Spontaneous magnetization under a pseudovector interaction between quarks in high density quark matter

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Spontaneous magnetization and magnetic susceptibility originated from the pseudovector-type four-point interaction between quarks are calculated in quark matter with zero temperature and finite quark chemical potential by using the two-flavor Nambu-Jona-Lasinio model. It is shown that both the chiral condensate and spin polarized condensate coexist in a narrow region of the quark chemical potential. And then, it is also shown that, in this narrow region, the spontaneous magnetization appears. Also, the magnetic susceptibility due to quarks with the positive energy is evaluated in the spin polarized phase.

§1. Introduction

One of recent interests to understand the world governed by the quantum chromodynamics (QCD) may be to clarify the phase structure in the plane spanned by the temperature and baryon chemical potential.[1)] In the region of high temperature and zero density, the numerical simulation by using the lattice QCD gives a useful information about the phase structure. However, in the region with low temperature and large quark chemical potential, the lattice simulation does not work until now. In that region, it has been remarked that various phases may appear such as the color superconducting phase,[2)–4)] the quarkyonic phase,[5)] the inhomogeneous chiral condensed phase,[6)] the quark ferromagnetic phase,[7)] the color-ferromagnetic phase,[8)] the spin polarized phase due to the axial vector interaction[9),10)] or due to the tensor interaction[11)–18)] and so forth.

Furthermore, it may be interesting to investigate magnetic properties in quark matter in the region of low temperature and large baryon chemical potential. The reason is as follows: In the ultrarelativistic heavy-ion collisions, it has been remarked that a strong magnetic field may be created in the early stage of nucleus-nucleus collisions,[19)] for example, $|eB| \sim m_\pi^2$ at the Relativistic Heavy-Ion Collider (RHIC) experiment at Brookhaven, where e, B and m_π represent the elementary electric charge, the magnetic flux density or magnetic field and pion mass, respectively, and maybe even stronger at the Large Hadron Collider (LHC) experiment at CERN. In astrophysical fields, compact stars such as neutron stars, especially magnetars,[20),21)] show a very strong magnetic field. Thus, the investigation of magnetic properties in quark matter is one of the interesting and important problems of QCD.[22),23)]

Recently, the present authors have investigated the phase structure of high density quark matter under a strong external magnetic field.[24)] By using the Nambu-Jona-Lasinio (NJL) model with tensor or pseudovector interaction between quarks, we have shown that a quark spin polarized phase may exist in high density quark matter under a strong external magnetic field in a certain model parameter regions. Similarly, the effects of a strong magnetic field in the model with the tensor interaction have been investigated in one-flavor NJL model at finite temperature with zero chemical potential.[25),26)] Also, in Ref. 27), the effects of the axial-vector interaction under a strong magnetic field on the spatially-modulated chiral condensed phase was investigated by means of the holographic technique. Thus, the physics of the strong interacting matter under a magnetic field becomes interesting and important subject in quark matter with various possible phases and many investigations are carried out recently.[28)]

In this paper, we investigate spontaneous magnetization in the finite quark chemical potential region or in high density quark matter at zero temperature by using the Nambu-Jona-Lasinio (NJL) model[29)–32)] with the pseudovector-type[9)] four-point interaction between quarks as an effective model of QCD. As for the tensor-type interaction, we have already investigated a possibility of a spontaneous magnetization.[16)] As a result, the tensor interaction does not reveal the spontaneous magnetization except for the existence of the anomalous magnetic moments of quarks, even if the spin polarized condensate exists. Thus, in this paper, we investigate the magnetic properties due to the pseudovector interaction between quarks in quark matter at zero temperature.

This paper is organized as follows: In the next section, a model under consideration is introduced as an extension of the original NJL model. In Sect. 3, the thermodynamic potential is evaluated under a weak external magnetic filed and the way how to calculate the spontaneous magnetization is explained. In Sect. 4, numerical results are shown for the chiral condensate, spin polarized condensate and the thermodynamic potential, and the spontaneous magnetization and the magnetic susceptibility are calculated. The last section is devoted to a summary and concluding remarks.

§2. Mean field approximation for the Nambu-Jona-Lasinio model with vector-pseudovector-type four-point interactions between quarks

Let us start from the two-flavor Nambu-Jona-Lasinio model with vector-pseudovector-type[9),10)] four-point interactions between quarks under an external magnetic field. The Lagrangian density can be expressed as

$$\begin{aligned}\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi + G_s[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2] \\ -G_p[(\bar{\psi}\gamma^\mu\vec{\tau}\psi)^2 + (\bar{\psi}i\gamma_5\gamma^\mu\vec{\tau}\psi)^2] \ ,\end{aligned} \tag{2.1}$$

where m_0 represents a current quark mass and D_μ represents the covariant derivative introduced as

$$D_\mu = \partial_\mu + iQA_\mu \ , \qquad A_\mu = \left(0,\ \frac{By}{2},\ -\frac{Bx}{2},\ 0\right) = (0,\ -\boldsymbol{A}) \ . \tag{2.2}$$

Here, $Q = 2e/3$ for up quark and $-e/3$ for down quark are the electric charges where e is the elementary charge. There is an external magnetic field B along z-axis.

