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Frames with Transitive Structures

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Abstract. A classical result in the theory of uniform spaces is that any topological space with a base of clopen sets admits a uniformity with a transitive base and the uniform topology of such a space has a base of clopen sets. This paper presents a pointfree generalization of this, both to uniform and quasi-uniform frames, together with various properties concerning total boundedness, compactifications and completions.

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1. Introduction

The study of transitivity has been important in the development of the theory of both uniform spaces and quasi-uniform spaces. Uniformities with a transitive base necessarily have a base of equivalence relations. These are called non-archimedean uniformities and have been investigated by B. Banaschewski [1], J. R. Isbell [14], P. Nyikos and H. C. Reichel [18] and others. Any zero-dimensional topological space, that is, with a base of open-closed (clopen) sets admits a uniformity with a transitive base and the uniform topology of such a space has a base of clopen sets. Transitivity has proved to be more fundamental in the study of quasi-uniform spaces than in the study of uniform spaces. In fact the usual proof that every topological space admits a quasi-uniformity is given by constructing a compatible quasi-uniformity with a base of transitive entourages [19]. Furthermore, P. Fletcher's construction of transitive quasi-uniformities has provided an important technique in the theory of quasi-uniform spaces [6]. For further information on the role of transitivity in quasi-uniform spaces the reader is referred to [10].

In this paper we investigate frame (quasi-)uniformities with transitive bases. We begin by extending the notions of (quasi-)uniformities with transitive bases to

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frames. As we demonstrate the theory of frame uniformities parallels the theory in the spatial case. If $(L, \mathcal{E})$ is a uniform frame in which $\mathcal{E}$ has a base of transitive entourages then L is zero-dimensional. Furthermore, each zero-dimensional frame admits a uniformity with a transitive base.

For frame quasi-uniformities, in order to recognize the similarity of our results with the spatial case we need to consider biframes [4] and bitopological spaces [17]. If $(X, \mathcal{T})$ is a T_1-space then the transitive structures given by Pervin [19] and Fletcher [6] each have the property that the conjugate structure is discrete, and hence the underlying bitopological space is pairwise zero-dimensional [23]. We prove that each zero-dimensional biframe (L_0, L_1, L_2) admits a transitive frame quasi-uniformity on its total part L_0 whose underlying biframe is (L_0, L_1, L_2). Furthermore, if $(L, \mathcal{E})$ is a quasi-uniform frame, and $\mathcal{E}$ has a base of transitive entourages, then the underlying biframe is zero-dimensional.

Finally, we establish a number of additional properties concerning total boundedness, the universal zero-dimensional compactification and completions.

This paper uses the Weil entourages of Picado [20]. We note that, as in the case of spaces, there are several different ways of describing frame uniformities. Examples are the covering uniformities of Isbell [15], the entourage uniformities of Fletcher and Hunsaker [7] and the Weil uniformities of Picado [20]. There are also three equivalent approaches to quasi-uniform frames: Frith [11], Fletcher, Hunsaker and Lindgren [8], and Picado [22]. The equivalence of the approaches to uniform (resp. quasi-uniform) frames was proved by Picado [20] and Fletcher and Hunsaker [7] (resp. Picado [22] and Fletcher, Hunsaker and Lindgren [9]).

2. Preliminaries and Notation

Recall that a *frame* L is a complete lattice satisfying the infinite distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s \mid s \in S\}$ for every $x \in L$ and every $S \subseteq L$, and a *frame homomorphism* $h : L \to M$ is a mapping preserving arbitrary joins (including the bottom element 0) and finite meets (including the top 1). The *pseudocomplement* of an $x \in L$ is $x^* = \bigvee\{a \in L \mid a \wedge x = 0\}$. If $x \vee x^* = 1$, then x is *complemented*. We denote by $\mathcal{B}L$ the sublattice of L of all complemented elements of L (the *Boolean part* of L). A frame L is *zero-dimensional* [2] provided that each element of L is a join of complemented elements. L is *compact* provided that, for any *cover* S of L, that is, for any $S \subseteq L$ such that $1 = \bigvee S$, there is a finite cover $F \subseteq S$. A standard reference for frames is Johnstone [16].

A *biframe* [4] is a triple (L_0, L_1, L_2) where L_1 and L_2 are subframes of the frame L_0, and each element of L_0 is the join of finite meets from $L_1 \cup L_2$. In the sequel, we use L_i, L_j to denote L_1 or L_2 always assuming that $i, j = 1, 2, i \neq j$.

A *biframe homomorphism* $(L_0, L_1, L_2) \to (M_0, M_1, M_2)$ is a frame homomorphism from L_0 to M_0 which maps L_i into M_i. A biframe (L_0, L_1, L_2) is *compact* provided that L_0 is a compact frame. Schauerte [24] introduces the following useful notation for a biframe (L_0, L_1, L_2):

For any $x \in L_i$, $x^\bullet = \bigvee\{y \in L_j \mid y \wedge x = 0\}$. The biframe (L_0, L_1, L_2) is *zero-dimensional* [2] provided that each element of L_i can be written as a join of elements $x \in L_i$ for which $x \vee x^\bullet = 1$.

Additional information concerning biframes may be found in [4].

For a subset A of a poset $(X, \leq)$, let $\downarrow A = \{x \in X \mid \exists\, a \in A : x \leq a\}$. The set A is said to be a *decreasing subset* of X if $\downarrow A = A$. For a frame L consider the frame $\mathcal{D}(L \times L)$ of all non-void decreasing subsets of $L \times L$, ordered by inclusion. The coproduct $L \oplus L$ will be represented, as usual (cf. [16]), as the subset of $\mathcal{D}(L \times L)$ consisting of all the *saturated* sets, that is, of those sets A which satisfy

$$\{x\} \times S \subseteq A \Rightarrow \left(x, \bigvee S\right) \in A$$

and

$$S \times \{y\} \subseteq A \Rightarrow \left(\bigvee S, y\right) \in A.$$

Since the premise is trivially satisfied if $S = \emptyset$, each saturated set A contains $\mathbf{O} := \{(0, a), (a, 0) \mid a \in L\}$, and $\mathbf{O}$ is the zero of $L \oplus L$. Obviously, each $x \oplus y = \downarrow(x, y) \cup \mathbf{O}$ is a saturated set and for each saturated set A one has $A = \bigvee\{x \oplus y \mid x \oplus y \leq A\} = \bigvee\{x \oplus y \mid (x, y) \in A\}$. The coproduct injections $u_i^L : L \to L \oplus L$ are defined by $u_1^L(x) = x \oplus 1$ and $u_2^L(x) = 1 \oplus x$ so that $x \oplus y = u_1^L(x) \wedge u_2^L(y)$. Consequently, the *codiagonal* $\nabla : L \oplus L \to L$ is given by $\nabla(A) = \bigvee\{x \wedge y \mid (x, y) \in A\}$.

For any frame homomorphism $h : L \to M$, the definition of coproduct ensures us the existence (and uniqueness) of a frame homomorphism $h \oplus h : L \oplus L \to M \oplus M$ such that $(h \oplus h) \cdot u_i^L = u_i^M \cdot h$ $(i = 1, 2)$.

