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REGULARITY OF A SEGREGATION PROBLEM WITH AN OPTIMAL CONTROL OPERATOR

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REGULARITY OF A SEGREGATION PROBLEM WITH AN OPTIMAL CONTROL OPERATOR

by

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Dedicated to Fernando Cavaco, my sisters Leonor and Rosario, and to the woodpeckers, dragonflies, parakeets, squirrels, eagle dancer and trees in Hyde park and Zilker park.

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It is the main goal of this thesis to study the regularity of solutions for a nonlinear elliptic system coming from population segregation, and the free boundary problem that is obtained in the limit as the competition parameter goes to infinity ($\epsilon \to 0$). The system is described by the following equations:

$$\begin{cases} \mathcal{M}^-(u_i^\epsilon) = \dfrac{1}{\epsilon} u_i^\epsilon \displaystyle\sum_{j \neq i} u_j^\epsilon, \qquad i = 1, \ldots, d, \quad \text{in } \Omega, \\ u_i^\epsilon = \phi_i, \quad i = 1, \ldots, d, \quad \text{on} \quad \partial\Omega. \end{cases}$$

The diffusion operator, which in the literature is usually taken to be a linear operator, is replaced by the nonlinear minimal Pucci operator $\mathcal{M}^-$. The results are similar to those obtained for the corresponding linear problem, work done by Caffarelli, Karakhanyan, and Fang-Hua Lin, but the techniques are substantially different. The main results are existence and Hölder regularity of solutions of the elliptic system, characterization of the limit as a free boundary problem, and Lipschitz regularity at the boundary for the limiting problem.

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Chapter 1

Introduction

In this dissertation we study the existence and regularity of solutions of a problem motivated by population dynamics. The motivation of the models and main results will be presented in the next chapter, but generally speaking, the goal of this work is to generalize the regularity results for the system

$$\begin{cases} \Delta u_i^\epsilon = \dfrac{1}{\epsilon} u_i^\epsilon \displaystyle\sum_{i \neq j} u_j^\epsilon \quad \text{in } \Omega,\ i = 1, \ldots, d, \\ u_i^\epsilon > 0 \quad \text{in } \Omega,\ i = 1, \ldots, d, \\ u_i^\epsilon(x) = \phi_i(x) \geq 0 \quad \text{on } \partial\,\Omega,\ i = 1, \ldots, d, \\ \phi_i\, \phi_j = 0 \quad \text{on } \partial\,\Omega,\ i \neq j \end{cases} \tag{1.1}$$

and

$$\begin{cases} \Delta u_i = 0 \quad \text{when } u_i > 0,\ i = 1, \ldots, d, \\ \Delta(u_i - \displaystyle\sum_{i \neq j} u_j) \leq 0 \quad \text{in } \Omega,\ i = 1, \ldots, d, \\ u_i(x) > 0 \quad \text{in } \Omega,\ i = 1, \ldots, d, \\ u_i\, u_j = 0 \quad \text{in } \Omega,\ i \neq j, \\ u_i = \phi_i \quad \text{on } \partial\Omega,\ i = 1, \ldots, d, \end{cases} \tag{1.2}$$

presented in Sections 1 and 2 of [4], to the following nonlinear elliptic system of equations

$$\begin{cases} \mathcal{M}^-(u_i^\epsilon) = \dfrac{1}{\epsilon} u_i^\epsilon \displaystyle\sum_{j\neq i} u_j^\epsilon, \quad \text{in } \Omega,\ i=1,\ldots,d, \\ u_i^\epsilon > 0 \quad \text{in } \Omega,\ i=1,\ldots,d, \\ u_i^\epsilon = \phi_i \quad \text{on } \partial\Omega,\ i=1,\ldots,d, \\ \phi_i\,\phi_j = 0 \quad \text{on } \partial\Omega,\ i\neq j, \end{cases}$$

where $\mathcal{M}^-$ denotes the extremal Pucci operator (see (1.8)), and to characterize the limit problem, analogous to (1.2), for this case.

We have chosen this problem, besides its intrinsic mathematical interest, in order to study a model that takes into account diffusion with preferential directions, so we are able to model situations with maximal diffusion. The choice of the operator is also related with its natural comparison with a non-divergence linear operator with measurable coefficients.

1.1 Motivation and main results

1.1.1 Some models of segregation of populations

The problem we study is motivated by the Gause-Lotka-Voltera model of extinction or coexistence of species that live in the same territory, can diffuse, and have high competition rates.

Consider the equation,

$$\frac{\partial u_i}{\partial t} = \underbrace{d_i \Delta u_i}_{\text{diffusion term}} + R_i u_i - a_i {u_i}^2 - \sum_{i\neq j} b_{ij} u_i u_j \quad \text{in } \Omega,$$

which models populations of different species in competition, where

$u_i(x,t)$ is the density of the population i at time t and position x;

R_i is the intrinsic rate of growth of species i;

d_i is the diffusion rate for species i;

a_i is a positive number that characterizes the intraspecies competition for the species i;

b_{ij} is a positive number that characterizes the interspecies competition between the species i and j.

In the papers [25, 27] this model was studied initially without diffusion. These papers studied how species can survive or get extinct with time, depending on the interactions among them. Upon adding diffusion, Mimura, Ei and Fang proved that the existence of a stable solution depends on the shape of the domain and on the relations between the coefficients in the equation. The characterization for two species has been proven to be easier, while the three species interactions remain to be fully understood in these papers.

In the sequence of papers by Dancer and Du [15, 16, 20] the authors decided to first understand better the steady case (time independent) in order to obtain results for the parabolic problem. In these papers, one can find sufficient and necessary conditions for the existence of positive solutions (u_1, u_2)

and (u_1, u_2, u_3) with explicit conditions on the coefficients R_i, a_i, b_{ij} for the following problem:

$$\begin{cases} -\Delta u_i = R_i u_i - a_i {u_i}^2 - \sum_{i \neq j} b_{ij} u_i u_j \quad \text{in } \Omega, \\ u_i = 0 \quad \text{on } \partial\Omega, \\ u_i > 0 \quad \text{in } \Omega, \end{cases} \tag{1.3}$$

with $i = 1, 2$ and $i = 1, 2, 3$, respectively.

The spatial segregation obtained in the limit as $b_{ij} \to \infty$ of the competition-diffusion system was associated with a free boundary problem by Dancer, Hilhorst, Mimura, and Peletier in [19] (i.e. in the case of high competition between the species). In [17] the existence and uniqueness of the solution to Problem (1.3) with just two populations has been studied using variational methods.

Later in [9], Conti, Terracini and Verzini proved that the limit problem is related with the optimal partition problem in N dimensional domains. Since then, several papers by Conti, Felli, Terracini and Verzini [10, 11, 12, 13, 14] studied with a general formulation, the existence, uniqueness, and regularity for the asymptotic limit of the following system,

$$\begin{cases} -\Delta u_1^\epsilon = f(u_1^\epsilon) - \frac{1}{\epsilon} u_1^\epsilon u_2^\epsilon \quad \text{in } \Omega, \\ -\Delta u_2^\epsilon = f(u_2^\epsilon) - \frac{1}{\epsilon} u_1^\epsilon u_2^\epsilon \quad \text{in } \Omega, \\ u_i = \phi_i \quad \text{on } \partial\Omega, \ i = 1, 2. \end{cases}$$

where $\phi_i(x)\phi_j(x) = 0$, for $i \neq j$. In these papers, the existence of a limit pair of functions (u_1, u_2) such that $(u_1^\epsilon, u_2^\epsilon) \to (u_1, u_2)$ when $\epsilon \to 0$ is shown to have a tight connection with two different mathematical problems. Namely,

a) to find the solution of a free boundary problem characterized by the conditions:

$$\begin{cases} -\Delta u_i = f(u_i)\chi_{\{u_i>0\}} \quad i = 1,2, \\ u_i(x) > 0 \quad \text{in}\, \Omega,\ i = 1,2, \\ u_1(x)\, u_2(x) = 0 \quad \text{in}\, \Omega, \\ u_i = \phi_i \quad \text{on}\, \partial\, \Omega,\ i = 1,2, \end{cases}$$

b) to find the solution for an optimal partition problem. Optimal partition problems are problems like, for example, (from [7]):

> Let Ω be a bounded, smooth domain in $\mathbb{R}^n$, and let $m \geq 1$ be a positive integer. One seeks for a partition of Ω into m, mutually disjoint subsets, Ω_j, $j = 1, \ldots, m$ such that Ω is its union and that it minimizes the sum of the first Dirichlet eigenvalue of the Laplacian on Ω_j with zero Dirichlet boundary condition on $\partial\Omega_j$. (1.4)

The existence and uniqueness of solution for a type of free boundary problem of the form

$$\begin{cases} -\Delta u = f(u)\chi_{\{u>0\}} \\ u(x) > 0 \quad \text{in}\, \Omega, \\ u = 0 \quad \text{on}\, \partial\Omega \end{cases}$$

with u bounded, was studied using variational methods by Dancer [18].

Then the regularity of solutions for the free boundary problem

$$\begin{cases} \Delta u_i^\epsilon = \dfrac{1}{\epsilon} u_i^\epsilon \displaystyle\sum_{i \neq j} u_j^\epsilon \quad \text{in } \Omega, \ i = 1, \ldots, d, \\ u_i^\epsilon > 0 \quad \text{in } \Omega, \ i = 1, \ldots, d, \\ u_i^\epsilon(x) = \phi_i(x) \geq 0 \quad \text{on } \partial\Omega, \ i = 1, \ldots, d, \\ \phi_i \, \phi_j = 0 \quad \text{on } \partial\Omega, \ i \neq j \end{cases} \tag{1.5}$$

was studied by Caffarelli, Karakhanyan and Lin in [3, 4] with the viscosity approach. More specifically, in [4] the authors proved that the singular perturbed elliptic system (1.1) has as limit when $\epsilon \to 0$, the following free boundary problem

$$\begin{cases} \Delta u_i = 0 \quad \text{when } u_i > 0, \ i = 1, \ldots, d, \\ \Delta(u_i - \displaystyle\sum_{i \neq j} u_j) \leq 0 \quad \text{in } \Omega, \ i = 1, \ldots, d, \\ u_i(x) > 0 \quad \text{in } \Omega, \ i = 1, \ldots, d, \\ u_i \, u_j = 0 \quad \text{in } \Omega, \ i \neq j, \\ u_i = \phi_i \quad \text{on } \partial\Omega, \ i = 1, \ldots, d. \end{cases} \tag{1.6}$$

They also proved that the limit solutions u_i are Hölder continuous and have linear growth from a free boundary point. Also that the set of interfaces $\{x : \mathbf{u}(x) = 0\}$ consists of two parts: a singular set of Hausdorff dimension $n-2$ where three or more species can concur; and a family of analytic surfaces, level surfaces of harmonic functions.

The segregation model that we study in this thesis has the diffusion operator replaced by the nonlinear minimal Pucci operator $\mathcal{M}^-$. Besides the inherent interest of the extension of these results to the nonlinear setting, we think that this work may be relevant to those interested in non-standard diffusion models. In this work, we were able to extend to the nonlinear setting the regularity results for the solutions proven by Caffarelli, Karakhanyan and Lin. The statement of the problem and main results are presented in the next section.

1.1.2 Set-up of the problem and main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain where d populations co-exist. Consider the following system of fully nonlinear elliptic equations with Dirichlet boundary data for

$$\begin{cases} \mathcal{M}^-(u_i^\epsilon) = \dfrac{1}{\epsilon} u_i^\epsilon \displaystyle\sum_{j \neq i} u_j^\epsilon, \qquad i = 1, \ldots, d, \quad \text{in } \Omega, \\ u_i^\epsilon = \phi_i, \quad i = 1, \ldots, d, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.7}$$

where $u_i^\epsilon, (i = 1, \ldots, d)$ are non-negative functions defined in Ω that can be seen as a density of the population i, and the parameter $\frac{1}{\epsilon}$ characterizes the level of competition between species.

Each ϕ_i is a non-negative Hölder continuous function defined on $\partial\Omega$ such that $\phi_i(x)\phi_j(x) = 0$ for $i \neq j$, meaning that they have disjoint supports.

Here $\mathcal{M}^-$ denotes the extremal Pucci operator, defined as

$$\mathcal{M}^-(\omega) := \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} a_{ij} D_{ij}(\omega(x)) = \Lambda \sum_{e_i < 0} e_i + \lambda \sum_{e_i > 0} e_i, \tag{1.8}$$

where $\mathcal{A}_{\lambda,\Lambda}$ is the set of symmetric $n \times n$ real matrices with eigenvalues in $[\lambda, \Lambda]$, for some fixed constants $0 < \lambda < \Lambda$, and e_i are the eigenvalues of the matrix $D^2\omega(x)$.

We assume that u_i^ϵ are bounded, $0 \leq u_i^\epsilon \leq N$, for all i. Note that $\lambda\Delta\omega \geq \mathcal{M}^-(\omega)$, thus u_i^ϵ are subharmonic, for all i.

Our results in this thesis are the following:

Theorem (Existence)**.** *Let $\epsilon > 0$ constant, and Ω be a Lipschitz domain. Let ϕ_i be non-negative Hölder continuous functions defined on $\partial\Omega$. Then there exist continuous functions $(u_1^\epsilon, \cdots, u_d^\epsilon)$ depending on the parameter ϵ such that u_i^ϵ is a viscosity solution of Problem (1.7).*

Theorem (Regularity of solutions)**.** *Let $\epsilon > 0$ constant and ϕ_i be non-negative Hölder continuous functions defined on $\partial\Omega$. Let $\boldsymbol{u}^\epsilon = (u_1^\epsilon, \cdots, u_d^\epsilon)$ be solutions of Problem (1.7) in $B_1(0)$. Then there exist a constant α, $0 < \alpha < 1$, such that for any ϵ, $\boldsymbol{u}^\epsilon \in \left(C^\alpha\left(B_1(0)\right)\right)^d$ and*

$$\|\boldsymbol{u}^\epsilon\|_{\left(C^\alpha\left(B_{\frac{1}{2}}\right)\right)^d} \leq C(N),$$

with $N = \sup_j \left\|u_j^\epsilon\right\|_{L^\infty(B_1(0))}$ and $C(N)$ independent of ϵ.

In the limit as $\epsilon \to 0$, this model forces the populations to segregate, meaning that in the limit the supports of the functions are disjoint and

$$\frac{u_i^\epsilon u_j^\epsilon}{\epsilon} \rightharpoonup \mu \quad \text{in the sense of measures, when } \epsilon \to 0.$$

The measure μ has support on the free boundary. Recall that the support of a measure μ is the complementary of the set

$$\{E : E \quad \text{the biggest open set such that} \quad \mu(E) = 0\}.$$

Theorem (Characterization of the limit problem)**.** *Let ϕ_i be non-negative Hölder continuous functions defined on $\partial\Omega$. If $\boldsymbol{u} \in (C^\alpha)^d$ is the limit of solutions of (1.7), then*

1. $\mathcal{M}^-\left(u_i - \sum_{k\neq j} u_k\right) \leq 0$;
2. $(\mathrm{supp}\, u_i)^o \cap \left(\mathrm{supp}\,(\sum_{k\neq i} u_k)\right)^o = \emptyset$ *for* $i = 1, \ldots d$;
3. $\mathcal{M}^-(u_i) = 0$, *when* $u_i(x) > 0$, *for* $x \in \Omega \quad i = 1, \ldots, d$;
4. $u_i(x) = \phi_i(x)$, *for* $x \in \partial\Omega, \quad i = 1, \ldots, d$.

Theorem (Lipschitz regularity for the free boundary problem)**.** *If $\boldsymbol{u}$ belonging to $(C^\alpha(B_1(0)))^d$, is the limit of solutions of (1.7) in $B_1(0)$, and x_0 belongs to the set $\partial\,(\mathrm{supp}\, u_1) \cap B_{\frac{1}{2}}(0)$, then, without loss of generality, the growth of u_1 near the boundary of its support is controlled in a linear way and u_1 is Lipschitz. More precisely, there exist a universal constant C such that for any solution $\boldsymbol{u}$, for any point x_0 on the free boundary:*

1. $\sup_{B_R(x_0)} u_1 \leq C\, R$,
2. $\|u_1\|_{Lip(B_R(x_0))} \leq C$,

where $C = C(n, \|\boldsymbol{u}\|_{L^2(B_1)})$ *and* $R \leq \frac{1}{4}$.

Although the last three results are similar in spirit to the ones proved in [4] for the elliptic linear system of equations, our proofs use different techniques.

The organization of this thesis is the following.

In Chapter 2 we review the definition and properties of the Pucci operator, some of the results necessary for this work from the viscosity theory for fully nonlinear elliptic differential equations. Some properties of subharmonic functions and the Fabes and Strook inequality are presented. These results are essential tools for this work.

Each main result is developed in a separate chapter.

Chapter 3 is dedicate to prove existence of solution for Problem (1.7). The proof of Hölder regularity up to the boundary for an equation of the type $\mathcal{M}^{-}(u) = f(x)$ with Hölder boundary values in Lipschitz domain necessary for the existence proof is also contained in this chapter.

Then, in Chapter 4, we prove Hölder regularity uniform in ϵ for $\mathbf{u}^{\epsilon}$ and this allows us to characterize the limit problem that is, in fact, a new system of equations that constitute a free boundary problem. We will recall here some free boundary type of problems and state the essential monotonicity formula introduced in [1].

To study the regularity of the free boundary problem we need to study independently the regularity of the solution and of the free boundary set. In this work, we present the regularity of the solution; the regularity of the free boundary remains an open problem for which there are no tools.

The main result, the linear decay from the free boundary, is developed in Chapter 6.

Chapter 2

Preliminaries and important tools

2.1 Some properties of subharmonic functions

In this thesis we use some known properties of subharmonic functions that are presented here with proof. The first result is a very useful inequality that can be obtained from Green's Identity with a special Green function.

Lemma 2.1.1. *Let u be subharmonic function, that is, $\Delta u \geq 0$. Then,*

$$\left(\frac{r}{2}\right)^2 ⨍_{B_{\frac{r}{2}}(x_o)} \Delta u \mathrm{dx} \leq C ⨍_{\partial B_r(x_o)} \left(u(x) - u(x_0)\right) \mathrm{dS} \tag{2.1}$$

Proof. Consider $\tilde{\Gamma}$ to be the Green function:

$$\tilde{\Gamma}(x) = \frac{1}{n\omega_n(2-n)}\left(\frac{1}{|x-x_0|^{n-2}} - \frac{1}{r^{n-2}}\right),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Then

$$\int_{B_r(x_o)} \Delta u(x)\tilde{\Gamma}(x) - \Delta\tilde{\Gamma}(x)u(x)\mathrm{dx} = \int_{\partial B_r(x_o)} \underbrace{\frac{\partial u}{\partial \nu}\tilde{\Gamma}(x)}_{=0} - \frac{\partial \tilde{\Gamma}(x_0)}{\partial \nu}u(x)\mathrm{dS}.$$

And

$$\int_{B_r(x_o)} \Delta u(x)\tilde{\Gamma}(x)\mathrm{dx} = ⨍_{\partial B_r(x_o)} u(x_0)\mathrm{dS} + \int_{\partial B_r(x_o)} \left(-\frac{\partial\tilde{\Gamma}(x)}{\partial \nu}u(x)\right)\mathrm{dS}$$

$$= ⨍_{\partial B_r(x_o)} u(x_0)\mathrm{dS} + \int_{\partial B_r(x_o)} \left(-\frac{(2-n)}{n\omega_n(2-n)}\frac{1}{|x-x_0|^{n-1}}u(x)\right)\mathrm{dS}$$

$$= \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x_o)} u(x_0) - u(x)\mathrm{dS}.$$

Therefore,

$$\frac{1}{n\omega_n(n-2)}\int_{B_r(x_o)}\Delta u\left(\frac{1}{|x-x_0|^{n-2}}-\frac{1}{r^{n-2}}\right)\mathrm{dx}=⨍_{\partial B_r(x_o)}(u(x)-u(x_0))\,\mathrm{dS}.$$

Since u is subharmonic, $\left(\frac{1}{|x-x_0|^{n-2}}-\frac{1}{r^{n-2}}\right)$ is nonnegative in $B_r(x_0)$, and since for $x\in B_{\frac{r}{2}}(x_0)$ we have the inequality,

$$\left(\frac{1}{(\frac{r}{2})^{n-2}}-\frac{1}{r^{n-2}}\right)\leq\left(\frac{1}{|x-x_0|^{n-2}}-\frac{1}{r^{n-2}}\right),$$

we obtain,

$$\frac{1}{n\omega_n(n-2)}\int_{B_{\frac{r}{2}}(x_0)}\Delta u(x)\left(\frac{1}{(\frac{r}{2})^{n-2}}-\frac{1}{r^{n-2}}\right)\mathrm{dx}\leq⨍_{\partial B_r(x_o)}(u(x)-u(x_0))\,\mathrm{dS}.$$

Therefore,

$$\frac{1}{n\omega_n(n-2)}\left(1-\frac{1}{2^{n-2}}\right)\frac{1}{(\frac{r}{2})^{n-2}}\int_{B_{\frac{r}{2}}(x_o)}\Delta u(x)\mathrm{dx}\leq⨍_{\partial B_r(x_0)}(u(x)-u(x_0))\,\mathrm{dS}$$

that is,

$$\left(\frac{r}{2}\right)^2⨍_{B_{\frac{r}{2}}(x_0)}\Delta u\mathrm{dx}\leq C⨍_{\partial B_r(x_0)}(u(x)-u(x_0))\,\mathrm{dS}.\qquad\square$$

The second result is related with the growth of subharmonic functions in thin domains. More precisely, we will need the following results.