Hereafter, we treat the model within the mean field approximation. In order to consider the spin polarization under the mean field approximation, the pseudovector condensate $\langle\bar{\psi}\gamma_5\gamma^3\tau_3\psi\rangle$ is taken into account. Then, the Lagrangian density reduces to

$$\begin{aligned}\mathcal{L}_{MF} &= \bar{\psi}(i\gamma^\mu D_\mu - M_q)\psi + U_A\bar{\psi}\gamma_5\gamma^3\tau_3\psi - \frac{M^2}{4G_s} - \frac{U_A^2}{4G_p} \\ &= \bar{\psi}(i\gamma^\mu D_\mu - M_q)\psi - U_A\psi^\dagger\Sigma_3\tau_3\psi - \frac{M^2}{4G_s} - \frac{U^2}{4G_p} \ ,\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}&\Sigma_3 = -\gamma^0\gamma_5\gamma^3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \\ &M_q = m_0 + M \ , \qquad M = -2G_s\langle\bar{\psi}\psi\rangle \ , \\ &U_A = 2G_p\langle\bar{\psi}\gamma_5\gamma^3\tau_3\psi\rangle = -2G_p\langle\psi^\dagger\Sigma_3\tau_3\psi\rangle \equiv U\tau_f \ .\end{aligned} \tag{2.4}$$

Here, $\tau_f = 1$ for up quark and -1 for down quark denote the eigenvalues of τ_3. Also, σ_3 is the third component of the Pauli spin matrices.

Introducing the quark chemical potential μ in order to consider finite density quark matter, the Hamiltonian density can be obtained from the Lagrangian density within the mean field approximation as

$$\begin{aligned}\mathcal{H}_{MF} - \mu\mathcal{N} = \bar{\psi}\left(-i\boldsymbol{\gamma}\cdot(\boldsymbol{\nabla} - iQ\boldsymbol{A}) + M_q - \mu\gamma^0 + U_A\gamma^0\Sigma_3\tau_3\right)\psi \\ +\frac{M^2}{4G_s} + \frac{U^2}{4G_p} \ ,\end{aligned} \tag{2.5}$$

where $\mathcal{N}$ represents the quark number density, $\psi^\dagger\psi$.

§3. Thermodynamic potential

The Hamiltonian density (2.5) can be rewritten as

$$\mathcal{H}_{MF,A} - \mu\mathcal{N} = \psi^\dagger(h_A - \mu)\psi + \frac{M^2}{4G_s} + \frac{U^2}{4G_p} \ , \tag{3.6}$$

$$h_A = -i\gamma^0\boldsymbol{\gamma}\cdot(\boldsymbol{\nabla} - iQ\boldsymbol{A}) + \gamma^0 M_q + U_A\Sigma_3\tau_3 \ . \tag{3.7}$$

In order to obtain the eigenvalues of h_A, namely the energy eigenvalues of a single quark, it is necessary to diagonalize h_A, the eigenvalues of which can be obtained easily as

$$E^f_{A,p\nu\eta} = \begin{cases} E^u_{A,p\nu\sigma} = \sqrt{2Q_uB\nu + \left(\sqrt{p_z^2+M_q^2}+\sigma U\right)^2}\ , \begin{cases} \nu = 0,1,2,\cdots & \text{for} \ \ \sigma = 1 \\ \nu = 1,2,\cdots & \text{for} \ \ \sigma = -1 \end{cases} \\ E^d_{A,p\nu\sigma} = \sqrt{-2Q_dB\nu + \left(\sqrt{p_z^2+M_q^2}-\sigma U\right)^2}\ , \begin{cases} \nu = 1,2,\cdots & \text{for} \ \ \sigma = 1 \\ \nu = 0,1,2,\cdots & \text{for} \ \ \sigma = -1 \end{cases} \end{cases}$$

$$= \sqrt{2|Q_f|B\nu + \left(\eta U + \sqrt{p_z^2+M_q^2}\right)^2}\ . \quad \begin{cases} \nu = 0,1,2,\cdots & \text{for} \ \ \eta = 1 \\ \nu = 1,2,\cdots & \text{for} \ \ \eta = -1 \end{cases} \tag{3.8}$$

The thermodynamic potential can be expressed as

$$\begin{aligned} \varPhi_A =& \sum_{f,\alpha}\int\frac{dp_z}{2\pi}\frac{|Q_f|B}{2\pi}\left(E^f_{A,p\ \nu=0\ \eta=1}-\mu\right)\theta(\mu - E^f_{A,p\ \nu=0\ \eta=1}) \\ &+\sum_{\eta,f,\alpha}\int\frac{dp_z}{2\pi}\sum_{\nu=1}^{E_A<\mu}\frac{|Q_f|B}{2\pi}\left(E^f_{A,p\nu\eta}-\mu\right)\theta(\mu-E^f_{A,p\nu\eta}) \\ &-\sum_{f,\alpha}\int^{\varLambda}\frac{dp_z}{2\pi}\frac{|Q_f|B}{2\pi}E^f_{A,p\ \nu=0\ \eta=1} - \sum_{\eta,f,\alpha}\int^{\varLambda}\frac{dp_z}{2\pi}\sum_{\nu=1}^{E_A<\varLambda}\frac{|Q_f|B}{2\pi}E^f_{A,p\nu\eta} \\ &+\frac{M^2}{4G_s}+\frac{U^2}{4G_p}\ . \end{aligned} \tag{3.9}$$

The first and the second lines represent the positive-energy contribution of quarks and the third line represents the vacuum contribution. It should be noted that the single quark energy does not depend on the flavor in the lowest Landau level with $\nu = 0$.

In the thermodynamic potential (3.9), the quantum number ν, which labels the Landau level, has to be summed up. However, since it is interesting to consider the spontaneous magnetization, it may be assumed that the external magnetic field B is small and finally B becomes 0. Therefore, let us replace the sum with respect to ν by an integration approximately.[11] In general, let us consider a function $f(x)$. Here, we introduce a small quantity a and let us consider the Tailor expansion around $x = a\nu$ as follows:

$$\begin{aligned} \int_{a(\nu-1)}^{a\nu} dx f(x) &= \int_{a(\nu-1)}^{a\nu} dx \left[f(a\nu) + \left.\frac{df}{dx}\right|_{x=a\nu}(x-a\nu) + \frac{1}{2}\left.\frac{d^2f}{dx^2}\right|_{x=a\nu}(x-a\nu)^2 + \cdots\right] \\ &= af(a\nu) - \frac{1}{2}a^2f'(a\nu) + \frac{1}{6}a^3f''(a\nu) + \cdots\ . \end{aligned} \tag{3.10}$$