For a frame L, a *Weil entourage* [20] is an $E \in L \oplus L$ for which the codiagonal factorizes through $(-) \cap E : L \oplus L \to \downarrow\{E\}$:

$$\begin{tikzcd} L \oplus L \arrow[r, two heads, "(-)\cap E"] \arrow[d, "\nabla"'] & \downarrow\{E\} \arrow[dl, "\exists f"] \\ L & \end{tikzcd}$$

This is equivalent to saying that $\nabla(E) = 1$, that is, $\bigvee\{x \in L \mid (x, x) \in E\} = 1$. The collection $W\,\mathrm{Ent}(L)$ of all Weil entourages of L may be partially ordered by inclusion. This is a partially ordered set with finitary meets (including a unit $1 = L \oplus L$).

For any decreasing sets E, F of $L \oplus L$ define

$$E \circ F := \bigvee\{x \oplus y \mid \exists z \in L \setminus \{0\} : (x, z) \in E, (z, y) \in F\}.$$

In particular, we are defining the composition for Weil entourages.

For the basic properties of the operation $\circ$ see [21].

The behavior of entourages in the frame setting is similar to their behavior in spaces because of the following fundamental property, proved in [20]:

LEMMA 2.1.
$$\Big(\bigvee_{i\in I}(a_i \oplus b_i)\Big) \circ \Big(\bigvee_{j\in J}(c_j \oplus d_j)\Big) = \Big(\bigcup_{i\in I}(a_i \oplus b_i)\Big) \circ \Big(\bigcup_{j\in J}(c_j \oplus d_j)\Big).$$

A Weil entourage E is *symmetric* if it coincides with $E^{-1} := \{(y, x) \mid (x, y) \in E\}$. Further we have:
For any $\mathcal{E} \subseteq L \oplus L$ and $x, y \in L$, $x \overset{\mathcal{E}}{\lhd} y$ means that

$$E \circ (x \oplus x) \subseteq y \oplus y \quad \text{for some } E \in \mathcal{E}. \tag{2.1}$$

Of course, when $\mathcal{E}$ is symmetric (that is, $E \in \mathcal{E}$ implies $E^{-1} \in \mathcal{E}$) this is also equivalent to saying that

$$(x \oplus x) \circ E \subseteq y \oplus y \quad \text{for some } E \in \mathcal{E}. \tag{2.2}$$

The elements $st(x, E)$ satisfy the following properties [21]:

$$x \leq st(x, E) \tag{2.3}$$

$$st(st(x, E), F) \leq st(x, E \circ F) \tag{2.4}$$

$$st(x, E) \leq y \Rightarrow st(y^*, E) \leq x^*. \tag{2.5}$$

A set $\mathcal{E} \subseteq W\,\mathrm{Ent}(L)$ is called *admissible* if, for every $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\overline{\mathcal{E}}}{\lhd} x\}$, where $\overline{\mathcal{E}} := \mathcal{E} \cup \{E^{-1} : E \in \mathcal{E}\}$. A (*Weil*) *uniformity* on L is an admissible filter $\mathcal{E}$ of $(W\,\mathrm{Ent}(L), \subseteq)$ satisfying:

(1) For each $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that $F \circ F \subseteq E$.
(2) For each $E \in \mathcal{E}$, $E^{-1} \in \mathcal{E}$.

A (*Weil*) *uniform frame* is a pair $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is a uniformity on L. If $(L, \mathcal{E})$ and $(M, \mathcal{F})$ are uniform frames, a *uniform homomorphism* $h : (L, \mathcal{E}) \to (M, \mathcal{F})$ is a frame homomorphism $h : L \to M$ such that $(h \oplus h)(E) \in \mathcal{F}$ whenever $E \in \mathcal{E}$. The resulting category WUFrm contains the category KRFrm of compact regular frames as a full subcategory in virtue of the fact that, for any compact regular frame L, $W\,\mathrm{Ent}(L)$ is its unique uniformity [13]. Further, each uniform frame has a coreflection to KRFrm, called the *Samuel compactification* [5]. On the other hand, there is a notion of completeness for uniform frames, determined as follows:

A frame homomorphism $h : L \to M$ is said to be *dense* if $h(x) = 0$ implies $x = 0$. A uniform homomorphism $h : (L, \mathcal{E}) \to (M, \mathcal{F})$ is called a *dense surjection* if it is onto, dense and $\mathcal{F}$ is generated by the image entourages $(h \oplus h)(E)$, $E \in \mathcal{E}$. A uniform frame $(M, \mathcal{F})$ is called *complete* if any dense surjection $(L, \mathcal{E}) \to (M, \mathcal{F})$ is an isomorphism, and a *completion* of $(L, \mathcal{E})$ is a complete $(M, \mathcal{F})$ together with a dense surjection $(M, \mathcal{F}) \to (L, \mathcal{E})$. Isbell [15] established that each uniform frame has an essentially unique completion which thus provides the coreflection to complete uniform frames.

We point out that, for any uniformity $\mathcal{E}$ on L, $x \overset{\mathcal{E}}{\triangleleft} y$ if and only if there exists $E \in \mathcal{E}$ such that

$$st(x, E) := \bigvee\{z \in L \mid (z, z) \in E, z \wedge x \neq 0\} \leq y \quad [20].$$

By just dropping the symmetry condition (2) in the definition of uniform frame we get the category of *quasi-uniform frames*. With the lack of symmetry the equivalence between conditions (2.1) and (2.2) is no longer valid; whence, in the place of $\overset{\mathcal{E}}{\triangleleft}$ we have two partial orders

$$x \overset{\mathcal{E}}{\triangleleft}_1 y \equiv E \circ (x \oplus x) \subseteq y \oplus y, \quad \text{for some } E \in \mathcal{E},$$
$$x \overset{\mathcal{E}}{\triangleleft}_2 y \equiv (x \oplus x) \circ E \subseteq y \oplus y, \quad \text{for some } E \in \mathcal{E},$$

which in turn, lead to two subframes of L,

$$L^1 := \left\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft}_1 x\}\right\} \quad \text{and}$$
$$L^2 := \left\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft}_2 x\}\right\}.$$

Notice that the admissibility condition is equivalent to saying that the triple (L, L^1, L^2) is a biframe [22].

As in the symmetric case, $\overset{\mathcal{E}}{\triangleleft}_1$ and $\overset{\mathcal{E}}{\triangleleft}_2$ may be characterized in the following way:

- $x \overset{\mathcal{E}}{\triangleleft}_1 y$ if and only if there exists $E \in \mathcal{E}$ such that
$$st_1(x, E) := \bigvee\{z \in L \mid (z, w) \in E, w \wedge x \neq 0\} \leq y;$$
- $x \overset{\mathcal{E}}{\triangleleft}_2 y$ if and only if there exists $E \in \mathcal{E}$ such that
$$st_2(x, E) := \bigvee\{z \in L \mid (w, z) \in E, w \wedge x \neq 0\} \leq y.$$

Moreover we have [21]:

$$x \leq st_i(x, E), \tag{2.6}$$
$$st_i(st_i(x, E), F) \leq st_i(x, E \circ F). \tag{2.7}$$

A Weil entourage E is said to be *finite* [13] if there exists a finite cover $x_1, x_2, \ldots, x_n$ of L such that $\bigvee_{i=1}^{n}(x_i \oplus x_i) \subseteq E$. A (quasi-)uniformity is *totally bounded* if it has a base of finite entourages.