Lemma 2.1.2 (L^∞ decay for subharmonic functions supported in small domains)**.** *Let u be a non-negative subharmonic function in a domain that contains $B_1(0)$. If for some small $\epsilon_0>0$,*

$$\sup_{B_1(0)}u\leq 1\quad\textit{and}\quad\frac{|\{u\neq 0\}\cap B_1(0)|}{|B_1(0)|}\leq\epsilon_0$$

then,

$$\sup_{B_{\frac{1}{2}}(0)}u\leq\epsilon_0\,2^n.$$

Proof. Let y be an arbitrary point in the ball $B_{\frac{1}{2}}(0)$. Due to the subharmonicity and the fact that u is a non-negative function

$$u(y) \leq \fint_{B_{\frac{1}{2}}(y)} u(x)\mathrm{dx} \leq \frac{1}{\left|B_{\frac{1}{2}}(y)\right|} \int_{B_1(0)} u(x)\mathrm{dx} \leq \frac{|B_1(0) \cap \operatorname{supp} u|}{\left|B_{\frac{1}{2}}(y)\right|} \sup_{B_1(0)} u(x)$$

and so by hypotheses,

$$u(y) \leq \frac{\epsilon_0\, \omega_n}{\left(\frac{1}{2}\right)^n \omega_n} = \epsilon_0\, 2^n,$$

which gives the result. □

Proposition 2.1.3. *Let u be a non-negative subharmonic function in a domain that contains $B_1(0)$. If for some $\rho \leq 1$ and for some constants $N, \epsilon_0 \geq 0$*

$$\sup_{x \in B_\rho(0)} u(x) \leq N\rho \quad \text{and} \quad \frac{|\{u \neq 0\} \cap B_\rho(0)|}{|B_\rho(0)|} \leq \epsilon_0,$$

then,

$$\sup_{x \in B_{\frac{\rho}{2}}(0)} u(x) \leq N\, \rho\, \epsilon_0\, 2^n.$$

Proof. Consider the function v defined on $B_1(0)$ by

$$v(x) = \frac{1}{N\rho} u(\rho\, x).$$

The new function v satisfies

$$\sup_{x \in B_1(0)} v(x) \leq 1,$$

and so, by Lemma 2.1.2, we have that

$$\sup_{x \in B_{\frac{1}{2}}(0)} v(x) \leq \epsilon_0\, 2^n.$$

Substituting v by its definition in terms of u gives that,

$$\sup_{x\in B_{\frac{1}{2}}(0)} \frac{1}{N\rho} u(\rho\, x) \leq \epsilon_0\, 2^n \Leftrightarrow \sup_{y\in B_{\frac{\rho}{2}}(0)} u(y) \leq N\, \rho\, \epsilon_0\, 2^n,$$

which gives the final result. □

2.2 Pucci Operators: General properties

In this section we present the definition of viscosity solutions and review the general properties of Pucci operators. We also introduce the spaces $\underline{S}(\lambda, \Lambda, f)$ and $\overline{S}(\lambda, \Lambda, f)$ and some results from the fully nonlinear elliptic theory that are used in this work (see [2] for the proofs and more detail). Definition 2.2.2, Remark 2.2.1, and Proposition 2.2.5 are valid in a more general setting for a fully nonlinear elliptic operator $F(D^2u)$ (see [2] for more details).

Here $\mathcal{M}^+$ and $\mathcal{M}^-$ will denote the extremal Pucci operators,

$$\mathcal{M}^+(u) := \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} a_{ij}D_{ij}(u) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(AD^2u) = \lambda \sum_{e_i<0} e_i + \Lambda \sum_{e_i>0} e_i,$$

$$\mathcal{M}^-(u) := \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} a_{ij}D_{ij}(u) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(AD^2u) = \Lambda \sum_{e_i<0} e_i + \lambda \sum_{e_i>0} e_i,$$

where $\mathcal{A}_{\lambda,\Lambda}$ is the set of symmetric $(n \times n)$ matrices with eigenvalues in $[\lambda, \Lambda]$ for $0 < \lambda < \Lambda$, and e_i are the eigenvalues of the matrix D^2u.

To see this, consider a fixed x. Let $M = D^2u$ and $\{v_i\}_i$ be the basis of $\mathbb{R}^n$ of the eigenvectors of M with eigenvalues e_i. Let O be the orthogonal matrix that change the variables to the basis of eigenvectors of M. Then

$$OMO^t = [\delta_{ij}e_i]_{ij} = D.$$

As the trace is invariant under change of coordinates,

$$\mathrm{Tr}(AM) = \mathrm{Tr}(AMO^tO) = \mathrm{Tr}(OAMO^t) = \mathrm{Tr}(OA\,O^tO\,MO^t) = \mathrm{Tr}(\underbrace{OAO^t}_{\tilde{A}}\;D),$$

the Pucci operator does not fix A, $A \in \mathcal{A}_{\lambda,\Lambda}$. Note that A and $\tilde{A}$ have the same eigenvalues. So, let e_i are the eigenvalues of the matrix D^2u for a certain x and A be the extreme case possible in order to maximize (or minimize) the trace:

$$Ae_i = \begin{cases} \Lambda e_i, & e_i > 0 \\ \lambda e_i, & e_i < 0 \end{cases}$$

Then

$$\mathcal{M}^+(u) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(AD^2u) = \lambda \sum_{e_i<0} e_i + \Lambda \sum_{e_i>0} e_i$$

The analog is valid to $\mathcal{M}^-$.

These two operators are a special case of nonlinear uniformly elliptic operators:

Definition 2.2.1. Let $F : \mathcal{S} \times \Omega \to \mathbb{R}$, where $\mathcal{S}$ is the space of real $n \times n$ symmetric matrices and $\Omega \subset \mathbb{R}^n$. We say that

- F is a uniform elliptic operator if there are two positive constants $\lambda \leq \Lambda$, called ellipticity constants, such that, for any $M \in \mathcal{S}$ and $x \in \Omega$

  $$\lambda \, \|N\| \leq F(M+N) - F(M) \leq \Lambda \, \|N\|$$

  for all nonnegative definite matrix $N \in \mathcal{S}$, where $\|N\| = \sup_{|x|=1} |Nx|$ is the value of the maximum eigenvalue of N if $N \geq 0$.

- F is concave (convex) if it is concave (convex) as a function of $M \in \mathcal{S}$.

When dealing with these operators we consider solutions in the viscosity sense.

Definition 2.2.2. Let f be a continuous function defined in Ω and $0 < \lambda < \Lambda$ two constants. We denote by $\underline{S}(\lambda, \Lambda, f)$ the space of continuous functions u defined in Ω that are viscosity subsolutions of $\mathcal{M}^+(u) = f(x)$ in Ω, meaning that if $x_0 \in \Omega$, A is a neighborhood of x_0, and P a paraboloid (see Remark 2.2.2) that touches u from above at x_0, i.e.

$$P(x) \geq u(x) \qquad \forall x \in A \qquad \text{and} \qquad P(x_0) = u(x_0),$$

then

$$\mathcal{M}^+(P(x_0)) \geq f(x_0).$$

In similar way, we denote by $\overline{S}(\lambda, \Lambda, f)$ the space of continuous functions u defined in Ω that are viscosity supersolutions of $\mathcal{M}^-(u) = f(x)$ in Ω, meaning that if $x_0 \in \Omega$, A is a neighborhood of x_0, and P a paraboloid (see Remark 2.2.2) that touches u from below at x_0, i.e.

$$P(x) \leq u(x) \qquad \forall x \in A \qquad \text{and} \qquad P(x_0) = u(x_0),$$

then

$$\mathcal{M}^-(P(x_0)) \leq f(x_0).$$

Remark 2.2.1. As in [2] we will denote by $S^\star(\lambda, \Lambda, f)$ the set of viscosity solutions

$$\underline{\mathrm{S}}(\lambda, \Lambda, -|f|) \cap \overline{S}(\lambda, \Lambda, |f|).$$

Remark 2.2.2. A paraboloid P is a polynomial in $(x_1, x_2, \cdots, x_n)$ of second degree:

$$P(x) = l_0 + l(x) + \frac{1}{2} x^t A x$$

where l_0 is a constant, $A = D^2 P$ is a symmetric matrix and $l(x)$ is a linear function.

Remark 2.2.3. Observe now that if $\omega(z)$ defined on $B_d(y)$ is a solution to $\mathcal{M}^\pm(\omega) = 0$ we have invariance under translation by y (2.2), rotation by R (2.3) and dilation by $\frac{1}{d}$ and rescaling by d (2.4):

$$\overline{\omega}(x) = \omega(\underbrace{x+y}_{z}), \quad x \in B_d(0) \Rightarrow \mathcal{M}^\pm(\overline{\omega}(x)) = \mathcal{M}^\pm(\omega(z)), x \in B_d(0), \tag{2.2}$$

$$\overline{\omega}(x) = \omega(Rx), \quad x \in B_d(0) \Rightarrow \mathcal{M}^\pm(\overline{\omega}(x)) = \mathcal{M}^\pm(\omega(z)), \quad x \in B_d(0), \tag{2.3}$$

$$\overline{\omega}(x) = \frac{1}{d}\omega(dx), \quad x \in B_1(0) \Rightarrow \mathcal{M}^\pm(\overline{\omega}(x)) = d\mathcal{M}^\pm(\omega(z)), x \in B_1(0), \tag{2.4}$$

and so, $\overline{\omega}(x)$ defined in $B_1(0)$ is still a solution to $\mathcal{M}^\pm(\overline{\omega}) = 0$ with the direction e_n as we want.

Remark 2.2.4. Observe that

$$\inf_{A \in \mathcal{A}_{\lambda,\Lambda}} a_{ij} D_{ij}(u(x)) \leq a_{ij}(x) D_{ij}(u(x)) \leq \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} a_{ij} D_{ij}(u(x)).$$

Before proceeding, let us call the attention upon for the following properties of Pucci operators. Those properties follow easily from the definition and previous remarks. Let u, v be smooth functions and $0 < \lambda < 1 < \Lambda$:

- $\mathcal{M}^-(u) \leq \Lambda \Delta u \leq \mathcal{M}^+(u)$;

- $\mathcal{M}^-(-u) = -\mathcal{M}^+(u)$;

- $\mathcal{M}^-(u)+\mathcal{M}^-(v) \leq \mathcal{M}^-(u+v) \leq \mathcal{M}^+(u)+\mathcal{M}^-(v)$ and so $\mathcal{M}^-$ is concave;

- $\mathcal{M}^+(u)+\mathcal{M}^-(v) \leq \mathcal{M}^+(u+v) \leq \mathcal{M}^+(u)+\mathcal{M}^+(v)$ and so $\mathcal{M}^+$ is convex;

- $0 \leq \mathcal{M}^-(u_i^\epsilon) \leq \Lambda \Delta u_i^\epsilon \leq \mathcal{M}^+(u_i^\epsilon) \Rightarrow\ u_i^\epsilon$ is subharmonic in the viscosity sense;

- $0 \leq \sum_i \mathcal{M}^-(u_i^\epsilon) \leq \mathcal{M}^-(\sum_i u_i^\epsilon) \leq \Delta\left(\sum_i u_i^\epsilon\right)$ which implies that $\sum_i u_i^\epsilon$ is subharmonic in the viscosity sense;

Remark 2.2.5. Observe that if u is continuous and subharmonic in the viscosity sense then u is subharmonic in the distributional sense, meaning that

$$\int_\Omega \Delta u\, \phi\, \mathrm{d}\mathbf{x} := \int_\Omega u\, \Delta\phi\, \mathrm{d}\mathbf{x} \geq 0 \qquad \forall \phi \geq 0,\ \phi \in C_0^\infty.$$

Now, we recall the comparison principle for viscosity solutions, Corollary 3.7 in [2], that states that a viscosity subsolution that is negative on the boundary has to remain negative in whole domain, and that a viscosity supersolution that is positive on the boundary has to remain positive in whole domain:

Proposition 2.2.1. *Assume that* $u \in C(\overline{\Omega})$*. Then,*

1. $u \in \underline{S}(\lambda, \Lambda, 0)$ *and* $u \leq 0$ *on* $\partial\Omega$ *imply* $u \leq 0$ *in* Ω.

2. $u \in \overline{S}(\lambda, \Lambda, 0)$ *and* $u \geq 0$ *on* $\partial\Omega$ *imply* $u \geq 0$ *in* Ω.

The following compactness result (Proposition 4.11 in [2]) follows from the closedness of the family of viscosity solutions of Problem (2.5) under the uniform convergence and the Ascoli-Arzela theorem.

Proposition 2.2.2. *Let* $\{F_k\}_{k\geq 1}$ *be a sequence of uniformly elliptic operators with ellipticity constants* λ, Λ *and let* $\{u_k\}_{k\geq 1} \subset C(\Omega)$ *be viscosity solutions in* Ω *of*

$$F_k(D^2u_k, x) = f(x), \tag{2.5}$$

with f *continuous and bounded. Assume that* F_k *converges uniformly in compact sets of* $\mathcal{S} \times \Omega$ *to* F*, where* $\mathcal{S}$ *is the space of real symmetric matrices, and that* u_k *is uniformly bounded in compact sets of* Ω*. Then there exist* $u \in C(\Omega)$ *and a subsequence of* $\{u_k\}_{k\geq 1}$ *that converges uniformly to* u *in compact sets of* Ω*. Moreover,* $F(D^2u, x) = f(x)$ *in the viscosity sense in* Ω.

Below is the L^ϵ Lemma that follows from Lemma 4.6 in [2], using a standard covering argument. Note that it is enough to consider f^+ instead of $|f|$ due to the Alexandroff-Bakelman-Pucci estimate, Theorem 3.2 in [2].

Lemma 2.2.3. *If* $u \in \overline{S}(f^+)$ *in* $B_1(0)$, $u \in C(\overline{B_1(0)})$, f *a is continuous and bounded function in* $B_1(0)$*, and they satisfy:*

1. $\inf_{B_{\frac{1}{2}}(0)} u(x) \leq 1$

2. $u(x) \geq 0$ *in* $B_1(0)$

3. $\|f^+\|_{L^n(B_1(0))} \leq \epsilon_0$

Then, if ϵ_0 is sufficiently small, there exist d and ϵ positive universal constants such that:

$$\left|\{x \in B_{\frac{1}{4}}(0) : u(x) \geq t\}\right| \leq dt^{-\epsilon}, \quad \text{for all} \quad t > 0.$$

Now, we recall the inequality that gives interior Hölder regularity and that follows from the Harnack inequality for viscosity solutions:

Proposition 2.2.4. *Let $\omega \in \overline{S}(\lambda, \Lambda, |f|) \cap \underline{S}(\lambda, \Lambda, -|f|)$ with f a continuous and bounded function in $B_1(0)$. Then, there exists a universal constant $\mu < 1$ such that*

$$\mathrm{osc}_{B_{\frac{1}{2}}(0)}\omega \leq \mu \mathrm{osc}_{B_1(0)}\omega + \|f\|_{L^n(B_1(0))}.$$

The interior Hölder regularity that we will use is a particular case of Theorem 7.1 in [2], and Sobolev embedding:

Proposition 2.2.5. *Let ω be a bounded viscosity solution of $\mathcal{M}^-(\omega) = f(x)$ in $B_1(0)$, with f a continuous bounded function in $B_1(0)$. Then there exists a positive constant C depending only on n, λ, Λ such that $\omega \in W^{2,p}(B_{\frac{1}{2}}(0))$, for any $p < \infty$, and so $\omega \in C^{1,\tilde{\alpha}}(B_{\frac{1}{2}}(0))$ for any $\tilde{\alpha} < 1$, and we have*

$$\|\omega\|_{C^{1,\tilde{\alpha}}(B_{\frac{1}{2}}(0))} \leq C\left(\|\omega\|_{L^\infty(B_1(0))} + \|f\|_{L^p(B_1(0))}\right).$$

Remark 2.2.6. (1) The same result under the same hypothesis is also valid for a general uniformly elliptic operator, concave or convex.
(2) Observe that if $f \in C^{\tilde{\alpha}}$ then $\omega \in C^{2,\tilde{\alpha}}$.

2.3 Existence of barriers

One of the essential tools in this thesis is the existence of "fundamental" sub and supersolutions for the extremal Pucci operators that work as a barrier. Since the radii of the balls in which we need existence of barriers has to be arbitrary we construct those on the most general setting as is stated below.

Lemma 2.3.1. *Given constants* $0 < \lambda < \Lambda$ *and* $M, r, a, b, \rho \geq 0$, $\frac{a\,r}{b} < r < \rho$ *there exist a smooth function defined on* $B_\rho(0)\backslash B_{\frac{a\,r}{2b}}(0)$ *and a constant* c, $c(\alpha, a, b)$, *and a universal constant* α *such that:*

1. $\psi(x) = r\,M$ *for* $x \in \partial B_{\frac{a\,r}{b}}(0)$
2. $\psi(x) = 0$ *for* $x \in \partial B_r(0)$
3. $\mathcal{M}^-(\psi) \geq 0$ *for* $x \in B_r(0)\backslash B_{\frac{a\,r}{2b}}(0)$
4. $\frac{\partial\psi}{\partial\nu} = cM$ *when* $x \in \partial B_r(0)$, *and* $c = -\alpha\frac{a^\alpha}{b^\alpha - a^\alpha}$.

Proof. Consider first $r = 1$ and $M = 1$, we will rescale afterwards. Let $\alpha, M_2 > 0$, $\alpha > n-2$ and

$$\varphi(x) = M_1 + M_2\frac{1}{|x|^\alpha}.$$

where M_1, M_2 and α are such that the following conditions are satisfied:

1. $\varphi(x) = 0$ when $|x| = 1$
2. $\varphi(x) = 1$ when $|x| = \frac{a}{b} < 1$
3. $\mathcal{M}^-(\varphi) \geq 0$ in $B_1(0)\backslash B_{\frac{a}{2b}}(0)$

In detail,

1. $\varphi(x) = 0$ when $|x| = 1 \Rightarrow M_1 = -M_2$

2. $\varphi(x) = 1$ when $|x| = \frac{a}{b} \Rightarrow M_2 = \frac{1}{\left(\frac{a}{b}\right)^{\alpha} - 1} = \frac{a^{\alpha}}{b^{\alpha} - a^{\alpha}} \geq 0$

and so have that,

$$\varphi(x) = -M_2 + M_2 \frac{1}{|x|^{\alpha}} \quad \text{with} \quad M_2 = \frac{a^{\alpha}}{b^{\alpha} - a^{\alpha}}.$$

Note that if $\frac{a}{b}$ is very small then M_2 is very small too. On the other hand, the second derivatives of φ are given by:

$$\partial_{ij}\varphi(x) = -\alpha M_2 |x|^{-\alpha-2} \delta_{ij} - \alpha(-\alpha - 2) M_2 x_i x_j |x|^{-\alpha-4}.$$

Evaluating the Hessian of φ at a point $(r, 0, \cdots, 0)$ one obtains:

$$\partial_{ij}\varphi = 0 \quad i \neq j$$

$$\partial_{11}\varphi = M_2 \alpha(\alpha + 1) r^{-\alpha-2}$$

$$\partial_{ii}\varphi = -\alpha M_2 r^{-\alpha-2} \quad i > 1$$

And so by radial symmetry and rotational invariance the Pucci operator is given by

$$\begin{aligned} \mathcal{M}^{-}(\varphi(x)) &= M_2 \lambda \alpha(\alpha + 1) |x|^{-\alpha-2} - \Lambda (n-1) \alpha M_2 |x|^{-\alpha-2} \\ &= M_2 \alpha \, |x|^{-\alpha-2} \left(\lambda(\alpha + 1) - \Lambda (n-1) \right). \end{aligned}$$

In order to satisfy (3) one needs that:

$$\alpha \geq \frac{\Lambda(n-1) - \lambda}{\lambda},$$

which gives that

$$\varphi(x) = -M_2 + M_2 \frac{1}{|x|^\alpha} \quad \text{with } \; M_2 = \frac{a^\alpha}{b^\alpha - a^\alpha} \text{ and } \quad \alpha \geq \frac{\Lambda(n-1) - \lambda}{\lambda}.$$

Notice that the normal derivative is:

$$\frac{\partial \varphi}{\partial \nu}(x) = -\alpha\, M_2 \frac{1}{|x|^{\alpha+1}}.$$

And so when $r = |x| = 1$,

$$\frac{\partial \varphi}{\partial \nu}(x) = -\alpha\, M_2 = -\alpha \frac{a^\alpha}{b^\alpha - a^\alpha}.$$

Let $c = -\alpha \frac{a^\alpha}{b^\alpha - a^\alpha}$.