Thus, the following relations is obtained :

$$\begin{aligned} \sum_{\nu=\nu_m+1}^{\nu_M}\int_{a(\nu-1)}^{a\nu} dx f(x) &\equiv \int_{a\nu_m}^{a\nu_M} dx f(x) \\ &= a\sum_{\nu=\nu_m+1}^{\nu_M} f(a\nu) - \frac{a^2}{2}\sum_{\nu=\nu_m+1}^{\nu_M} f'(a\nu) + \frac{a^3}{6}\sum_{\nu=\nu_m+1}^{\nu_M} f''(a\nu) + \cdots\ . \end{aligned} \tag{3.11}$$

Here, it should be noted that the definition of integral can be used when a is infinitesimally small, namely,

$$a\sum_{\nu=\nu_m+1}^{\nu_M} f'(a\nu) = \int_{a\nu_m}^{a\nu_M} dx\ f'(x) + \frac{a^2}{2}\sum_{\nu=\nu_m+1}^{\nu_M} f''(a\nu) + \cdots$$
$$= f(a\nu_M) - f(a\nu_m) + \frac{a^2}{2}\sum_{\nu=\nu_m+1}^{\nu_M} f''(a\nu) + \cdots\ , \qquad (3.12)$$

and so on. Thus, by using the above formula repeatedly, useful approximate formula is obtained as follows:

$$a\sum_{\nu=\nu_m+1}^{\nu_M} f(a\nu)$$
$$= \int_{a\nu_m}^{a\nu_M} dx f(x) + \frac{a}{2}\left[f(a\nu_M) - f(a\nu_m)\right] + \frac{a^2}{12}\left[f'(a\nu_M) - f'(a\nu_m)\right] + \cdots \qquad (3.13)$$

In (3.9), we separate the sum over ν into a part with $\nu = 0$ and another one with $\nu > 0$. As for the positive-energy part with $\eta = 1$, we obtain

$$\nu_m = 0\ , \qquad \nu_M \equiv \nu_M^{(1)} = \left[\frac{\mu^2 - \left(\sqrt{p_z^2 + M_q^2} + U\right)^2}{2|Q_f|B}\right]\ , \qquad (3.14)$$

where $[\cdots]$ represents the Gauss symbol. Also, $\nu_M \geq 0$, we obtain

$$|p_z| \leq \sqrt{(\mu - U)^2 - M_q^2}\ . \qquad (3.15)$$

Similarly, for $\eta = -1$, we obtain

$$\nu_m = 0\ , \qquad \nu_M \equiv \nu_M^{(-1)} = \left[\frac{\mu^2 - \left(\sqrt{p_z^2 + M_q^2} - U\right)^2}{2|Q_f|B}\right]\ ,$$
$$|p_z| \leq \sqrt{(\mu + U)^2 - M_q^2}\ . \qquad (3.16)$$

As for the vacuum contributions, the three-momentum cutoff Λ is, as usually, introduced as

$$p_x^2 + p_y^2 + p_z^2 \leq \Lambda^2\ . \qquad (3.17)$$

In the case under consideration, the Landau quantization in the x-y-plane is carried out and $p_x^2 + p_y^2$ should be replaced to $2|Q_f|B\nu$. Thus, we get the maximum integer of ν as

$$\nu_M \equiv \nu_M^{\rm vac} = \left[\frac{\Lambda^2 - p_z^2}{2|Q_f|B}\right]\ , \qquad |p_z| \leq \Lambda\ . \qquad (3.18)$$

Thus, the thermodynamic potential (3.9) can be evaluated, for example, up to order of B as follows:

$$\Phi = \Phi_0 + \Phi_1 + \Phi_{-1} \ , \tag{3.19}$$

$$\begin{aligned}
\Phi_0 &= \sum_{f,\alpha} \int \frac{dp_z}{2\pi} \frac{|Q_f|B}{2\pi} \left(E^f_{A,p\ \nu=0\ \eta=1} - \mu \right) \theta(\mu - E^f_{A,p\ \nu=0\ \eta=1}) \\
&\quad - \sum_{f,\alpha} \int \frac{dp_z}{2\pi} \frac{|Q_f|B}{2\pi} E^f_{A,p\ \nu=0\ \eta=1} + \frac{M^2}{4G_s} + \frac{U^2}{4G_p} \\
&= \frac{3eB}{4\pi^2} \left[-(\mu - U)\sqrt{(\mu-U)^2 - M_q^2} + M_q^2 \ln \frac{\mu - U + \sqrt{(\mu-U)^2 - M_q^2}}{M} \right] \\
&\qquad \times \theta(\mu - (U + M_q)) \\
&\quad - \frac{3eB}{4\pi^2} \left[\Lambda \sqrt{\Lambda^2 + M_q^2} + M_q^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + M_q^2}}{M} + 2U\Lambda \right] + \frac{M^2}{4G_s} + \frac{U^2}{4G_p} \ ,
\end{aligned}$$