3. Transitive Entourages: Basic Results

In this section, we collect a number of preliminary results needed later on.

For a frame L, we say that $E \in L \oplus L$ is *transitive* if $E^2 := E \circ E = E$.

We begin by checking that, for any transitive Weil entourage E, the elements $st(x, E)$ are complemented:

PROPOSITION 3.1. *Let L be a frame and let E be a transitive Weil entourage of L. For every $x \in L$, $st(x, E) \vee st(x, E)^* = 1$.*

Proof. Let $(y, y) \in E$ with $y \wedge st(x, E) \neq 0$. Then there exists $(\alpha, \alpha) \in E$ such that $\alpha \wedge x \neq 0$ and $\alpha \wedge y \neq 0$. Thus (α, y) and (y, α) belong to $E^2 = E$ and, consequently, $(\alpha \vee y, \alpha \vee y) \in E$. Since $(\alpha \vee y) \wedge x \neq 0$, we may conclude that $st(x, E) \geq \alpha \vee y \geq y$. Hence $st(x, E) \vee st(x, E)^* \geq (\bigvee\{y \in L \mid (y, y) \in E, y \wedge st(x, E) \neq 0\}) \vee (\bigvee\{y \in L \mid y \wedge st(x, E) = 0\}) \geq \bigvee\{y \in L \mid (y, y) \in E\} = 1$. □

Similarly, for any transitive entourage E of the total part L_0 of a biframe (L_0, L_1, L_2), the elements $st_i(x, E)$ are complemented. Moreover, if all elements $st_i(x, E)$ belong to L_i then they are complemented, with complement in L_j. For this we need the following lemma:

LEMMA 3.2. *Let L be a frame and let E be a transitive Weil entourage of L. For every $x \in L$, if $st_i(x, E) \wedge y = 0$ then $st_i(x, E) \wedge st_j(y, E) = 0$.*

Proof. We prove only for $i = 1$ and $j = 2$ (the case $i = 2$ and $j = 1$ is analogous):

Computing $st_1(x, E) \wedge st_2(y, E)$ we get

$$\begin{aligned} st_1(x, E) \wedge st_2(y, E) \;=\; & \bigvee\{\alpha \wedge \beta \mid (\alpha, z) \in E,\ (w, \beta) \in E, \\ & z \wedge x \neq 0, w \wedge y \neq 0\}. \end{aligned}$$

This join is indeed 0 because, whenever (α, z) and (w, β) belong to E with $\alpha \wedge \beta \neq 0$, we have $(w, z) \in E^2 = E$ and then, if $z \wedge x \neq 0$, we obtain, by hypothesis, $w \wedge y \neq 0$. □

PROPOSITION 3.3. *Let (L_0, L_1, L_2) be a biframe and let E be a transitive entourage of L_0 such that, for any $x \in L_0$, $st_i(x, E) \in L_i$. Then, for any $x \in L_0$, $st_i(x, E) \vee st_i(x, E)^\bullet = 1$.*

Proof. Let us show that $st_1(x, E) \vee st_1(x, E)^\bullet = 1$ by proving that

$$\begin{aligned} & \bigvee\{y \in L_0 \mid (y, z) \in E, z \wedge x \neq 0\} \vee \bigvee\{y \in L_2 \mid y \wedge st_1(x, E) = 0\} \\ & \quad \geq \bigvee\{y \in L_0 \mid (y, y) \in E\} = 1. \end{aligned}$$

Consider $y \in L_0$ such that $(y, y) \in E$.

If $y \wedge st_1(x, E) = 0$ then we have $y \leq st_2(y, E) \in L_2$ and, by Lemma 3.2, $st_1(x, E) \wedge st_2(y, E) = 0$.

Otherwise, the existence of $(\alpha, \beta) \in E$ such that $x \wedge \beta \neq 0$ and $y \wedge \alpha \neq 0$ implies that $(y, \beta) \in E^2 = E$ with $x \wedge \beta \neq 0$.

The proof for $i = 2$ is similar.

Respectively by (2.4) and (2.7) we immediately have:

PROPOSITION 3.4. *Let L be a frame and let $\mathcal{E} \subseteq W\,\mathrm{Ent}(L)$. For every $x \in L$ and every transitive $E \in \mathcal{E}$, $st(x, E) \overset{\mathcal{E}}{\lhd} st(x, E)$.*

PROPOSITION 3.5. *Let (L_0, L_1, L_2) be a biframe and let $\mathcal{E} \subseteq W\,\mathrm{Ent}(L_0)$. For every $x \in L_i$ and every transitive $E \in \mathcal{E}$, $st_i(x, E) \overset{\mathcal{E}}{\lhd}_i st_i(x, E)$.*

4. Transitive Uniformities

We say that a (quasi-)uniform frame $(L, \mathcal{E})$ is *transitive* if $\mathcal{E}$ has a base of transitive entourages. We now derive the results concerning transitive uniform frames.

PROPOSITION 4.1. *Let $(L, \mathcal{E})$ be a transitive uniform frame. Then L is zero-dimensional.*

Proof. Let $x \in L$. Then $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$. But $y \overset{\mathcal{E}}{\lhd} x$ if and only if there exists a transitive $E \in \mathcal{E}$ such that $st(y, E) \leq x$. Thus, since $y \leq st(y, E)$,

$$x \leq \bigvee\{st(y, E) \mid E \in \mathcal{E},\ \text{transitive},\ y \in L \text{ and } st(y, E) \leq x\} \leq x.$$

So, every x in L is the join of some $st(y, E)$ with transitive $E \in \mathcal{E}$, which are, by Proposition 3.1, complemented elements. $\square$

Conversely, every zero-dimensional frame admits a transitive uniformity. As a step in this direction, we need to introduce the following:

For any $a \in L$ let E_a denote the element $(a \oplus a) \vee (a^* \oplus a^*)$ of $L \oplus L$.

