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The Gauge of a Uniform Frame

JORGE PICADO
Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal

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Abstract. This note characterizes gauge structures for frames, i.e., families of metric diameters which completely describe frame uniformities. Some applications are presented.

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One of the most important results of the theory of uniform spaces states that any uniform cover in a uniform space can be 'approximated' by a pseudometric (see, for instance, [2]). Thus every uniformity on a set X gives rise to a family of pseudometrics on X. Moreover, this family of pseudometrics can be used to recover the original uniformity. From this it follows that uniform spaces can be described in terms of those families of pseudometrics, usually called 'gauge' structures. Such a description was given by Bourbaki in [1]. The efficiency of this tool can be observed in [3]. This is one of the three well-known equivalent ways of describing a uniformity on a set X: as a collection of relations on X (*à la* Weil [12]), as a collection of covers of X (*à la* Tukey [11]), or as a collection of pseudometrics on X (*à la* Bourbaki), where each of these collections satisfies a respective set of axioms. The approaches of Weil and Tukey have already been considered in the more general context of pointless spaces (frames) [4, 6, 7]. Owing to current interest in uniform frames, it would seem desirable to complete the picture by extending gauge structures to frames and by getting a characterization of frame uniformities that is analogous to the spatial one given in terms of pseudometrics.

The main purpose of this note is to present such characterization. The classical pseudometric is here replaced by the notion of metric diameter of Pultr [8].

1. Preliminaries

1.1. FRAMES

A *frame* (see [5] for details) is a complete lattice L satisfying the infinite distribution law $x \wedge (\bigvee S) = \bigvee\{x \wedge s \mid s \in S\}$ for all $x \in L$ and $S \subseteq L$.

If L and M are frames, a mapping $f : L \to M$ is said to be a *frame homomorphism* if it preserves joins (including the zero 0) and finite meets (including the unit 1). The resulting category `Frm` is called the category of frames. $K \subseteq L$ is a *subframe* of L if $\{0, 1\} \subseteq K$ and K is closed under finite meets and arbitrary joins. Recall also the notation $U \cdot x := \bigvee\{u \mid u \in U, u \wedge x \neq 0\}$, for $U \subseteq L$ and $x \in L$.

A cover of a frame L is a subset $U \subseteq L$ such that $\bigvee U = 1$. The set $Cov\, L$ of all covers of L can be preordered: a cover U *refines* a cover V, written $U \leq V$, if for each $u \in U$ there is $v \in V$ with $u \leq v$. This is a preordered set with meets: take for $U \wedge V$ the cover $\{u \wedge v \mid u \in U, v \in V\}$, which is obviously a common refinement of U and V.

1.2. METRIC FRAMES

The set of all non-negative reals augmented by $+\infty$ will be denoted by $\mathbb{R}^+$. A *prediameter* on a frame L (cf. [8, 9, 10]) is a mapping $d : L \to \mathbb{R}^+$ such that

(D1) $d(0) = 0$,
(D2) $d(x) \leq d(y)$ if $x \leq y$, and
(D3) $d(x \vee y) \leq d(x) + d(y)$ if $x \wedge y \neq 0$.

For any prediameter d on L and any $\epsilon > 0$, let U_ϵ^d denote the set

$$\{x \in L \mid d(x) < \epsilon\}.$$

A prediameter d on L is said to be *compatible* if

(D4) for each $x \in L$, $x \leq \bigvee\{y \in L \mid y \overset{d}{\triangleleft} x\}$,

where $y \overset{d}{\triangleleft} x$ means that $U_\epsilon^d \cdot y \leq x$ for some $\epsilon > 0$.

A *star-prediameter* is a prediameter that satisfies

($*$) if $S \subseteq L$ is *strongly connected* (i.e., $x \wedge y \neq 0$ for every $x, y \in S$), then

$$d\Big(\bigvee S\Big) \leq 2 \sup\{d(x) \mid x \in S\}.$$

A prediameter is *metric* if

(M) for any $\alpha < d(x)$ and $\epsilon > 0$, there exist $y, z \leq x$ such that $d(y), d(z) < \epsilon$ and $\alpha < d(y \vee z)$.

Condition (M) implies ($*$). In fact, (M) even implies the stronger property

($*'$) for each $x \in L$ and each $S \subseteq L$ such that $x \wedge y \neq 0$ for all $y \in S$

$$d\Big(x \vee \bigvee S\Big) \leq d(x) + \sup\{d(y) + d(z) \mid y, z \in S, y \neq z\}.$$

A prediameter which, moreover, satisfies

(D5) for each $\epsilon > 0$, U_ϵ^d is a cover,

is called a *diameter*. Observe that when d is a compatible diameter the equality holds in (D4).