Now let us consider a dilation and obtain the result for general r and $M = 1$. Let $\tilde{\varphi}(x) = r\varphi(\frac{x}{r})$. Then $\tilde{\varphi}$ defined on $B_r(0)$ satisfies:

1. $\tilde{\varphi}(x) = 0$ when $|x| = r$
2. $\tilde{\varphi}(x) = r$ when $|x| = \frac{a\,r}{b}$
3. $\mathcal{M}^-(\tilde{\varphi}) \geq 0$ in $B_r(0) \backslash B_{\frac{a\,r}{2b}}(0)$
4. $\frac{\partial \tilde{\varphi}}{\partial \nu}(x) = c$ when $|x| = r$

Finally for an arbitrary M, let $\psi(x) = M\tilde{\varphi}(x)$. Now it is easy to check that the barrier function ψ satisfies:

1. $\psi(x) = 0$ when $|x| = r$
2. $\psi(x) = rM$ when $|x| = \frac{a\,r}{b}$

3. $\mathcal{M}^{-}(\psi) \geq 0$ in $B_r(0) \backslash B_{\frac{a\,r}{2b}}(0)$

4. $\frac{\partial \psi}{\partial \nu}(x) = cM$ when $|x| = r$

$\square$

Lemma 2.3.2. *Given constants $0 < \lambda < \Lambda$ and $M, r, a, b, \rho \geq 0$, $\frac{a}{b} r < r < \rho$, there exist a smooth function defined on $B_\rho(0) \backslash B_{\frac{a\,r}{2b}}(0)$, a constant c, $c(\alpha, a, b)$, and a universal constant α such that:*

1. *$\psi(x) = r\,M$ for $x \in \partial B_r(0)$*

2. *$\psi(x) = 0$ for $x \in \partial B_{r\frac{a}{b}}(0)$*

3. *$\mathcal{M}^{+}(\psi) \leq 0$ for $x \in B_r(0) \backslash B_{r\frac{a}{2b}}(0)$*

4. *$\frac{\partial \psi}{\partial \nu} = cM$ when $x \in \partial B_{r\frac{a}{b}}(0)$, where $c = \alpha \frac{1}{\frac{a}{b} - \left(\frac{a}{b}\right)^{\alpha+1}}$*

Proof. Consider first $r = 1$ and $M = 1$, we will rescale afterwards. Let $\alpha, M_2 > 0$ $\alpha > n - 2$ and

$$\varphi(x) = M_1 - M_2 \frac{1}{|x|^\alpha}.$$

where M_1, M_2, and α are such that the following conditions are satisfied:

1. $\varphi(x) = 0$ when $|x| = \frac{a}{b}$

2. $\varphi(x) = 1$ when $|x| = 1$

3. $\mathcal{M}^{+}(\varphi) \leq 0$ in $B_1(0) \backslash B_{\frac{a}{2b}}(0)$

In detail, (1) and (2) imply that

$$M_1 = 1 + M_2 \quad \text{and } M_2 = \frac{a^\alpha}{b^\alpha - a^\alpha}$$

and so have that,

$$\varphi(x) = 1 + M_2 - M_2 \frac{1}{|x|^\alpha} \quad \text{with} \quad M_2 = \frac{a^\alpha}{b^\alpha - a^\alpha}.$$

On the other hand, the second derivatives of φ are given by:

$$\partial_{ij}\varphi(x) = \alpha M_2 \left|x\right|^{-\alpha-2} \delta_{ij} + \alpha(-\alpha-2) M_2 x_i x_j \left|x\right|^{-\alpha-4}.$$

Evaluating the Hessian of φ at a point $(r, 0, \cdots, 0)$ one obtains:

$$\partial_{ij}\varphi = 0 \quad i \neq j$$

$$\partial_{11}\varphi = -M_2 \alpha(\alpha+1) r^{-\alpha-2}$$

$$\partial_{ii}\varphi = \alpha M_2 r^{-\alpha-2} \quad i > 1$$

And so by radial symmetry and rotational invariance the Pucci operator is given by

$$\begin{aligned} \mathcal{M}^+(\varphi(x)) &= \Lambda\,(n-1)\,\alpha M_2 \left|x\right|^{-\alpha-2} - \lambda M_2 \alpha(\alpha+1) \left|x\right|^{-\alpha-2} \\ &= M_2 \alpha \left|x\right|^{-\alpha-2} \left(\Lambda\,(n-1) - \lambda(\alpha+1)\right). \end{aligned}$$

In order to satisfy (3) one needs that:

$$\Lambda\,(n-1) - \lambda(\alpha+1) \leq 0 \Leftrightarrow \alpha \geq \frac{\Lambda(n-1) - \lambda}{\lambda},$$

which gives that

$$\varphi(x) = 1 + M_2 - M_2 \frac{1}{|x|^\alpha} \quad \text{with } M_2 = \frac{a^\alpha}{b^\alpha - a^\alpha} \text{ and } \quad \alpha \geq \frac{\Lambda(n-1) - \lambda}{\lambda}.$$

Notice that the normal derivative is:

$$\frac{\partial \varphi}{\partial \nu}(x) = \alpha\, M_2 \frac{1}{|x|^{\alpha+1}}.$$

And so when $|x| = \frac{a}{b}$,

$$\frac{\partial \varphi}{\partial \nu}(x) = \alpha\, M_2 \frac{1}{\left(\frac{a}{b}\right)^{\alpha+1}} = \frac{\alpha}{\frac{a}{b} - \left(\frac{a}{b}\right)^{\alpha+1}}.$$

Let $c = \frac{\alpha}{\frac{a}{b} - \left(\frac{a}{b}\right)^{\alpha+1}}$.

Now let us consider a dilation and obtain the result for general r and $M = 1$. Let $\tilde{\varphi}(x) = r\varphi(\frac{x}{r})$. Then $\tilde{\varphi}$ defined on $B_r(0)$ satisfies:

1. $\tilde{\varphi}(x) = 0$ when $|x| = r\frac{a}{b}$,

2. $\tilde{\varphi}(x) = r$ when $|x| = r$,

3. $\mathcal{M}^+(\tilde{\varphi}) \leq 0$ in $B_r(0) \backslash B_{r\frac{a}{2b}}(0)$,

4. $\frac{\partial \tilde{\varphi}}{\partial \nu}(x) = c$ when $|x| = r\frac{a}{b}$.

Finally for an arbitrary M, let $\psi(x) = M\tilde{\varphi}(x)$. Now, it is easy to check that the barrier function ψ satisfies

1. $\psi(x) = 0$ when $|x| = r\frac{a}{b}$,

2. $\psi(x) = rM$ when $|x| = r$,

3. $\mathcal{M}^+(\psi) \leq 0$ in $B_r(0) \backslash B_{r\frac{a}{2b}}(0)$,

4. $\frac{\partial \psi}{\partial \nu}(x) = cM$ when $|x| = r\frac{a}{b}$.

□

Remark 2.3.1. Observe that the barrier on Lemma 2.3.2 could have been obtained from the barrier on Lemma 2.3.1 by doing $-\psi(x)+1$, since $\mathcal{M}^-(\psi) \geq 0$ implies $\mathcal{M}^+(-\psi+1) \leq 0$.

2.4 Fabes and Strook inequality

To prove the uniform Hölder regularity for the solutions of the elliptic system we need to use the Fabes and Strook inequality that we state below. For the proof of the Lemma see [22] (for more details see also [6, 21]). This inequality relates the value of the integral of the generalized Green function in a ball with the integral in a non-trivial subset. The generalized Green functions allows us to have a representation formula for viscosity solutions of linear operators in non divergence form and measurable coefficients. For more details about the generalized Green functions see [8].

Lemma 2.4.1 (Fabes and Strook inequality)**.** *Let $G(x,y)$ denote the Green's function for a linear operator L with measurable coefficients $Lu = a_{ij}(x)D_{ij}u$, $[a_{ij}(x)] \in \mathcal{A}_{\lambda,\Lambda}$. Then, there exist universal constants C and β such that whenever $E \subset B_r$, and $B_r \subset B_{\frac{1}{2}}$ the following holds:*

$$\left[\frac{|E|}{|B_r|}\right]^{\beta} \int_{B_r} G(x,y)\mathrm{dy} \leq C \int_E G(x,y)\mathrm{dy} \qquad \textit{for all } x \in B_1.$$

Chapter 3

Existence of solutions

3.1 Preliminaries

To prove the existence theorem, Theorem 3.3.1, we will need a fixed-point argument that can be found in [23], pg 280. We recall the result here for the sake of completeness:

Proposition 3.1.1. *Let σ be a closed, convex subset of a Banach space B. Let $T : \sigma \to \sigma$ be a continuous function such that $T(\sigma)$ is a pre-compact set. Then T has a fixed point.*

To apply the fixed point theorem, we need an existence result and regularity up to the boundary for a Bellman-type equation. In the following results, we denote by G the operator

$$\begin{aligned} G[\omega_i] := G(D^2\omega_i, x) &= \inf_{\substack{a_{st} \in \mathbb{Q} \\ [a_{st}] \in \mathcal{A}_{\lambda,\Lambda}}} \left(a_{st} D_{st}\omega_i - \frac{1}{\epsilon}\omega_i \sum_{j \neq i} u_j^\epsilon \right) \\ &= \mathcal{M}^-(\omega_i) - \frac{1}{\epsilon}\omega_i \sum_{j \neq i} u_j^\epsilon, \end{aligned}$$

with u_j^ϵ fixed positive continuous functions and $\mathcal{A}_{\lambda,\Lambda}$ the set of symmetric $n \times n$ real matrices with eigenvalues in $[\lambda, \Lambda]$, for $0 < \lambda < \Lambda$. The existence result is

Theorem 17.18 in [23], but we also state it below in the adequate form for our purpose.

Proposition 3.1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the exterior sphere condition for all $x \in \partial\Omega$. Let u_j^ϵ, $j \neq i$, be given functions, and a_{st} symmetric matrices. Suppose that, for all s, t, j, there exists a positive constant μ such that:*

$$a_{st} \in C^2(\Omega) \text{ and } \|a_{st}\|_{C^2(\Omega)} \leq \mu\lambda;$$

$$\frac{1}{\epsilon} u_j^\epsilon \in C^2(\Omega) \text{ and } \left\|\frac{1}{\epsilon} u_j^\epsilon\right\|_{C^2(\Omega)} \leq \mu\lambda;$$

and

$$0 \leq \lambda |\xi|^2 \leq a_{st}\xi_s\xi_t \leq \Lambda |\xi|^2, \quad \text{and} \quad u_j^\epsilon \geq 0.$$

Then, for any $\phi_i \in C(\partial\Omega)$, there exists a unique solution $\omega_i \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} G(D^2\omega_i, x) = 0, & \text{in } \Omega \\ \omega_i = \phi_i & \text{on } \partial\Omega. \end{cases}$$

We also need a generalization of the comparison principle that the reader can find on page 443 in [23] and comment on page 446:

Proposition 3.1.3. *Let u, $v \in C(\overline{\Omega}) \cap C^2(\Omega)$. If $G[u] \geq G[v]$ in Ω and $u \leq v$ in $\partial\Omega$, then $u \leq v$ in Ω.*

3.2 An improvement: Hölder regularity up to the boundary.

The next Proposition is the Hölder regularity up to the boundary for a viscosity solution of an equation of the type $\mathcal{M}^-(\omega) = f(x)$ in a Lipschitz domain. The proof we present here uses the comparison principle and an inductive construction of barriers. A different proof, by Luis Escauriaza, can be found in [21], Lemma 3. Escauriaza uses the Fabes and Stroock inequality (see Lemma 2.4.1) to estimate the Hölder norm of the solution up to the boundary in terms of the L^q norm of the right-hand side.

This result is an improvement of Proposition 4.12 and 4.13 in [2] for general Lipschitz domains.

Proposition 3.2.1. *Let Ω be an Lipschitz domain. Let $\omega \in C^2(\Omega) \cap C(\overline{\Omega})$ be a viscosity solution of*

$$\begin{cases} \mathcal{M}^-(\omega) = f(x) \quad \text{in} \quad \Omega, \\ \omega = \phi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

with $f \geq 0$ and $f \in C(\overline{\Omega}) \cap C^2(\Omega)$, $\phi \in C^\beta(\partial\Omega)$. Then $\omega \in C^\gamma(\overline{\Omega})$, where $\gamma = \min(\tilde{\alpha}, \alpha)$ and $\tilde{\alpha}, \alpha$ are the universal Hölder exponents for the interior regularity and boundary regularity, respectively.

The proof of this proposition follows the same lines of the proof of Proposition 4.13, in [2]. Once the interior regularity and the regularity for an arbitrary point in the boundary is guaranteed the proof is basically the interplay of these two results depending on how close two points are, compared to the maximum of their distance to the boundary. The interior regularity comes from Proposition 2.2.5. To prove the regularity for an arbitrary point on the boundary, Proposition 3.2.5, we need the couple of Lemmas that follow.

The first Lemma establishes the decay of a subsolution of $\mathcal{M}^+$ in concentric balls centered at an external point in the outside cone. The proof uses a standard comparison argument and the use of a barrier function.

Lemma 3.2.2. *Let Ω be an Lipschitz domain and C an external cone centered at $x_0 \in \partial\Omega$, with some universal opening. Let $y \in C$ be the center of the balls $B^1 \subset B^2 \subset B^3$ such that $B^1 \in C$, and that $\operatorname{dist}(\partial B^2 \cap \Omega, x_0) > \delta > 0$. Let u be a solution of $\mathcal{M}^+(u) \geq 0$ in the viscosity sense in Ω, such that $u \leq 1$ on $B^3 \cap \Omega$ and $u \leq 0$ on $B^3 \cap \partial\Omega$. Then, there exist $\lambda > 0$ such that*

$$u(x) \leq \lambda < 1 \quad in \quad B^2 \cap \Omega.$$

Proof. Since the domain is Lipschitz there exists a cone C with opening equal to ρ, such that for any point of the boundary we can place the cone with opening ρ and vertex at that point such that $C \cap \Omega = \emptyset$.

Without loss of generality, take the cone with origin at $x_0 \in \partial\Omega$ and with axis e_n,

$$C = \left\{ x : (x - x_0) \cdot e_n < -\rho \sqrt{\sum_{i=1}^{n-1} (x - x_0)_i^2} \right\},$$

and consider the origin as the center of the balls B^1, B^2, B^3 as illustrated in Figure 3.1.

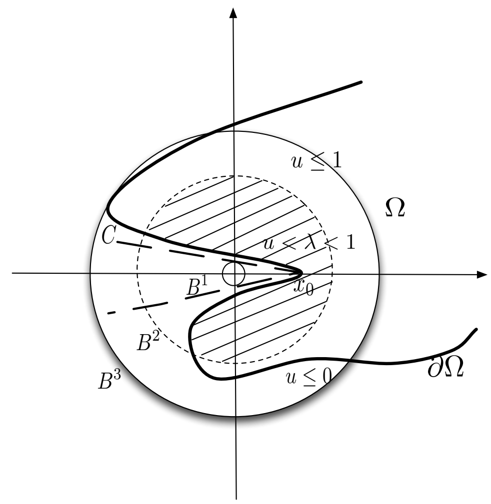

Figure 3.1: Reference decay.

Applying Lemma 2.3.2 with $M = \frac{1}{r}$ where r is the radius of B^3 and $\frac{ar}{b}$ equal to the radius of B^1 we obtain a supersolution ψ, $\mathcal{M}^+(\psi) \leq 0$ in $B^3 \backslash B^1$, such that $\psi(x) = 0$ in ∂B^1 and $\psi(x) = 1$ in ∂B^3. And so

$$u(x) \leq \psi(x) \quad \text{on} \quad \partial(B^3 \cap \Omega)$$

Applying the comparison principle stated in Proposition 2.2.1 we can conclude that

$$u(x) \leq \psi(x) \quad \text{in} \quad B^3 \cap \Omega$$

Let $\lambda = \psi(x) < 1$ for $x \in \partial B^2 \cap \Omega$. As ψ is an increasing function, see Figure 3.2, we can conclude that for $x \in B^2 \cap \Omega$ we have that $u(x) \leq \lambda < 1$ as we claim. $\square$

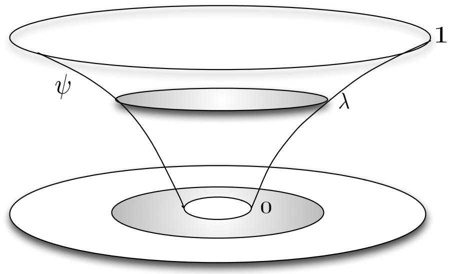

Figure 3.2: Upper bound for u due to the supersolution ψ.

The next Lemma is important for the iteration construction. Basically we prove that if the boundary data is bounded from above in half of the unit ball centered at a boundary point, and the function is bounded in the unit ball then in one fourth of the ball, the function decays by a fixed value.

Lemma 3.2.3. *Let Ω be a Lipschitz domain and C an external cone with universal opening. Let $0 \in \partial\Omega$ be the origin of the cone. Let v be a solution of $\mathcal{M}^+(u) \geq 0$ in the viscosity sense in Ω, such that $v \leq 1$ on $B_1(0) \cap \Omega$, $v(0) = 0$ and $v(x) \leq \left(\frac{1}{2}\right)^\beta$ on $B_{\frac{1}{2}}(0) \cap \partial\Omega$, for some $\beta > 0$. Then, there exists a constant $\left(\frac{1}{2}\right)^\beta < \mu < 1$ such that*

$$v(x) \leq \mu \quad in \quad B_{\frac{1}{4}}(0) \cap \Omega.$$

Proof. By hypothesis,

$$v(x) \leq \left(\frac{1}{2}\right)^\beta, \quad x \in \partial\Omega \cap B_{\frac{1}{2}}(0).$$

Let

$$\omega(x) = \frac{v(\frac{x}{2}) - (\frac{1}{2})^\beta}{1 - (\frac{1}{2})^\beta} \quad \text{for} \quad x \in B_1(0) \cap \Omega_{\frac{1}{2}},$$

where $\Omega_{\frac{1}{2}} = \{2x : x \in \Omega\}$. ω satisfies all the hypotheses of Lemma 3.2.2 with:

1. $x_0 = 0$;
2. C the uniform external cone with axis without loss of generality equals to e_n axis;
3. $B^1 = B_r(y)$ with $y = (0, \cdots, 0, -\frac{1}{10})$ and $r \leq \text{dist}(y, \partial C), r = \frac{\rho}{10}$;
4. $B^2 = B_{\frac{7}{10}}(y)$;
5. $B^3 = B_{\frac{4}{5}}(y)$.

Observe that $B^3 \subset B_1(0)$ and that $B_{\frac{1}{2}}(0) \subset B^2$. Then by Lemma 3.2.2

$$\omega(x) \leq \lambda < 1 \quad \text{in} \quad B^2 \cap \Omega_{\frac{1}{2}}$$

and so we have also that, for $x \in B_{\frac{1}{2}}(0) \cap \Omega_{\frac{1}{2}}$,

$$\frac{v(\frac{x}{2}) - (\frac{1}{2})^\beta}{1 - \left(\frac{1}{2}\right)^\beta} \leq \lambda \Leftrightarrow v(\frac{x}{2}) \leq \lambda(1 - \left(\frac{1}{2}\right)^\beta) + \left(\frac{1}{2}\right)^\beta = \underbrace{\lambda + (1-\lambda)\left(\frac{1}{2}\right)^\beta}_{\mu},$$

and $\left(\frac{1}{2}\right)^\beta \leq \mu \leq 1$. Then, we obtain that

$$v(z) \leq \mu \qquad z \in B_{\frac{1}{4}}(0) \cap \Omega. \quad \square$$

3.2.1 Iterative decay

Now we will be able to prove an iterative decay illustrated in Figure 3.3.

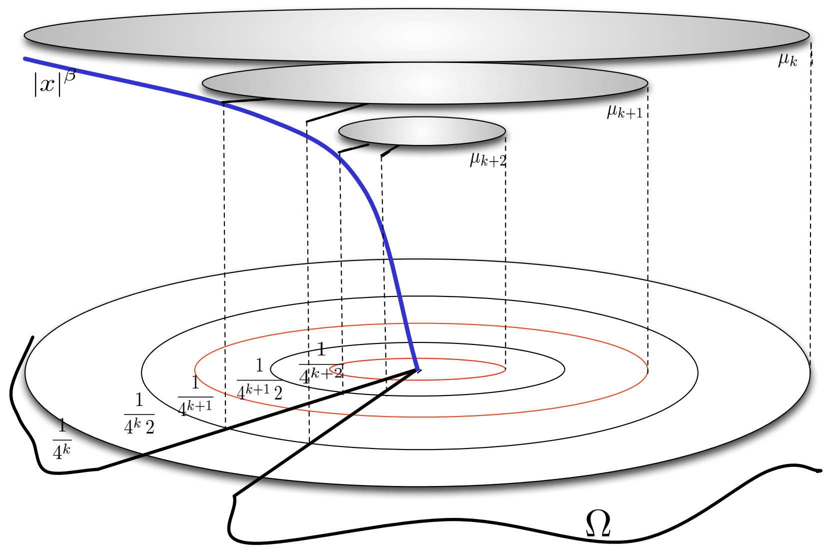

Figure 3.3: Iterative decay.

