1),

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y) \leq \lambda r+(1-\lambda)s,$$

which implies that

$$\lambda(x,r)+(1-\lambda)(y,s)=(\lambda x+(1-\lambda)y, \lambda r+(1-\lambda)s) \in \mathrm{Epi}(f). \tag{4.2}$$

If $x \in \mathrm{Dom}(f) \setminus \mathrm{dom}(f)$ then, from (4.1),

$$f(\lambda x+(1-\lambda)y) \leq -\infty,$$

Which implies that $f(\lambda x+(1-\lambda)y)=-\infty$,. Thus, also in this case, (4.2) holds. Thus, $\mathrm{Epi}(f)$ is convex. □

Other authors, with specific explanation use different inequalities.

Theorem 4.1.3 *f is convex if and only if*

$$f(\lambda x+(1-\lambda)y)<\lambda A+(1-\lambda)B, \tag{4.3}$$

for every $x,y \in \mathrm{Dom}(f)$, every $\lambda \in (0,1)$ and every $(A,B) \in \mathbb{R}^2$ such that $(A,B)>(f(x),f(y))$.

Proof: ($\Rightarrow$)

Let $x,y \in \mathrm{Dom}(f)$, $\lambda \in (0,1)$ and $(A,B)>(f(x),f(y))$, arbitrary. Then, for some $\delta>0$, $(x,A-\delta),(y,B-\delta) \in \mathrm{Epi}(f)$. Since $\mathrm{Epi}(f)$ is convex,

$$\lambda(x,A-\delta)+(1-\lambda)(y,B-\delta)=(\lambda x+(1-\lambda)y, \lambda A+(1-\lambda)B-\delta) \in \mathrm{Epi}(f),$$

from where (4.3) follows.

($\Leftarrow$)

Let $(x,r),(y,s) \in \mathrm{Epi}(f)$ and $\lambda \in (0,1)$, arbitrary. Then, $x,y \in \mathrm{Dom}(f)$ and $(r+\delta,s+\delta)>(f(x),f(y))$, for every $\delta>0$. From (4.3),

$$(\lambda x+(1-\lambda)y, \lambda(r+\delta)+(1-\lambda)(s+\delta))=(\lambda x+(1-\lambda)y, \lambda r(1-\lambda)s+\delta) \in \mathrm{Epi}(f),$$

for every $\delta>0$. Since $\mathrm{Epi}(f)$ is an epigraph set,

$$(\lambda x+(1-\lambda)y, \lambda r(1-\lambda)s)=\lambda(x,r)+(1-\lambda)(y,s) \in \mathrm{Epi}(f).$$

Thus, $\mathrm{Epi}(f)$ is convex. $\square$

The follows two results of convex function will be used later.

Proposition 4.1.4 *f is convex if and only if, for every $x, d \in \mathbb{R}^n$, the restriction $g\colon \mathbb{R} \to [-\infty, +\infty]$ defined by $g(t) \equiv f(x+td)$ is convex.*

Proof: $(\Rightarrow)$

Let x, d fixed, $\mathrm{Epi}(f)$ is convex. Where $\mathrm{Epi}(f) = \{(x,s)\colon\ (x+td, s) \in \mathrm{Epi}(f)\}$. Let $(t_1, s_1), (t_2, s_2) \in \mathrm{Epi}(f)$ and $\lambda \in [0,1]$. For, $(x+t_1 d, s_1), (x+t_2 d, s_2) \in \mathrm{Epi}(f)$ as $\mathrm{Epi}(f)$ is convex, then

$$\lambda(x+t_1, s_1) + (1-\lambda)(x+t_2 d, s_2) = \Big(x + (\lambda t_1 + (1-\lambda)t_2)d, \lambda s_1 + (1-\lambda)s_2\Big) \in \mathrm{Epi}(f).$$

Hence,

$$(\lambda t_1 + (1-\lambda)t_2, \lambda s_1 + (1-\lambda)s_2) = \lambda(t_1, s_1) + (1-\lambda)(t_2, s_2) \in \mathrm{Epi}(f)$$

$(\Leftarrow)$ Let $(x_1, s_1), (x_2, s_2) \in \mathrm{Epi}(f)$ and $\lambda \in [0,1]$. Let,

$$\begin{aligned}\lambda(x_1, s_1) + (1-\lambda)(x_2, s_2) &= (\lambda x_1 + (1-\lambda)x_2, \lambda s_1 + (1-\lambda)s_2) \\ &= (x_2 + \lambda(x_1 - x_2), s_2 + \lambda(s_1 - s_2)) \in \mathrm{Epi}(f)\end{aligned}$$

Let $g(t) \equiv f(x_2 + \lambda(x_1 - x_2)), t \in \mathbb{R}$, then $(1, s_1), (0, s_2) \in \mathrm{Epi}(g)$. Where, $\mathrm{Epi}(g)$ is convex, then

$$\lambda(1, s_1) + (1-\lambda)(0, s_2) = (\lambda, \lambda s_1 + (1-\lambda)s_2) \in \mathrm{Epi}(g).$$

Hence,

$$(x_2 + \lambda(x_1 - x_2), \lambda s_1 + (1-\lambda)s_2) = \lambda(x_1, s_1) + (1-\lambda)(x_2, s_2) \in \mathrm{Epi}(f).$$

$\square$

Note that the proposition above use epigraph sets not Jensen's Inequalities.

Theorem 4.1.5 *If f is convex and $x \in \mathrm{dom}(f)$ then, for any $d \in \mathbb{R}^n$,*

$$\frac{f(x+\lambda d) - f(x)}{\lambda} \tag{4.4}$$

is nondecreasing in $\lambda > 0$.

Proof: First, observe that (4.4) equals $\left(h \boxdot \frac{1}{\lambda}\right)(y)$, where h denotes the function defined by $h(y) \equiv f(x+y) - f(x)$.

The epigraph of h is the set

$$\mathrm{Epi}(f) - (x, f(x)),$$

which is convex, because f is convex, and contains the origin.
The epigraph of $h \boxdot (1/\lambda)$ is $(1/\lambda)\,\mathrm{Epi}(h)$ (see Proposition 2.3.5). From Lemma 1.1.5, the set $(1/\lambda)\,\mathrm{Epi}(h)$ is decreasing in $\lambda > 0$. Hence, for any fixed $d \in \mathbb{R}^n$, (4.4) is increasing in $\lambda > 0$. $\square$

What is the aspect of convex function with $-\infty$ values.

Proposition 4.1.6 *Let f be convex. If there is $\hat{x} \in \operatorname{core}(\operatorname{dom}(f))$ such that $f(\hat{x}) > -\infty$ then $f(x) > -\infty$, for every $x \in \mathbb{R}^n$.*[1]

Proof: By contradiction, suppose $(x,s) \in Epi(f)$, for every real s. Since $\hat{x} \in \operatorname{core}(\operatorname{dom}(f))$, there is some real $t > 0$ with $\hat{x} + t(\hat{x} - x)$ in $\operatorname{dom}(f)$, and hence a real r with $(\hat{x} + t(\hat{x} - x), r)$ in $Epi(f)$. Since $\operatorname{Epi}(f)$ is convex,

$$\frac{1}{1+t}(\hat{x} + t(\hat{x} - x), r) + \frac{t}{1+t}(x,s) = \left(\hat{x}, \frac{r+ts}{1+t}\right) \in \operatorname{Epi}(f),$$

for every real s. But, this implies that $(\hat{x}, S) \in \operatorname{Epi}(f)$, for every real S, a contradiction. □

Theorem 4.1.7 *Let f be convex function. If f is improper, then $f(x) = -\infty$, for every $x \in \operatorname{ri}(\operatorname{Dom} f)$.*