$$\begin{aligned}
\Phi_1 &= \sum_{f,\alpha} \int \frac{dp_z}{2\pi} \sum_{\nu=1}^{\nu_M^1} \frac{|Q_f|B}{2\pi} \left(E^f_{A,p\nu\ \eta=1} - \mu \right) \theta(\mu - E^f_{A,p\nu\ \eta=1}) + \Phi_1^{\rm vac} \\
&= \frac{3}{4\pi^2} \left[\frac{1}{6} \sqrt{(\mu-U)^2 - M_q^2} \left(-2\mu^3 + 2\mu^2 U + 2\mu U^2 - 2U^3 - 13 M_q^2 U + 5\mu M_q^2 \right) \right. \\
&\qquad \left. - \frac{M_q^2}{2} \left(M_q^2 + 4U^2 - 4\mu U \right) \ln \frac{\mu - U + \sqrt{(\mu-U)^2 - M_q^2}}{M_q} \right] \\
&\qquad \times \theta(\mu - (M_q + U)) \\
&\quad + \frac{3eB}{8\pi^2} \left[(\mu - U)\sqrt{(\mu-U)^2 - M_q^2} - M_q^2 \ln \frac{\mu - U + \sqrt{(\mu-U)^2 - M_q^2}}{M_q} \right] \\
&\qquad \times \theta(\mu - (M_q + U)) \\
&\quad + \Phi_1^{\rm vac} \ , \\
\Phi_1^{\rm vac} &= - \sum_{f,\alpha} \int \frac{dp_z}{2\pi} \sum_{\nu=1}^{\nu_M^{\rm vac}} \frac{|Q_f|B}{2\pi} E_{A,p\nu\ \eta=1} \\
&= -\frac{1}{\pi^2} \int_0^\Lambda dp_z \left[\left(\Lambda^2 - p_z^2 + \left(\sqrt{p_z^2 + M_q^2} + U \right)^2 \right)^{\frac{3}{2}} - \left(\sqrt{p_z^2 + M_q^2} + U \right)^3 \right] \\
&\quad - \frac{3eB}{4\pi^2} \int_0^\Lambda dp_z \left[\sqrt{\Lambda^2 - p_z^2 + \left(\sqrt{p_z^2 + M_q^2} + U \right)^2} - \left(\sqrt{p_z^2 + M_q^2} + U \right) \right] \ ,
\end{aligned}$$

$$
\begin{aligned}
\varPhi_{-1} &= \sum_{f,\alpha}\int\frac{dp_z}{2\pi}\sum_{\nu=1}^{\nu_M^{-1}}\frac{|Q_f|B}{2\pi}\left(E^f_{A,p\nu\ \eta=-1}-\mu\right)\theta(\mu-E^f_{A,p\nu\ \eta=-1})+\varPhi_{-1}^{\rm vac}\\
&= \frac{3}{4\pi^2}\left[\frac{1}{6}\sqrt{(\mu+U)^2-M_q^2}\left(-2\mu^3-2\mu^2U+2\mu U^2+2U^3+13M_q^2U+5\mu M_q^2\right)\right.\\
&\qquad\left.-\frac{M_q^2}{2}\left(M_q^2+4U^2+4\mu U\right)\ln\frac{\mu+U+\sqrt{(\mu+U)^2-M_q^2}}{M_q}\right]\\
&\qquad\times\theta(\mu-(M_q-U))\\
&\quad+\frac{3eB}{8\pi^2}\left[(\mu+U)\sqrt{(\mu+U)^2-M_q^2}-M_q^2\ln\frac{\mu+U+\sqrt{(\mu+U)^2-M_q^2}}{M_q}\right]\\
&\qquad\times\theta(\mu-(M_q-U))\\
&\quad+\varPhi_{-1}^{\rm vac}\ ,\\
\varPhi_{-1}^{\rm vac} &= -\sum_{f,\alpha}\int\frac{dp_z}{2\pi}\sum_{\nu=1}^{\nu_M^{\rm vac}}\frac{|Q_f|B}{2\pi}E_{A,p\nu\ \eta=-1}\\
&= -\frac{1}{\pi^2}\int_0^{\varLambda}dp_z\left[\left(\varLambda^2-p_z^2+\left(\sqrt{p_z^2+M_q^2}-U\right)^2\right)^{\frac{3}{2}}-\left(\sqrt{p_z^2+M_q^2}-U\right)^3\right]\\
&\quad-\frac{3eB}{4\pi^2}\int_0^{\varLambda}dp_z\left[\sqrt{\varLambda^2-p_z^2+\left(\sqrt{p_z^2+M_q^2}-U\right)^2}-\left(\sqrt{p_z^2+M_q^2}-U\right)\right]
\end{aligned}
\tag{3.20}
$$

Here, the integrations $(\sqrt{p_z^2+M_q^2}+U)^n$ in $\varPhi_1^{\rm vac}$ and $\varPhi_{-1}^{\rm vac}$ can be performed. As a result, the thermodynamic potential $\varPhi_A$ is arranged with the term independent of B and dependent of B:

$$
\varPhi_A=\varPhi_{B=0}+\varPhi_B+\varPhi_{B^2}+O(B^3)\ , \tag{3.21}
$$

$$
\begin{aligned}
\varPhi_{B=0} &= \frac{3}{4\pi^2}\left[\frac{1}{6}\sqrt{(\mu-U)^2-M_q^2}\left(-2\mu^3+2\mu^2U+2\mu U^2-2U^3-13M_q^2U+5\mu M_q^2\right)\right.\\
&\qquad\left.-\frac{M_q^2}{2}(M_q^2+4U^2-4\mu U)\ln\frac{\mu-U+\sqrt{(\mu-U)^2-M_q^2}}{M_q}\right]\theta(\mu-(M_q+U))\\
&\quad+\frac{3}{4\pi^2}\left[\frac{1}{6}\sqrt{(\mu+U)^2-M_q^2}\left(-2\mu^3-2\mu^2U+2\mu U^2+2U^3+13M_q^2U+5\mu M_q^2\right)\right.\\
&\qquad\left.-\frac{M_q^2}{2}(M_q^2+4U^2+4\mu U)\ln\frac{\mu+U+\sqrt{(\mu+U)^2-M_q^2}}{M_q}\right]\theta(\mu-(M_q-U))
\end{aligned}
$$

$$+\frac{1}{4\pi^2}\left[\Lambda\sqrt{\Lambda^2+M_q^2}(5M_q^2+2\Lambda^2+12U^2)+3M_q^2(M_q^2+4U^2)\ln\frac{\Lambda+\sqrt{\Lambda^2+M_q^2}}{M_q}\right]$$
$$-\frac{1}{\pi^2}\int_0^{\Lambda}dp_z\left[\left(\Lambda^2-p_z^2+\left(\sqrt{p_z^2+M_q^2}-U\right)^2\right)^{\frac{3}{2}}\right.$$
$$\left.+\left(\Lambda^2-p_z^2+\left(\sqrt{p_z^2+M_q^2}+U\right)^2\right)^{\frac{3}{2}}\right]+\frac{M^2}{4G_s}+\frac{U^2}{4G_p}\ , \qquad (3.22)$$