Lemma 3.2.4. *Let Ω be a Lipschitz domain and C an external cone with universal opening. Let $0 \in \partial\Omega$ be the origin of the cone. Let v be a solution of $\mathcal{M}^+(u) \geq 0$ in the viscosity sense in Ω, such that $v \leq \mu_k$ on $B_{\frac{1}{4^k}}(0) \cap \Omega$, ($\mu_0 = 1$), $v(0) = 0$ and $v(x) \leq \left(\frac{1}{4^k 2}\right)^\beta$ on $B_{\frac{1}{4^k 2}}(0) \cap \partial\Omega$. Then, there exist a constant, μ_{k+1}, $\mu_{k+1} := \lambda\mu_k + (1-\lambda)\left(\frac{1}{4^k 2}\right)^\beta$ for some $\lambda \in (0,1)$ universal, such that*

$$v(x) \leq \mu_{k+1} \quad \textit{in} \quad B_{\frac{1}{4^{k+1}}}(0) \cap \Omega.$$

Proof. By scaling and dilation, define for $x \in B_1(0)$

$$\omega(x) = \frac{v(\frac{x}{4^k}) - \left(\frac{1}{4^k 2}\right)^\beta}{\mu_k - \left(\frac{1}{4^k 2}\right)^\beta}.$$

Since ω satisfies the hypotheses of Lemma 3.2.3, considering this dilation and scaling, we see that

$$\omega(x) \leq \lambda < 1 \quad \text{in} \quad B_{\frac{1}{4}}(0) \cap \Omega$$

and so, like in the previous proof, we have also that

$$\frac{v(\frac{x}{4^k}) - \left(\frac{1}{4^k\,2}\right)^{\beta}}{\mu_k - \left(\frac{1}{4^k\,2}\right)^{\beta}} \leq \lambda \Leftrightarrow v(\frac{x}{4^k}) \leq \underbrace{\lambda\mu_k + (1-\lambda)\left(\frac{1}{4^k\,2}\right)^{\beta}}_{\mu_{k+1}} \qquad \text{for} \quad x \in B_{\frac{1}{4}}(0) \cap \Omega.$$

Therefore,

$$v(y) \leq \mu_{k+1} \qquad \text{for} \quad y \in B_{\frac{1}{4^{k+1}}}(0) \cap \Omega,$$

with $\left(\frac{1}{4^k\,2}\right)^{\beta} \leq \mu_{k+1} \leq \mu_k$. This finishes the proof. □

Remark 3.2.1. Observe that the decay at each step of this iteration is constant and equal to $C_0\left(\frac{1}{4^k}\right)^{\alpha}$ for α much smaller than β and C_0 a large positive constant. In fact, there exit constants C_0 and α such that $\mu_k \leq C_0\left(\frac{1}{4^k}\right)^{\alpha}$. By induction, for $k = 0$ the result is true by Lemma 3.2.3 for $C_0 \geq 1$. Assuming the result valid for a general k, we have that

$$\mu_{k+1} = \lambda\mu_k + (1-\lambda)\left(\frac{1}{4^k\,2}\right)^{\beta} \leq \frac{1}{4^{\alpha(k+1)}}\left(4^{\alpha}\lambda C_0 + (1-\lambda)\frac{1}{2^{\beta-2\alpha}4^{k(\beta-\alpha)}}\right)$$

Take $\alpha = \epsilon\beta$ and such that $4^{\alpha}\lambda < 1$ then

$$\left(4^{\alpha}\lambda C_0 + (1-\lambda)\frac{1}{2^{\beta-2\alpha}4^{k(\beta-\alpha)}}\right) \leq C_0$$

for C_0 large constant since,

$$4^{\alpha}\lambda C_0 + (1-\lambda)\frac{1}{2^{\beta-2\alpha}4^{k(\beta-\alpha)}} = \underbrace{4^{\alpha}\lambda}_{<1} C_0 + (1-\lambda)\frac{1}{2^{\beta(1-2\epsilon)}4^{k\beta(1-\epsilon)}} \leq C_0$$

$$\Leftrightarrow C_0 \geq (1-\lambda)\frac{1}{(1-4^{\alpha}\lambda)2^{\beta(1-2\epsilon)}4^{k\beta(1-\epsilon)}}.$$

Remark 3.2.2. Note that Lemmas 3.2.2 - 3.2.4 are valid for v a viscosity solution of Problem (3.1).

Proposition 3.2.5 (Hölder regularity up to the boundary)**.** *Let Ω be an Lipschitz domain and C an external cone with universal opening that only depends on the domain. Let $x_0 \in \partial\Omega$ be the origin of the cone. Let v be a viscosity solution of Problem (3.1) such that $|v(x)| \leq 1$ on $B_1(x_0) \cap \Omega$, $v(x_0) = 0$ and $|v(x)| \leq |x|^{\beta}$ on $B_1(x_0) \cap \partial\Omega$. Then, there exist constants $C > 0$ and $\alpha << \beta$ such that*

$$\sup_{x \in \overline{B}_1(x_0) \cap \overline{\Omega}} \frac{|v(x) - v(x_0)|}{|x - x_0|^{\alpha}} \leq C$$

and $C = C(\|f\|_{L^{\infty}})$.

Remark 3.2.3. The constant C in the previous proposition would depend on ϵ if this result was to be applied to our main problem. The uniform Hölder regularity in ϵ will be proved in the next section and does not depend on this result.

Proof. Assume by translation invariance that $x_0 = 0$. Note that $v(0) = v(x_0) = 0$. So if we prove that

$$|v(x)| \leq C\,|x|^{\alpha},$$

the result follows. Observe that on the boundary the regularity of the boundary data gives the result directly. Let k be such that

$$\frac{1}{4^{k+1}} \leq |x| \leq \frac{1}{4^k}.$$

Using Lemma 3.2.3 followed by Lemma 3.2.4 (see Remark 3.2.2) we can assume that for x such that $|x| = \rho < \frac{1}{4^k}$ we have:

$$v(x) \leq \mu_k.$$

Then, taking in account the previous remark, we also have,

$$v(x) \leq C_0 \left(\frac{1}{4^k}\right)^\alpha$$

but then

$$v(x) \leq C_0 4^\alpha \left(\frac{1}{4^{k+1}}\right)^\alpha \leq C_0 4^\alpha |x|^\alpha .$$

To obtain the other inequality observe that $-v(x) \leq 1$. So if we consider instead of $-v$ the function

$$\omega(x) = \frac{\|f\|_{L^\infty}}{2n\lambda} |x|^2 - v(x).$$

we have that, for $x \in B_1(0)$,

$$\mathcal{M}^+(\omega(x)) \geq \mathcal{M}^- \left(\frac{\|f\|_{L^\infty}}{2n\lambda} |x|^2\right) - \mathcal{M}^-(v) \geq \|f\|_{L^\infty} - f(x) \geq 0.$$

Observe that we have as well, for $x \in \partial\Omega \cap B_1(0)$ that

$$\omega(x) = \frac{\|f\|_{L^\infty}}{2n\lambda} |x|^2 - \phi(x) \leq \left(\frac{\|f\|_{L^\infty}}{2n\lambda} + C\right) |x|^\beta .$$

Therefore, we can apply the comparison principle for ω and a barrier function as in Lemma 3.2.3. Repeating the same construction as in the proof of Lemma 3.2.3 and Lemma 3.2.4 we obtain also that

$$\omega(x) \leq \overline{C} |\rho|^\alpha$$

and so, in an analogous way, we obtain that

$$-v(x) \leq \tilde{C}\,|\rho|^{\alpha}\,.$$

with $\tilde{C}$ depending of $\|f\|_{L^\infty}$. This completes the proof. □

With this result that is the analogous of Proposition 4.12 in [2], the proof of Proposition 3.2.1 follows as the proof of Propositon 4.13 in [2].

3.3 Proof of Theorem 3.3.1

In this section we finally present the proof of the existence of solution for the nonlinear elliptic system.

Theorem 3.3.1 (Existence)**.** *Let $\epsilon > 0$ constant, and Ω be a Lipschitz domain. Let ϕ_i be a non-negative Hölder continuous functions defined on $\partial\Omega$. Then there exist continuous functions $(u_1^\epsilon, \cdots, u_d^\epsilon)$ depending on the parameter ϵ such that u_i^ϵ is a viscosity solution of Problem (1.7).*

Proof. Let B be the Banach space of bounded continuous vector-valued functions defined on a domain Ω with the norm

$$\|(u_1^\epsilon, u_2^\epsilon, \cdots, u_d^\epsilon)\|_B = \max_i \left(\sup_{x\in\Omega} |u_i(x)|\right)$$

Let σ be the subset of bounded continuous functions that satisfy prescribed boundary data, and are bounded from above and from below as is stated below:

$$\sigma = \left\{ (u_1^\epsilon, u_2^\epsilon, \cdots, u_d^\epsilon) : \begin{array}{c} u_i^\epsilon \text{ is continuous, } u_i^\epsilon(x) = \phi_i(x) \text{ when } x \in \partial\Omega, \\ 0 \leq u_i^\epsilon(x) \leq \sup_i \|\phi_i\|_{L^\infty} \end{array} \right\}$$

σ is a closed and convex subset of B. Let T be the operator that is defined in the following way: $T\left((u_1^\epsilon, u_2^\epsilon, \cdots, u_d^\epsilon)\right) = (v_1^\epsilon, v_2^\epsilon, \cdots, v_d^\epsilon)$ if $(u_1^\epsilon, u_2^\epsilon, \cdots, u_d^\epsilon)$ and $(v_1^\epsilon, v_2^\epsilon, \cdots, v_d^\epsilon)$ are such that,

$$\begin{cases} \mathcal{M}^-(v_i^\epsilon) = \dfrac{1}{\epsilon}\displaystyle\sum_{j\neq i} v_i^\epsilon u_j^\epsilon \quad i = 1, \ldots, d \qquad \text{in} \quad \Omega \\ v_i^\epsilon = \phi_i, \quad i = 1, \ldots, d, \qquad \text{in} \quad \partial\Omega, \end{cases}$$

in the viscosity sense, where u_j^ϵ, $j \neq i$ are fixed. Let $g(\mathbf{u}^\epsilon) = \frac{1}{\epsilon}\sum_{j\neq i} u_j^\epsilon$ and observe that each of the previous equations of the vector $\mathbf{u}^\epsilon = (u_1^\epsilon, u_2^\epsilon, \cdots, u_d^\epsilon)$ take the form:

$$\mathcal{M}^-(v_i^\epsilon) = v_i^\epsilon \, g(\mathbf{u}^\epsilon).$$

Observe that if T has a fixed point, then

$$T\left((u_1^\epsilon, u_2^\epsilon, \cdots, u_d^\epsilon)\right) = (u_1^\epsilon, u_2^\epsilon, \cdots, u_d^\epsilon)$$

meaning that $u_i^\epsilon = \phi_i$ on the boundary and that

$$\mathcal{M}^-(u_i^\epsilon) = \frac{1}{\epsilon}\sum_{j\neq i} u_i^\epsilon u_j^\epsilon \qquad i = 1, \ldots, d$$

in Ω, which proves the existence as desired.

So in order for T to have a fixed point we need to prove that it satisfies the hypotheses of Proposition 3.1.1:

1. $T(\sigma) \subset \sigma$: We need to prove that there exists a regular solution for each one of the equations

$$\begin{cases} \mathcal{M}^-(v_k^\epsilon) - v_k^\epsilon \, g(\mathbf{u}^\epsilon) = 0, \quad k = 1, \ldots, d, \quad \text{in} \quad \Omega \\ v_k^\epsilon = \phi_k, \qquad k = 1, \ldots, d, \quad \text{in} \quad \partial\Omega. \end{cases}$$

Observe that if such a regular solution exists, the comparison principle is valid by Proposition 3.1.3.

To use Proposition 3.1.2 we can rewrite the differential equation in the form

$$F(v_k^\epsilon) = \inf_{\substack{a_{ij} \in \mathbb{Q} \\ [a_{ij}] \in \mathcal{A}_{\lambda,\Lambda}}} (a_{ij} D_{ij} v_k^\epsilon - v_k^\epsilon \, g(\mathbf{u}^\epsilon)) = 0, \qquad k = 1, \ldots, d.$$

since, by density, taking the infimum in $\mathbb{Q}$ is equal to taking the infimum over $\mathbb{R}$. Let $(\Omega_l)_{l\in\mathbb{N}}$ be a family of smooth domains contained in Ω such that $\Omega_l \nearrow \Omega$ as $l \to \infty$. Since we do not have the desired regularity on the coefficients of v_k^ϵ we will first consider the boundary value problem in each smooth domain Ω_l with the following regularized equation,

$$F^\delta(v_k^\epsilon) = \inf_{\substack{a_{ij} \in \mathbb{Q} \\ [a_{ij}] \in \mathcal{A}_{\lambda,\Lambda}}} (a_{ij} D_{ij} v_k^\epsilon - v_k^\epsilon \, (g(\mathbf{u}^\epsilon) \star \rho_\delta)) = 0, \qquad k = 1, \ldots, d,$$

where ρ_δ is an approximation of the identity. And we will consider suitable boundary data that converges to the original boundary data when we approach the original domain.

So now, by Proposition 3.1.2, there exist a unique solution $(v_k^\epsilon)_l^\delta$ in $C^2(\Omega) \cap C^0(\overline{\Omega})$ with

$$(v_k^\epsilon)_l^\delta = (\phi_k)_l \quad \text{on } \partial\Omega_l, \quad k = 1, \cdots, d.$$

Taking the limit in l we obtain the existence of $(v_k^\epsilon)^\delta$, solutions in Ω with boundary data equal to ϕ_k. From Proposition 3.2.1 we also have that there exists a universal exponent γ such that $(v_k^\epsilon)^\delta$ is Hölder continuous up to the boundary, $(v_k^\epsilon)^\delta \in C^\gamma(\overline{\Omega})$.

As we have $F^\delta \to F$ uniformly due to the properties of identity approximation, we can conclude that

$$F^\delta({v_k^\epsilon}^\delta) \stackrel{\delta\to 0}{\longrightarrow} F(v_k^\epsilon)$$

by Proposition 2.2.2, this is, that $v_k^\epsilon \in C^\gamma(\overline{\Omega})$ is the solution for our problem $F(v_k^\epsilon) = 0$ in Ω, for all k.

We need also to prove that for each k, $0 \leq v_k^\epsilon \leq \sup_i \|\phi_i\|_{L^\infty}$.

Suppose, by contradiction that there exists x_0 such that $v_k^\epsilon(x_0) < 0$. Since $v_k^\epsilon = \phi_k$ on $\partial\Omega$ and ϕ_k is non-negative, x_0 must be an interior point. But, if v_k^ϵ has an interior minimum attained at x_0 then there exists a paraboloid P touching v_k^ϵ from above at x_0 and such that

$$\mathcal{M}^-(P(x_0)) > 0.$$

Since, $g(\mathbf{u}^\epsilon) \geq 0$, on the other hand we have that

$$\mathcal{M}^-(v_k^\epsilon(x_0)) = v_k^\epsilon(x_0)\, g(\mathbf{u}^\epsilon(x_0)) \leq 0,$$

and so we have a contradiction.

Analogously, suppose, by contradiction that there exists x_0 such that $v_k^\epsilon(x_0) > \sup_i \|\phi_i\|_{L^\infty}$. Then, by the same reason as before x_0 must be an interior point and

$$\mathcal{M}^-(v_k^\epsilon(x_0)) < 0.$$

Since, $g(\mathbf{u}^\epsilon) \geq 0$, we have that

$$\mathcal{M}^-(v_k^\epsilon(x_0)) = v_k^\epsilon(x_0)\, g(\mathbf{u}^\epsilon(x_0)) \geq 0,$$

and so we have a contradiction.

2. *T is continuous:* Let us assume that for each fixed ϵ we have

$$((u_1^\epsilon)_n, \cdots, (u_d^\epsilon)_n) \to (u_1^\epsilon, \cdots, u_d^\epsilon) \in [C(\Omega)]^d$$

meaning that when n tends to ∞,

$$\max_{1<i<d} \|(u_i^\epsilon)_n - u_i^\epsilon\|_{L^\infty} \to 0.$$

We need to prove that

$$\|T\left((u_1^\epsilon)_n, \cdots, (u_d^\epsilon)_n\right) - T\left(u_1^\epsilon, \cdots, u_d^\epsilon\right)\|_{[C(\Omega)]^d} \to 0$$

when $n \to \infty$. Since,

$$T\left((u_1^\epsilon)_n, \cdots, (u_d^\epsilon)_n\right) = \left((v_1^\epsilon)_n, \cdots, (v_d^\epsilon)_n\right)$$

if we prove that there exists a constant C, independent of i, so that we have the estimate

$$\|(v_i^\epsilon)_n - v_i^\epsilon\|_{L^\infty} \leq C \max_j \left\|(u_j^\epsilon)_n - u_j^\epsilon\right\|_{L^\infty}$$

the result follows. For all $x \in \Omega$, let ω_n be the function

$$\omega_n(x) = (v_i^\epsilon)_n(x) - v_i^\epsilon(x),$$

and suppose by contradiction that there exists $y \in \Omega$ such that

$$\omega_n(y) > r^2 K \max_j \left\|(u_j^\epsilon)_n - u_j^\epsilon\right\|_{L^\infty}, \tag{3.2}$$

for some large $K > 0$, where r is such that $\Omega \subset B_r(0)$. We want to prove that this is impossible if K is sufficiently large. Let h_n be the concave radially symmetric function,

$$h_n(x) = \gamma(r^2 - |x|^2),$$

with $\gamma = K \max_i \|(u_i^\epsilon)_n - u_i^\epsilon\|_{L^\infty}$. Observe that:

(a) $h_n(x) = 0$ on $\partial B_r(0)$;

(b) $h_n(x) \leq r^2\, K\, \max_j \left\|(u_j^\epsilon)_n - u_j^\epsilon\right\|_{L^\infty}$ for all x in $B_r(0)$;

(c) $0 = \omega_n(x) \leq h_n(x)$ on $\partial\Omega$, since $(v_i^\epsilon)_n$ and v_i^ϵ are solutions with the same boundary data.

So, since we are assuming (3.2), there exists a negative minimum of $h_n - \omega_n$. Let x_0 be the point where the negative minimum value of $h_n - \omega_n$ is attained, $h_n(x_0) - \omega_n(x_0) \leq 0$. Then we have that for any matrix $A \in \mathcal{A}_{\lambda,\Lambda}$,

$$a_{ij} D_{ij}\left(\left(h_n - \omega_n\right)(x_0)\right) \geq 0 \quad \text{and} \quad a_{ij} D_{ij} h_n(x) = \sum_i -2\,\gamma\, a_{ii} \leq 0.$$

Moreover,

$$\begin{aligned}
\mathcal{M}^+(\omega_n) &\geq \mathcal{M}^-((v_i^\epsilon)_n) - \mathcal{M}^-(v_i^\epsilon) \\
&= \frac{1}{\epsilon}\left(((v_i^\epsilon)_n - v_i^\epsilon)\sum_{j\neq i}(u_j^\epsilon)_n - v_i^\epsilon \sum_{j\neq i}\left(u_j^\epsilon - (u_j^\epsilon)_n\right)\right) \\
&\geq \frac{1}{\epsilon}\left(((v_i^\epsilon)_n - v_i^\epsilon)\sum_{j\neq i}(u_j^\epsilon)_n - v_i^\epsilon (d-1)\max_j \left\|(u_j^\epsilon)_n - u_j^\epsilon\right\|_{L^\infty}\right)
\end{aligned}$$

adding and substrating $\frac{1}{\epsilon}v_i^\epsilon \sum_{j\neq i}(u_j^\epsilon)_n$. Then, if a_{ij}^ω are the coefficients associated to ω_n, i.e.

$$a_{ij}^{\omega_n} D_{ij}\omega_n = \mathcal{M}^+(\omega_n),$$

then

$$\begin{aligned}
0 &\leq a_{ij}^{\omega_n} D_{ij}\left(h_n - \omega_n\right)(x_0) \\
&\leq \sum_i -2\,\gamma\, a_{ii}^{\omega_n}(x_0) - \frac{1}{\epsilon}\left((v_i^\epsilon)_n - v_i^\epsilon\right)(x_0)\sum_{j\neq i}(u_j^\epsilon)_n(x_0) \\
&\quad + \frac{1}{\epsilon} v_i^\epsilon(x_0)(d-1)\max_j \left\|(u_j^\epsilon)_n - u_j^\epsilon\right\|_{L^\infty} \\
&\leq \sum_i -2\,K \max_j \left\|(u_j^\epsilon)_n - u_j^\epsilon\right\|_{L^\infty}\; a_{ii}^{\omega_n}(x_0) \\
&\quad + \frac{1}{\epsilon} v_i^\epsilon(x_0)(d-1)\max_j \left\|(u_j^\epsilon)_n - u_j^\epsilon\right\|_{L^\infty},
\end{aligned}$$

because $0 < h_n(x_0) \leq \omega_n(x_0)$ and $\sum_{j\neq i}(u_j^\epsilon)_n(x_0) \geq 0$ and so

$$-\frac{1}{\epsilon}\left((v_i^\epsilon)_n - v_i^\epsilon\right)(x_0)\sum_{j\neq i}(u_j^\epsilon)_n(x_0) \leq 0.$$

Taking $K > \frac{d-1}{\lambda\epsilon}\sup_i \|\phi_i\|_{L^\infty}$, we obtain that

$$0 \;\leq\; a_{ij}^{\omega_n} D_{ij}\left(h_n - \omega_n\right)(x_0) < 0$$

which is a contradition.

3. $T(\sigma)$ *is precompact:* This a consequence of (1), since the solutions to the equation are Hölder continuous, and this set is precompact in σ.

This concludes the proof. □

Chapter 4

Regularity of solutions

To obtain Hölder regularity for the solutions we want to construct a barrier function of the type $c\,|x - x_0|^\alpha$ and prove that it is a upper and lower bound for the solution for all x_0. One way of doing so is to prove that there exists a uniform decay on the oscillation of the solution over dyadic balls. Meaning that, we will prove that there exist two constants $0 < \lambda, \tilde{\mu} < 1$, independent of ϵ, such that for all $i = 1, \ldots, d$, we have

$$\mathrm{osc}_{x \in B_\lambda(0)} u_i^\epsilon(x) \leq \tilde{\mu}\, \mathrm{osc}_{x \in B_1(0)} u_i^\epsilon(x).$$

The C^α regularity of each function will follow from here in a standard way using Lemma 8.23, in [23]. We will follow this strategy to prove the regularity following the proof presented in [4]. We will prove first the decay of the oscillation under certain particular hypotheses and then use this result to prove the uniform C^α regularity for a solution of Problem (1.7). Instead of the Green representation formula for the Laplace operator used in [4] we need to use the Fabes and Strook inequality and the generalized green functions for second order elliptic operators with measurable coefficients (for more details see [8, 21]).