The proof of the following proposition uses Lemma 2.1:

PROPOSITION 4.2. *For every $a \in L$, $E_a^2 = E_a$.*

Proof. Consider $(\alpha, \beta), (\beta, \gamma) \in (a \oplus a) \cup (a^* \oplus a^*)$ with $\alpha, \beta, \gamma \neq 0$. Clearly, the cases $(\alpha, \beta), (\beta, \gamma) \in a \oplus a$ or $(\alpha, \beta), (\beta, \gamma) \in a^* \oplus a^*$ are the only possible ones. In each one, $(\alpha, \gamma) \in (a \oplus a) \cup (a^* \oplus a^*)$. $\square$

For any complemented a, E_a is obviously a Weil entourage. Moreover, we have (for a proof see Remark IV.6.3 of [21]):

PROPOSITION 4.3. *E_a is a Weil entourage if and only if a is complemented.*

THEOREM 4.4. *Let L be a zero-dimensional frame. The family $\mathcal{S} = \{E_a \mid a \in \mathcal{B}L\}$ is a subbase for a transitive, totally bounded, uniformity on L.*

Proof. Let $\mathcal{E}$ be the filter of $(W\,\mathrm{Ent}(L), \subseteq)$ generated by $\mathcal{S}$, that is,

$$\mathcal{E} = \left\{ E \in W\,\mathrm{Ent}(L) \mid \exists a_1, \ldots, a_n \in \mathcal{B}L : \bigcap_{i=1}^{n} E_{a_i} \subseteq E \right\}.$$

This is indeed a transitive, totally bounded, uniformity on L. We first check the admissibility condition:

By the zero-dimensionality, every $x \in L$ is a join of complemented elements; but, for any $c \in \mathcal{B}L$, $E_c \circ (c \oplus c) \subseteq c \oplus c$; indeed, if $(\alpha, \beta) \in (c \oplus c) \cup (c^* \oplus c^*)$ and $(\beta, \gamma) \in c \oplus c$ with $\beta \neq 0$, then $(\alpha, \beta) \in c \oplus c$ and thus $(\alpha, \gamma) \in c \oplus c$. Therefore $c \overset{\mathcal{E}}{\lhd} c$ and, consequently, we may conclude that $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$.

The total boundedness is obvious because each $\bigcap_{i=1}^{n} E_{a_i}$ is a finite Weil entourage. The proof of the other conditions is straightforward. □

In the sequel, given a zero-dimensional frame, we denote by $\mathcal{E}_P$ the transitive uniformity on L constructed in the proof of Theorem 4.4.

LEMMA 4.5. *Let $h : L \to M$ be a frame homomorphism between zero-dimensional frames. Then $h : (M, \mathcal{E}_P^M) \to (L, \mathcal{E}_P^L)$ is uniform.*

Proof. Since $(h \oplus h)(E_a) = (h \oplus h)((a \oplus a) \vee (a^* \oplus a^*)) = (h(a) \oplus h(a)) \vee (h(a^*) \oplus h(a^*))$ we have $(h \oplus h)(E_a) = E_{h(a)}$ whenever a is complemented. Consequently, $(h \oplus h)(E) \in \mathcal{E}_P^L$ for every $E \in \mathcal{E}_P^M$. □

THEOREM 4.6. *Let L be a zero-dimensional frame and let $(CL, C\mathcal{E}_P)$ be the uniform completion of $(L, \mathcal{E}_P)$. Then $(CL, C\mathcal{E}_P)$ is the universal zero-dimensional compactification of L.*

Proof. By [5, Proposition 3], $(CL, C\mathcal{E}_P)$ is compact and coincides with the Samuel compactification $\mathcal{R}(L, \mathcal{E}_P)$ of $(L, \mathcal{E}_P)$.

Now for any frame homomorphism $h : M \to L$ with M compact and zero-dimensional, h is uniform by Lemma 4.5. Then, in the diagram

$$\begin{tikzcd} \mathcal{R}(M, \mathcal{E}_P) \arrow[r, "\mathcal{R}h"] \arrow[d, "\rho_M"'] & \mathcal{R}(L, \mathcal{E}_P) \arrow[d, "\rho_L"] \\ M \arrow[r, "h"] & L \end{tikzcd}$$

ρ_M is an isomorphism and we have $h = \rho_L \mathcal{R}h \rho_M^{-1}$, which shows that $\mathcal{R}(L, \mathcal{E}_P)$ is the compact zero-dimensional coreflection of L with coreflection map $\rho_L : \mathcal{R}(L, \mathcal{E}_P) \to L$. □

It is easy to verify that if $h : M \to L$ is a zero-dimensional compactification of a frame L, then the image by h of the unique uniform structure on M is a totally bounded transitive uniformity. From this and the comments of B. Banaschewski [3, p. 115] it follows that there is a one to one correspondence between zero-dimensional compactifications of a frame L and transitive totally bounded uniformities on L. Therefore, symmetric transitive structures give us a method for representing all zero-dimensional compactifications.

THEOREM 4.7. *The totally bounded coreflection* $(L, \mathcal{E}_*)$ *of any transitive uniform frame* $(L, \mathcal{E})$ *is transitive.*

Proof. Consider

$$\mathcal{F} = \left\{ E \in W\,\mathrm{Ent}(L) \mid \exists c_1, c_2, \ldots, c_n \in L : c_i \overset{\mathcal{E}}{\lhd} c_i \text{ and } \bigcap_{i=1}^{n} E_{c_i} \subseteq E \right\}.$$

We first prove that $\mathcal{F}$ is a uniformity on L. Clearly, $\mathcal{F}$ is a filter of $W\,\mathrm{Ent}(L)$ and each intersection $\bigcap_{i=1}^{n} E_{c_i}$ is symmetric. In order to show the admissibility, it suffices to prove that $x \overset{\mathcal{E}}{\lhd} y$ implies $x \overset{\mathcal{F}}{\lhd} y$. Let $x \overset{\mathcal{E}}{\lhd} y$, that is, $st(x, E) \leq y$ for some symmetric and transitive $E \in \mathcal{E}$. By Proposition 3.4, $E_{st(x,E)} \in \mathcal{F}$. Further, $E_{st(x,E)} \circ (x \oplus x) \subseteq y \oplus y$:

Indeed, for any

$$(\alpha, \beta) \in \downarrow(st(x, E), st(x, E)) \cup \downarrow(st(x, E)^*, st(x, E)^*)$$

and $(\beta, \gamma) \leq (x, x)$, we have $(\alpha, \beta) \leq (st(x, E), st(x, E))$ whenever $\beta \neq 0$, because $x \leq st(x, E)$. Since $st(x, E) \leq y$ we have $(\alpha, \gamma) \leq (y, y)$. Consequently $x \overset{\mathcal{F}}{\lhd} y$. Therefore $\mathcal{F}$ is a uniformity on L. Notice that, if $c \overset{\mathcal{E}}{\lhd} c$, then c is equal to $st(c, E)$ for some transitive $E \in \mathcal{E}$ and, therefore, c is complemented and $E_c \in W\,\mathrm{Ent}(L)$. These Weil entourages form a subbase for $\mathcal{F}$. From

$$\bigcap_{i=1}^{n} E_{c_i} \circ \bigcap_{i=1}^{n} E_{c_i} \subseteq \bigcap_{i=1}^{n} (E_{c_i} \circ E_{c_i}) = \bigcap_{i=1}^{n} E_{c_i},$$

it follows that $\mathcal{F}$ is transitive.

$\mathcal{F}$ is also totally bounded. Indeed, whenever $c \overset{\mathcal{E}}{\lhd} c$ and $d \overset{\mathcal{E}}{\lhd} d$,

$$\begin{aligned} E_c \cap E_d \;=\; & (c \wedge d \oplus c \wedge d) \vee (c \wedge d^* \oplus c \wedge d^*) \vee (c^* \wedge d \oplus c^* \wedge d) \\ & \vee (c^* \wedge d^* \oplus c^* \wedge d^*) \end{aligned}$$

and, since $\{c \wedge d, c \wedge d^*, c^* \wedge d, c^* \wedge d^*\}$ is a cover of L, $E_c \cap E_d$ is finite.