Proof: The result holds trivially when $\operatorname{ri}(\operatorname{Dom} f) = \emptyset$. So, suppose $\operatorname{ri}(\operatorname{Dom} f) \neq \emptyset$. In particular, $\operatorname{Dom}(f) \neq \emptyset$. If $f(x) \in \mathbb{R}$, for every $x \in \operatorname{Dom}(f)$ then f is proper, a contradiction. So, let $x \in \operatorname{Dom}(f)$ be such that $f(x) = -\infty$, let $z \in \operatorname{ri}(\operatorname{Dom} f)$, arbitrary but distinct from x. Then, there is $\mu > 1$ such that $y \equiv x + \mu(z - x) \in \operatorname{Dom}(f)$. Therefore,

$$z = \left(1 - \frac{1}{\mu}\right)x + \frac{1}{\mu}y \in \operatorname{Dom}(f).$$

Since $x, y \in \operatorname{Dom}(f)$, there is $(A,B) > (f(x), f(y))$. By Theorem 4.1.3,

$$f(z) < \left(1 - \frac{1}{\mu}\right)A + \frac{1}{\mu}B. \tag{4.5}$$

If $f(z) \in \mathbb{R}$ then we may choose A small enough, because $f(x) = -\infty$, to contradict (4.5). Hence, $f(z) = -\infty$. □

4.2 Closed convex functions

Proposition 4.2.1 *If f is convex then $\operatorname{cl} f$ is convex.*

Proof: The epigraph of $\operatorname{cl} f$ is the $\operatorname{Epi}(f)$, since the closed of convex set is also convex, then, $\operatorname{cl} f$ is convex. □

Our next results characterizes $\operatorname{cl} f$ when f is not closed. But need this lemma first.

Lemma 4.2.2 *If f is convex then*

$$\operatorname{ri}(\operatorname{Epi} f) = \{(x,t) \colon x \in \operatorname{ri}(\operatorname{dom} f), f(x) < t\}$$

[1] The *core* of a convex set C is the set of points $x \in C$ such that for any $d \in \mathbb{R}^n$, $x + td \in C$, for all $t > 0$ small enough. This set clearly contains the interior of C but it may be larger.

Proof: From Lemma 1.1.3 in the Introduction,

$$\mathrm{ri}(\mathrm{Dom}\, f) = A \cdot \mathrm{ri}(\mathrm{Epi}\, f),$$

where A stands for the projection onto $\mathbb{R}^n$, which is a linear operator from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. Thus,

$$\mathrm{ri}(\mathrm{Epi}\, f) = \bigcup_{x \in \mathrm{ri}(\mathrm{Dom}\, f)} \{x\} \times R(x), \tag{4.6}$$

where, for each x, $R(x) \subseteq \mathbb{R}$. It remains to be shown that $R(x) = \{t \colon f(x) < t\}$. Let $x \in \mathrm{ri}(\mathrm{Dom}\, f)$, arbitrary. Then,

$$\begin{aligned}
\{x\} \times R(x) &= \{x\} \times \mathbb{R} \cap \mathrm{ri}\,\mathrm{Epi}(f) \\
&= \mathrm{ri}\,(\{x\} \times \mathbb{R}) \cap \mathrm{ri}(\mathrm{Epi}\, f) \\
&= \mathrm{ri}\,(\{x\} \times \mathbb{R} \cap \mathrm{Epi}(f)), \qquad (4.7) \\
&= \begin{cases} (f(x), +\infty) & \text{if } f(x) \in \mathbb{R} \\ \mathbb{R} & \text{if } f(x) = -\infty \end{cases}
\end{aligned}$$

where (4.7) follows from Lemma 1.1.4 in the Introduction. In any case, $r \in R(x)$ if and only if $r > f(x)$, which concludes the proof. □

Proposition 4.2.3 *Let f convex, and let $y \in \mathrm{ri}(\mathrm{Dom}\, f)$. Then,*

$$(\mathrm{cl}\, f)(x) = \lim_{t \to 0^+} f(x + t(y - x)) \tag{4.8}$$

for every $x \in \mathbb{R}^n$.

Proof: Since $x_t \stackrel{\text{def}}{=} x + t(y - x) \to x$ when $t \in 0^+$, we certainly have

$$\begin{aligned}
(\mathrm{cl}\, f)(x) &= \liminf_{z \to x} f(z) \\
&\leq \liminf_{t \to 0^+} f(x + t(y - x))
\end{aligned} \tag{4.9}$$

Now proving the converse inequality by showing that

$$\beta \leq \limsup_{t \to 0^+} f(x + t(y - x))$$

for every $\beta \geq (\mathrm{cl}\, f)(x)$.

Let $(y, \alpha) \in \mathrm{ri}(\mathrm{Epi}\, f)$ and another $\beta \geq (\mathrm{cl}\, f)(x)$. Thus

$$(x, \beta) \in \mathrm{Epi}(\mathrm{cl}\, f) = \mathrm{cl}(\mathrm{Epi}\, f),$$

then

$$(x, \beta) + t\Big((y, \alpha) - (x, \beta)\Big) \in \mathrm{ri}(\mathrm{Epi}\, f)$$

for every $t \in (0,1]$. Therefore,

$$\Big(x + t(y-x), \beta + t(\alpha - \beta)\Big) \in \mathrm{Epi}(f) \tag{4.10}$$

for every $t \in (0,1]$. From (4.10) and (4.9) implies (4.8). $\square$

Chapter 5

The Recession and the Perspective (of a convex function)

In this chapter we are interested in the behaviour of the function in infinity. And we will see some results that will be necessary for the analysis of that conditions, see [8].

5.1 The recession cone

Definition 5.1.1 *The recession cone of a nonempty convex set $C \subseteq \mathbb{R}^n$ is the following subset of $\mathbb{R}^n$,*

$$C_\infty = \bigcap_{x \in C} C_\infty(x),$$

where $C_\infty(x) \equiv \{d \in \mathbb{R}^n \colon x + \lambda d \in C, \text{ for all } \lambda \geq 0\}$, for a given $x \in C$.

The set $C_\infty(x)$ is the set of directions d along which and departing from $x \in C$ the points remain in C. Thus, C_∞ is the set of directions d along which independently of the chosen departing point $x \in C$ the points remain in C. In particular, C_∞ contains the origin.
It is easy to verify that $C_\infty(x)$ is a convex cone so that, being the intersection of convex cones, the set C_∞ is also a convex cone. An alternative characterization of C_∞ is as follows.

Theorem 5.1.2 $C_\infty = \{d \in \mathbb{R}^n \colon C + d \subseteq C\}$.

Proof: If $d \in C_\infty$ then, for every $x \in C$, $x + d \in C$. Thus, $C + d \subseteq C$. Reciprocally, take any $d \in \mathbb{R}^n$ such that $C + d \subseteq C$ and any $x \in C$. Then, for every $\lambda \geq 0$,

$$x + \lambda d = x + (\lfloor \lambda \rfloor + \lambda) d = (x + \lfloor \lambda \rfloor d) + \delta d = y + \delta d = \delta(y + d) + (1 - \delta) y \in C.$$

where $y \in C$ and $\delta \in [0, 1)$. Thus, $d \in C_\infty(x)$, for every $x \in C$. $\square$

If C is not closed, C_∞ need not be closed. For example, if $C = \{(x, y) \colon x, y > 0\} \cup \{(0, 0)\}$ then $C_\infty = C$, which is not closed. Another example, $D = C \cup \{(0, y) \colon y > 0\}$ then $D_\infty = \{(0, 0)\}$, shows that a larger C does not imply a larger C_∞. The same example shows that C_∞ may be singleton even when C is unbounded. The same examples also show that if C is not closed, the sets $C_\infty(x)$ may differ, accross all $x \in C$.

Theorem 5.1.3 *If C is closed then C_∞ is closed.*

Proof: Let $\{d_k\} \subseteq C_\infty$ be a convergent sequence to d. For every $x \in C$ and every $\lambda \geq 0$,

$$x + \lambda d = \lim_k x + \lambda d_k \in C,$$

because the sequence $\{x + \lambda d_k\} \subseteq C$ is convergent and C is closed. $\square$

If C is closed then $C_\infty(x)$ is closed across all $x \in C$, as results shows.