4.1 Some Lemmas

We first establish some conditions under which the maximum of a function in a smaller ball is lower.

Lemma 4.1.1. *Let $\boldsymbol{u}^\epsilon$ be a solution of Problem (1.7). Let $M_i = \max_{x \in B_1(0)} u_i^\epsilon(x)$ and $O_i = \mathrm{osc}_{x \in B_1(0)} u_i^\epsilon(x)$. If for some positive constant γ_0 one of the following hypotheses is verified*

1. $\left| \{x \in B_{\frac{1}{4}}(0) : u_i^\epsilon(x) \leq M_i - \gamma_0 O_i\} \right| \geq \gamma_0$
2. $\left| \{x \in B_{\frac{1}{4}}(0) : \mathcal{M}^-(u_i^\epsilon(x)) \geq \gamma_0\, O_i\} \right| \geq \gamma_0$
3. $\left| \{x \in B_{\frac{1}{4}}(0) : \mathcal{M}^-(u_i^\epsilon(x)) \geq \gamma_0\, u_i^\epsilon(x)\} \right| \geq \gamma_0$

then there exist a small positive constant $c_0 = c_0(\gamma_0)$ such that the following decay estimate is valid:

$$\max_{x \in B_{\frac{1}{4}}(0)} u_i^\epsilon(x) \leq M_i - c_0\, O_i$$

and so

$$\mathrm{osc}_{B_{\frac{1}{4}}(0)} u_i^\epsilon(x) \leq \tilde{c}_0\, O_i, \qquad \textit{with} \quad \tilde{c}_0 < 1.$$

Proof.

1. By contradiction, assume that for all c_0 small, exist a point $x_0 \in B_{\frac{1}{4}}(0)$ such that

$$u_i^\epsilon(x_0) \geq M_i - c_0\, O_i$$

and let

$$v_i(x) = \frac{M_i - u_i^{\epsilon}(x)}{O_i}.$$

v_i satisfy the following properties, with $f_i(x) = -\frac{u_i^{\epsilon}(x)}{\epsilon c_0 \, O_i} \sum_j u_j^{\epsilon}(x)$:

(a) $\inf_{B_{\frac{1}{2}}(0)} \frac{v_i(x)}{c_0} \leq \inf_{B_{\frac{1}{4}}(0)} \frac{v_i(x)}{c_0} \leq 1$

(b) $\mathcal{M}^{-}(\frac{v_i}{c_0}) \leq f_i(x)$

(c) $v_i(x) \geq 0$ in $B_1(0)$

(d) $\left\| f_i^{+} \right\|_{L^n} = 0$ since $f_i^{+}(x) = 0$.

Then we can apply the $L^{\epsilon}-$ Lemma stated as Lemma 2.2.3, to conclude that

$$\left| \{ x \in B_{\frac{1}{4}}(0) : \frac{v_i(x)}{c_0} \geq t \} \right| \leq dt^{-\delta},$$

for d, δ universal constants and for all $t > 0$. If $t = \frac{\gamma_0}{c_0}$ then we have

$$\left| \{ x \in B_{\frac{1}{4}}(0) : v_i(x) \geq \gamma_0 \} \right| \leq d(\frac{\gamma_0}{c_0})^{-\delta}$$

But, taking in account the hypothesis we have:

$$\gamma_0 \leq \left| \{ x \in B_{\frac{1}{4}}(0) : v_i(x) \geq \gamma_0 \} \right| \leq d(\frac{\gamma_0}{c_0})^{-\delta}$$

Since c_0 is arbitrary, if $c_0^{\delta} < \frac{\gamma_0^{\delta+1}}{d}$ we have the contradiction.

2. If u_i^{ϵ} is a solution for $\mathcal{M}^{-}(u_i^{\epsilon}) = f(u_i^{\epsilon})$ there exists a symmetric matrix with coefficients $a_{ij}(x)$ with

$$\lambda \left| \xi \right|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda \left| \xi \right|^2$$

such that

$$a_{kl}(x)D_{kl}u_i^\epsilon(x) = f(u_i^\epsilon(x))$$

For that particular matrix consider the linear problem with measurable coefficients in $B_1(0)$:

$$L_a\left(v(x)\right) = g(x),$$

and let $G(x,\cdot)$ be the respective Green function on $B_1(0)$ such that for all $x \in B_1(0)$,

$$v(x) = -\int_{B_1(0)} G(x,y)g(y)\mathrm{dy} + \text{boundary terms}$$

(see [8, 21]). Since

$$a_{kl}(x)D_{kl}(M_i - u_i^\epsilon(x)) = -f(u_i^\epsilon(x)),$$

and since the boundary values are positive and $G(x,y)f(u_i^\epsilon(y)) \geq 0$ for all y, we conclude that

$$\begin{aligned} M_i - u_i^\epsilon(x) &\geq -\int_{B_1(0)} G(x,y)(-f(u_i^\epsilon(y)))\mathrm{dy} \\ &\geq \int_{A_i} (G(x,y))(f(u_i^\epsilon(y)))\mathrm{dy} \\ &\geq \gamma_0\, O_i \int_{A_i} G(x,y)\mathrm{dy}, \end{aligned}$$

for $A_i := \{x \in B_{\frac{1}{4}}(0) : \mathcal{M}^-(u_i^\epsilon(x)) \geq \gamma_0\, O_i\}$. Since by hypothesis

$$|A_i| := \left|\{x \in B_{\frac{1}{4}}(0) : \mathcal{M}^-(u_i^\epsilon(x)) \geq \gamma_0\, O_i\}\right| \geq \gamma_0$$

and due to Fabes-Strook inequality, (see Lemma 2.4.1 and Theorem 2 in [22] for more details) there exist γ and c, universal constants such that for $\frac{1}{4} < r < \frac{1}{2}$,

$$\begin{aligned}\gamma_0\, O_i \int_{A_i} G(x,y)\mathrm{dy} &\geq c\,\gamma_0\, O_i \left(\frac{|A_i|}{|B_r(0)|}\right)^{\gamma} \int_{B_r(0)} G(x,y)\mathrm{dy} \\ &\geq c\,\gamma_0\, O_i \left(\frac{\gamma_0}{|B_r(0)|}\right)^{\gamma} \int_{B_r(0)} G(x,y)\mathrm{dy}.\end{aligned}$$

We claim that for $x \in B_{\frac{1}{4}}(0)$ there exists an universal constant C such that:

$$\int_{B_r(0)} G(x,y)\mathrm{dy} \geq C. \tag{4.1}$$

So again by hypothesis and due to the claim (that we will prove later) we have

$$c\,\gamma_0\, O_i \left(\frac{\gamma_0}{|B_r(0)|}\right)^{\gamma} \int_{B_r(0)} G(x,y)\mathrm{dy} \geq c\,\gamma_0\, O_i \left(\frac{\gamma_0}{|B_r(0)|}\right)^{\gamma} C$$

and finally

$$M_i - u_i^{\epsilon}(x) \geq c\,\gamma_0\, O_i \left(\frac{\gamma_0}{|B_r(0)|}\right)^{\gamma} C = c_0 O_i,$$

with $c_0 < 1$.

To prove claim (4.1) we argue that for $x \in B_{\frac{1}{4}}(0)$

$$\begin{aligned}\int_{B_r(0)} G(x,y)\mathrm{dy} &= \frac{1}{2n\Lambda} \int_{B_r(0)} G(x,y)(2n\Lambda)\mathrm{dy} \\ &\geq \frac{1}{2n\Lambda} \int_{B_r(0)} G(x,y) \left(2 \sum_i a_{ii}\right) \mathrm{dy} \\ &\geq \frac{1}{2n\Lambda} \int_{B_r(0)} \tilde{G}(x,y) \left(2 \sum_i a_{ii}\right) \mathrm{dy} \\ &= \frac{1}{2n\Lambda} \left(r^2 - |x|^2\right) \geq C,\end{aligned}$$

where $\tilde{G}$ is the Green function for L_a on $B_r(0)$, since x is interior moreover

$$a_{ij}(x)D_{ij}(r^2 - |x|^2) = -2\sum_i a_{ii}(x)$$

and $2n\lambda \leq 2\sum_i a_{ii}(x) \leq 2n\Lambda$.

3. Let

$$A_i = \left\{x \in B_{\frac{1}{4}}(0) : \mathcal{M}^-(u_i^\epsilon(x)) \geq \gamma_0\, u_i^\epsilon(x)\right\}$$

and

$$H_i = \left\{x \in A_i : u_i^\epsilon(x) \leq \frac{M_i}{2}\right\}$$

and consider the two possible cases:

$$(a)\ |A_i \backslash H_i| \geq \frac{1}{2}|A_i| \quad \text{and} \quad (b)\ |A_i \backslash H_i| < \frac{1}{2}|A_i|.$$

(a) If $|A_i \backslash H_i| \geq \frac{1}{2}|A_i|$ then as

$$\left\{x \in A_i \backslash H_i : \mathcal{M}^-(u_i^\epsilon(x)) \geq \gamma_0\, \tfrac{M_i}{2}\right\}$$

$$\cap$$

$$\left\{x \in B_{\frac{1}{4}}(0) : \mathcal{M}^-(u_i^\epsilon(x)) \geq \tfrac{\gamma_0\, O_i}{2}\right\}$$

since $\frac{O_i}{2} \leq \frac{M_i}{2}$, we can conclude that

$$\left|\left\{x \in B_{\frac{1}{4}}(0) : \mathcal{M}^-(u_i^\epsilon(x)) \geq \gamma_0\, \frac{O_i}{2}\right\}\right| \geq |A_i \backslash H_i| \geq \frac{1}{2}|A_i| \geq \frac{\gamma_0}{2}.$$

Then we have the decay by (2) with γ_0 replaced by $\frac{\gamma_0}{2}$.

(b) If $|A_i \backslash H_i| < \frac{1}{2}|A_i|$ then as

$$H_i = \left\{ x \in A_i : u_i^\epsilon(x) \le \frac{M_i}{2} \right\} \subset \left\{ x \in B_{\frac{1}{4}}(0) : u_i^\epsilon(x) \le M_i - \beta_0 O_i \right\}$$

for $\beta_0 \le \frac{M_i}{2O_i}$ and

$$|H_i| = |A_i \backslash (A_i \backslash H_i)| = |A_i| - |(A_i \backslash H_i)| \ge \frac{|A_i|}{2} \ge \frac{\gamma_0}{2}$$

we have

$$\left| \left\{ x \in B_{\frac{1}{4}}(0) : u_i^\epsilon(x) \le M_i - \tilde{\gamma_0} O_i \right\} \right| \ge \frac{\tilde{\gamma_0}}{2}$$

for $\tilde{\gamma_0} = \min(\beta_0, \gamma_0)$. The decay follows by (1).

□

Next Lemma states that if all the oscillations are tiny compared to just one that remains big (see Figure 4.1), then the largest oscillation has to decay due to an increase of the minimum in a smaller ball.

Lemma 4.1.2. *Let $\boldsymbol{u}^\epsilon$ be a solution of Problem (1.7) in $B_1(0)$. Let*

$$O_i^1 = \text{osc}_{x \in B_1(0)} u_i^\epsilon(x).$$

Assume that for some $\delta > 0$, sufficiently small,

$$\sum_{j \ne 1} O_j^1 \le \delta O_1^1.$$

then O_1^1 must decay in $B_{\frac{1}{2}}(0)$, that is, there exist $\mu < 1$ such that

$$O_1^{\frac{1}{2}} \le \mu O_1^1.$$

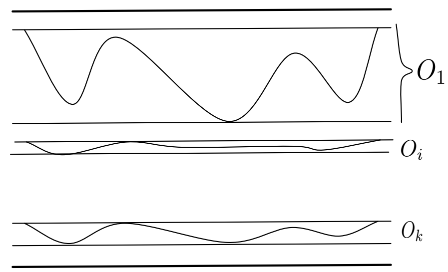

Figure 4.1: Illustration of our hypotheses about oscillations.

Proof. Let ω be the solution of the problem

$$\begin{cases} \mathcal{M}^-(\omega(x)) = 0, & x \in B_1(0) \\ \omega(x) = u_1^\epsilon(x), & x \in \partial B_1(0) \end{cases}$$

Since $u_1^\epsilon - \omega$ is a subsolution for the positive Pucci extremal operator

$$\mathcal{M}^+(u_1^\epsilon - \omega) \geq \mathcal{M}^-(u_1^\epsilon) + \mathcal{M}^+(-\omega) \geq 0,$$

and $u_1^\epsilon - \omega + \sum_{i\neq 1}(M_i - u_i^\epsilon)$ is a supersolution for the negative Pucci extremal operator,

$$\begin{aligned} \mathcal{M}^-\left(u_1^\epsilon + \sum_{i\neq 1}(M_i - u_i^\epsilon) - \omega\right) &\leq \mathcal{M}^-\left(u_1^\epsilon + \sum_{i\neq 1}(M_i - u_i^\epsilon)\right) + \mathcal{M}^+(-\omega) \\ &\leq \mathcal{M}^-(u_1^\epsilon) + \sum_{i\neq 1}\mathcal{M}^+(M_i - u_i^\epsilon) \\ &\leq \mathcal{M}^-(u_1^\epsilon) - \sum_{i\neq 1}\mathcal{M}^-(u_i^\epsilon) \leq 0, \end{aligned}$$

by the maximum principle for viscosity solutions and our hypotheses we get,

$$u_1^\epsilon(x) \leq \omega(x) \leq u_1^\epsilon(x) + \sum_{i\neq 1}(M_i - u_i^\epsilon(x)) \leq u_1^\epsilon + \delta O_1^1.$$

Since $\omega \in S^\star(\lambda, \Lambda, 0)$, ω decays, namely

$$\text{osc}_{x\in B_{\frac{1}{2}}(0)}\omega(x) \leq \mu \, \text{osc}_{x\in B_1(0)}\omega(x)$$

and from this inequalities, since

$$\max \omega - \min \omega \leq \max u_1^\epsilon + \delta O_1^1 - \min \omega \leq \max u_1^\epsilon + \delta O_1^1 - \min u_1^\epsilon,$$

we can conclude that

$$\text{osc}_{B_r(0)}\omega \leq \text{osc}_{B_r(0)} u_1^\epsilon + \delta O_1^1.$$

In a analogous way,

$$\text{osc}_{B_r(0)} u_1^\epsilon \leq \text{osc}_{B_r(0)}\omega + \delta O_1^1.$$

So, then,

$$\begin{aligned} \text{osc}_{B_{\frac{1}{2}}(0)} u_1^\epsilon &\leq \text{osc}_{B_{\frac{1}{2}}(0)}\omega + \delta O_1^1 \leq \mu \, \text{osc}_{x\in B_1(0)}\omega(x) + \delta O_1^1 \\ &\leq \mu \left(\text{osc}_{B_1(0)} u_1^\epsilon + \delta O_1^1\right) + \delta O_1^1. \end{aligned}$$

Simplifying,

$$\text{osc}_{B_{\frac{1}{2}}(0)} u_1^\epsilon \leq (\mu (1+\delta) + \delta) \, O_1^1,$$

which conclude the proof by taking δ sufficiently small. $\square$

4.2 Uniform in ϵ Hölder regularity: proof of Theorem 4.2.1

In this section we finally present the proof of the regularity Theorem

Theorem 4.2.1 (Regularity of solutions). *Let ϵ and ϕ_i be as in Theorem 3.3.1. Let $\boldsymbol{u}^\epsilon = (u_1^\epsilon, \cdots, u_d^\epsilon)$ be solutions of Problem (1.7) in $B_1(0)$. Then there exist a constant α, $0 < \alpha < 1$, such that for any ϵ, $\boldsymbol{u}^\epsilon \in \left(C^\alpha\left(B_1(0)\right)\right)^d$ and*

$$\|\boldsymbol{u}^\epsilon\|_{\left(C^\alpha\left(B_{\frac{1}{2}}\right)\right)^d} \leq C(N),$$

with $N = \sup_j \left\|u_j^\epsilon\right\|_{L^\infty(B_1(0))}$ and $C(N)$ independent of ϵ.

Proof. We prove this theorem iteratively. We will prove that the oscillation of $\mathbf{u}^\epsilon$ will decay, by some constant factor $\tilde{\mu} < 1$, independent of ϵ when it goes from $B_1(0)$ to $B_\lambda(0)$ for some $\lambda < 1$ also independent of ϵ. Meaning that, we will prove that there exist two constants $0 < \lambda, \tilde{\mu} < 1$, independent of ϵ, such that for all $i = 1, \ldots, d$, we have

$$\operatorname{osc}_{x \in B_\lambda(0)} u_i^\epsilon(x) \leq \tilde{\mu} \operatorname{osc}_{x \in B_1(0)} u_i^\epsilon(x).$$

The C^α regularity of each function will follow from here in a standard way using Lemma 8.23, in [23]. Since, what matters is the ratio between oscillations

$$\frac{\operatorname{osc}_{B_\lambda(0)} u_i^\epsilon(x)}{\operatorname{osc}_{B_1(0)} u_i^\epsilon(x)} \leq \tilde{\mu},$$

the result will hold true if we prove this decay for the normalized functions, which satisfy the same equation with a different value of ϵ. Thus, consider u_1^ϵ, $u_2^\epsilon, \cdots, u_d^\epsilon$ solutions of Problem (1.7) on $B_1(0)$ and the renormalized functions $\overline{u_1^\epsilon}, \cdots, \overline{u_d^\epsilon}$,

$$\overline{u_i^\epsilon}(x) = \rho\, u_i^\epsilon\left(x\right), \quad x \in B_1(0), \qquad i = 1, \cdots, d,$$

with $\rho = \frac{1}{max_{x\in B_1(0),k=1,\dots,d}\, u_k^\epsilon(x)}$. These functions are bounded from above by one and satisfy

$$\mathcal{M}^-(\overline{u_i^\epsilon}(x)) = \underbrace{\frac{1}{\rho\,\epsilon}}_{\overline{\epsilon}}\,\overline{u_i^\epsilon}(x)\sum_{i\neq l}\overline{u_l^\epsilon}(x), \qquad i = 1,\cdots,d.$$

Briefly, the iterative process consists of the following. We prove that in $B_{\frac{1}{4}}(0)$ at least the largest oscillation decays. Without loss of generality consider $\overline{u_1^\epsilon}$ the function with the largest oscillation. Then, there exists $\overline{\mu} < 1$ such that

$$\mathrm{osc}_{x\in B_{\frac{1}{4}}(0)}\overline{u_1^\epsilon}(x) \leq \overline{\mu}\,\mathrm{osc}_{x\in B_1(0)}\overline{u_1^\epsilon}(x).$$

Then, we consider the renormalization by the dilation in x :

$$\overline{\overline{u_i^\epsilon}}(x) = \overline{\rho}\,\overline{u_i^\epsilon}\left(\frac{1}{4}x\right), \quad x \in B_1(0), \qquad i = 1,\cdots,d,$$

with

$$\overline{\rho} = \frac{1}{\max_{x\in B_{\frac{1}{4}}(0),k=1,\dots,d}\ \overline{u_k^\epsilon}(x)} > 1.$$

Observe that these functions are solutions of the system

$$\mathcal{M}^-\left(\overline{\overline{u_i^\epsilon}}(x)\right) = \underbrace{\frac{1}{\overline{\rho}\,4^2\,\overline{\epsilon}}}_{\overline{\overline{\epsilon}}}\overline{\overline{u_i^\epsilon}}(x)\sum_{i\neq l}\overline{\overline{u_l^\epsilon}}(x), \qquad i = 1,\cdots,d.$$

So, basically, they are the solutions of an equivalent system with a different ϵ, still defined on $B_1(0)$.

We start all over to prove that we have again the reduction of the next largest oscillation, when we are in $B_{\frac{1}{4}}(0)$. We call the new function with the

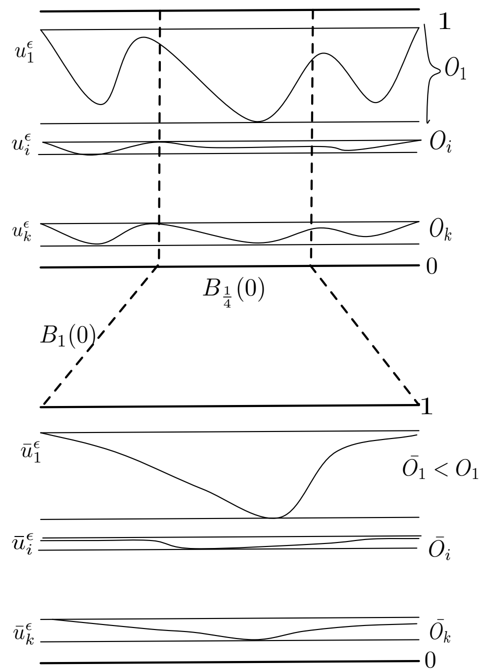

Figure 4.2: Decay iteration in Theorem 4.2.1. After the renormalization the oscillation of the first function decays while the others remain the same. In the original configuration we register that decay and we proceed with the next renormalization.

largest oscillation $\overline{\overline{u_i^\epsilon}}$. So we have that there exists $\overline{\overline{\mu}} < 1$, independent of ϵ, such that

$$\operatorname{osc}_{x\in B_{\frac{1}{4}}(0)}\overline{\overline{u_i^\epsilon}}(x) \leq \overline{\overline{\mu}}\operatorname{osc}_{x\in B_1(0)}\overline{\overline{u_i^\epsilon}}(x) \Rightarrow \operatorname{osc}_{x\in B_{\frac{1}{4^2}}(0)}\overline{u_i^\epsilon}(x) \leq \overline{\overline{\mu}}\operatorname{osc}_{x\in B_{\frac{1}{4}}(0)}\overline{u_i^\epsilon}(x).$$

Considering what has happened in the previous step we can have several scenarios. Either we have the reduction of the oscillation from $B_1(0)$ to $B_{\frac{1}{4^2}}(0)$ of just one of the functions, or two or more. (Later, when there is no possible confusion, the function with the largest oscillation at each iteration will always be denoted by $\overline{\overline{u_1^\epsilon}}$).