We conclude the proof that $\mathcal{E}_*$ is transitive by showing that $\mathcal{E}_* = \mathcal{F}$. By the well-known isomorphism between the categories of totally bounded uniform frames and proximal frames it suffices to show that the corresponding proximities $\overset{\mathcal{E}_*}{\lhd}$ and $\overset{\mathcal{F}}{\lhd}$ do coincide.

First we prove that $\mathcal{F} \subseteq \mathcal{E}$. Consider E_c with $c \overset{\mathcal{E}}{\lhd} c$. By (2.5), $c^* \overset{\mathcal{E}}{\lhd} c^*$. Therefore there exist $E_1, E_2 \in \mathcal{E}$ for which $E_1 \circ (c \oplus c) \subseteq c \oplus c$ and $E_2 \circ (c^* \oplus c^*) \subseteq c^* \oplus c^*$. Thus $E_1 \cap E_2 \subseteq (E_1 \cap E_2) \circ E_c \subseteq E_c$, which implies that $E_c \in \mathcal{E}$.

Now, the totally boundedness of $\mathcal{F}$ implies that $\mathcal{F} \subseteq \mathcal{E}_*$, hence $\overset{\mathcal{F}}{\lhd} \subseteq \overset{\mathcal{E}_*}{\lhd}$.

The opposite inclusion follows from the fact, already proved, that $\overset{\mathcal{E}}{\lhd} \subseteq \overset{\mathcal{F}}{\lhd}$. □

It is well-known that a frame is uniformizable iff it is completely regular. As in the spatial situation, it is natural to consider under what conditions the collection of all entourages of a given completely regular frame form a uniformity, and if not, to characterize the "finest" uniformity on the given frame.

Let L be a completely regular frame, and consider the union $\mathcal{S}$ of all uniformities on L. This is a subbase for a uniformity. Indeed, the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ form a base for a uniformity:

- Let $E = E_1 \cap E_2 \cap \cdots \cap E_n$ with $E_1, E_2, \ldots, E_n \in \mathcal{S}$. For each $i \in \{1, 2, \ldots, n\}$ there exists $F_i \in \mathcal{S}$ such that $F_i^2 \subseteq E_i$. Then $(F_1 \cap F_2 \cap \cdots \cap F_n)^2 \subseteq E$ and $F_1 \cap F_2 \cap \cdots \cap F_n \in \mathcal{B}$.
- Let $E = E_1 \cap E_2 \cap \cdots \cap E_n \in \mathcal{B}$ with $E_1, E_2, \ldots, E_n \in \mathcal{S}$. For each $i \in \{1, 2, \ldots, n\}$ $E_i^{-1} \in \mathcal{S}$. Then $E_1^{-1} \cap \cdots \cap E_n^{-1} \in \mathcal{S}$ and $(E_1^{-1} \cap \cdots \cap E_n^{-1})^{-1} = E$.
- Since for each uniformity $\mathcal{E}$ on L, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\} \leq \bigvee\{y \in L \mid y \overset{\mathcal{B}}{\lhd} x\}$, we have $x = \bigvee\{y \in L \mid y \overset{\mathcal{B}}{\lhd} x\}$ for every $x \in L$.

This uniformity is called the *fine uniformity* on L.

We say that a sequence of entourages $E_1, E_2, \ldots, E_n, \ldots$ is a *normal sequence* if $E_{n+1}^2 \subseteq E_n$. An entourage E is a *normal entourage* if $E = E_1$ in some normal sequence.

LEMMA 4.8. *The fine uniformity on L coincides with the collection of all normal entourages of L.*

Proof. By the square root property for uniformities, each entourage in the fine uniformity is a normal entourage.

On the other hand, take any normal entourage E of L, say with $E = E_1$ in some normal sequence $\{E_n\}$, and let $\mathcal{E}$ be any uniformity on L. Then it is easy to see that $\mathcal{E} \cup \{E_n\} \cup \{E_n^{-1}\}$ is a subbase for some uniformity on L. Hence E belongs to the fine uniformity on L. □

Recall that a frame is *Lindelöf* if each cover has a countable subcover.

THEOREM 4.9. *If L is a zero-dimensional Lindelöf frame then the fine uniformity of L is transitive.*

Proof. Trivially any zero-dimensional frame is completely regular. By Lemma 4.8, it suffices to show that any normal entourage E_1 contains a transitive entourage.

Let $(x, x) \in E_1$. Then $x = \bigvee x_\alpha$ where each x_α is complemented. Let E be the join of all $x_\alpha \oplus x_\alpha$ where x_α is complemented and appears in a join which generates some x with $(x, x) \in E_1$. Clearly E is a Weil entourage contained in E_1.

By the Lindelöf property there exists a countable cover $\{x_i \mid i \in \mathbb{N}\}$ of L such that $(x_i, x_i) \in E$ for every $i \in \mathbb{N}$. Put $y_1 = x_1$, $y_2 = x_2 \wedge x_1^*$, $y_3 = x_3 \wedge x_1^* \wedge x_2^*, \ldots, y_n = x_n \wedge (\bigwedge_{i=1}^{n-1} x_i^*)$ and let $F = \bigvee_{i\in\mathbb{N}}(y_i \oplus y_i) \subseteq E$. Notice that, for every $n \in \mathbb{N}$, $\bigvee_{i=1}^{n} y_i = \bigvee_{i=1}^{n} x_i$. Therefore $\bigvee_{i\in\mathbb{N}} y_i = \bigvee_{i\in\mathbb{N}} x_i = 1$, which means that F is a Weil entourage. Also, for $m > n$,

$$y_n \wedge y_m = \left(x_n \wedge \bigwedge_{i=1}^{n-1} x_i^*\right) \wedge \left(x_m \wedge \bigwedge_{i=1}^{m-1} x_i^*\right) \le x_n \wedge x_n^* = 0.$$

This implies that F is transitive. Indeed: if $(\alpha, \beta) \in y_i \oplus y_i$ and $(\beta, \gamma) \in y_j \oplus y_j$ with $\alpha, \beta, \gamma \neq 0$, then $\beta \le y_i \wedge y_j$ which implies $j = i$ and $(\alpha, \gamma) \in y_i \oplus y_i$. This shows that $F \circ F = F$ because

$$F \circ F = \bigcup_{i\in\mathbb{N}}(y_i \oplus y_i) \circ \bigcup_{i\in\mathbb{N}}(y_i \oplus y_i). \qquad \square$$

5. Transitive Quasi-Uniformities

We close with the results concerning transitive quasi-uniform frames.