Theorem 5.1.4 $(\mathrm{cl}\, C)_\infty = (\mathrm{cl}\, C)_\infty(x)$*, for all $x \in \mathrm{cl}\, C$.*

Proof: Take $x \in \mathrm{cl}\, C$, arbitrary. From the definition, $(\mathrm{cl}\, C)_\infty \subseteq (\mathrm{cl}\, C)_\infty(x)$. Reciprocally, let $d \in (\mathrm{cl}\, C)_\infty(x)$, arbitrary. Take any $y \in \mathrm{cl}\, C$ and any $\lambda \geq 0$. Then, for any $\mu \in (0,1]$,

$$\mu \left(x + \frac{\lambda}{\mu} d \right) + (1-\mu) y \in \mathrm{cl}\, C.$$

Moreover,

$$\lim_{\mu \to 0^+} \mu x + (1-\mu) y + \lambda d = \lim_{\mu \to 0^+} \mu \left(x + \frac{\lambda}{\mu} d \right) + (1-\mu) y = y + \lambda d \in \mathrm{cl}\, C,$$

because $\mathrm{cl}\, C$ is closed. Thus $d \in (\mathrm{cl}\, C)_\infty(y)$, for all $y \in \mathrm{cl}\, C$, which implies that $(\mathrm{cl}\, C)_\infty(x) \subseteq (\mathrm{cl}\, C)_\infty$. $\square$

Theorem 5.1.5 $(\mathrm{ri}\, C)_\infty = (\mathrm{cl}\, C)_\infty$.

Proof: Both sets $\mathrm{ri}(C)$ and $\mathrm{cl}(C)$ are nonempty and convex [8, p45,Th6.2]. Assume $d \in (\mathrm{ri}\, C)_\infty$ and take any $x \in \mathrm{ri}(C)$. Then, for every $\lambda \geq 0$, $x + \lambda d \in \mathrm{ri}(C)$. Since $x \in \mathrm{ri}(C) \subseteq \mathrm{cl}(C)$, we conclude that $d \in (\mathrm{cl}\, C)_\infty(x) = (\mathrm{cl}\, C)_\infty$. Reciprocally, assume $d \in (\mathrm{cl}\, C)_\infty$ and take any $x \in \mathrm{ri}(C)$. Then, for every $\lambda > 0$, $x + \lambda d \in \mathrm{cl}(C)$. Thus, from [8, p45,Th6.1], $x + \mu d \in \mathrm{ri}(C)$, for every $\mu \in [0, \lambda)$. Thus, $x + \lambda d \in \mathrm{ri}(C)$, for every $\lambda \geq 0$, which implies $d \in (\mathrm{ri}\, C)_\infty$. $\square$

Now we will show, that, if C is closed, C_∞ is singleton if and only if C is bounded. But we need the next lemmma first.

Lemma 5.1.6 *Let $d \in \mathbb{R}^n$ such that $\|d\| = 1$. Then, $d \in (\mathrm{cl}\, C)_\infty$ if and only if there exists a sequence $\{x_k\} \subseteq \mathrm{cl}\, C$ such that $\{\|x_k\|\}$ diverges to $+\infty$ and $\{x_k / \|x_k\|\}$ converges to d.*

Proof: Assume $d \in (\mathrm{cl}\, C)_\infty$ and consider the sequence $\{x_k \equiv x + kd\} \subseteq \mathrm{cl}\, C$. Since

$$\|x + kd\| \geq \|kd\| - \|x\| \geq k\|d\|,$$

from where we conclude that $\{\|x_k\|\}$ diverges to $+\infty$. Furthermore,

$$\frac{x+kd}{\|x+kd\|} = \frac{x}{\|x+kd\|} + \frac{d}{\|x/k+d\|}.$$

We conclude that $\{x_k/\|x_k\|\}$ converges to d. Reciprocally, assume the existence of such a sequence $\{x_k\} \subseteq \mathrm{cl}\,C$. Take $x \in \mathrm{cl}\,C$ and $\lambda > 0$, both arbitrary. Then, for all k large enough, $\lambda/\|x_k\| \in [0,1]$. Therefore, for all k large enough,

$$\left(1 - \frac{\lambda}{\|x_k\|}\right)x + \left(\frac{\lambda}{\|x_k\|}\right)x_k \in \mathrm{cl}\,C,$$

which implies that

$$x + \lambda d = \lim_k x + \lambda\frac{x_k}{\|x_k\|} - \lambda\frac{x}{\|x_k\|} = \lim_k \left(1 - \frac{\lambda}{\|x_k\|}\right)x + \left(\frac{\lambda}{\|x_k\|}\right)x_k \in \mathrm{cl}\,C.$$

□

Theorem 5.1.7 *The set* $\mathrm{cl}\,C$ *is bounded if and only if* $(\mathrm{cl}\,C)_\infty$ *is singleton.*

Proof: If $(\mathrm{cl}\,C)_\infty$ contains other vectors besides the origin, from Lemma 5.1.6, $\mathrm{cl}\,C$ must be unbounded. Reciprocally, if $\mathrm{cl}\,C$ is unbounded then there exists an unbounded sequence $\{x_k\} \subseteq \mathrm{cl}\,C$, that does not contain the origin. The sequence $\{x_k/\|x_k\|\}$ is bounded and, therefore, it contains a convergent subsequence. The limit of such sequence must be different from the origin, as all its elements have norm one. □

The equivalent form of Lemmma 5.1.6 is as follows:

Lemma 5.1.8 *Let* $d \in \mathbb{R}^n$*. Then,* $d \in (\mathrm{cl}\,C)_\infty$ *if and only if there exists a sequence of scalars* $\{\lambda_k\} \downarrow 0$ *and a sequence* $\{x_k\} \subseteq \mathrm{cl}\,C$ *such that* $\{\lambda_k x_k\}$ *converges to* d*.*

Proof: Assume $d \in (\mathrm{cl}\,C)_\infty$ and consider the sequence of scalars $\{\lambda_k \equiv 1/k\} \downarrow 0$ and the sequence $\{x_k \equiv x + kd\} \subseteq \mathrm{cl}\,C$, for any fixed $x \in \mathrm{cl}\,C$. Then,

$$\lim_k \lambda_k x_k = \lim_k \frac{1}{k}(x+kd) = \lim_k \frac{x}{k} + d = d.$$

Reciprocally, assume the existence of a sequence of scalars $\{\lambda_k\} \downarrow 0$ and a sequence $\{x_k\} \subseteq \mathrm{cl}\,C$ such that $\{\lambda_k x_k\}$ converges to d. Take any $x \in \mathrm{cl}\,C$ and any $\lambda > 0$. Then, for all k large enough, $\lambda\lambda_k \in [0,1]$. Therefore, for all k large enough,

$$(1 - \lambda\lambda_k)x + (\lambda\lambda_k)x_k \in \mathrm{cl}\,C,$$

which implies that

$$x + \lambda d = \lim_k x + \lambda(\lambda_k x_k) = \lim_k (x + \lambda\lambda_k x_k) - \lambda\lambda_k x = \lim_k (1 - \lambda\lambda_k)x + (\lambda\lambda_k)x_k \in \mathrm{cl}\,C.$$

□

Proposition 5.1.9 *Let $\mathscr{C}$ be a family of nonempty, convex and closed subsets of $\mathbb{R}^n$ such that $\bigcap_{C\in\mathscr{C}} C$ is nonempty. Then,*

$$\left(\bigcap_{C\in\mathscr{C}} C\right)_\infty = \bigcap_{C\in\mathscr{C}} C_\infty.$$

Proof: Denote $D \equiv \cap_{C\in\mathscr{C}} C$. Let $d \in D_\infty$ and take any $x \in D$. From definiton, $d \in D_\infty(x)$. Thus, for any $C \in \mathscr{C}$, we have that, not only $x \in C$ but also $x+\lambda d \in C$, for all $\lambda \geq 0$. Hence, $d \in C_\infty(x)$, for any $C \in \mathscr{C}$. Since, every $C \in \mathscr{C}$ is closed, we have that

$$d \in \bigcap_{C\in\mathscr{C}} C_\infty(x) = \bigcap_{C\in\mathscr{C}} C_\infty.$$

Reciprocally, let $d \in \cap_{C\in\mathscr{C}} C_\infty$. Take any $x \in D$ and any $C \in \mathscr{C}$. Since C is closed, $d \in C_\infty = C_\infty(x)$. Thus, $x+\lambda d \in C$, for all $\lambda \geq 0$. Hence,

$$x+\lambda d \in \bigcap_{C\in\mathscr{C}} C = D, \text{ for all } \lambda \geq 0.$$

Thus, $d \in D_\infty(x)$, for any $x \in D$, which implies $d \in D_\infty$. □

5.2 Recession functions

Building on the concept of recession cone, we are now interested in understanding the behavior of limit in ∞.