$$\Phi_B=\frac{3eB}{8\pi^2}\left[-(\mu-U)\sqrt{(\mu-U)^2-M_q^2}+M_q^2\ln\frac{\mu-U+\sqrt{(\mu-U)^2-M_q^2}}{M_q}\right]$$
$$\times\theta(\mu-(M_q+U))$$
$$+\frac{3eB}{8\pi^2}\left[(\mu+U)\sqrt{(\mu+U)^2-M_q^2}-M_q^2\ln\frac{\mu+U+\sqrt{(\mu+U)^2-M_q^2}}{M_q}\right]$$
$$\times\theta(\mu-(M_q-U))$$
$$-\frac{3eB}{2\pi^2}\Lambda U$$
$$-\frac{3eB}{4\pi^2}\int_0^{\Lambda}dp_z\left[\sqrt{\Lambda^2-p_z^2+\left(\sqrt{p_z^2+M_q^2}+U\right)^2}+\sqrt{\Lambda^2-p_z^2+\left(\sqrt{p_z^2+M_q^2}-U\right)^2}\right]\ , \qquad (3.23)$$

$$\Phi_{B^2}=\frac{5e^2B^2}{72\pi^2}\left[\frac{\sqrt{(\mu-U)^2-M_q^2}}{\mu}-\int_0^{\sqrt{(\mu-U)^2-M_q^2}}dp_z\frac{1}{\sqrt{p_z^2+M_q^2}+U}\right]$$
$$\times\theta\left(\mu-(M_q+U)\right)$$
$$+\frac{5e^2B^2}{72\pi^2}\left[\frac{\sqrt{(\mu+U)^2-M_q^2}}{\mu}-\int_0^{\sqrt{(\mu+U)^2-M_q^2}}dp_z\frac{1}{\sqrt{p_z^2+M_q^2}-U}\right]$$
$$\times\theta\left(\mu-(M_q-U)\right)$$
$$-\frac{5e^2B^2}{72\pi^2}\int_0^{\Lambda}dp_z\left[\left(\Lambda^2-p_z^2+\left(\sqrt{p_z^2+M_q^2}+U\right)^2\right)^{-\frac{1}{2}}\right.$$
$$\left.+\left(\Lambda^2-p_z^2+\left(\sqrt{p_z^2+M_q^2}-U\right)^2\right)^{-\frac{1}{2}}\right]$$
$$+\frac{5e^2B^2}{72\pi^2}\int_0^{\Lambda}dp_z\left[\frac{1}{\sqrt{p_z^2+M_q^2}+U}+\frac{1}{\sqrt{p_z^2+M_q^2}-U}\right]\ . \qquad (3.24)$$

Here, it is found that no problem arises. We define a spontaneous magnetization $\mathcal{M}$ as

$$\mathcal{M} = -\left.\frac{\partial \Phi_A}{\partial B}\right|_{B=0} . \tag{3.25}$$

Here, in (3.23), even if $U = 0$ and $\mu < M_q$, $\mathcal{M}$ appears because the last two integrations in the last line in (3.23) survives, which leads to $\Phi_B = -3eB/(2\pi^2) \cdot \Lambda\sqrt{\Lambda^2 + M_q^2}$. When we sum up ν over the Landau level, we have introduced the maximum value of ν. Then, the maximum value of ν, namely $\nu_M^{\rm vac}$, has been replaced to the boundary value $(\Lambda^2 - p_z^2)/(2|Q_f|B)$ which is not always integer. So, we subtract the following Φ_R in order to delete this artificial contribution.

$$\begin{aligned}
&\Phi = \Phi_A - \Phi_R , \\
&\Phi_R = \sum_{f,\alpha}\int_{-\Lambda}^{\Lambda}\frac{dp_z}{2\pi}\frac{|Q_f|B}{2\pi}(-E_\Lambda) = -\frac{3eB}{2\pi^2}\Lambda\sqrt{\Lambda^2 + M_q^2} ,
\end{aligned} \tag{3.26}$$

where $E_\Lambda = \sqrt{\Lambda^2 + M_q^2}$. As a result, $\mathcal{M}$ disappears when $U = 0$ and $\mu < M_q$.

§4. Numerical results

4.1. *Spontaneous magnetization*

First, we set up $U = 0$. Then, $\Phi_{B=0}$ is written as

$$\begin{aligned}
\Phi_{B=0}(U=0) = &\frac{3}{4\pi^2}\left[\frac{1}{3}\sqrt{\mu^2 - M_q^2}(-2\mu^3 + 5\mu M_q^2) - M_q^4 \ln\frac{\mu + \sqrt{\mu^2 - M_q^2}}{M_q}\right] \\
&\times\theta(\mu - M_q) \\
&-\frac{3}{4\pi^2}\left[\Lambda\sqrt{\Lambda^2 + M_q^2}(2\Lambda^2 + M_q^2) - M_q^4 \ln\frac{\Lambda + \sqrt{\Lambda^2 + M_q^2}}{M_q}\right] \\
&+\frac{M^2}{4G_s} .
\end{aligned} \tag{4.27}$$

When we adopt the chiral limit, namely $M_q = M$ with $m_0 = 0$, the gap equation is derived as

$$\begin{aligned}
\frac{\partial \Phi_{B=0}(U=0)}{\partial M} &= -\frac{3M}{\pi^2}\left[\Lambda\sqrt{\Lambda^2 + M^2} - M^2 \ln\frac{\Lambda + \sqrt{\Lambda^2 + M^2}}{M} - \frac{\pi^2}{6G_s}\right] \\
&= 0 .
\end{aligned} \tag{4.28}$$