Let $(L, \mathcal{E})$ be a quasi-uniform frame and let (L, L^1, L^2) be its underlying biframe. By Proposition III 4.3 of [21], for every transitive entourage E of L and every $x \in L$,

$$st_i(x, E) \in L^i. \tag{5.1}$$

So, in this case, the conditions of Proposition 3.3 are satisfied. Moreover: since $st_i(x, E) \le y$ implies $st_j(y^\bullet, E) \wedge x = 0$ for every $y \in L^i$, (5.1) also implies that, for any $x, y \in L^i$ and any transitive E,

$$st_i(x, E) \le y \Rightarrow st_j(y^\bullet, E) \le x^\bullet. \tag{5.2}$$

PROPOSITION 5.1. *Let $(L, \mathcal{E})$ be a transitive quasi-uniform frame. Then the biframe (L, L^1, L^2) is zero-dimensional.*

Proof. Let $x \in L^i$ $(i = 1, 2)$. For any $y \in L$ such that $y \overset{\mathcal{E}}{\lhd}_i x$ take $E \in \mathcal{E}$, transitive, satisfying $st_i(y, E) \le x$. From $st_i(st_i(y, E), E) \le st_i(y, E^2) = st_i(y, E) \le x$ it follows that $y \le st_i(y, E) \overset{\mathcal{E}}{\lhd}_i x$. Thus

$$\begin{aligned} x &= \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd}_i x\} \\ &= \bigvee\{y \in L^i \mid y \overset{\mathcal{E}}{\lhd}_i x\} \\ &\le \bigvee\{st_i(y, E) \mid E \in \mathcal{E}, E^2 = E, st_i(y, E) \le x\} \le x. \end{aligned}$$

The conclusion now follows from Proposition 3.3. $\square$

In particular, this implies that the frame L is zero-dimensional.

Now, let L be a frame. For any $a \in L$, let us denote by E_a^r and E_a^l, respectively, the elements $(1 \oplus a) \vee (a^* \oplus a^*)$ and $(a \oplus 1) \vee (a^* \oplus a^*)$ of $L \oplus L$. Clearly, $(E_a^l)^{-1} = E_a^r$ and $E_a^l \cap E_a^r = E_a$.

The following result is quite obvious:

PROPOSITION 5.2. *For any $a \in L$, E_a^l and E_a^r are transitive.*

PROPOSITION 5.3. *For any $a \in L$, E_a^l and E_a^r are Weil entourages of L if and only if a is complemented.*

Proof. Clearly, if E_a^r and E_a^l are Weil entourages, $E_a^r \cap E_a^l = E_a$ is also a Weil entourage. Thus, by Proposition 4.3, a is complemented.

Conversely, if a is complemented, $\bigvee\{x \in L \mid (x, x) \in E_a^r\} \geq a \vee a^* = 1$ and, similarly, for E_a^l. □

The following auxiliary result will be useful in the sequel.

LEMMA 5.4. *Let $E_k = (a_1^k \oplus b_1^k) \vee (a_2^k \oplus b_2^k)$ $(k = 1, 2, \ldots, n)$. Then*

$$st_1\left(x, \bigcap_{k=1}^{n} E_k\right) = \bigvee\{a_{t_1}^1 \wedge a_{t_2}^2 \wedge \cdots \wedge a_{t_n}^n \mid t_k \in \{1,2\}, \\ k \in \{1, 2, \ldots, n\}, b_{t_1}^1 \wedge b_{t_2}^2 \wedge \cdots \wedge b_{t_n}^n \wedge x \neq 0\}$$

and

$$st_2\left(x, \bigcap_{k=1}^{n} E_k\right) = \bigvee\{b_{t_1}^1 \wedge b_{t_2}^2 \wedge \cdots \wedge b_{t_n}^n \mid t_k \in \{1,2\}, \\ k \in \{1, 2, \ldots, n\}, a_{t_1}^1 \wedge a_{t_2}^2 \wedge \cdots \wedge a_{t_n}^n \wedge x \neq 0\}.$$

Proof. Since

$$\begin{aligned} \bigcap_{k=1}^{n} E_k &= \bigvee_{t_1=1}^{2} \bigvee_{t_2=1}^{2} \cdots \bigvee_{t_n=1}^{2} \big((a_{t_1}^1 \oplus b_{t_1}^1) \cap (a_{t_2}^2 \oplus b_{t_2}^2) \cap \cdots \cap (a_{t_n}^n \oplus b_{t_n}^n)\big) \\ &= \bigvee_{t_1=1}^{2} \bigvee_{t_2=1}^{2} \cdots \bigvee_{t_n=1}^{2} \big((a_{t_1}^1 \wedge a_{t_2}^2 \wedge \cdots \wedge a_{t_n}^n) \oplus (b_{t_1}^1 \wedge b_{t_2}^2 \wedge \cdots \wedge b_{t_n}^n)\big) \end{aligned}$$

we have that $st_1(x, \bigcap_{k=1}^{n} E_k)$ is equal to

$$\begin{aligned} &st_1\left(x, \textstyle\bigcup_{t_1=1}^{2} \bigcup_{t_2=1}^{2} \cdots \bigcup_{t_n=1}^{2} \downarrow(a_{t_1}^1 \wedge a_{t_2}^2 \wedge \cdots \wedge a_{t_n}^n, b_{t_1}^1 \wedge b_{t_2}^2 \wedge \cdots \wedge b_{t_n}^n)\right) \\ &\quad = \bigvee\{a_{t_1}^1 \wedge a_{t_2}^2 \wedge \cdots \wedge a_{t_n}^n \mid t_1, t_2, \ldots, t_n \in \{1,2\}, \\ &\qquad\qquad a_{t_1}^1 \wedge a_{t_2}^2 \wedge \cdots \wedge a_{t_n}^n \wedge x \neq 0\}. \end{aligned}$$

The proof of the other assertion is similar. □

THEOREM 5.5. *Let (L_0, L_1, L_2) be a zero-dimensional biframe. The family*

$$\begin{aligned}\mathcal{S} \;=\; &\{E_a^l \mid a \in L_1 \text{ has complement in } L_2\}\\ &\cup\{E_a^r \mid a \in L_2 \text{ has complement in } L_1\}\end{aligned}$$

is a subbase for a transitive, totally bounded, quasi-uniformity $\mathcal{E}_F$ on L_0, for which $L_0^i = L_i$ $(i = 1, 2)$.

Proof. Let $\mathcal{E}_F$ be the filter of $(W\,\mathrm{Ent}(L), \subseteq)$ generated by $\mathcal{S}$.

We prove the admissibility condition by showing, equivalently, that (L_0, L_0^1, L_0^2) is a biframe. In order to do this it suffices to see that $L_i = L_0^i$ $(i = 1, 2)$.

For $i = 1$ we have

$$\begin{aligned}L_0^1 \;&=\; \Big\{x \in L_0 \mid x = \bigvee\{y \in L_0 \mid y \overset{\mathcal{E}_F}{\lhd_1} x\}\Big\}\\ &\supseteq\; \{c \in L_1 \mid c \text{ has complement in } L_2\}\end{aligned}$$

since, for these c, $E_c^l \circ (c \oplus c) \subseteq c \oplus c$, that is, $c \overset{\mathcal{E}_F}{\lhd_1} c$.

The inclusion $L_1 \subseteq L_0^1$ now follows from the facts that L_0^1 is a subframe of L_0 and that (L_0, L_1, L_2) is zero-dimensional.