Theorem 5.2.1 *Let E be a nonempty convex subset of $\mathbb{R}^n$. If E is an epigraph set then E_∞ is also an epigraph set.*

Proof: Let $(d,t) \in E_\infty$. Then, $(x,u)+\lambda(d,t) = (x+\lambda d, u+\lambda t) \in E$, for for every $(x,u) \in E$ and every $\lambda \geq 0$. Now, consider $s > t$. Take any $(x,u) \in E$ and any $\lambda > 0$. Then, $(x,u)+\lambda(d,s) = (x+\lambda d, u+\lambda s) \in E$, because $u+\lambda s > \lambda t$ and E is an epigraph set. Hence, $(d,s) \in E_\infty$.

Now, consider a sequence of scalars $\{t_k\}$ converging to $\bar{t}$ in the set

$$\{t \colon (d,t) \in E_\infty\}. \tag{5.1}$$

In particular, $(x,u)+\lambda(d,t_k) = (x+\lambda d, u+\lambda t_k) \in E$, for every $(x,u) \in E$ and every $\lambda \geq 0$. The sequence of scalars $\{u+\lambda t_k\}$ converges to $u+\lambda\bar{t}$ and, since E is an epigraph set, $(x+\lambda d, u+\lambda\bar{t}) = (x,u)+\lambda(d,\bar{t}) \in E$, for every $(x,u) \in E$ and every $\lambda \geq 0$. Hence, $\bar{t}$ belongs to (5.1), which is, then, closed, and, henceforth, E_∞ is an epigraph set. □

Recall that the epigraph of f is nonempty if and only if f is not $+\infty$ everywhere.

Definition 5.2.2 *Let f be convex and not $+\infty$ everywhere. Then, the recession function of f is the function $f_\infty \colon \mathbb{R}^n \to [-\infty,\infty]$ such that*

$$\mathrm{Epi}(f_\infty) = (\mathrm{Epi}\, f)_\infty.$$

The recession function f_∞ enjoys the following basic properties.

Theorem 5.2.3 *The function f_∞ is positively homogeneous and not $+\infty$ everywhere. Moreover, for every $d \in \mathbb{R}^n$,*

$$f_\infty(d) = \begin{cases} +\infty & \text{if there } \exists x \,\text{s.t.}\, f(x) = -\infty,\ f(x+d) > -\infty, \\ \sup\limits_{x \in \mathrm{dom}(f)} \{f(x+d) - f(x)\} & \text{otherwise.} \end{cases}$$

Proof: From the definition of recession cone, $(\mathrm{Epi}\, f)_\infty$ is a cone and contains the origin. Now, since $(0,0) \in (\mathrm{Epi}\, f)_\infty = \mathrm{Epi}(f_\infty)$, then $f_\infty(0) \le 0$. Thus, f_∞ is not $+\infty$ everywhere. Moreover, since $(\mathrm{Epi}\, f)_\infty = \mathrm{Epi}(f_\infty)$ is a cone, f_∞ is positively homogeneous, *i.e.*, $f_\infty(\lambda d) = \lambda f_\infty(d)$, for every $d \in \mathbb{R}^n$, and every $\lambda > 0$ - see Proposition 2.1.5. Moreover, from Theorem 5.1.2,

$$\begin{aligned} f_\infty(d) &= \inf\{t\colon (d,t) \in \mathrm{Epi}(f_\infty)\} \\ &= \inf\{t\colon (d,t) \in (\mathrm{Epi}\, f)_\infty\} \\ &= \inf\{t\colon \mathrm{Epi}(f) + (d,t) \subseteq \mathrm{Epi}(f)\} \\ &= \inf\{t\colon (x,u) + (d,t) \in \mathrm{Epi}(f),\ \forall (x,u) \in \mathrm{Epi}(f)\} \\ &= \inf\{t\colon f(x+d) \le u + t,\ \forall (x,u) \in \mathrm{Epi}(f)\}\,. \end{aligned}$$

Now, we observe that $\mathrm{Epi}(f)$ is the disjoint union of two sets,

$$\{(x,u) \in \mathrm{Epi}(f), f(x) \in \mathbb{R}\}\,, \qquad \{(x,u) \in \mathrm{Epi}(f), f(x) = -\infty\}$$

(the second set is empty if f is proper). Now, since

$$\begin{aligned} (x,u) \in \mathrm{Epi}(f),\ f(x) \in \mathbb{R} &\iff x \in \mathrm{dom}(f),\ f(x) \le u, \\ (x,u) \in \mathrm{Epi}(f),\ f(x) = -\infty &\iff x \in \mathrm{Dom}(f) \setminus \mathrm{dom}(f),\ u \in \mathbb{R} \end{aligned}$$

we have that

$$\begin{aligned} f_\infty(d) &= \inf\left\{t\colon \begin{array}{ll} f(x+d) \le u+t, & \forall x \in \mathrm{dom}(f),\ f(x) \le u, \\ f(x+d) \le u+t, & \forall x \in \mathrm{Dom}(f) \setminus \mathrm{dom}(f),\ \forall u \in \mathbb{R} \end{array}\right\} \\ &= \inf\left\{t\colon \begin{array}{ll} f(x+d) \le f(x)+t, & \forall x \in \mathrm{dom}(f) \\ f(x+d) \le u+t, & \forall x \in \mathrm{Dom}(f) \setminus \mathrm{dom}(f),\ \forall u \in \mathbb{R} \end{array}\right\}. \end{aligned}$$

Thus, if there exists $x \in \mathrm{Dom}(f) \setminus \mathrm{dom}(f)$ such that $f(x+d) > -\infty$ then, $f_\infty(d) = +\infty$. Otherwise, $f(x+d) = -\infty$, for every $x \in \mathrm{Dom}(f) \setminus \mathrm{dom}(f)$ and so,

$$\begin{aligned} f_\infty(d) &= \inf\{t\colon f(x+d) \le f(x) + t,\ \forall x \in \mathrm{dom}(f)\} \\ &= \sup\{f(x+d) - f(x)\colon x \in \mathrm{dom}(f)\}\,. \end{aligned}$$

□

Note that, if f is proper then $\mathrm{dom}(f) = \mathrm{Dom}(f) \neq \emptyset$. Hence, in this case f_∞ is simply given through

$$f_\infty(d) = \sup\{f(x+d) - f(x) \colon x \in \mathrm{dom}(f)\},$$

which cannot be $-\infty$. Thus, since f_∞ is not $+\infty$ everywhere, f_∞ is also proper.

