Positivity **8:** 209–213, 2004.

Resolution of the Symmetric Nonnegative Inverse Eigenvalue Problem for Matrices Subordinate to a Bipartite Graph*

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(Received 2 March 1999; accepted 27 November 2000)

Abstract. There is a symmetric nonnegative matrix A, subordinate to a given bipartite graph G on n vertices, with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ if and only if $\lambda_1 + \lambda_n \geqslant 0$, $\lambda_2 + \lambda_{n-1} \geqslant 0, \ldots, \lambda_m + \lambda_{n-m+1} \geqslant 0$, $\lambda_{m+1} \geqslant 0, \ldots, \lambda_{n-m} \geqslant 0$, in which m is the matching number of G. Other observations are also made about the symmetric nonnegative inverse eigenvalue problem with respect to a graph.

1. Nonnegative Inverse Eigenvalue Problems

The *nonnegative inverse eigenvalue problem* (NIEP) asks which collections of n complex numbers (repeats allowed) occur as the eigenvalues of an n-by-n, entry-wise nonnegative matrix. This problem has attracted considerable attention over 50+ years [7] and, despite many exciting partial results, remains quite unresolved. The companion *symmetric nonnegative inverse eigenvalue problem* (SNIEP) in which the realizing nonnegative matrix is required to be symmetric and the eigenvalues are (of course) real is also open and has also been the subject of attention e.g. [2, 6], etc. The intermediate *real nonnegative inverse eigenvalue problem* (RNIEP) asks which collections of n real numbers occur as the eigenvalues of an n-by-n nonnegative matrix and is now known to be a properly different problem from the SNIEP [6].

2. Graph Theoretic Versions

Mathematically, it is natural to consider graph theoretic versions (of the nonnegative inverse eigenvalue problems), in which a non-edge requires a 0 entry in the realizing matrix. Specifically, given a directed (undirected) graph G on n vertices, which may be taken to be $\{1, \ldots, n\}$, we say that an n-by-n matrix $A = (a_{ij})$ is *subordinate* to G if $a_{ij} \neq 0$, $i \neq j$, implies that (i, j) (resp. $\{(i, j)\}$) is an edge of

* This research was supported in part by Project 574/94 of Fundação Luso Americana para o Desenvolvimento, by CMUC/FCT, by Projecto PRAXIS 2/2.1/MAT/458/94 and by PRAXIS XXI/BCC/11955/97 grant.

G. (The diagonal entries of A are free.) The nonnegative inverse eigenvalue problem *relative to* G (G-NIEP) then just asks which collections of n complex numbers occur as the eigenvalues of a nonnegative matrix subordinate to the directed graph G. The G-SNIEP and G-RNIEP are defined analogously. For example in the G-SNIEP, G is undirected and the n numbers are real, while the realizing matrices are symmetric. Because, we only require that the realizing matrix be subordinate to G (rather than having graph exactly G), the solution to each of our problems is always a closed set. The alternative versions of our problems, in which the realizing matrix is required to have graph exactly G, are also of interest, but the topological nature of the solution set can be quite subtle. The union of the solution sets of such problems for graphs contained in G gives the solution to (one of) our problems for G.

3. Bipartite Graphs and the G-SNIEP

An undirected graph G is *bipartite* if its vertices may be partitioned into two sets (the 'parts') in such a way that all edges have a vertex in each part. The *complete bipartite graph* for a given such partition has all allowed edges. An important parameter of a (bipartite) graph is the matching number $m = m(G)$: The maximum number of vertex-disjoint edges of G. For a bipartite graph G, $m(G)$ is never more than the smaller of the cardinalities of its two parts, and equality is attained for complete (and other), bipartite graphs. All trees are bipartite, and it is a simple exercise to see that when G is a tree, the solutions to each of G-NIEP, G-SNIEP and G-RNIEP are the same. Thus, when we solve the G-SNIEP, we also solve the other two problems for trees. (Here, as usual and throughout, we identify as convenient, an undirected graph with the directed graph having two oppositely directed edges in place of each undirected edge.) The first instance of study of the G-SNIEP for particular G's (though not stated in these terms) seems to have been the content of [3] in which paths (tridiagonal matrices) were considered. Paths are trees, and a path on n vertices has matching number k if $n = 2k$ or $n = 2k + 1$, and the length of the path (even or odd) played a role in the observations of [3].