The proof for $i = 2$ can be done similarly because, for $c \in L_2$ with complement in L_1, $(c \oplus c) \circ E_c^r \subseteq c \oplus c$.

Since, for each complemented a, E_a^l and E_a^r are finite, transitive, Weil entourages, $\mathcal{E}_F$ is a totally bounded transitive quasi-uniformity on L_0.

Finally let us show that $L_0^1 \subseteq L_1$. Let $y \overset{\mathcal{E}_F}{\lhd_1} x$. It suffices to show the existence of $z \in L_1$ such that $y \leq z \overset{\mathcal{E}_F}{\lhd_1} x$. Let $E \in \mathcal{E}_F$ be a transitive entourage such that $st_1(y, E) \leq x$. As we have already seen, $st_1(y, E) \overset{\mathcal{E}_F}{\lhd_1} x$. The following lemma ensures the existence of that z:

LEMMA 5.6. *For any $x \in L_0$ and $E \in \mathcal{E}_F$ there exists $E' \in \mathcal{E}_F$ such that $st_i(x, E') \leq st_i(x, E)$ and $st_i(x, E') \in L_i$ $(i = 1, 2)$.*

Proof. Any $E \in \mathcal{E}_F$ contains an $E' \in \mathcal{E}_F$ of the form $E' = (\bigcap_{j=1}^n E_{a_j}^l) \cap (\bigcap_{j=1}^m E_{b_j}^r)$. Let us show that $st_1(x, E') \in L_1$.

By Lemma 5.4, $st_1(x, E')$ is equal to

$$\begin{aligned}\bigvee\big\{&\alpha_{t_1}^1 \wedge \cdots \wedge \alpha_{t_n}^n \wedge \gamma_{u_1}^1 \wedge \cdots \wedge \gamma_{u_m}^m \mid t_1, \ldots, t_n, u_1, \ldots, u_m \in \{1, 2\},\\ &\beta_{t_1}^1 \wedge \cdots \wedge \beta_{t_n}^n \wedge \delta_{u_1}^1 \wedge \cdots \wedge \delta_{u_m}^m \wedge x \neq 0\big\},\end{aligned}$$

where, for each $i = 1, \ldots, n$,

$$\alpha_1^i = a_i^*, \qquad \alpha_2^i = a_i,$$
$$\beta_1^i = a_i^*, \qquad \beta_2^i = 1$$

and, for each $i = 1, \ldots, m$,

$$\gamma_1^i = b_i^*, \qquad \gamma_2^i = 1,$$
$$\delta_1^i = \beta_i^*, \qquad \delta_2^i = \beta_i.$$

The only problem is when there is some $i \in \{1, \ldots, n\}$ such that

$$x \wedge \beta^1_{t_1} \wedge \beta^2_{t_2} \wedge \cdots \wedge \beta^i_1 \wedge \cdots \wedge \beta^n_{t_n} \wedge \delta^1_{u_1} \wedge \cdots \wedge \delta^m_{u_m} \neq 0,$$

because then $\alpha^i_1 = a^*_i \in L_2$ appears in the join $st_1(x, E')$. But

$$x \wedge \beta^1_{t_1} \wedge \beta^2_{t_2} \wedge \cdots \wedge \beta^i_1 \wedge \cdots \wedge \beta^n_{t_n} \wedge \delta^1_{u_1} \wedge \cdots \wedge \delta^m_{u_m} \neq 0$$

implies that

$$x \wedge \beta^1_{t_1} \wedge \beta^2_{t_2} \wedge \cdots \wedge \beta^i_2 \wedge \cdots \wedge \beta^n_{t_n} \wedge \delta^1_{u_1} \wedge \cdots \wedge \delta^m_{u_m} \neq 0$$

thus

$$\alpha^1_{t_1} \wedge \alpha^{i-1}_{t_{i-1}} \wedge \alpha^i_2 \wedge \alpha^{i+1}_{t_{i+1}} \wedge \cdots \wedge \alpha^n_{t_n} \wedge \gamma^1_{u_1} \wedge \cdots \wedge \gamma^m_{u_m}$$

also appears in the join. Therefore, since the join of

$$\alpha^1_{t_1} \wedge \alpha^{i-1}_{t_{i-1}} \wedge a^*_i \wedge \alpha^{i+1}_{t_{i+1}} \wedge \cdots \wedge \alpha^n_{t_n} \wedge \gamma^1_{u_1} \wedge \cdots \wedge \gamma^m_{u_m}$$

and

$$\alpha^1_{t_1} \wedge \alpha^{i-1}_{t_{i-1}} \wedge a_i \wedge \alpha^{i+1}_{t_{i+1}} \wedge \cdots \wedge \alpha^n_{t_n} \wedge \gamma^1_{u_1} \wedge \cdots \wedge \gamma^m_{u_m}$$

is equal to

$$\alpha^1_{t_1} \wedge \alpha^{i-1}_{t_{i-1}} \wedge \alpha^{i+1}_{t_{i+1}} \wedge \cdots \wedge \alpha^n_{t_n} \wedge \gamma^1_{u_1} \wedge \cdots \wedge \gamma^m_{u_m}$$

we get rid of a^*_i.

By repeating this process we can get rid of all the a^*_i that appear in the join, which shows that this join belongs to L_1.

This completes the proofs of Lemma 5.6 and Theorem 5.5. □

J. Frith [11, Proposition 4.2] constructs a quasi-uniform structure compatible with the congruence lattice of a given frame. The construction of $\mathcal{E}_F$ given in Theorem 5.5 generalizes Frith's construction. For this reason we shall call $\mathcal{E}_F$ the *Frith structure* of the total part L_0 of a given zero-dimensional biframe (L_0, L_1, L_2).

THEOREM 5.7. *The totally bounded coreflection* $(L, \mathcal{E}_*)$ *of any transitive quasi-uniform frame* $(L, \mathcal{E})$ *is transitive.*

Proof. As in the proof of Theorem 4.7, one can easily show that

$$\{E^l_c \mid c \in L^1, c \overset{\mathcal{E}}{\lhd}_1 c\} \cup \{E^r_c \mid c \in L^2, c \overset{\mathcal{E}}{\lhd}_2 c\}$$

is a subbase for a transitive, totally bounded, quasi-uniformity $\mathcal{F}$ on L. It suffices then to check that the corresponding quasi-proximities (strong inclusions in [24]) $(\overset{\mathcal{E}_*}{\lhd}_1, \overset{\mathcal{E}_*}{\lhd}_2)$ and $(\overset{\mathcal{F}}{\lhd}_1, \overset{\mathcal{F}}{\lhd}_2)$ do coincide. Now this follows from (5.2) and Proposition 3.5 (in a very similar way to the proof of the analogous identity in Theorem 4.7). □

Recall that for any frame homomorphism $h : L \to M$, there is its *right adjoint* $h_* : M \to L$ such that $h(x) \leq y$ if and only if $x \leq h_*(y)$, explicitly given by $h_*(y) = \bigvee\{x \in L \mid h(x) \leq y\}$. It satisfies $hh_* \leq \mathrm{id}_M$ and $h_*h \geq \mathrm{id}_L$. If h maps onto M, then $hh_* = \mathrm{id}_M$. Clearly, h_* is dense whenever h is onto.