Theorem 5.2.4 *If f is closed then f_∞ is closed. Moreover, for every $d \in \mathbb{R}^n$,*

$$f_\infty(d) = \begin{cases} -\infty & \text{if there } \exists x \,\text{s.t.}\; f(x+\lambda d) = -\infty, \forall \lambda \geq 0, \\ \sup\limits_{\lambda>0} \dfrac{f(x+\lambda d) - f(x)}{\lambda} & \text{if } x \in \mathrm{dom}(f), \\ +\infty & \text{if there } \exists x \text{ and } \lambda > 0 \,\text{s.t.}\; f(x) = -\infty, f(x+\lambda d) > -\infty \end{cases}$$

Proof: Take any $d \in \mathbb{R}^n$. Since f is closed, see Proposition 5.1.2,

$$\begin{aligned} f_\infty(d) &= \inf\{t \colon (d,t) \in \mathrm{Epi}(f_\infty)\} \\ &= \inf\{t \colon (d,t) \in (\mathrm{Epi}\, f)_\infty\} \\ &= \inf\{t \colon (x,u) + \lambda(d,t) \in \mathrm{Epi}\, f, \forall \lambda \geq 0\} \\ &= \inf\{t \colon f(x+\lambda d) \leq u + \lambda t,\ \forall \lambda \geq 0\}, \end{aligned}$$

for any $(x,u) \in \mathrm{Epi}\, f$. If $x \in \mathrm{dom}(f)$ then we may choose $u = f(x)$ and so,

$$\begin{aligned} f_\infty(d) &= \inf\{t \colon (x, f(x)) + \lambda(d,t) \in \mathrm{Epi}\, f, \forall \lambda \geq 0\} \\ &= \inf\{t \colon (x, f(x)) + \lambda(d,t) \in \mathrm{Epi}\, f, \forall \lambda > 0\} \\ &= \inf\{t \colon f(x+\lambda d) \leq f(x) + \lambda t,\ \forall \lambda > 0\} \\ &= \inf\left\{t \colon \frac{f(x+\lambda d) - f(x)}{\lambda} \leq t,\ \forall \lambda > 0\right\} \\ &= \sup_{\lambda>0} \frac{f(x+\lambda d) - f(x)}{\lambda}. \end{aligned}$$

Otherwise, $x \in \mathrm{Dom}(f) \setminus \mathrm{dom}(f)$.

If $f(x+\lambda d) = -\infty$, for all $\lambda > 0$, then $f_\infty(d) = -\infty$. Otherwise, $f(x+\lambda d) > -\infty$, for some $\lambda > 0$ and so, $f_\infty(d) = +\infty$. □

Theorem 5.2.5 *If f is convex and closed then, for every $x \in \mathrm{dom}(f)$,*

$$\sup_{\lambda>0} \frac{f(x+\lambda d) - f(x)}{\lambda} = \lim_{\lambda \to +\infty} \frac{f(x+\lambda d) - f(x)}{\lambda} \tag{5.2}$$

Proof: Let $x \in \mathrm{Dom}\, f$.

$$(\mathrm{Epi}\, f)_\infty = \{(d,\mu) \in \mathbb{R}^n \times \mathbb{R} | (x, f(x)) + \lambda(d,\mu) \in \mathrm{Epi}\, f, \forall\, \lambda > 0\}$$

and hence $(d,\mu) \in (\mathrm{Epi}\, f)_\infty$ if and only if for any $x \in \mathrm{Dom}\, f$ we have

$$f(x+\lambda d) \leq f(x) + \lambda\mu, \forall\, \lambda > 0,$$

which means exactly that

$$g(d) := \sup_{\lambda \to 0} \frac{f(x+\lambda d) - f(x)}{\lambda}$$

and hence $(\operatorname{Epi} f)_\infty = \operatorname{Epi} g$, for every $x \in \operatorname{Dom} f$, proving the formula (5.2), that the limit in λ coincides with the supremum in $\lambda > 0$ simply follows by recalling that the convexity of implies that fixed $x, d \in \mathbb{R}^n$, for any $s > 0$, the function

$$s \to \frac{f(x+sd) - f(x)}{s}$$

is nondecreasing. □

5.3 The perspective function

Definition 5.3.1 *Given f, let $\tilde{f} \colon \mathbb{R}^n \times \mathbb{R} \to [-\infty, +\infty]$ defined by*

$$\tilde{f}(x,t) = t f\left(\frac{x}{t}\right),$$

when $t > 0$, and $+\infty$ otherwise. This function $\tilde{f}$ is called the perspective *of f. Note that*

$$\begin{aligned} \operatorname{Dom}(\tilde{f}) &= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \colon t > 0, x/t \in \operatorname{Dom}(f)\}, \\ \operatorname{Epi}(\tilde{f}) &= \{(x,t,\delta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \colon t > 0, t f(x/t) \leq \delta\}. \end{aligned}$$

Theorem 5.3.2 *If f is convex then, $\tilde{f} \colon \mathbb{R}^n \times \mathbb{R} \to [-\infty, +\infty]$ is convex.*

Proof: We will use Theorem 4.1.2 to show the $\tilde{f}$ is convex. Let $(x_1,t_1),(x_2,t_2) \in \operatorname{Dom}(\tilde{f})$ and $\lambda \in (0,1)$, arbitrary. Since $\lambda t_1 + (1-\lambda)t_2 > 0$,

$$\begin{aligned} &\tilde{f}(\lambda(x_1,t_1) + (1-\lambda)(x_2,t_2)) = \\ &\quad = \tilde{f}(\lambda x_1 + (1-\lambda)x_2, \lambda t_1 + (1-\lambda)t_2) \\ &\quad = (\lambda t_1 + (1-\lambda)t_2) f\left(\frac{\lambda x_1 + (1-\lambda)x_2}{\lambda t_1 + (1-\lambda)t_2}\right) \\ &\quad = (\lambda t_1 + (1-\lambda)t_2) f\left(\frac{\lambda t_1}{\lambda t_1 + (1-\lambda)t_2} \cdot \frac{x_1}{t_1} + \frac{(1-\lambda)t_2}{\lambda t_1 + (1-\lambda)t_2} \cdot \frac{x_2}{t_2}\right). \end{aligned}$$

Since $x_1/t_1, x_2/t_2 \in \operatorname{Dom}(f)$ and $\operatorname{Dom}(f)$ is convex, then

$$\frac{\lambda t_1}{\lambda t_1 + (1-\lambda)t_2} \frac{x_1}{t_1} + \frac{(1-\lambda)t_2}{\lambda t_1 + (1-\lambda)t_2} \frac{x_2}{t_2} \in \operatorname{dom}(f)$$

Thus, $\lambda(x_1,t_1)+(1-\lambda)(x_2,t_2)\in \mathrm{dom}(\tilde{f})$. Moreover,

$$\begin{aligned}
&\tilde{f}(\lambda(x_1,t_1)+(1-\lambda)(x_2,t_2)) = \\
&\leq \quad (\lambda t_1+(1-\lambda)t_2)\left[\left(\frac{\lambda t_1}{\lambda t_1+(1-\lambda)t_2}\right)f\left(\frac{x_1}{t_1}\right)+\left(\frac{\lambda t_2}{\lambda t_1+(1-\lambda)t_2}\right)f\left(\frac{x_2}{t_2}\right)\right] \\
&= \quad \lambda t_1 f\left(\frac{x_1}{t_1}\right)+(1-\lambda)t_2 f\left(\frac{x_2}{t_2}\right) \\
&= \quad \lambda\tilde{f}(x_1,t_1)+(1-\lambda)\tilde{f}(x_2,t_2).
\end{aligned}$$

Hence, $\tilde{f}$ is convex. □

5.4 The convex hull of the union of convex sets

Consider a set $K\subseteq\mathbb{R}^n$ defined by functions $g_i\colon \mathbb{R}^n\to[-\infty,+\infty]$ there,

$$K=\{x\in\mathbb{R}^n\colon\, g_i(x)\leq 0, i=1,2,\dots,m\} \tag{5.3}$$

If all g_i are convex (and closed) then K is convex (and closed). It will on interest to characterize in algebraic terms the set

$$P=\mathrm{cl}\,\mathrm{conv}(K^1\cup K^2)$$

where K^1 and K^2 are both defined on the terms (5.3). Namely,

$$K^1=\{x\in\mathbb{R}^n\colon\, G^1(x)\leq 0\} \text{ and } K^2=\{x\in\mathbb{R}^n\colon\, G^2(x)\leq 0\}$$

where G^1,G^2 are vector of function g_i closed and convex (not necessarily of the same number). The theorem below provides a higher-dimensional algebra characterization of P.

Theorem 5.4.1 *If the set K^1,K^2 are nonempty then x and P if and only if*

$$\begin{aligned}
x &= x^1+x^2 \\
\mathrm{cl}\,\tilde{G}^1(\lambda_1,x^1)\leq 0,\ &\mathrm{cl}\,\tilde{G}^2(\lambda_2,x^2)\leq 0 \\
\lambda_1+\lambda_2 &= 1,\ \lambda_1,\lambda_2\geq 0.
\end{aligned}$$

where $\mathrm{cl}\,\tilde{G}^i(\lambda,x)$ denotes the vector mapping the conjugate are the closeness of the perpective of each conjugate of G^i.