There is a solution except for $M = 0$ in the vacuum $\mu = 0$. For example, if we adopt the model parameters $\Lambda = 0.631$ GeV and $G_s = 5.5$ GeV^{-2}, then, the dynamical quark mass M is obtained as $M = 0.322$ GeV. If we introduce the current quark

μ / GeV	M / GeV ($M \neq 0,\ U = 0$)	$\Phi(M, U = 0)$ / GeV^4	M /GeV	U /GeV	$\Phi(M, U)$ /GeV^4	$\Phi(M = 0, U = 0)$ /GeV^4
0.0	0.322387	$\underline{-0.0246944}$	0.279373	0.138605	-0.0246559	-0.0240940
0.1	0.322387	$\underline{-0.0246944}$	0.279373	0.138605	-0.0246559	-0.0240991
0.1408	0.322387	$\underline{-0.0246944}$	0.279373	0.138605	-0.0246559	-0.0241139
0.1409	0.322387	$\underline{-0.0246944}$	–	–	–	-0.0241140
0.1999	0.322387	$\underline{-0.0246944}$	–	–	–	-0.0241749
0.2	0.322387	$\underline{-0.0246944}$	0.286078	0.11997	-0.0246606	-0.0241751
0.3	0.322387	$\underline{-0.0246944}$	0.311389	0.0431851	-0.0246902	-0.0245044
0.32	0.322387	$\underline{-0.0246944}$	0.318850	0.0152899	-0.0246941	-0.0246252
0.322387	0.322387	$\underline{-0.0246944}$	0.320032	0.0103812	-0.0246943	-0.0246412
0.3224	–	–	0.320039	0.0103560	$\underline{-0.0246943}$	-0.0246413
0.323	–	–	0.320391	0.00885381	$\underline{-0.0246944}$	-0.0246454
0.324	–	–	0.321140	0.00560727	$\underline{-0.0246944}$	-0.0246523
0.3246	–	–	0.321754	0.00289703	$\underline{-0.0246944}$	-0.0246564
0.3247	–	–	–	–	–	$\underline{-0.0246564}$
0.4	–	–	–	–	–	$\underline{-0.0253909}$
μ (> 0.4)	–	–	–	–	–	$\underline{-\frac{\mu^4+3\Lambda^4}{2\pi^2}}$

Table I. The numerical results for the quark mass M, the pseudovector condensate U and the thermodynamic potential $\Phi(M, U)$ are shown as a function of the quark chemical potential μ. The underline for the numerical values of the thermodynamic potential means the lowest value of the thermodynamic potential in a few branch of the solutions.

mass $m_0 = 0.005$ GeV, the constituent quark mass $M_q = 0.335$ GeV is obtained under the same model parameters.

Next, let us assume $U \geq 0$. If the quark chemical potential μ is large, then, the dynamical quark mass becomes zero because the chiral symmetry is restored. Then, if $M = 0$ and $\mu > U$ in the chiral limit, the thermodynamic potential with $B = 0$ is written as

$$\Phi_{B=0}(M=0) = \frac{1}{2\pi^2}(-\mu^4 + U^4 + \Lambda^4 + 6\Lambda^2 U^2) + \frac{1}{5\pi^2}\frac{1}{U}\left((\Lambda - U)^5 - (\Lambda + U)^5\right) + \frac{U^2}{4G_p} . \tag{4.29}$$

Then, the gap equation for U is obtained as

$$\frac{\partial \Phi_{B=0}(M=0)}{\partial U} = \frac{U}{5\pi^2}\left(2U^2 - 10\Lambda^2 + \frac{5\pi^2}{2G_P}\right) = 0 \tag{4.30}$$

Thus, we obtain the solution

$$U = 0 , \qquad \text{or} \qquad U = \sqrt{5\Lambda^2 - \frac{5\pi^2}{4G_p}} . \tag{4.31}$$

In the following parts of this section, we adopt $G_p = 2G_s$. Under these parameters with $\Lambda = 0.631$ GeV and $G_s = 5.5$ GeV^{-2}, a non-trivial solution of the gap equation gives 0.932 GeV for U. This value is larger than the cutoff Λ and also the condition $\mu > U$ is not satisfied. Thus, if $M = 0$, then only $U = 0$ may be a possible solution.

μ / GeV	M / GeV	U / GeV	$\Phi(M,U)$ /GeV^4	$\mathcal{M}\times 10^{18}$ / (C/ms)
0.3224	0.320039	0.0103560	−0.0246943	1.15849
0.3226	0.320153	0.00987126	−0.0246943	1.1046
0.3228	0.320269	0.00937387	−0.0246944	1.04871
0.3230	0.320391	0.00885481	−0.0246944	0.990414
0.3232	0.320518	0.00830905	−0.0246944	0.929145
0.3234	0.320653	0.00772916	−0.0246944	0.864079
0.3236	0.320797	0.00710339	−0.0246944	0.793903
0.3238	0.320956	0.00641091	−0.0246944	0.716291
0.3240	0.321140	0.00560727	−0.0246944	0.62679
0.3242	0.321380	0.00454891	−0.0246944	0.507837
0.3244	0.321717	0.00305037	−0.0246944	0.339844
0.3246	0.321754	0.00289703	−0.0246944	0.321587

Table II. The numerical results for the quark mass M, the pseudovector condensate U, the thermodynamic potential $\Phi(M,U)$ and the spontaneous magnetization per unit volume $\mathcal{M}$ are shown as a function of the quark chemical potential μ in the range from $\mu = 0.3224$ GeV to 0.3246 GeV, in which the phase with $M \neq 0$ and $U \neq 0$ is realized.