LEMMA 5.8. *Let $h : L \to M$ be a dense frame homomorphism. Then*:

(a) *For every $x, y \in M$, $(h \oplus h)_*(x \oplus y) = h_*(x) \oplus h_*(y)$.*

(b) *For every $E \in M \oplus M$, $(h \oplus h)_*(E) = \bigvee_{(x,y)\in E}(h_*(x) \oplus h_*(y))$.*

(c) *For every $E \in M \oplus M$, $(h \oplus h)_*(E)^2 \subseteq \bigvee\{h_*(x) \oplus h_*(w) \mid (x, y), (z, w) \in E, y \wedge z \neq 0\}$. Moreover, if h_* is dense then the equality holds.*

Proof. (a) Since $(h \oplus h)(h_*(x) \oplus h_*(y)) = hh_*(x) \oplus hh_*(y) \subseteq x \oplus y$, we have $h_*(x) \oplus h_*(y) \subseteq (h \oplus h)_*(x \oplus y)$.

On the other hand, for any $(\alpha, \beta) \in E$, $\alpha \neq 0$, $\beta \neq 0$, with $(h\oplus h)(E) \subseteq x\oplus y$, we have $(h(\alpha), h(\beta)) \in x \oplus y$. Since h is dense, $h(\alpha) \neq 0$ and $h(\beta) \neq 0$, thus $h(\alpha) \leq x$ and $h(\beta) \leq y$. Hence $(\alpha, \beta) \in h_*(x) \oplus h_*(y)$. We have proved that $\bigcup\{E \in L \oplus L \mid (h \oplus h)(E) \subseteq x \oplus y\} \subseteq h_*(x) \oplus h_*(y)$ and the inclusion $(h \oplus h)_*(x \oplus y) \subseteq h_*(x) \oplus h_*(y)$ follows.

(b) Let $(\alpha, \beta) \in F$ with $(h \oplus h)(F) \subseteq E$. Then $(h(\alpha), h(\beta)) \in E$ and, since $(\alpha, \beta) \leq (h_*h(\alpha), h_*h(\beta))$, the inclusion $(h \oplus h)_*(E) \subseteq \bigvee_{(x,y)\in E}(h_*(x) \oplus h_*(y))$ holds.

The reverse inclusion is a consequence of the inclusion $h_*(x) \oplus h_*(y) \subseteq (h \oplus h)_*(x \oplus y)$ in (a).

Note that here the denseness of h is not needed.

(c) By Lemma 2.1

$$\begin{aligned}(h \oplus h)_*(E)^2 \;=\; & \bigcup\{F \in L \oplus L \mid (h \oplus h)(F) \subseteq E\} \circ \\ & \bigcup\{F \in L \oplus L \mid (h \oplus h)(F) \subseteq E\}.\end{aligned}$$

Consider $(\alpha, \beta) \in F_1$ and $(\beta, \gamma) \in F_2$ with $(h \oplus h)(F_1), (h \oplus h)(F_2) \subseteq E$ and $\beta \neq 0$. Then $(\alpha, \gamma) \leq (h_*h(\alpha), h_*h(\gamma))$ with $(h(\alpha), h(\beta)), (h(\beta), h(\gamma)) \in E$ and, by denseness, $h(\beta) \neq 0$, which proves the desired inclusion.

On the other hand, if $(x, y), (z, w) \in E$ with $y \wedge z \neq 0$, then also $h_*(y) \wedge h_*(z) \neq 0$ since h_* is dense, and $(h_*(x), h_*(y))$ and $(h_*(z), h_*(w))$ belong to $(h \oplus h)_*(E)$; it follows that $(h_*(x), h_*(w)) \in (h \oplus h)_*(E)^2$. □

PROPOSITION 5.9. *Let $h : L \to M$ be a dense frame homomorphism. If $E \in M \oplus M$ is transitive then $(h \oplus h)_*(E)$ is transitive.*

Proof. Let $E \in M \oplus M$ be transitive. By Lemma 5.8(c),

$$(h \oplus h)_*(E)^2 \subseteq \bigvee\{h_*(x) \oplus h_*(w) \mid (x, y), (z, w) \in E, y \wedge z \neq 0\}$$

thus $(h \oplus h)_*(E)^2$ is contained in $\bigvee\{h_*(x) \oplus h_*(w) \mid (x, w) \in E\}$ which, by Lemma 5.8(b), is equal to $(h \oplus h)_*(E)$. □

Let us recall that a quasi-uniform frame $(M, \mathcal{F})$ is *complete* if every dense quasi-uniform surjection $(L, \mathcal{E}) \to (M, \mathcal{F})$ is an isomorphism, and a *completion* of $(L, \mathcal{E})$ is a complete $(M, \mathcal{F})$ together with a dense surjection $(M, \mathcal{F}) \to (L, \mathcal{E})$. In [12, Theorem 3.4] the authors prove that each quasi-uniform frame has a unique completion. In the language of entourage quasi-uniformities [8], if $h : (CL, C\mathcal{V}) \to (L, \mathcal{V})$ is the completion of $(L, \mathcal{V})$ then $\{v_{h_*} : v \in \mathcal{V}\}$ is a base for $C\mathcal{V}$. Let $v \in \mathcal{V}$ and let F be the Weil entourage corresponding to v in the equivalent Weil quasi-uniform structure on L. Then $(h \oplus h)_*(F)$ corresponds to v_{h_*}.

COROLLARY 5.10. *The completion $(CL, C\mathcal{E})$ of any transitive quasi-uniform frame $(L, \mathcal{E})$ is transitive.*

Proof. By the remarks preceding this corollary it suffices to apply Proposition 5.9. □

B. Banaschewski [2, Proposition 3] gives the universal zero-dimensional compactification of a zero-dimensional biframe. The following theorem gives an alternative proof of Banaschewski's result.

THEOREM 5.11. *The quasi-uniform completion of the Frith structure yields the universal zero-dimensional compactification of a given zero-dimensional biframe.*

Proof. Let $L = (L_0, L_1, L_2)$ be a zero-dimensional biframe and let $\mathcal{E}_F$ be the Frith structure on L_0. Let $M = (M_0, M_1, M_2)$ be any compact zero-dimensional biframe, and let $h : M \to L$ be any biframe map.

The biframe M with the Frith structure is totally bounded and compact hence complete. So, by Theorem 3.7 of [12], there exists $\overline{h}$ such that the diagram

$$\begin{tikzcd} & (CL_0, C\mathcal{E}_F) \arrow[d, "\phi_L"] \\ M_0 \arrow[ur, "\overline{h}"] \arrow[r, "h"'] & L_0 \end{tikzcd}$$

commutes. It follows from [9, Proposition 3.7] that $\overline{h}$ is a biframe map. It remains to prove uniqueness: suppose $g : M_0 \to (CL_0, C\mathcal{E}_F)$ such that $\phi_L g = \phi_L \overline{h}$. The map ϕ_L is dense hence monic for frames [5, Lemma 6]. Therefore $g = \overline{h}$. □

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