For example, consider $f\colon\,\mathbb{R}^2\to[-\infty,+\infty]$ defined by

$$f(x,y)=\begin{cases} a/y-x & \text{whenever } y\in[c,d], \\ +\infty & \text{otherwise .}\end{cases}$$

where $a>0$ and $0<c<d$. Since $\mathrm{dom}(f)=[c,d]\times\mathbb{R}$ we conclude that f is proper. We may check that f is convex (it is in the sense of convex functions). Moreover, its epigraph is

$$\begin{aligned}\mathrm{Epi}(f) &= \{(x,y,t)\colon f(x,y) \le t,\ y \in [c,d]\} \\ &= \{(x,y,t)\colon a/y - x \le t,\ y \in [c,d]\}\,.\end{aligned}$$

which is closed, so f is also closed. Now, consider the perspective of f defined by

$$\begin{aligned}\tilde{f}(\lambda,x,y) &= \begin{cases} \lambda f(x/\lambda, y/\lambda) & \text{whenever } \lambda > 0 \text{ and } y/\lambda \in [c,d], \\ +\infty & \text{otherwise .}\end{cases} \\ &= \begin{cases} \lambda^2 a/y - x & \text{whenever } \lambda > 0 \text{ and } y \in [\lambda c, \lambda d], \\ +\infty & \text{otherwise .}\end{cases}\end{aligned}$$

The epigraph of $\tilde{f}$ is

$$\begin{aligned}\mathrm{Epi}(\tilde{f}) &= \left\{(\lambda,x,y,t)\colon \tilde{f}(\lambda,x,y) \le t\right\} \\ &= \left\{(\lambda,x,y,t)\colon \lambda^2 a/y - x \le t,\ \lambda c \le y \le \lambda d,\ \lambda > 0\right\}.\end{aligned}$$

Thus, the closure of the epigraph of $\tilde{f}$ is

$$\mathrm{cl}(\mathrm{Epi}\,\tilde{f}) = \left\{(\lambda,x,y,t)\colon \lambda^2 a/y - x \le t,\ \lambda c \le x \le \lambda d,\ \lambda \ge 0\right\}.$$

Thus, the closure of $\tilde{f}$ is

$$\mathrm{cl}\,\tilde{f}(\lambda,x,y) = \begin{cases} \lambda^2 a/y - x & \text{whenever } \lambda \ge 0 \text{ and } y \in [\lambda c, \lambda d], \\ +\infty & \text{otherwise .}\end{cases}$$

Chapter 6

(Application) the lower convex envelope of x/y

In this chapter, we show example of how to obtain a convex envelope, given a nonconvex functions, for more examples, see [9] and [10].

6.1 Bivariate function over rectangle

Consider $f\colon \mathbb{R}^2 \to [-\infty,+\infty]$ defined by x/y on the square $[a,b]\times[c,d] \subset \mathbb{R}^2$, with $0<a<b$ and $0<c<d$, and by $+\infty$ outside the same square. Epigraph and domain are given by

$$\mathrm{Epi}(f) = \{(x,y,t)\colon x/y \le t,\ a \le x \le b,\ c \le y \le d\}, \quad \mathrm{Dom}(f) = [a,b]\times[c,d].$$

Hence, the function is proper and closed. For fixed $x \in [a,b]$, $f(\cdot,y)$ is convex strictly decreasing, while, for fixed $y \in [c,d]$, $f(x,\cdot)$ is affine increasing. Nevertheless, the function is not convex, as our nest result shows. The nonconvexity of f - see Figure 6.1 is hardly noticeable. Interestingly, the function x^2/y, on the same domain, is convex.

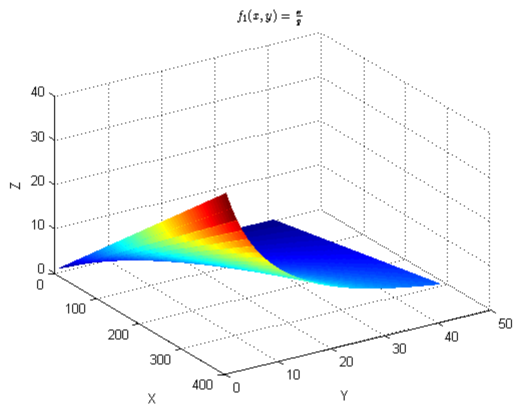


Fig. 6.1 The fractional term x/y in the positive orthant.

Proposition 6.1.1 *Let $P \equiv (a,c)$ denote the lower left corner of the domain $\mathrm{Dom}(f) = [a,b] \times [c,d]$ and let $Q \equiv P + \alpha d$, for some $\alpha > 0$ and direction $d \equiv (c, \sqrt{a^2+c^2} - a) > 0$, be any point in the same square $[a,b] \times [c,d]$. Then,*

$$f\left((P+Q)/2\right) > \left(f(P) + f(Q)\right)/2. \tag{6.1}$$

and, so, f is not convex.

Proof: The midpoint between P and Q has coordinates

$$\left(a + (\alpha/2)c, c + (\alpha/2)\left(\sqrt{a^2+c^2} - a\right)\right).$$

Hence,

$$f\left(\frac{1}{2}P + \frac{1}{2}Q\right) - \left(\frac{1}{2}f(P) + \frac{1}{2}f(Q)\right) = \frac{a + (\alpha/2)c}{c + (\alpha/2)z} - \frac{1}{2}\frac{a}{c} - \frac{1}{2}\frac{a + \alpha c}{c + \alpha z},$$

where to simplify notation we employ $z \equiv \sqrt{a^2+c^2} - a$. Now, we perform algebraic operations to get a single denominator. The denominator is positive, the numerator is

$$\begin{aligned}
&\left(a + \frac{\alpha}{2}c\right)(c)(c+\alpha z) - \left(\frac{a}{2}\right)\left(c + \frac{\alpha}{2}z\right)(c+\alpha z) - \left(\frac{a}{2} + \frac{\alpha}{2}c\right)\left(c + \frac{\alpha}{2}z\right)(c) = \\
&= \left(\frac{a}{2}\right)(c)(c+\alpha z) + \left(\frac{a}{2} + \frac{\alpha}{2}c\right)(c)\left(\frac{\alpha}{2}z\right) - \left(\frac{a}{2}\right)\left(c + \frac{\alpha}{2}z\right)(c+\alpha z) \\
&= \left(\frac{a}{2}\right)(c+\alpha z)\left(-\frac{\alpha}{2}z\right) + \left(\frac{a}{2}\right)(c)\left(\frac{\alpha}{2}z\right) + \left(\frac{\alpha}{2}c\right)(c)\left(\frac{\alpha}{2}z\right) \\
&= \left(\frac{a}{2}\right)(\alpha z)\left(-\frac{\alpha}{2}z\right) + \left(\frac{\alpha}{2}c\right)(c)\left(\frac{\alpha}{2}z\right) \\
&= \left(\frac{\alpha^2}{4}z\right)\left(c^2 - az\right)
\end{aligned}$$

Now, we observe that

$$c^2 - az = c^2 - a\left(\sqrt{a^2+c^2} - a\right) = c^2\left(1 - \left(\frac{a}{c}\right)\left(\sqrt{\left(\frac{a}{c}\right)^2 + 1} - \left(\frac{a}{c}\right)\right)\right) > 0,$$

because $x\left(\sqrt{x^2+1} - x\right) < 1/2$, for every $x > 0$. Simply note that

$$\left(2x^2+1\right)^2 = 4x^4 + 4x^2 + 1 > 4x^4 + 4x^2 = 4x^2\left(x^2+1\right) = \left(2x\sqrt{x^2+1}\right)^2,$$

from where we deduce that $2x\left(\sqrt{x^2+1} - x\right) = 2x\sqrt{x^2+1} - 2x^2 < 1.$ □

We want to find a characterization a $\mathrm{conv}(f)$ which by definition is the best convex underestimator of f. We will do so by first characterizing $\mathrm{conv}\,\mathrm{epi}(f)$ which is the same as $\mathrm{epi}\,\mathrm{conv}(f)$.