The numerical results for the quark mass M, the pseudovector condensate U and the thermodynamic potential $\Phi(M,U)$ are summarized in Table I as a function of the quark chemical potential μ. The underline for the numerical values of the thermodynamic potential represent the lowest value of the thermodynamic potential in a few branch of the solutions and hyphen represents no solution. Usually, $\mu \leq 0.3224$ GeV $(= \mu_{\rm cr,1})$, the chiral symmetry is broken and the non-trivial solution of the gap equation for chiral condensate or dynamical quark mass exists. Also, in $\mu > \mu_{\rm cr,1}$, the chiral symmetry is restored and only $M = 0$ has a true solution. However, in the case with the pseudovector interaction, other solutions exist. Namely, in larger region of the quark chemical potential $\mu > \mu_{\rm cr,1}$, the chiral symmetry is still

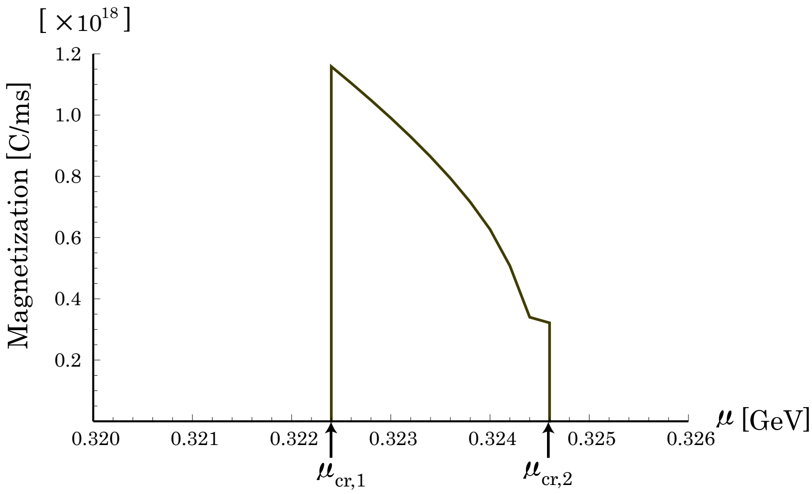

Fig. 1. The spontaneous magnetization per unit volume $\mathcal{M}$ is depicted as a function of the quark chemical potential μ.

not restored and the solution of the gap equation for the dynamical quark mass with $M \neq 0$ appears with $U \neq 0$. The window of the quark chemical potential where the non-trivial solutions with $M \neq 0$ and $U \neq 0$ exist is very narrow. In the region with $\mu > 0.3247$ GeV $(= \mu_{\rm cr,2})$, only the trivial solution with $M = U = 0$ exists, which corresponds to the free quark gas. It should be noted here that the critical baryon density from the pseudovector condensed phase or spin polarized phase with $M \neq 0$ and $U \neq 0$ to the chiral symmetric phase with $M = U = 0$ corresponds to $1.78\rho_0$ where $\rho_0 = 0.17$ fm^{-3} is the normal nuclear density. However, the two critical chemical potentials, $\mu_{\rm cr1}$ and $\mu_{\rm cr,2}$ correspond to rather small quark number densities, 0.0116 fm^{-3} and 0.0030 fm^{-3}, respectively. According to Ref. 33) where the first order phase transition was studied in the ρ-T and μ-T space, these small densities lie inside the low density metastable before the onset of the spinodal region.

In this narrow window, the spontaneous magnetization per unit volume (3.25) with Φ in (3.26) instead of Φ_A appear. The numerical results are summarized in Table II. The order of magnitude of the spontaneous magnetization is about 10^{17} and/or 10^{18} C/ms. These magnitude leads to the magnetic flux density with 10^{13} or 10^{14} Gauss in the surface of compact stars.[16)] Also, the result of the spontaneous magnetization per unit volume is shown in Fig.1. It should be noted here that, if the coupling strength G_p is smaller than the value adopted here, the window, in which the spontaneous magnetization occurs, does not open. Namely, there is no pseudovector condensate or spin polarized condensate. On the other hand, if G_p is rather large, the local minimum of the thermodynamic potential with respect to finite U and M changes to the saddle point as G_p increases. Thus, the pseudovector condensate only exists in very narrow region in the parameter space, G_p.

4.2. *Magnetic susceptibility*

We calculate the magnetic susceptibility in the same way as spontaneous magnetization. First, we define the magnetic susceptibility as

$$\chi = \mu_0 \left.\frac{\partial \mathcal{M}}{\partial B}\right|_{B=0} = -\mu_0 \left.\frac{\partial^2 \Phi_A}{\partial B^2}\right|_{B=0} , \tag{4.32}$$

where μ_0 represents the vacuum permeability. From (3.21), specifically, it is obtained as follows:

$$\begin{aligned}
\Delta\chi \equiv\ & \chi - \chi_{(\mu=0)} \\
=\ & -\frac{5e^2\mu_0}{36\pi^2}\left[\frac{\sqrt{(\mu-U)^2-M_q^2}}{\mu} - \int_0^{\sqrt{(\mu-U)^2-M_q^2}} dp_z \frac{1}{\sqrt{p_z^2+M_q^2}+U}\right] \\
& \qquad \times\theta\left(\mu-(M_q+U)\right) \\
& -\frac{5e^2\mu_0}{36\pi^2}\left[\frac{\sqrt{(\mu+U)^2-M_q^2}}{\mu} - \int_0^{\sqrt{(\mu+U)^2-M_q^2}} dp_z \frac{1}{\sqrt{p_z^2+M_q^2}-U}\right] \\
& \qquad \times\theta\left(\mu-(M_q-U)\right)
\end{aligned}$$