We repeat this process taking the renormalizations

$$\overline{\overline{u_i^\epsilon}}(x) = \frac{1}{max_{z\in B_1(0),j}\, \overline{u_j^\epsilon}\left(\frac{1}{4^k}z\right)}\,\overline{u_i^\epsilon}\left(\frac{1}{4^k}x\right), \quad x \in B_1(0), \qquad i = 1, \cdots, d,$$

until eventually we have the reduction of all the oscillations, or until the largest oscillation is much larger then the other oscillations. If that is the case, then eventually we will have that, for $O_j = \mathrm{osc}_{B_1}\overline{\overline{u_j^\epsilon}}(x)$,

$$\sum_{j\neq 1} O_j \leq \delta O_1.$$

In that case, using the Lemma 4.1.2, O_1 must also decay and we have the result. Thus, after repeating this iterative process a finite number of times we obtain that for some $\lambda < 1$, $\tilde{\mu} < 1$ and every $i = 1, \ldots, d$,

$$\mathrm{osc}_{x\in B_\lambda(0)}\overline{u_i^\epsilon}(x) \leq \tilde{\mu}\, \mathrm{osc}_{x\in B_1(0)}\overline{u_i^\epsilon}(x).$$

For simplicity, we will still refer to the renormalized functions bounded by one, as u_i^ϵ, and also $\frac{1}{\rho\, 4^k\, \overline{\epsilon}}$ will still be denoted by ϵ in each new step. Although, ϵ can be bigger than the one in the first steps, depending on the number of steps needed, it will eventually remain smaller than one since the renormalization after each dilation will be a multiplication by a factor smaller than one.

In what follows $u_{j_0}^\epsilon$ denotes at each renormalization, the function that achieves the maximum value 1 and u_1^ϵ is the function that has maximum oscillation. Naturally, they can be or not the same function.

Below follows the proof of decay for the renormalized functions in all possible cases.

Case 1: Let us assume that $\frac{1}{\epsilon} > 1$.

We are going to use the following argument: observe that, if there exists k such that:

$$\left|\{x \in B_{\frac{1}{4}} : u_k^\epsilon(x) \geq \gamma_0\}\right| \geq \gamma_0,$$

then since for any $j \neq k$

$$\mathcal{M}^-(u_j^\epsilon(x)) = \frac{1}{\epsilon} u_j^\epsilon(x)(u_k^\epsilon + \cdots)$$

we have,

$$\left|\left\{x \in B_{\frac{1}{4}} : \mathcal{M}^-(u_j^\epsilon) \geq \gamma_0 u_j^\epsilon\right\}\right| \geq \gamma_0.$$

And so by (3) in Lemma 4.1.1 we can conclude that O_j, for all $j \neq k$ decays. Let $0 < \gamma < \frac{1}{4}$ be a fixed constant.

1. If $\max_x u_1^\epsilon \geq \gamma$ and O_1 does not decay then by (1) in Lemma 4.1.1 we can conclude that there exists γ_0 a positive small constant such that

$$\left|\left\{x \in B_{\frac{1}{4}} : u_1^\epsilon(x) \geq \max_x u_1^\epsilon - \gamma_0 O_1 \geq \frac{\gamma}{2}\right\}\right| \geq 1 - \gamma_0 \geq \frac{1}{2},$$

   so then as we observed before, we have for all $j \neq 1$ that,

$$\left|\left\{x \in B_{\frac{1}{4}} : \mathcal{M}^-(u_j^\epsilon) \geq \frac{\gamma}{2} u_j^\epsilon\right\}\right| \geq \frac{\gamma}{2}.$$

   And so by (3) in Lemma 4.1.1 we can conclude that O_j, for all $j \neq 1$ decays.

2. If $\max_x u_1^\epsilon < \gamma$ then, since $O_1 \leq \max_x u_1^\epsilon < \gamma$, all oscillations are smaller than γ (see Figure 4.3).

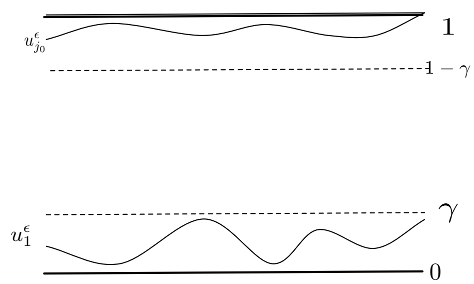

Figure 4.3: All oscillations are smaller than γ.

Then in particular the function $u^\epsilon_{j_0}$ has oscillation smaller than γ. Thus,

$$u^\epsilon_{j_0}(x) \geq 1-\gamma$$

and so for all $j \neq j_0$, we have for all $j \neq j_0$,

$$\mathcal{M}^-(u^\epsilon_j) \geq (1-\gamma)u^\epsilon_j.$$

So again by (3) in Lemma 4.1.1 we can conclude that for all $j \neq j_0$, O_j decays. In particular, the largest oscillation decayed.

Case 2: Let us assume that $\frac{1}{\epsilon} < 1$. Let $\theta = \frac{1}{\epsilon}$.

Observe that, since all the functions are bounded from above by one and are positive, we have that for all $i \neq i_0$, and i_0 an arbitrarily fixed indix,

$$\theta u^\epsilon_{i_0} u^\epsilon_i \leq \mathcal{M}^-(u^\epsilon_i) \leq \theta u^\epsilon_i (d-1). \tag{4.2}$$

In a more general point of view we have for any $i = 1, \ldots, d$,

$$0 \leq \mathcal{M}^-(u^\epsilon_i) \leq d, \tag{4.3}$$

We will use one or another expression according to convenience. There are two possible cases: either (1) $\frac{1}{4} \leq O_1 \leq 1$ or (2) $O_1 < \frac{1}{4}$. We prove that in the first case all the big oscillations decay. So after a finite number of steps, (and if all the functions did not decay yet in the mean time) we will have that all oscillations are less than $\frac{1}{4}$. In this case, since the function that attains the maximum will have also oscillation less than $\frac{1}{4}$, we can prove that all the other functions decay. In this way, we will eventually be again in the case of

$$\sum_{j \neq j_0} O_j \leq \delta O_{j_0}.$$

and as before we have the result. The proof of the decay in each case follows.

1. Assume that $\frac{1}{4} < O_1 \leq 1$. There exists an interior point $y \in B_{\frac{1}{4}}(0)$, such that

   $$\min u_1^\epsilon + \frac{1}{8} \leq u_1^\epsilon(y) \leq \max u_1^\epsilon - \frac{1}{8}.$$

   By (4.3) we conclude that the equation for u_1^ϵ has right hand side continuous and bounded. Then, by regularity, Proposition 2.2.5, we can conclude that there exists a universal constant C such that for all $x \in B_{\frac{1}{16\,C}}(y)$

   $$|u_1^\epsilon(x) - u_1^\epsilon(y)| \leq \|\nabla u_1^\epsilon\|_{L^\infty} \frac{1}{16\,C} \leq \|u_1^\epsilon\|_{C^{1,\alpha}} \frac{1}{16\,C} \leq \frac{1}{16}.$$

   And so, for all $x \in B_{\frac{1}{16\,C}}(y)$ we have that

   $$u_1^\epsilon(x) \leq \max u_1^\epsilon - \frac{1}{16}.$$

Then by (1) of Lemma 4.1.1, we can conclude that

$$\mathrm{osc}_{B_{\frac{1}{4}}(0)} u_1^\epsilon \leq \mu \, \mathrm{osc}_{B_1(0)} u_1^\epsilon \qquad \text{with} \quad \mu < 1.$$

With this argument we can conclude that all the functions with oscillations bigger than $\frac{1}{4}$ decay. Moreover, repeating the same argument we can conclude that all the functions with oscillations bigger than $\frac{1}{8}$ decay.

2. Assume that $O_1 \leq \frac{1}{4}$. Observe that in this case $u_{j_0}^\epsilon(x) \geq \frac{3}{4}$ for all $x \in B_1(0)$

   Then for all $i \neq j_0$ the function u_i^ϵ satisfies the equation

$$\frac{3}{4}\theta \, u_i^\epsilon \leq \mathcal{M}^-(u_i^\epsilon) \leq \theta \, u_i^\epsilon(d-1)$$

   Meaning that,

$$\mathcal{M}^-(u_i^\epsilon) \sim \theta \, u_i^\epsilon$$

   Consider v_i, $i \neq j_0$, the renormalized functions u_i^ϵ, by the renormalization

$$v_i(x) = \frac{1}{\max_{x \in B_1(0)} \; u_i^\epsilon(x)} \, u_i^\epsilon(x) \, .$$

   Observe that v_i has maximum 1 and satisfies in $B_1(0)$,

$$\theta \frac{3}{4} v_i \leq \mathcal{M}^-(v_i) \leq \theta v_i (d-1).$$

   For each function v_i, $i \neq j_0$, we can have two situations. Either (a) $v_i(x) \geq \frac{1}{2}$ for all x or (b) $v_i(x) < \frac{1}{2}$ for some x in $B_1(0)$. In both cases we will prove that v_i, $i \neq j_0$ decay in $B_{\frac{1}{4}}(0)$.

(a) If $v_i(x) \geq \frac{1}{2}$ for all $x \in B_1(0)$. Then

$$O_i = \text{osc}_{x \in B_1(0)} v_i(x) \leq \frac{1}{2}.$$

We claim that: there exists a universal constant N such that $\theta \leq NO_i$.

Assuming that the claim is true, observe that

$$0 \leq \mathcal{M}^-(v_i) \leq dNO_i,$$

which implies that we can consider the function ω with oscillation 1 in $B_1(0)$ defined by

$$\omega(x) = \frac{v_i(x) - \min_{z \in B_1(0)} v_i(z)}{O_i}.$$

and that satisfies in $B_1(0)$

$$0 < \mathcal{M}^-(\omega) \leq dN.$$

By regularity, Proposition 2.2.5, there exists a universal constant C depending on N and d such that,

$$|\omega(x) - \omega(y)| \leq \|\omega\|_{C^{1,\alpha}} |x - y| \leq C(N, d) |x - y|.$$

Assuming that ω didn't decay, let $y \in B_{\frac{1}{4}}(0)$ be such that $\omega(y) < \frac{1}{2}$. Then, if $\sigma = \min\left(\text{dist}(y, \partial B_{\frac{1}{4}}), \frac{1}{4C(N,d)}\right)$, for all $x \in B_\sigma(y)$

$$|\omega(x) - \omega(y)| \leq \frac{1}{4}$$

And so, for $x \in B_\sigma(y)$,

$$\omega(x) \leq \frac{1}{4} + \frac{1}{2} \leq \frac{3}{4}.$$

Then by (1) of Lemma 4.1.1, we can conclude that for all $i \neq j_0$, there exist $\mu < 1$

$$\text{osc}_{B_{\frac{1}{4}}(0)} v_i(x) \leq \mu \text{osc}_{B_1(0)} v_i(x) \Rightarrow \text{osc}_{B_{\frac{1}{4}}(0)} u_i^\epsilon(x) \leq \mu \text{osc}_{B_1(0)} u_i^\epsilon(x)(x).$$

Proof of the claim: consider by contradiction that $\theta > 2n\lambda\, 24\, O_i$. So, for all $x \in B_1(0)$,

$$\frac{3}{4} v_i(x)\theta > \frac{3}{8}\theta > 2n\lambda\, 9\, O_i.$$

Observe that, if $m = \min_{z \in B_1(0)} v_i(z)$, $m < t < m + \text{osc}_{B_1(0)} v_i(x)$ is a positive constant to be chosen later and

$$P(x) = 8\, O_i\, |x|^2 + t$$

we have that for any x, and any t,

$$\mathcal{M}^-(v_i) \geq \frac{3}{4} v_i(x)\theta > 2n\lambda\, 9\, O_i > \mathcal{M}^-(P(x)) = 2n\lambda\, 8\, O_i.$$

and so P can not touch v_i from above at any y, since it would contradict that the v_i is a subsolution. But since for $t = m$ and $|x| = 1$,

$$P(x) \geq m + 8O_i \quad \text{and} \quad P(0) = m,$$

P crosses the function v_i, and so it is possible to find t such that P would touch the function v_i from above at some point, which is a contradiction.

(b) If $v_i(y) < \frac{1}{2}$ for some interior point y of $B_{\frac{1}{4}}(0)$ we proceed as before and use regularity. If the function never attains a value less than $\frac{1}{2}$ in $B_{\frac{1}{4}}(0)$ then the function has decayed.

$\square$

With this uniform bound in the Banach space $C^{\alpha}_{,}$ one can conclude that these sequence converges uniformly (up to a subsequence) to a vector of functions $\mathbf{u}$.

We will prove in the next chapter that this limit vector function and its support are in fact the solutions of a free boundary problem.

Chapter 5

Characterization of limit problem: a free boundary problem

5.1 Free boundary problems. Monotonicity formula

Free boundary problems consist in problems where the function and part of the boundary of the support of the function (so called free boundary) are both unknowns.

Typically the problem consists in an equation that is satisfied in a subset of the domain, a boundary condition for the fixed boundary of the problem and a condition to be satisfied across the free boundary, or and equation that is satisfied in all the domain (as here).

In the study of the regularity of the solution of a free boundary problem two aspects are important: the regularity of the solution on the interior of its support and across the free boundary; and to characterize the geometry of the free boundary set itself.

For the similar linear problems, variational and non variational form (problem (1.4) in [7] and problem (1.2) in [4], respectively), the regularity results for the free boundary were very similar: that the set of interfaces $\{x : \mathbf{u}(x) = 0\}$ consists of two parts: a singular set of Hausdorff dimension

$n-2$ where three or more species may concur; and a family of level surfaces of $C^{2,\alpha}$ and harmonic functions, respectively.

For the nonlinear problem in this thesis we study the regularity of the solution across the free boundary. The regularity of the free boundary is still an open problem, since the existent tools, like the Almgren's monotonicity formula or similar frequency functions, work just in the linear setting.

The literature for free boundary problems is vast but one can find in the books [5, 26] essential tools and results about free boundary problems. We will just recall here the essential tool that we use in this thesis to prove the Lipchitz regularity of the solution across the free boundary: the Alt-Caffarelli-Friedman monotonicity formula introduced in [1].

Lemma 5.1.1. *Let u, v be two Hölder continuous functions defined on $B_1(0)$, nonnegative, subharmonic when positive and with disjoints supports. Assume that z_0 is an interior point and belongs to the intersection of the boundary of their support. Then, the following quantity is increasing with the radius ρ and uniformly bounded,*

$$\left(\frac{1}{\rho^2}\int_{B_\rho(z_0)}\frac{|\nabla u|^2}{|x-z_0|^{n-2}}\mathrm{dx}\right)\left(\frac{1}{\rho^2}\int_{B_\rho(z_0)}\frac{|\nabla v|^2}{|x-z_0|^{n-2}}\mathrm{dx}\right)\leq N,$$

since the norm of the densities is bounded, say by the fixed constant N;

$$C(n)\,\|(u,v)\|^4_{L^2(B_1(0))}\leq N.$$

For the proof see page 214 on [5] and the proof of Lemma 7 (a) in [4].

5.2 Limit problem

In this chapter we will assume without loss of generality that $\lambda = 1$. Observe that if $\mathbf{u}^\epsilon$ is a viscosity solution of Problem (1.7) then there exists a subsequence still indexed by ϵ and function $\mathbf{u} \in (C^\alpha)^d$ such that

$$\mathbf{u}^\epsilon \to \mathbf{u} \qquad \text{uniformly.}$$

We will prove that in the limit as $\epsilon \to 0$, this model forces the populations to segregate, meaning that in the limit the supports of the functions are disjoint and

$$\frac{u_i^\epsilon u_j^\epsilon}{\epsilon} \rightharpoonup \mu \quad \text{in the sense of measures, when } \epsilon \to 0.$$

The measure μ has support on the free boundary. Recall that the support of a measure μ is the complementary of the set

$$\{E : E \quad \text{the biggest open set such that} \quad \mu(E) = 0\}.$$

The following Lemma characterizes the Laplacian of the limit solution.

Lemma 5.2.1. *If $\boldsymbol{u} \in (C^\alpha)^d$ is the limit of a solution of (1.7) then Δu_i are positive measures.*

Proof. Let ϕ be positive a test function. Then

$$0 \leq \int \mathcal{M}^-(u_i^\epsilon)\phi \leq \int \Delta u_i^\epsilon \phi = \int \Delta\phi u_i^\epsilon \to \int \Delta\phi u_i = \int \phi\Delta u_i$$

and so Δu_i is a positive distribution which implies that it is a nonnegative Radon measure. □

Now we are ready to prove the following characterization Theorem:

Theorem 5.2.2 (Characterization of the limit problem)**.** *Let ϕ_i be as in Theorem 3.3.1. If*

$\boldsymbol{u} \in (C^\alpha)^d$ *is the limit of solutions of (1.7), then*

1. $\mathcal{M}^-\left(u_i - \sum_{k\neq j} u_k\right) \leq 0$;
2. $(\text{supp } u_i)^o \cap \left(\text{supp } (\sum_{k\neq i} u_k)\right)^o = \emptyset$ *for* $i = 1, \ldots d$;
3. $\mathcal{M}^-(u_i) = 0$, *when* $u_i(x) > 0$, *for* $x \in \Omega \quad i = 1, \ldots, d$;
4. $u_i(x) = \phi_i(x)$, *for* $x \in \partial\Omega, \quad i = 1, \ldots, d$.

Proof. (1) Observe that

$$\begin{aligned} \mathcal{M}^-\left(u_i^\epsilon - \sum_{k\neq i} u_k^\epsilon\right) &\leq \mathcal{M}^-(u_i^\epsilon) + \mathcal{M}^+\left(-\sum_{k\neq i} u_k^\epsilon\right) \\ &\leq \mathcal{M}^-(u_i^\epsilon) - \mathcal{M}^-\left(\sum_{k\neq i} u_k^\epsilon\right) \\ &\leq \mathcal{M}^-(u_i^\epsilon) - \sum_{k\neq i} \mathcal{M}^-(u_k^\epsilon) \leq 0. \end{aligned}$$

As $u_i^\epsilon - \sum_{k\neq i} u_k^\epsilon \to u_i - \sum_{k\neq i} u_k$ when $\epsilon \to 0$ uniformly and $\overline{S}(\lambda, \Lambda, 0)$ is closed under uniform convergence, $u_i - \sum_{k\neq i} u_k$ is a supersolution of $\mathcal{M}^-$:

$$\mathcal{M}^-\left(u_i - \sum_{k\neq i} u_k\right) \leq 0$$

(2) If $u_i(x_0) = \alpha_0 > 0$ for any i, $i = 1, \ldots, d$ then for $\delta < \frac{\alpha_0}{2}$ there exists an ϵ_0 such that for $\epsilon < \epsilon_0$, $\frac{\alpha_0}{2} < \alpha_0 - \delta < u_i^\epsilon(x_0) < \alpha_0 + \delta < \frac{3\alpha_0}{2}$. So by Hölder continuity there exist $h > 0$ such that:

a) $|u_i^\epsilon(y) - u_i^\epsilon(x_0)| \leq \frac{\alpha_o}{4}$ in $B_{2h}(x_0)$;

b) $u_i^\epsilon(y) > \frac{\alpha_0}{4}$ in a ball of radius $2h$ and center x_0.

Observe that applying Green's Identity with a function u_i^ϵ and the fundamental solution

$$\tilde{\Gamma}(x_0) = \frac{1}{n\omega_n(2-n)}\left(\frac{1}{|x-x_0|^{n-2}} - \frac{1}{|2\,h|^{n-2}}\right),$$

we obtain the inequality (see Lemma 2.1.1)

$$h^2 \fint_{B_h(x_0)} \Delta u_i^\epsilon \,\mathrm{dx} \leq C \fint_{\partial B_{2h}(x_0)} (u_i^\epsilon(x) - u_i^\epsilon(x_0))\,\mathrm{dS},$$

where C is a constant just depending on n. From the equation for u_i^ϵ, we obtain

$$\begin{aligned}
\fint_{B_h(x_0)} \frac{\alpha_0 \sum_{k\neq i} u_k^\epsilon}{4\epsilon}\mathrm{dx} &\leq \fint_{B_h(x_0)} \frac{u_i^\epsilon \sum_{k\neq i} u_k^\epsilon}{\epsilon}\mathrm{dx} = \fint_{B_h(x_0)} \mathcal{M}^-(u_i^\epsilon)\mathrm{dx} \\
&\leq \fint_{B_h(x_0)} \Delta u_i^\epsilon \mathrm{dx} \leq \frac{C}{h^2} \fint_{\partial B_{2h}(x_o)} (u_i^\epsilon(y) - u_i^\epsilon(x_o))\,\mathrm{dS} \\
&\leq \frac{C\,\alpha_0}{4\,h^2}.
\end{aligned}$$

Thus,

$$\fint_{B_h(x_o)} \frac{\alpha_0 \sum_{k\neq i} u_k^\epsilon}{4}\mathrm{dx} \leq \frac{\epsilon\, C\,\alpha_0}{4\,h^2},$$

which implies that,

$$\fint_{B_h(x_o)} \sum_{k\neq i} u_k^\epsilon \to 0$$

when $\epsilon \to 0$. By subharmonicity,

$$\sum_{k \neq i} u_k^\epsilon(x_0) \leq \fint_{B_h(x_0)} \sum_{k \neq i} u_k^\epsilon \to 0$$

and so $\sum_{k \neq i} u_k(x_0) = 0$ and this proves the result.