Lemma 6.1.2 *For every $P_1 \equiv (x_1/y_1, x_1, y_1), P_2 \equiv (x_2/y_2, x_2, y_2) \in \mathrm{epi}(f)$, the relative interior of the line segment $\overline{P_1P_2}$ is either empty or the whole set. Moreover, it is nonempty if and only if*

$$(y_2 - y_1)\left(\frac{x_1}{y_1} - \frac{x_2}{y_2}\right) \geq 0$$

Proof: First, note that, for any pair of points (t_1,x_1,y_1) and (t_2,x_2,y_2) in $\operatorname{epi}(f)$, and for any $\lambda \in [0,1]$,

$$\begin{aligned}
&(\lambda t_1+(1-\lambda)t_2)\,(\lambda y_1+(1-\lambda)y_2)-(\lambda t_1 y_1+(1-\lambda)t_2 y_2) = \\
&\quad = \lambda^2 t_1 y_1+\lambda(1-\lambda)t_1 y_2+\lambda(1-\lambda)t_2 y_1+(1-\lambda)^2 t_2 y_2-\lambda t_1 y_1-(1-\lambda)t_2 y_2 \\
&\quad = -\lambda(1-\lambda)t_1 y_1+\lambda(1-\lambda)t_1 y_2+\lambda(1-\lambda)t_2 y_1-\lambda(1-\lambda)t_2 y_2 \\
&\quad = \lambda(1-\lambda)(y_2-y_1)(t_1-t_2) \\
&\quad = \lambda(1-\lambda)\,(y_2-y_1)\,(t_1-t_2)\,.
\end{aligned} \tag{6.2}$$

Now, every point in the relative interior of the segment $\overline{P_1P_2}$ is of the following form for some $\lambda \in (0,1)$,

$$\begin{bmatrix} t \\ x \\ y \end{bmatrix} \equiv \lambda \begin{bmatrix} x_1/y_1 \\ x_1 \\ y_1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} x_2/y_2 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \lambda(x_1/y_1)+(1-\lambda)(x_2/y_2) \\ \lambda x_1+(1-\lambda)x_2 \\ \lambda y_1+(1-\lambda)y_2 \end{bmatrix}.$$

Clearly $x \in [a,b]$ and $y \in [c,d]$. Moreover, from (6.2),

$$\begin{aligned}
ty-x &= (\lambda(x_1/y_1)+(1-\lambda)(x_2/y_2))\,(\lambda y_1+(1-\lambda)y_2)-(\lambda x_1+(1-\lambda)x_2) \\
&= \lambda(1-\lambda)(y_2-y_1)(x_1/y_1-x_2/y_2).
\end{aligned}$$

Hence, the sign of $ty-x$ does not depend on λ, from where the desired result follows. $\square$

6.2 The union of two sets

Consider the following sets $K^1, K^2 \subseteq \mathbb{R}^3$, where $0<a<b$ and $0<c<d$,

$$\begin{aligned}
K^1 &= \{(a,y,t)\colon a/y \le t,\ c \le y \le d\} \\
K^2 &= \{(b,y,t)\colon b/y \le t,\ c \le y \le d\}\,.
\end{aligned}$$

Clearly, these two disjoint sets are closed and convex. We will be interested in characterizing

$$\operatorname{cl}\operatorname{conv}\left(K^1 \cup K^2\right)$$

From [4, Theorem 1], $(x,y,t) \in \operatorname{cl}\operatorname{conv}\left(K^1 \cup K^2\right)$ if and only if

$$\begin{aligned}
x &= \lambda_1 a+\lambda_2 b \\
y &= z_1+z_2 \\
t &= s_1+s_2
\end{aligned}$$

for any $\lambda = (\lambda_1,\lambda_2)$ that satisfies

$$\lambda_1,\lambda_2 \ge 0, \quad \lambda_1+\lambda_2 = 1,$$

and any (z_1,s_1) e (z_2,s_2) that satisfies

$$\begin{aligned}\lambda_1^2 a/z_1 - s_1 &\leq 0, \quad \lambda_1 c \leq z_1 \leq \lambda_1 d\\ \lambda_2^2 b/z_2 - s_2 &\leq 0, \quad \lambda_2 c \leq z_2 \leq \lambda_2 d.\end{aligned}$$

Lemma 6.2.1 *Assume $0 < l \leq u < y$. If $0 < B \leq A$ then, the (global) minimizer of $f(z) \equiv A/z + B/(y-z)$ on $[l,u] \subset \mathbb{R}$ is*

$$t = \text{mid}\left(l, \frac{y}{1+\sqrt{B/A}}, u\right). \tag{6.3}$$

Proof: The derivative of f is $f'(z) \equiv -A/z^2 + B/(y-z)^2$; the second derivative is $f''(z) \equiv 2A/z^3 + 2B/(y-z)^3 > 0$, for every $z \in [l,u]$. Moreover, $f'(z) = 0$ if and only if

$$(A-B)z^2 - (2Ay)z + (Ay^2) = 0. \tag{6.4}$$

Thus, if $A = B$, the minimizer of f on $[l,u]$ is mid$(l, y/2, u)$; if $A > B$, the quadratic function in (6.4) has the two roots:

$$r \equiv \frac{A-\sqrt{AB}}{A-B}y, \quad r_2 \equiv \frac{A+\sqrt{AB}}{A-B}y.$$

Since $0 < r < y < r_2$, then the minimizer of f on $[l,u]$ is mid(l,r,u). Since

$$\frac{A-\sqrt{AB}}{A-B} = \frac{(\sqrt{A})(\sqrt{A}-\sqrt{B})}{(\sqrt{A}+\sqrt{B})(\sqrt{A}-\sqrt{B})} = \frac{\sqrt{A}}{\sqrt{A}+\sqrt{B}} = \frac{1}{1+\sqrt{B/A}}$$

the desired result follows. □

Lemma 6.2.2 *Assume $S(y) \equiv \{(z_1,z_2) \subseteq [C,D] \times [E,F] \colon z_1 + z_2 = y\}$ is nonempty, where $0 < C \leq D < y$ and $0 < E \leq F < y$. Then, $0 < l \equiv \max(C, y-F) \leq \min(D, y-E) \equiv u < y$ and, if $0 < B \leq A$, the (global) minimizer of $f(z_1,z_2) \equiv A/z_1 + B/z_2$ on $S(y)$ is $(\bar{z}_1, \bar{z}_2) = (t, y-t)$, where t defined by (6.3).*

Proof: The (global) minimimum value of f on $S(y)$ coincides with the minimum value of $f(z_1) \equiv A/z_1 + B/(y-z_1)$ on $[l,u]$. Now, the desired result follows from Lemma 6.2.1. □

Note that, if $0 < A < B$ then Lemma 6.2.2 still applies with the roles of z_1 and z_2 interchanged.

For a given $(x,y) \in (a,b) \times [c,d]$, the value of the convex envelope of x/y on $(a,b) \times [c,d]$ is the minimum value of the following optimization problem

$$\begin{aligned}\min \quad & \lambda_1 (a/y_1) + \lambda_2 (b/y_2)\\ \text{s.t.} \quad & x = \lambda_1 a + \lambda_2 b,\ \lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 \geq 0\\ & y = \lambda_1 y_1 + \lambda_2 y_2,\ c \leq y_1 \leq d,\ c \leq y_2 \leq d.\end{aligned} \tag{6.5}$$

Thus, the values of (λ_1, λ_2) are uniquely determined. In fact, through Cramer's rule,

$$\lambda_1 = \frac{b-x}{b-a} \in (0,1), \qquad \lambda_2 = \frac{x-a}{b-a} \in (0,1).$$

Therefore, with these variables fixed, (6.5) becomes equivalent to

$$\min \quad \overbrace{\lambda_1^2 a}^{A}/z_1 + \overbrace{\lambda_2^2 b}^{B}/z_2 \tag{6.6}$$

$$\text{s.t.} \quad y = z_1 + z_2,\ \underbrace{\lambda_1 c}_{C} \le z_1 \le \underbrace{\lambda_1 d}_{D},\ \underbrace{\lambda_2 c}_{E} \le z_2 \le \underbrace{\lambda_2 d}_{F}, \tag{6.7}$$

where $(z_1, z_2) \equiv (\lambda_1 y_1, \lambda_2 y_2)$. Note that

$$\lambda_1^2 a - \lambda_2^2 b = \left(\frac{1}{a}\right)\left(\lambda_1 a - \lambda_2\sqrt{ab}\right)\left(\lambda_1 a + \lambda_2\sqrt{ab}\right) = \frac{(\sqrt{ab}-x)(\sqrt{ab}+x)}{b-a},$$

which implies that $\lambda_1^2 a - \lambda_2^2 b$ is: positive when $x \in (a, \sqrt{ab})$; zero when $x = \sqrt{ab}$; and, negative when $x \in (\sqrt{ab}, b)$.