μ / GeV	M / GeV	U / GeV	$\Delta\chi \times 10^{-6}$
0.3222	0.322387	0	0
0.3224	0.320039	0.0103560	−5.80492
0.3226	0.320153	0.00987126	−5.42174
0.3228	0.320269	0.00937387	−5.03584
0.3230	0.320391	0.00885481	−4.64583
0.3232	0.320518	0.00830905	−4.24974
0.3234	0.320653	0.00772916	−3.84460
0.3236	0.320797	0.00710339	−3.42554
0.3238	0.320956	0.00641091	−2.98378
0.3240	0.321140	0.00560727	−2.50005
0.3242	0.321380	0.00454891	−1.91054
0.3244	0.321717	0.00305037	−1.09987
0.3246	0.321754	0.00289703	−0.60179
0.3248	0	0	−336.097

Table III. The numerical results for the quark mass M, the pseudovector condensate U and the magnetic susceptibility $\Delta\chi$ by the positive energy quarks are shown as a function of the quark chemical potential μ in the range from $\mu = 0.3222$ GeV to 0.3248 GeV.

$$
\begin{aligned}
&+\frac{5e^2\mu_0}{36\pi^2}\int_0^{\Lambda} dp_z \left[\left(\Lambda^2 - p_z^2 + \left(\sqrt{p_z^2+M_q^2}+U\right)^2\right)^{-\frac{1}{2}}\right. \\
&\qquad\qquad\qquad \left. + \left(\Lambda^2 - p_z^2 + \left(\sqrt{p_z^2+M_q^2}-U\right)^2\right)^{-\frac{1}{2}}\right] \\
&-\frac{5e^2\mu_0}{36\pi^2}\int_0^{\Lambda} dp_z \left[\frac{1}{\sqrt{p_z^2+M_q^2}+U} + \frac{1}{\sqrt{p_z^2+M_q^2}-U}\right] - \chi_{(\mu=0)}. \qquad (4.33)
\end{aligned}
$$

Here we subtract the following $\chi_{(\mu=0)}$ in order to investigate the contribution only of

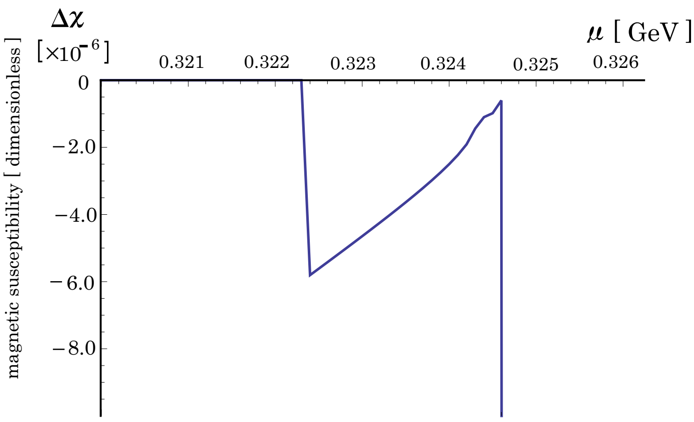

Fig. 2. The magnetic susceptibility compared with vacuum value $\chi_{(\mu=0)}$, $\Delta\chi$, is depicted as a function of the quark chemical potential μ.

the positive-energy particles due to the pesudvector-type interaction between quarks:

$$\chi_{(\mu=0)} = -\frac{5e^2\mu_0}{18\pi^2}\left[-\frac{\Lambda}{\sqrt{\Lambda^2+M_q^2(\mu=0)}}+\ln\frac{\Lambda+\sqrt{M_q^2(\mu=0)+\Lambda^2}}{M_q(\mu=0)}\right] . \tag{4.34}$$

In the region with $\mu < \mu_{\rm cr,1}$, the solutions of gap equations have the values $M \neq 0$ and $U = 0$, so $\chi_{\mu<\mu_{\rm cr,1}} \equiv \chi_<$ is obtained as

$$\begin{aligned}\Delta\chi_< &= -\frac{5e^2\mu_0}{18\pi^2}\left[-\frac{\Lambda}{\sqrt{\Lambda^2+M_q^2}}+\ln\frac{\Lambda+\sqrt{M_q^2+\Lambda^2}}{M_q}\right]-\chi_{(\mu=0)}\\ &= 0\end{aligned} \tag{4.35}$$

due to a subtraction of the contribution of the vacuum $\chi_{(\mu=0)}$. In the same way, in the region with $\mu > \mu_{\rm cr,2}$, the solutions with $M = U = 0$ exist. Then $\chi_{\mu>\mu_{\rm cr,2}} \equiv \chi_>$ is obtained as

$$\Delta\chi_> = -\frac{5e^2\mu_0}{18\pi^2}\ln\frac{\Lambda}{\mu}-\chi_{(\mu=0)} . \tag{4.36}$$

The numerical results are summarized in Table III. The order of magnitude is about 10^{-6} in the region with $\mu_{\rm cr,1} \lesssim \mu \lesssim \mu_{\rm cr,2}$. Also, the result of the magnetic susceptibility is shown in Fig.2.

§5. Summary and concluding remarks

It has been shown that the spontaneous magnetization occurs due to the pseudovector-type four-point interaction between quarks in quark matter at zero temperature within the NJL model. In the narrow region of the quark chemical potential, both the chiral condensate and pseudovector condensate, namely spin polarized condensate, coexist, which leads to the spontaneous magnetization. On the contrary, in the tensor-type four-point interaction between quarks, the spin polarization occurs above a certain quark chemical potential, that is in the high density quark matter. However, the spontaneous magnetization does not appear in the case of the tensor interaction except for the existence of the anomalous magnetic moment of quarks.[16] Also, we have calculated the magnetic susceptibility by expanding the thermodynamic potential up to the second order of the external magnetic field. As a result, the Landau diamagnetism may be revealed because only the contribution of the positive-energy quarks being free quasi-particles is considered.

In this paper, we ignore the effects of current quark mass. It was pointed out that the region in which the pseudovector condensate has non-zero value enlarges, if the current quark mass is introduced.[34] Further, the effects of the strange quark is missing in this work. These are interesting future problems which are left in order to clarify the magnetic properties of high density quark matter.

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