(3) To prove that $\mathcal{M}^-(u_i) = 0$, when $u_i > 0$ assume the set up in the beginning of (2) for $u_i(x_0)$. We need to prove that $\frac{u_i^\epsilon \sum_{k \neq i} u_k^\epsilon}{\epsilon} \to 0$ uniformly when $\epsilon \to 0$ in order to use the closedness of $S(\lambda, \Lambda)$. Since,

$$\Delta \left(\frac{\sum_{k \neq i} u_k^\epsilon}{\epsilon} \right) \geq \mathcal{M}^- \left(\frac{\sum_{k \neq i} u_k^\epsilon}{\epsilon} \right) \geq \frac{1}{\epsilon} \sum_{k \neq i} \mathcal{M}^- u_k^\epsilon \geq 0,$$

$\frac{\sum_{k \neq i} u_k^\epsilon}{\epsilon}$ is subharmonic. So, if we prove that $\frac{\sum_{k \neq i} u_k^\epsilon}{\epsilon} \to 0$ in $L^1(B_h(x_0))$, then for $y \in \overline{B}_{h-\delta'}(x_o)$,

$$\frac{\sum_{k \neq i} u_k^\epsilon(y)}{\epsilon} \leq \fint_{B_{\delta'}(y)} \frac{\sum_{k \neq i} u_k^\epsilon}{\epsilon} \mathrm{dx} \to 0,$$

and so $\frac{\sum_{k \neq i} u_k^\epsilon}{\epsilon} \to 0$ convergences uniformly in a compact set contained in $B_h(x_0)$. Recall that we proved in (2) that $\sum_{k \neq i} u_k^\epsilon \to 0$ uniformly in a compact set contained in $B_h(x_0)$ and we have that

$$\sum_{k \neq i} u_k^\epsilon \to 0 \Rightarrow \Delta \left(\sum_{k \neq i} u_k^\epsilon \right) \to 0 \text{ in the sense of distributions.}$$

So, since

$$\begin{aligned} \frac{\alpha_0}{4} \frac{\sum_{k \neq i} u_k^\epsilon}{\epsilon} &\leq \frac{u_i^\epsilon \sum_{k \neq i} u_k^\epsilon}{\epsilon} = \mathcal{M}^-(u_i^\epsilon) \leq \sum_{k \neq i} \mathcal{M}^- u_k^\epsilon \\ &\leq \mathcal{M}^- \left(\sum_{k \neq i} u_k^\epsilon \right) \leq \Delta \left(\sum_{k \neq i} u_k^\epsilon \right), \end{aligned}$$

we conclude that

$$\frac{\sum_{k\neq i} u_k^\epsilon}{\epsilon} \to 0 \ \text{ in } L^1.$$

Then, as we said, we have that

$$\frac{\sum_{k\neq i} u_k^\epsilon}{\epsilon} \to 0 \text{ uniformly in a compact set contained in } B_h(x_0).$$

As $u_i^\epsilon \to u_i$ uniformly and are bounded, we finally conclude that

$$u_i^\epsilon \frac{\sum_{k\neq i} u_k^\epsilon}{\epsilon} \to 0 \text{ uniformly in a compact set contained in } B_h(x_0).$$

Proceeding analogously with u_k, $k = 1, \cdots, n$ we conclude that the limit problem is

$$\mathcal{M}^-(u_i) = 0, \qquad u_i > 0 \qquad i = 1, \ldots, d.$$

(4) To prove the last statement we will construct an upper and lower barrier.

(a) Consider as upper barriers the solutions of the d problems, $(i = 1, \ldots, d)$:

$$\mathcal{M}^-(u_i^\star) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad u_i^\star(x) = \phi_i(x)\chi_{\{\phi_i(x)\neq 0\}} \quad \text{in } \partial\Omega$$

So we have that for all $i = 1, \ldots, d$,

$$\mathcal{M}^-(u_i^\star) = 0 \leq \frac{1}{\epsilon} u_i^\epsilon \sum_{k\neq i} u_k^\epsilon = \mathcal{M}^-(u_i^\epsilon) \quad \text{in } \Omega$$

and

$$u_i^\star(x) = u_i^\epsilon(x) \quad \text{for all} \quad x \in \partial\Omega.$$

So by the comparison principle, for all i and for all ϵ we have the upper bound

$$u_i^\star(x) \geq u_i^\epsilon(x) \quad \text{for all} \quad x \in \overline{\Omega}.$$

Taking limits in ϵ we can deduce that for all i

$$\phi_i(x) \geq u_i(x) \quad \text{for all} \quad x \in \partial\Omega.$$

(b) Now consider as a lower barrier the solution of the problem:

$$\mathcal{M}^-(\omega_i) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \omega_i(x) = \phi_i(x) - \sum_{j \neq i} \phi_j(x) \quad \text{in } \partial\Omega$$

So we have that for all $i = 1, \ldots, d$,

$$\mathcal{M}^-(u_i^\epsilon - \sum_{i \neq j} u_j^\epsilon) \leq 0 = \mathcal{M}^-(\omega_i) \quad \text{in } \Omega$$

and

$$u_i^\epsilon(x) - \sum_{i \neq j} u_j^\epsilon(x) = \omega_i(x) \quad \text{for all} \quad x \in \partial\Omega.$$

So by the comparison principle, for all i and for all ϵ we have the lower bound

$$u_i^\epsilon(x) - \sum_{i \neq j} u_j^\epsilon(x) \geq \omega_i(x) \quad \text{for all} \quad x \in \overline{\Omega}.$$

Taking the limit in ϵ we can deduce that for all i

$$u_i(x) - \sum_{i \neq j} \phi_j(x) \geq \phi_i(x) - \sum_{j \neq i} \phi_j(x) \quad \text{for all} \quad x \in \partial\Omega.$$

Since by hypothesis we know that ϕ_i have disjoint supports, when $\phi_i(x) \neq 0$,

$$u_i(x) \geq \phi_i(x),$$

and this proves the statement. □

Chapter 6

Lipschitz regularity for the free boundary problem

The regularity theorem of this chapter is the main result relating the growth of one of the functions in terms of the distance of the function to the free boundary. In the proof we will need to use barriers, properties of subharmonic functions (see Chapter 2) and the monotonicity formula introduced in [1] (see Chapter 5).

We start by presenting the linear decay to the boundary for subsolutions that are bounded and take the value zero in part of the boundary of the support. This fact is going to play a crucial role in the main proof and it uses the construction of fundamental barriers that was presented in Chapter 2. In the main proof we will also use the study of the L^∞ decay for subharmonic functions supported in a small domains that was presented in the next section.

6.1 Linear decay to the boundary

Lemma 6.1.1 (Linear decay to the boundary normalized). *Let v be a non-negative continuous function defined in $\Omega = B_\sigma(z_0)\backslash B_1(0)$, where $\sigma \leq \frac{1}{2}$ and z_0 is, without loss of generality, a point on $\partial B_1(0)$ with $\frac{z_0}{|z_0|} = e_n$, e_n the unit* *vector. Assume that,*

1. $\mathcal{M}^+(v) \geq 0$ *in* Ω,

2. $v(x) \leq U\sigma$ *in* Ω,

3. $v(x) = 0$ *on* $\partial B_1(0)$.

then, there exists a universal constant $\tilde{C}$, $\tilde{C} = \frac{8}{5} \frac{\alpha}{\frac{1}{5} - \left(\frac{1}{5}\right)^{\alpha+1}}$, *such that,*

$$v(x) \leq \tilde{C} U \mathrm{dist}(x, \partial B_1),$$

when $x \in S_\sigma$ *with* $S_\sigma := \left(B_{1+\frac{\sigma}{4}}(0) \backslash B_1(0)\right) \cap \{x = (x', x_n) : |x' - z_0'| < \frac{\sigma}{2}\}$. *In particular,*

$$v(x) \leq \tilde{C}\, U \mathrm{dist}(x, \partial B_1),$$

when $x \in B_{\frac{\sigma}{4}}(z_0)$.

Proof. Consider the function v extended by zero to all $B_\sigma(z_0)$. Observe that the extension still satisfies the hypotheses. We will take a barrier ϕ as in Lemma 2.3.2 with $r = \frac{5\sigma}{8}$, $\frac{a}{b} = \frac{1}{5}$ and $M = U\frac{8}{5}$, that will be used as a model and will be sliding tangentially along $\partial B_1(0)$ in order to construct a wall of barriers (see Figure 6.1). With that purpose, take a family of balls $\{B_{\frac{\sigma}{8}}(y)\}_y$ such that $y \in B_{\frac{3\sigma}{8}}(z_0) \cap \partial B_{1-\frac{\sigma}{8}}(0)$. So the balls $B_{\frac{\sigma}{8}}(y)$ are tangent to $\partial B_1(0)$ and are inside $B_1(0)$, where v is zero. For each ball consider ϕ is such that:

(a) $\phi(x) = U\sigma$ for $x \in \partial B_{\frac{5\sigma}{8}}(y)$;

(b) $\phi(x) = 0$ for $x \in \partial B_{\frac{\sigma}{8}}(y)$;

(c) $\mathcal{M}^+(\phi) \leq 0$ in $B_{\frac{5\sigma}{8}}(y) \backslash B_{\frac{\sigma}{16}}(y)$;

(d) $\frac{\partial \phi}{\partial \nu}(x) = U \frac{8}{5} \frac{\alpha}{\frac{1}{5} - \left(\frac{1}{5}\right)^{\alpha+1}}$ when $|x - y| = \frac{\sigma}{8}$.

Note that $\cup_y B_{\frac{5\sigma}{8}}(y) \subset B_\sigma(z_0)$ and so for all y defined previously and for $x \in \partial \left(\cup_y B_{\frac{5\sigma}{8}}(y) \right)$ we know by hypothesis that

$$v(x) \leq U\sigma.$$

We now apply the comparison principle for each barrier depending on y and respective ring $B_{\frac{5\sigma}{8}}(y) \backslash B_{\frac{\sigma}{8}}(y)$, since v is a subsolution and ϕ a supersolution for $\mathcal{M}^+$, and we obtain that

$$\phi(x) \geq v(x), \quad x \in \partial B_{\frac{5\sigma}{8}}(y) \cup \partial B_{\frac{\sigma}{8}}(y) \Rightarrow \phi(x) \geq v(x), \quad x \in B_{\frac{5\sigma}{8}}(y) \backslash B_{\frac{\sigma}{8}}(y).$$

Hence, repeating this for all y we obtain that

$$v(x) \leq \phi(x),$$

for all $x \in S_\sigma = \left(B_{1+\frac{\sigma}{4}}(0) \backslash B_1(0) \right) \cap \{x = (x', x_n) : |x' - z_0'| < \frac{\sigma}{2}\}$. Taking in account that ϕ is radially concave, we also obtain that,

$$v(x) \leq U \frac{8}{5} \frac{\alpha}{\frac{1}{5} - \left(\frac{1}{5}\right)^{\alpha+1}} \operatorname{dist}(x, \partial B_1(0)). \tag{6.1}$$

For the final remark, observe that $B_{\frac{\sigma}{4}}(z_0) \subset S_\sigma$. □

Corollary 6.1.2 (Linear decay to the boundary)**.** *Let v be a non-negative continuous function defined in $\Omega_{t_0} = B_{\tilde{\sigma}}(t_0 z_0) \backslash B_{t_0}(0)$, where $\tilde{\sigma} \leq \frac{t_0}{2}$ and $t_0 z_0$ is, without loss of generality, a point on $\partial B_{t_0}(0)$ with $\frac{z_0}{|z_0|} = e_n$, e_n the unit vector. Assume that,*

1. $\mathcal{M}^+(v) \geq 0$ *in* Ω_{t_0},

2. $v(x) \leq \tilde{U}\tilde{\sigma}$ *in* Ω_{t_0},

3. $v(x) = 0$ *on* $\partial B_{t_0}(0)$.

then, there exists a universal constant $\tilde{C}$, $\tilde{C} = \frac{8}{5}\frac{\alpha}{\frac{1}{5}-\left(\frac{1}{5}\right)^{\alpha+1}}$, *such that,*

$$v(x) \leq \tilde{C}\tilde{U}\mathrm{dist}(x, \partial B_{t_0}),$$

when $x \in S_{\tilde{\sigma}}$ *with* $S_{\tilde{\sigma}} := \left(B_{t_0+\frac{\tilde{\sigma}}{4}}(0)\backslash B_{t_0}(0)\right) \cap \{x = (x', x_n) : |x' - t_0 z_0'| < \frac{\tilde{\sigma}}{2}\}$. *In particular,*

$$v(x) \leq \tilde{C}\,\tilde{U}\mathrm{dist}(x, \partial B_{t_0}),$$

when $x \in B_{\frac{\tilde{\sigma}}{4}}(t_0 z_0)$.

Proof. Consider the function

$$u(x) = \frac{1}{\tilde{U}t_0} v(x\, t_0)$$

defined for $x \in \Omega = B_\sigma(z_0)\backslash B_1(0)$, with $\sigma = \frac{\tilde{\sigma}}{t_0}$. Observe that, u satisfies the hypotheses of Lemma 6.1.1 with $U = 1$ and notice that $\sigma \leq \frac{1}{2}$. Then,

$$u(x) \leq \tilde{C}\,\mathrm{dist}(x, \partial B_1),$$

when $x \in S_\sigma$. Substituting, we obtain

$$\frac{1}{\tilde{U}t_0} v(x\, t_0) \leq \frac{\tilde{C}}{t_0}\mathrm{dist}(t_0 x, \partial B_{t_0}) \quad \Leftrightarrow \quad v(y) \leq \tilde{C}\, U \mathrm{dist}(y, \partial B_{t_0}),$$

for $y \in S_{\tilde{\sigma}}$. □

6.2 Lipschitz regularity of the solutions: proof of Theorem 6.2.1

We finally present the proof of the main Theorem:

Theorem 6.2.1 (Lipschitz regularity for the free boundary problem)**.** *If* $\boldsymbol{u} \in (C^\alpha(B_1(0)))^d$ *is the limit of solutions of (1.7) in* $B_1(0)$, *and* x_0 *belongs to the set* $\partial\,(\operatorname{supp} u_1) \cap B_{\frac{1}{2}}(0)$, *then, without loss of generality, the growth of* u_1 *near the boundary of its support is controlled in a linear way and* u_1 *is Lipschitz. More precisely, there exist a universal constant* C *such that for any solution* $\boldsymbol{u}$, *for any point* x_0 *on the free boundary:*

1. $\sup_{B_R(x_0)} u_1 \leq C\,R$,
2. $\|u_1\|_{Lip(B_R(x_0))} \leq C$,

where $C = C(n, \|\boldsymbol{u}\|_{L^2(B_1)})$ *and* $R \leq \frac{1}{4}$.

Proof. (1) The proof is by contradiction. Let $x_0, z_0 \in \partial(\text{supp} u_1)$ be two interior points to be characterized later. We will find a smooth function η such that in $B_\delta(z_0) \subset B_R(x_0)$ touches the supersolution

$$u_1 - \sum_{k\neq 1} u_k$$

from below at z_0, and simultaneously satisfies

$$\mathcal{M}^-(\eta(z_0)) > 0,$$

and this contradicts the definition of supersolution. In fact, we will find a positive universal constant C_0 such that if $u_1(y) = CR$ for some $y \in B_R(x_0)$ and $C > C_0$ then we have a contradiction.

To simplify the notation let $u = u_1$ and $v = \sum_{k \neq 1} u_k$. Let us assume that the free boundary intersects the ball centered at the origin, $\partial(\operatorname{supp} u) \cap B_{\frac{1}{2}}(0) \neq \emptyset$.

By contradiction, assume without loss of generality that u grows above any linear function in a ball centered at x_0; more precisely, assume that for any constant M' and

$$x_0 \in \partial(\operatorname{supp} u) \cap B_{\frac{1}{2}}(0),$$

there exists y such that

$$y \in B_R(x_0) \qquad \text{and} \quad u(y) = M'R,$$

where $R < \frac{1}{4}$. Also, we may assume that $R = 2d$ where $d = \operatorname{dist}(y, \partial \operatorname{supp} u) > 0$ (if not we can always pick another x_0) and so we have

$$y \in B_R(x_0) \qquad \text{and} \quad u(y) = M'2d = Md.$$

For a later purpose we fix $z_0 \in \partial B_d(y) \cap \partial(\operatorname{supp} u)$ (the closest point to y in the free boundary).

As $u \geq 0$ and $u \in \mathcal{S}^*(\lambda, \Lambda, 0)$ we can apply Harnack for viscosity solutions on $B_d(y)$,

$$\sup_{B_{\frac{d}{2}}(y)} u \leq c \inf_{B_{\frac{d}{2}}(y)} u$$

and conclude that for any $x \in B_{\frac{d}{2}}(y)$

$$\frac{1}{c}\, M\, d \le u(x) \le c\, M\, d \tag{6.2}$$

As observed in Chapter 2, if $\omega(z)$ defined on $B_d(y)$ is a solution to $\mathcal{M}^-(\omega) = 0$, $\overline{\omega}(x)$ defined in $B_1(0)$ is still a solution to $\mathcal{M}^-(\overline{\omega}) = 0$ with the direction e_n as we want.

So we will prove this theorem using translation, rotation, dilation, and rescaling arguments on $(u_1, u_2, \cdots, u_d)$. In order to simplify the notation the new functions will always be denoted by the same name.

Consider the functions u, v and $u - v$ satisfying

$$\mathcal{M}^-(u) \ge 0, \quad \mathcal{M}^-(v) \ge 0, \quad \mathcal{M}^-((u-v)(x)) \le 0,$$

(see 2 in Lemma 5.2.1), and defined on an appropriate domain by translation (2.2), rotation (2.3), dilation and rescaling (2.4) such that y is the new origin, the direction $\frac{z_0 - y}{|z_0 - y|}$ is now the direction e_n and d is now 1. We will still call z_0 the point in $\partial B_1(0) \cap \partial \text{supp} u$. By (6.2) we now have

$$\frac{1}{d}\frac{1}{C}\, M\, d \le u(x) \le \frac{1}{d} C\, M\, d, \qquad x \in B_{\frac{1}{2}}(0),$$

that is,

$$\frac{1}{C}\, M \le u(x) \le C\, M, \qquad x \in B_{\frac{1}{2}}(0). \tag{6.3}$$

The rest of the proof consists of the following steps:

First step: we will prove that for a certain $\overline{M}$, a positive large constant,

$$u(x) \ge \overline{M}\, \underbrace{d(x, \partial B_1(0))}_{1-|x|} \qquad x \in B_1(0) \backslash B_{\frac{1}{2}}(0).$$

Second step: we will prove that, for a small ρ,

$$v(x) \leq \frac{C}{M} \underbrace{d(x, \partial B_1(0))}_{|x|-1} \qquad x \in \mathrm{S}_{\frac{\rho}{2}}, \tag{6.4}$$

where $S_{\frac{\rho}{2}} := \Big(B_{1+\frac{\rho}{8}}(0) \backslash B_1(0)\Big) \cap \{(x', x_n) \in \mathbb{R}^n : |x' - z_0'| < \frac{\rho}{4}\}$ (see Figure 6.1).

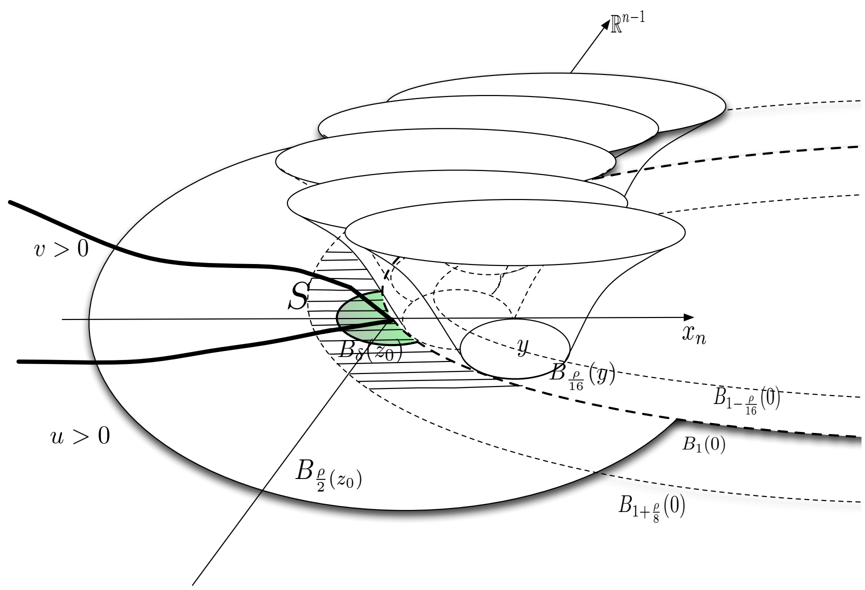

Figure 6.1: Barriers to control v. In the picture S is $S_{\frac{\rho}{2}}$ defined in Lemma 6.1.1.