4. Resolution of the G-SNIEP for Bipartite Graphs

If A is an n-by-n symmetric nonnegative matrix, we list the necessarily real eigenvalues of A as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Of course, $\lambda_1 \geqslant 0$ (and $\lambda_1 + \lambda_n \geqslant 0$), but the form of our solution is to give a complete set of inequalities on the λ_i's given only the information that A is subordinate to the bipartite graph G. Without knowing the solution, it does not seem obvious that the solution should be describable via simple, even linear, inequalities in the λ_i's. However, we show that the solution may be presented as follows (for $m = m(G)$):

$$\left\{\begin{array}{r} \lambda_1 + \lambda_n \geqslant 0 \\ \lambda_2 + \lambda_{n-1} \geqslant 0 \\ \vdots \\ \lambda_m + \lambda_{n-m+1} \geqslant 0 \\ \lambda_{m+1} \geqslant 0 \\ \vdots \\ \lambda_{n-m} \geqslant 0 \end{array}\right\} \tag{$*$}$$

Of course, if n is even and $m(G) = \frac{1}{2}n$, then the single λ inequalities do not appear. We first demonstrate the necessity of these inequalities.

LEMMA 1. *Let G be a bipartite graph on n vertices with matching number m and suppose that A is a symmetric nonnegative matrix subordinate to G. The eigenvalues of A satisfy the inequalities* $(*)$.

Proof. Write A as $A = D + B$, in which the diagonal entries of B are 0 and D is a nonnegative diagonal matrix. From the definition of matching number, it follows that, in the adjacency matrix of G, the maximum number of 1's, no two of which lie in the same row or column, is $2m(G)$, e.g. [1, p. 44]. So, the maximum number of nonzero entries of B, no two of which lie in the same row or column of B is no more than $2m(G)$. This means that any k-by-k submatrix of B, $k > 2m(G)$, will have zero determinant, and that rank $(B) \leqslant 2m(G)$. As B is similar to $-B$ via the signature matrix with 1's in diagonal entries corresponding to one of the parts of G and -1's in the other diagonal entries, we have

$$\begin{aligned} \mu_n &= -\mu_1 \\ \mu_{n-1} &= -\mu_2 \\ &\vdots \\ \mu_{n-m+1} &= -\mu_m \\ \mu_{m+1} &= 0 \\ &\vdots \\ \mu_{n-m} &= 0 \end{aligned}$$

in which $\mu_1 \geqslant \cdots \geqslant \mu_n$ are the eigenvalues of B. But, since D is positive semidefinite, we have, via e.g. the Weyl inequalities [4, Ch. 4], that

$$\lambda_i \geqslant \mu_i, i = 1, \ldots, n$$

in which $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of A. It follows that the λ_i's satisfy the inequalities $(*)$. □

We next show that the inequalities $(*)$ are sufficient for the existence of a symmetric nonnegative matrix subordinate to G.

LEMMA 2. *Let G be a bipartite graph on n vertices with matching number m and suppose that $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ satisfy the inequalities $(*)$. Then there exists a nonnegative symmetric n-by-n matrix subordinate to G with eigenvalues $\lambda_1, \ldots, \lambda_n$.*

Proof. First note that for any two real numbers α, β such $\alpha+\beta \geqslant 0$ there is a 2-by-2 symmetric nonnegative matrix whose eigenvalues are α and β. Now, consider a collection of edges of G that realize $m(G)$. Suppose, wlog, that they are $\{1,2\}$, $\{3,4\}, \ldots, \{2m-1, 2m\}$. Construct a 2-by-2 symmetric nonnegative matrix with eigenvalues λ_1, λ_n and call it $A_1, \ldots,$ construct a 2-by-2 symmetric nonnegative matrix with eigenvalues $\lambda_m, \lambda_{n-m+1}$ and call it A_m. Further, let the 1-by-1 matrix (λ_j) be $A_j, j=m+1, \ldots, n-m$. Now $A_1 \oplus \cdots \oplus A_m \oplus A_{m+1} \oplus \cdots \oplus A_{n-m}$ is a symmetric nonnegative n-by-n matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ that is subordinate to G. □

Taking the two lemmas together, we have our principal result.