If $x \in (a, \sqrt{ab})$ then, from Lemma 6.2.2, the optimal solution of (6.6) is $(\bar{z}_1, \bar{z}_2) = (t, y-t)$ for $t = \mathrm{mid}(l, r, u)$, where

$$\begin{aligned}
r &= y\frac{(\lambda_1^2 a) - \sqrt{(\lambda_1^2 a)(\lambda_2^2 b)}}{\lambda_1^2 a - \lambda_2^2 b} &&= y\frac{\lambda_1 a}{\lambda_1 a + \lambda_2\sqrt{ab}} \\
l &= \max(\lambda_1 c, y - \lambda_2 d) &&= \begin{cases} \lambda_1 c & \text{se } y \le \bar{y}_1 \equiv \lambda_1 c + \lambda_2 d \\ y - \lambda_2 d & \text{se } y > \bar{y}_1 \end{cases} \\
u &= \min(\lambda_1 d, y - \lambda_2 c) &&= \begin{cases} y - \lambda_2 c & \text{se } y \le \bar{y}_2 \equiv \lambda_2 c + \lambda_1 d \\ \lambda_1 d & \text{se } y > \bar{y}_2 \end{cases}
\end{aligned}$$

If $x = \sqrt{ab}$ then, from Lemma 6.2.2, the optimal solution of (6.6) is $(\bar{z}_1, \bar{z}_2) = (t, y-t)$ for $t = \mathrm{mid}(l, r, u)$, where $r = y/2$ and l, u are defined as above.

If $x \in (\sqrt{ab}, b)$ then, from Lemma 6.2.2, the optimal solution of (6.6) is $(\bar{z}_1, \bar{z}_2) = (y-t, t)$ for $t = \mathrm{mid}(l, r, u)$, where

$$\begin{aligned}
r &= y\frac{(\lambda_2^2 b) - \sqrt{(\lambda_1^2 a)(\lambda_2^2 b)}}{\lambda_2^2 b - \lambda_1^2 a} &&= y\frac{\lambda_2 b}{\lambda_1 a + \lambda_2\sqrt{ab}} \\
l &= \max(\lambda_2 c, y - \lambda_1 d) &&= \begin{cases} \lambda_2 c & \text{se } y \le \bar{y}_2 \\ y - \lambda_1 d & \text{se } y > \bar{y}_2 \end{cases} \\
u &= \min(\lambda_2 d, y - \lambda_1 c) &&= \begin{cases} y - \lambda_1 c & \text{se } y \le \bar{y}_1 \\ \lambda_2 d & \text{se } y > \bar{y}_1 \end{cases}
\end{aligned}$$

and $\bar{y}_1, \bar{y}_2$ defined before.

6.2.1 Numerical example

Consider the underestimator of x/y described on 6.5 over $(a,b)\times[c,d]$. We start off by looking at a uniform mesh divide into 20 equal parts. Comparing the minimum gap value, for the rectangle $(a,b)\times[c,d]$. The results is

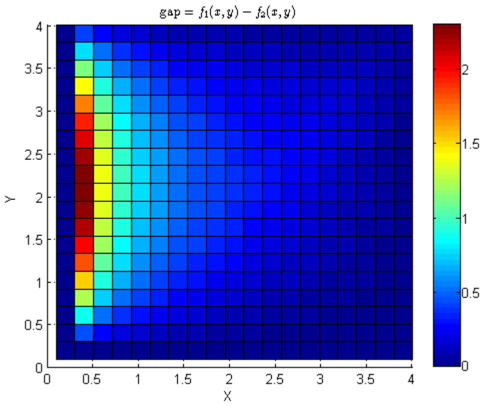
$[a,b]\times[c,d]$	point (x^*,y^*)	x^*/y^*	Maximum gap
$(0.1,0.5)\times[0.1,0.5]$	$(0.1421,0.2684)$	0.5294	0.2573
$(0.1,4.0)\times[0.1,0.5]$	$(0.7158,0.2684)$	2.6669	3.3620
$(0.1,0.5)\times[0.1,4.0]$	$(0.1211,2.1530)$	0.0562	0.1248
$(0.5,4.0)\times[0.5,4.0]$	$(0.6842,1.7890)$	0.3824	0.5101
$(0.1,4.0)\times[0.1,4.0]$	$(0.3053,1.9470)$	0.1568	2.3004

Table 6.1 Uniform mesh from near the origin

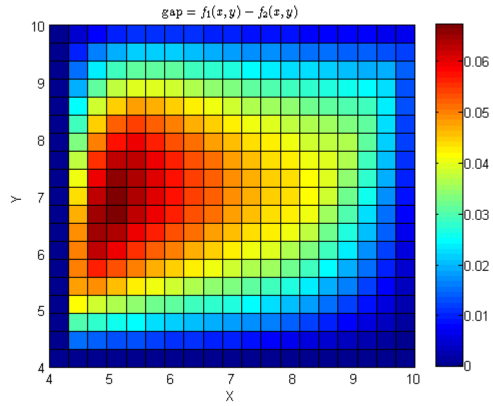
The maximum gap distant from the origin, for uniform mesh.

$[a,b]\times[c,d]$	point (x^*,y^*)	x^*/y^*	Maximum gap
$(4.0,5.0)\times[4.0,5.0]$	$(4.2630,4.5260)$	0.9419	0.0033
$(4.0,10.0)\times[4.0,5.0]$	$(6.8420,4.4720)$	1.5300	0.0557
$(4.0,5.0)\times[4.0,10.0]$	$(4.1050,7.1580)$	0.5735	0.0032
$(5.0,10.0)\times[5.0,10.0]$	$(5.7890,7.3680)$	0.7857	0.0359
$(4.0,10.0)\times[4.0,10.0]$	$(4.9470,6.8420)$	0.7230	0.0672

Table 6.2 Uniform mesh from distant of the origin



(a) Maximum gap on $(0.1,4.0)\times[0.1,4.0]$



(b) Maximum gap on $(4.0,10.0)\times[0.1,4.0]$

Therefore, we can conclude that we will have a better convex envelope if the rectangle is distant from the origin, as the second table shows, and the Figure 6.2b.

Chapter 7

Conclusion

In this chapter, we summarise the work and results presented.

This dissertation aimed to study the *convexification* of certain nonconvex functions over rectangle, showing in very simple way a geometric characterization.

We first made a review for some concepts of Algebra, Analysis and other sciences connected with this purpose. In ours searches we find residual books dealing with Convex Analysis. Convex Analysis is a basic prerequisite for students who want to understand issues related to linear programming and optimization, theoretical knowledge in solving problems related to optimization is practically alienated to the knowledge of convex sets and functions. Through the study of some definitions, theorems and their respective demonstrations creates a safe environment of being able to model the problems consistently and correctly, avoiding more complexity. In this sense, although there are many applications relating to sets and convex functions, it is perceived that the same concepts are not easy to understand. In this way, we hope that this material could, in a didactic way make the understanding of this type of study easy.

According to the first introduction of the epigraphic set in this matter gives us guarantees of perception of geometric form that is the purpose of this research work, without taking the focus on differentiable convex functions.

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