Third step: Finally we will construct the function η (see Figure 6.2) to obtain the contradiction, since we will have for $\delta \leq \frac{\rho}{8}$:

- $\mathcal{M}^-(u - v) \leq 0$ for all $x \in B_\delta(z_0)$;

- $\eta(x) \leq (u - v)(x)$, for all $x \in B_\delta(z_0)$;

- $\eta(z_0) = (u - v)(z_0)$;

- η is a smooth function such that $\mathcal{M}^-(\eta(z_0)) > 0$.

and by definition of supersolution this is impossible and the result follows.

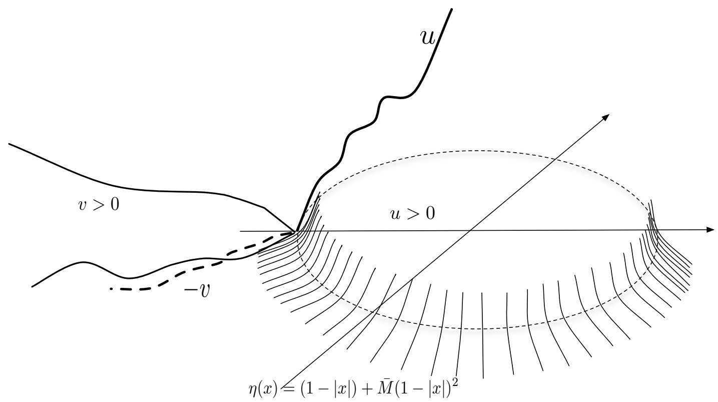

Figure 6.2: Barrier function that touches $u - v$ from below at z_0.

First step: By Lemma 2.3.1 ($r = 1, a = 1, b = 2$) there exist a subsolution ψ of $\mathcal{M}^-$ on the ring $B_1(0)\backslash B_{\frac{1}{2}}(0)$ such that:

(a) $\psi(x) = 0$ for $x \in \partial B_1(0)$;

(b) $\psi(x) = \frac{1}{C}M$ for $x \in \partial B_{\frac{1}{2}}(0)$;

(c) $\mathcal{M}^-(\psi) \geq 0$ for $x \in B_1(0)\backslash B_{\frac{1}{2}}(0)$;

(d) $\frac{\partial\psi(x)}{\partial\nu} = -\frac{\alpha}{2^\alpha - 1}\frac{M}{C}$ for $x \in \partial B_1(0)$, where ν is the outer normal direction.

Due to the comparison principle applied in the ring $B_1(0)\backslash B_{\frac{1}{2}}(0)$ and by (6.3) one can conclude that

$$u(x) \geq \psi(x), \quad x \in \partial B_1(0) \cup \partial B_{\frac{1}{2}}(0) \Rightarrow u(x) \geq \psi(x), \quad x \in B_1(0)\backslash B_{\frac{1}{2}}(0),$$

and also, since ψ is convex in the radial direction, and by (d),

$$u(x) \geq \underbrace{\frac{\alpha}{2^\alpha - 1} \frac{M}{C}}_{\overline{M}} \operatorname{dist}(x, \partial B_1(0)) \qquad x \in B_1(0) \backslash B_{\frac{1}{2}}(0). \tag{6.5}$$

<u>Second Step</u>: We do not know anything about the free boundary or the shape of the support of the other densities $v = \sum_{i \neq 1} u_i$. But we know that if z_0 is the point on the free boundary closest to 0 then by the monotonicity formula

$$\left(\frac{1}{\rho^2} \int_{B_\rho(z_0)} \frac{|\nabla u|^2}{|x - z_0|^{n-2}} \mathrm{dx} \right) \left(\frac{1}{\rho^2} \int_{B_\rho(z_0)} \frac{|\nabla v|^2}{|x - z_0|^{n-2}} \mathrm{dx} \right) \leq N$$

since the norm of the densities is bounded, say by the fixed constant N:

$$C(n) \left\| (u, v) \right\|^4_{L^2(B_1(0))} \leq N.$$

Our goal in this step is to prove that v has to grow very slowly below a linear function with small slope away from $\partial B_1(0) \cap B_\rho(z_0)$.

Let us consider the two possible cases around the point z_0:

<u>Second Step a)</u>: For a sequence of small radii, the measure of the intersection of the support of v with each ball is almost zero: there is a sequence of radii ρ_k, $\rho_k \to 0$, such that

$$\left| \{ v \neq 0 \} \cap B_{\rho_k}(z_0) \right| < \epsilon \left| B_{\rho_k}(z_0) \right|,$$

or,

Second Step b): the measure of the support of v contained in a ball with any radius is not small, say, if for any radius $\rho > 0$

$$|\{v \neq 0\} \cap B_\rho(z_0)| > \epsilon \, |B_\rho(z_0)| \, .$$

The proof proceeds separately in each case but in both cases we will prove (6.4).

Proof of Second Step a): If there exists a sequence of radius $(\rho_k)_{k\in\mathbb{N}}$, $\rho_k \to 0$, such that

$$|\{v \neq 0\} \cap B_{\rho_k}(z_0)| < \epsilon \, |B_{\rho_k}(z_0)| \, ,$$

we can consider a subsequence of $(\rho_k)_k$ such that

$$...\rho_{k+1} < \frac{\rho_k}{2} \leq \rho_k < \frac{\rho_{k-1}}{2} \leq \rho_{k-1} \cdots \leq \rho_1 \leq 1.$$

Since v is bounded in all domain, we have that there exists $\tilde{N}_1$ such that

$$\sup_{x \in B_{\rho_1}(z_0)} v(x) \leq \tilde{N}_1 \rho_1.$$

Then, since v is subharmonic, by Proposition 2.1.3, we have that

$$\sup_{x \in B_{\frac{\rho_1}{2}}(z_0)} v(x) \leq \epsilon \, \tilde{N}_1 \rho_1 2^n.$$

Then, since

$$\mathcal{M}^+(v) \geq \mathcal{M}^-(v) \geq \sum_{i \neq 1} \mathcal{M}^-(u_i) = 0,$$

by Corollary 6.1.2 with $\tilde{U} = \epsilon\,\tilde{N}_1 2^{n+1}$, $\tilde{\sigma} = \frac{\rho_1}{2}$ and $t_0 = 1$, we obtain that there exists a universal constant $\tilde{C}$:

$$v(x) \leq \tilde{C}\,\epsilon\,\tilde{N}_1\,2^{n+1}\mathrm{dist}(x, \partial B_1),$$

when $x \in S_{\frac{\rho_1}{2}}$ with $S_{\frac{\rho_1}{2}} := \Big(B_{1+\frac{\rho_1}{8}}(0)\backslash B_1(0)\Big) \cap \{x = (x', x_n) : |x' - z_0'| < \frac{\rho_1}{4}\}$. Let $\epsilon := \epsilon_0$ be such that

$$\epsilon_0\,\tilde{C}\,2^{n+1} \leq \frac{1}{2}. \tag{6.6}$$

Then, for $x \in S_{\frac{\rho_1}{2}}$

$$v(x) \leq \frac{\tilde{N}_1}{2}\mathrm{dist}(x, \partial B_1).$$

Take as the next radii in the subsequence ρ_2 such that $B_{\rho_2}(z_0) \subset S_{\frac{\rho_1}{2}}$. Then,

$$\sup_{x \in B_{\rho_2}(z_0)} v(x) \leq \frac{\tilde{N}_1}{2}\rho_2.$$

So we can repeat the process, and so, again by Proposition 2.1.3 with $\tilde{N}_2 = \frac{\tilde{N}_1}{2}$, we have that

$$\sup_{x \in B_{\frac{\rho_2}{2}}(z_0)} v(x) \leq \epsilon_0 \frac{\tilde{N}_1}{2}\rho_2\,2^n.$$

Then, by Corollary 6.1.2, with $\tilde{U} = \epsilon_0\,\tilde{N}_1\,2^n$, $\tilde{\sigma} = \frac{\rho_2}{2}$ and $t_0 = 1$, and by (6.6),

$$v(x) \leq \tilde{C}\epsilon_0\,\tilde{N}_1\,2^n\mathrm{dist}(x, \partial B_1) \leq \frac{\tilde{N}_1}{4}\mathrm{dist}(x, \partial B_1),$$

when $x \in S_{\frac{\rho_2}{2}}$ with $S_{\frac{\rho_2}{2}} := \Big(B_{1+\frac{\rho_2}{8}}(0)\backslash B_1(0)\Big) \cap \{x = (x', x_n) : |x' - z_0'| < \frac{\rho_2}{4}\}$.

Again take as the next radii in the subsequence ρ_3 such that $B_{\rho_3}(z_0) \subset S_{\frac{\rho_2}{2}}$. Then,

$$\sup_{x \in B_{\rho_3}(z_0)} v(x) \leq \frac{\tilde{N}_1}{4} \rho_3.$$

Repeating a finite number of times this process, we have that, for $x \in S_{\frac{\rho_k}{2}}$

$$v(x) \leq \frac{\tilde{N}_1}{2^k} \operatorname{dist}(x, \partial B_1).$$

Therefore, it is possible to find ρ_l such that $\frac{\tilde{N}_1}{2^l} \leq C \frac{1}{\overline{\overline{M}}}$, and so

$$v(x) \leq C \frac{1}{\overline{\overline{M}}} \underbrace{d(x, \partial B_1(0))}_{|x|-1} \qquad x \in \mathrm{S}_{\frac{\rho_l}{2}}.$$

In particular,

$$v(x) \leq C \frac{1}{\overline{\overline{M}}} \underbrace{d(x, \partial B_1(0))}_{|x|-1} \qquad x \in \mathrm{B}_{\frac{\rho_l}{8}}(\mathrm{z}_0). \tag{6.7}$$

<u>Proof of Second Step b)</u>: If for any $\rho > 0$ as small as we want

$$|\{v \neq 0\} \cap B_\rho(z_0)| > \epsilon \, |B_\rho(z_0)| \, .$$

Since

$$|\{v \neq 0\} \cap B_\rho(z_0)| = |\{u = 0\} \cap B_\rho(z_0)| > \epsilon \, |B_\rho(z_0)| \, ,$$

we can apply Poincaré-Sobolev inequality (see page 79 in [24]) to the function u. Also, by construction

$$|\{u \neq 0\}| = |\{v = 0\}| \geq |B_\rho(z_0) \cap B_1(0)| > \epsilon \, |B_\rho(z_0)| \, ,$$

so we can also apply the Poincaré-Sobolev inequality to v. Hence,

$$\int_{B_\rho(z_0)} v^2(x)\mathrm{dx} \leq C(n,\epsilon)\rho^2 \int_{B_\rho(z_0)} |\nabla v(x)|^2 \mathrm{dx},$$

$$\int_{B_\rho(z_0)} u^2(x)\mathrm{dx} \leq C(n,\epsilon)\rho^2 \int_{B_\rho(z_0)} |\nabla u(x)|^2 \mathrm{dx}.$$

Then,

$$\int_{B_\rho(z_0)} v^2(x)\mathrm{dx} \leq C(n,\epsilon)\rho^2 \int_{B_\rho(z_0)} |\nabla v(x)|^2 \frac{\rho^{n-2}}{|x-z_0|^{n-2}}\mathrm{dx},$$

$$\int_{B_\rho(z_0)} u^2(x)\mathrm{dx} \leq C(n,\epsilon)\rho^2 \int_{B_\rho(z_0)} |\nabla u(x)|^2 \frac{\rho^{n-2}}{|x-z_0|^{n-2}}\mathrm{dx},$$

and

$$\left(\int_{B_\rho(z_0)} v^2(x)\mathrm{dx}\right)\left(\int_{B_\rho(z_0)} u^2(x)\mathrm{dx}\right) \leq C(n,\epsilon)\,\rho^{2n-4}\rho^8\,N.$$

But we also know that u is controlled from below by the barrier function from the First Step, so if $A = B_1(0)\backslash B_{\frac{1}{2}}(0)$ then

$$\int_{B_\rho(z_0)\cap A} \psi^2(x)\mathrm{dx} \leq \int_{B_\rho(z_0)\cap A} u^2(x)\mathrm{dx} \leq \int_{B_\rho(z_0)} u^2(x)\mathrm{dx}.$$

To estimate the integral of the barrier function observe that by Hölder inequality

$$\frac{1}{|B_\rho(z_0)\cap A|}\left(\int_{B_\rho(z_0)\cap A} \psi(x)\mathrm{dx}\right)^2 \leq \int_{B_\rho(z_0)\cap A} \psi^2(x)\mathrm{dx},$$

and that

$$\psi(x) \geq C(x) \quad \text{for all} \quad x \in B_\rho(z_0)\cap A,$$

where C is a cone with base $B_\rho(z_0)\cap A$, vertex at $(1-\rho)\frac{z_0}{|z_0|}$, and height equal to $\rho\overline{M}$ with $\bar{M}$ the absolute value of the slope of the barrier function at z_0, namely $\bar{M}=\frac{\alpha}{2^\alpha-1}\frac{M}{C}$, (see (6.5)). Note that $\bar{M}$ is arbitrarily big and so $\frac{1}{\bar{M}}$ will be arbitrarily small. So, if c_1 is a constant such that $|B_\rho(z_0)\cap A| = c_1\rho^n\omega_n$, then,

$$\frac{\overline{M}^2\rho^2 c_1\rho^n\omega_n}{n^2} = \frac{(vol(C))^2}{|B_\rho(z_0)\cap A|} \leq \int_{B_\rho(z_0)\cap A}\psi^2(x)\mathrm{dx}.$$

Therefore,

$$\left(\int_{B_\rho(z_0)} v^2(x)\mathrm{dx}\right)\left(\frac{c_1\overline{M}^2\rho^2\rho^n\omega_n}{n^2}\right) \leq C(n,\epsilon)\,\rho^{2n-4}\rho^8 N,$$

so

$$\left(\int_{B_\rho(z_0)} v^2(x)\mathrm{dx}\right) \leq \frac{C(n,\epsilon)\,\rho^{2n-4}\rho^8 N}{c_1\overline{M}^2\rho^2\rho^n\omega_n} = C(\alpha,n,c_1,\epsilon)\rho^{n+2}\frac{N}{\overline{M}^2}.$$

Let $y\in B_{\frac{7\rho}{8}}(z_0)$ such that $B_{\frac{\rho}{16}}(y)\subset B_\rho(z_0)$. Due to the subharmonicity and positivity of v

$$v(y) \leq \fint_{B_{\frac{\rho}{16}}(y)} v(x)\mathrm{dx} \leq \left(\frac{\rho}{\frac{\rho}{16}}\right)^n \fint_{B_\rho(z_0)} v(x)\mathrm{dx}.$$

On the other hand, the Hölder inequality yields,

$$\left(\fint_{B_\rho(z_0)} v(x)\mathrm{dx}\right)^2 \leq \fint_{B_\rho(z_0)} v(x)^2\mathrm{dx}.$$

All together, for any $y\in B_{\frac{7\rho}{8}}(z_0)$, we have,

$$v^2(y) \leq 16^{2n}\fint_{B_\rho(z_0)} v(x)^2\mathrm{dx} = 16^{2n}\frac{1}{\rho^n\omega_n}\int_{B_\rho(z_0)} v(x)^2\mathrm{dx},$$

and hence

$$v^2(y) \le \frac{16^{2n}}{\rho^n \omega_n} C(\alpha, n, c_1, \epsilon) \rho^{n+2} \frac{N}{\overline{\overline{M}}^2} = \underbrace{16^{2n}\, C(\alpha, n, c_1, \epsilon)}_{c_2^2} \rho^2 \frac{N}{\overline{\overline{M}}^2}.$$

We conclude that $v(y) \le c_2 \frac{\sqrt{N}}{\overline{\overline{M}}} \frac{7\rho}{8}$, for $y \in B_{\frac{7\rho}{8}}(z_0)$. Note that doing the same type of argument for $r = \rho t$, $0 < t < 1$ we can conclude that

$$v(y) \le c_2 \sqrt{N} \frac{1}{\overline{\overline{M}}} \left| y - z_0 \right|. \tag{6.8}$$

Since, v a subsolution of the positive extremal Pucci operator

$$\mathcal{M}^+(v) \ge \mathcal{M}^-(v) \ge \sum_{i \ne 1} \mathcal{M}^-(u_i) = 0,$$

and we have by (6.8) that

$$v(y) \le c_2 \sqrt{N} \frac{1}{\overline{\overline{M}}} \frac{\rho}{2}, \qquad x \in B_{\frac{\rho}{2}}(z_0),$$

by Corollary 6.1.2, with $\tilde{U} = c_2 \frac{\sqrt{N}}{\overline{\overline{M}}}$, $\tilde{\sigma} = \frac{\rho}{2}$, and $t_0 = 1$, we obtain that

$$v(y) \le \tilde{C} \frac{c_2 \sqrt{N}}{\overline{\overline{M}}} \operatorname{dist}(y, \partial B_1(0)), \quad \text{when} \quad y \in S_{\frac{\rho}{2}}.$$

As before, $S_{\frac{\rho}{2}}$ is the portion of an annulus around $B_1(0)$ contained in $B_{\frac{\rho}{2}}(z_0)$,

$$S_{\frac{\rho}{2}} = \Big(B_{1+\frac{\rho}{8}}(0) \backslash B_1(0) \Big) \cap \{ (x', x_n) \in \mathbb{R}^n : |x' - z_0'| < \frac{\rho}{4} \}.$$

More explicitly, we have that for $y \in B_{\frac{\rho}{8}}(z_0)$,

$$v(y) \le \underbrace{c_2 \sqrt{N} \frac{8}{5} \frac{\alpha}{\frac{1}{5} - \left(\frac{1}{5}\right)^{\alpha+1}}}_{C} \frac{1}{\overline{\overline{M}}} \underbrace{\operatorname{dist}(y, \partial B_1(0))}_{|y|-1}. \tag{6.9}$$

Third step: Consider ρ the radii in Second Step a). Consider ϵ to be ϵ_0 in Second Step a), and C in (6.7) to be

$$C = c_2\sqrt{N}\frac{8}{5}\frac{\alpha}{\frac{1}{5} - \left(\frac{1}{5}\right)^{\alpha+1}}.$$

Putting together (6.5), (6.7) and (6.9), if $A = B_1(0) \cap B_\delta(z_0)$ and $B = S \cap B_\delta(z_0)$

$$u(x) \geq \overline{M}(1 - |x|) \quad x \in A \qquad \text{and} \qquad -v(x) \geq -\frac{C}{\overline{M}}(|x| - 1) \quad x \in B.$$

So letting η be the radial symmetric smooth function defined by (see Figure 6.2)

$$\eta(x) = (1 - |x|) + \overline{M}(1 - |x|)^2,$$

we can conclude that

$$(u - v)(x) \geq \eta(x) \qquad x \in B_\delta(z_0)$$

and that

$$(u - v)(z_0) = \eta(z_0).$$

But notice that the Hessian of η is given by

$$H(\eta)(x) = \begin{bmatrix} \frac{-1+2\overline{M}(r-1)}{r} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1+2\overline{M}(r-1)}{r} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{-1+2\overline{M}(r-1)}{r} & 0 \\ 0 & 0 & \cdots & 0 & 2\overline{M} \end{bmatrix}$$

(where $r = |x|$) and so at z_0 because $r = 1$ we obtain

$$\mathcal{M}^-(\eta(z_0)) = \lambda 2\overline{M} - \Lambda(n-1) > 0,$$

since $\overline{M}$ is as big as we wish. But this contradicts the fact that $u - v$ is a supersolution of $\mathcal{M}^-$and this finishes the proof of (1).

(2) Arguing in a similar way as in Proposition 4.13, in [2], we should consider:

a) When $2|x-y| < h$, $h = \max(\text{dist}(x, \partial\text{supp}u_1), \text{dist}(y, \partial\text{supp}u_1))$, $y \in B_{\frac{h}{2}}(x)$ so by interior regularity (Proposition 2.2.5 with $f = 0$) for properly scaled balls gives that

$$|u_1(x) - u_1(y)| \leq C \|u_1\|_{L^\infty(B_1(0))} |x-y|;$$

b) When $2|x-y| \geq h$ note that for x in the support of u_1 and $\overline{x}$ is the closest point to x on the free boundary we have from the previous result that

$$|u_1(x) - u_1(\overline{x})| \leq |u_1(x)| \leq Ch.$$

And so, by adding and subtracting $u_1(\overline{x})$ and $u_1(\overline{y})$, where $\overline{y}$ is the closest point to y on the free boundary, and by triangular inequality,

$$\begin{aligned} \frac{|u_1(x) - u_1(y)|}{|x-y|} &\leq \frac{|u_1(x) - u_1(\overline{x})|}{|x-y|} + \frac{|u_1(\overline{x}) - u_1(\overline{y})|}{|x-y|} + \frac{|u_1(\overline{y}) - u_1(y)|}{|x-y|} \\ &\leq \frac{2|u_1(x)|}{h} + \frac{2|u_1(y)|}{h} \leq 4C. \end{aligned}$$

Thus,

$$\|u_1\|_{Lip(B_{\frac{1}{4}}(x_0))} = \|u_1\|_{C^{0,1}(B_{\frac{1}{4}}(x_0))} \leq \tilde{C},$$

and the result follows. □

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Vita

Veronica Rita Antunes de Soares Quitalo was born in Lisbon, Portugal. Until the college years she divided her time between studies and sport. Her favorite sport and the one to which she did dedicate more time was roller skating. She did her undergraduate studies and Master degree in Pure Mathematics at the Science Faculty of Lisbon University while she also learned Aikido. She taught mathematics to first-year college students until 2007. When she could pursue her graduate studies, she had the precious opportunity to come to Austin. The 5 years of her PhD here in Austin were the most challenging, grounding and joyful years in her life up to this time. For that she will be always grateful.

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