THEOREM 1. *Let G be a bipartite graph on n vertices with matching number m. There is a symmetric nonnegative n-by-n matrix subordinate to G and with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ if and only if $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ satisfy the inequalities $(*)$.*

The inequalities $(*)$ imply, for example, the following.

COROLLARY 1. *An n-by-n symmetric nonnegative matrix subordinate to the bipartite graph G has at most $m(G)$ negative eigenvalues and at least $n-m(G)$ nonnegative eigenvalues.*

For general symmetric n-by-n nonnegative matrices, it has been known since [7] that there may be as many as $n-1$ negative eigenvalues (thus only the Perron root is nonnegative). However, as the corollary indicates, a sparsity pattern may guarantee more nonnegative eigenvalues. In fact, *any* sparsity pattern will guarantee more nonnegative eigenvalues. Using the independence number of an undirected graph, we may substantially broaden the corollary. The *independence number* $i=i(G)$ of an undirected graph G is the maximum number of vertices of G among which there are no edges (i.e. the subgraph induced by this independent set of vertices has no edges). It is a straightforward exercise that for *bipartite* G, $i(G)=n-m(G)$. Our further observation is the following.

THEOREM 2. *An n-by-n symmetric nonnegative matrix subordinate to the undirected graph G has at least $i(G)$ nonnegative eigenvalues.*

Proof. Let A be a symmetric nonnegative matrix subordinate to G. Suppose, wlog, that $\{1,2,\ldots,i\}$ is an independent set of vertices of G realizing $i(G)$. Then, the principal sub-matrix of A lying in rows and columns $\{1,2,\ldots,i\}$ of A, $A[\{1,\ldots,i\}]$, is a nonnegative diagonal matrix, which, therefore, has nonnegative eigenvalues $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_i$. But, the interlacing inequalities [4, Ch. 4] show that $\lambda_1 \geqslant a_1$, $\lambda_2 \geqslant a_2, \ldots, \lambda_i \geqslant a_i \geqslant 0$, in which $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of A. □

We note that, unless G is the complete graph, $i(G) \geqslant 2$, so that any nonnegative symmetric matrix with at least one off-diagonal 0 entry has at least two nonnegative eigenvalues. Is there a converse to theorem 2, i.e. for each undirected graph G is there a symmetric nonnegative matrix A, subordinate to G, that has only $i(G)$ nonnegative eigenvalues? or, is there a better statement in purely combinatorial terms?

We again note that if G is a tree, then the inequalities $(*)$ also characterize the solution to the G-NIEP and G-RNIEP as well.

QUESTION 1. It is interesting that, for bipartite graphs at least, the solution to the G-SNIEP is not only convex and polyhedral but is describable by such simple linear inequalities. Is the solution to the G-SNIEP always convex?, polyhedral?, describable via the nonnegativity of sums of the λ_i's? What about the G-NIEP and G-RNIEP?

We close with another observation that follows from our work herein. In [5] it was shown that for a tree T, the maximum multiplicity of an eigenvalue, in a symmetric matrix whose graph is T, is the *path cover number* $p(T)$ of T (the fewest vertex disjoint paths of T that cover all the vertices of T, a single vertex counting as a path). In our proof of Lemma 1 herein, it was noted that B, whose graph may be taken to be any tree T, has $n-2m(T)$ eigenvalues equal to 0. It follows that for any tree T, we have the purely graph theoretic inequality

$$n-2m(T) \leqslant p(T).$$

Though, the disparity can be large, equality often occurs, and this lower bound for the path covering number does not seem readily apparent by purely graph theoretic arguments.

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