It should be noted that diameters generalize the usual notion of the diameter of subsets of metric spaces and that they provide a pointfree expression of the usual notion of metrizations of spaces: in any metrizable space X, the compatible metric diameters d on its frame of open sets correspond exactly to the metrizations ρ of X by the relations $d(U) = \sup\{\rho(x, y) \mid x, y \in U\}$ and $\rho(x, y) = \inf\{d(U) \mid x, y \in U\}$ (see, e.g., [8, 9]).

A *prediametric frame* is a pair (L, d), where d is a compatible prediameter on the frame L. A *metric frame* is a pair (L, d), where d is a compatible metric diameter on L.

For prediametric frames (L_1, d_1), (L_2, d_2), a frame homomorphism $f\colon L_1 \to L_2$ is called *uniform* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $U^{d_2}_{\delta} \leq [U^{d_1}_{\epsilon}]$. Prediametric frames and uniform maps form the category of prediametric frames. Its full subcategory of metric frames will be denoted by MFrm.

For any star-diameter d on L, there is a unique metric diameter $\tilde{d}$, given by

$$\tilde{d}(x) = \inf_{\epsilon > 0} \sup\{d(y \vee z) \mid y, z \leq x, d(y), d(z) \leq \epsilon\},$$

such that $\frac{1}{2}d \leq \tilde{d} \leq d$ ([9, Proposition 3.4]). Moreover, $U^{\tilde{d}}_{\epsilon} \cdot x = U^{d}_{\epsilon} \cdot x$, for any $x \in L$ and $\epsilon > 0$.

1.3. UNIFORM FRAMES

A non-void filter $\mathcal{U}$ of $(Cov\, L, \leq)$ is said to be a *uniformity* ([4, 7]) provided that:

(U1) for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^* := \{V \cdot v \mid v \in V\} \leq U$;

(U2) for each $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{U}}{\lhd} x\}$, where $y \overset{\mathcal{U}}{\lhd} x$ means that $U \cdot y \leq x$ for some $U \in \mathcal{U}$.

A *uniformity basis* is just a filter basis of $(Cov\, L, \leq)$ satisfying (U1) and (U2).

A *uniform frame* is a pair $(L, \mathcal{U})$, where $\mathcal{U}$ is a uniformity on the frame L. For uniform frames $(L_1, \mathcal{U}_1)$, $(L_2, \mathcal{U}_2)$, a frame homomorphism $f : L_1 \to L_2$ is called *uniform* if, for every $U \in \mathcal{U}_1$, $f[U] \in \mathcal{U}_2$. The resulting category will be denoted by UFrm.

Let d be a compatible metric diameter on L. Since $\epsilon \leq \delta$ implies that $U^d_{\epsilon} \subseteq U^d_{\delta}$, (D4) and (D5) say that the U^d_{ϵ} generate a nearness, i.e., the U^d_{ϵ} form a filter basis of $(Cov\, L, \leq)$ satisfying (U2). This is actually a uniformity since $U^{d^*}_{\epsilon} \leq U^d_{3\epsilon}$. Conversely, any uniformity with a countable basis is obtained in this way from a compatible metric diameter:

THEOREM (Pultr [8, Theorem 4.6]). *$\mathcal{U}$ is a uniformity with a countable basis if and only if there is a compatible metric diameter d such that* $\mathcal{U} = \{U \subseteq L \mid \exists \epsilon > 0 : U^d_{\epsilon} < U\}$.

We also need to recall the following proposition from (Pultr [10]):

PROPOSITION. *Let $f : L \to M$ be a surjective frame homomorphism and d a metric diameter* (*resp. star diameter*) *on L. Then*

$$\overset{\circ}{d}(y) = \inf\{d(x) \mid x \in L, y \le f(x)\}$$

defines a metric diameter (*resp. star diameter*) *on M.*

2. Frame Uniformities in the Sense of Bourbaki

In the sequel L will always denote a frame.

LEMMA 2.1. *Let d_1 and d_2 be star-diameters on L. Then*

$$\begin{aligned} d_1 \vee d_2 : L &\to \mathbb{R}^+ \\ x &\mapsto \max(d_1(x), d_2(x)) \end{aligned}$$

is a star-diameter.

Proof. Conditions (D1), (D2), (D3) and the star-condition are trivially satisfied. To check condition (D5) just observe that, for any $\epsilon > 0$, $U_\epsilon^{d_1 \vee d_2} = U_\epsilon^{d_1} \cap U_\epsilon^{d_2} = U_\epsilon^{d_1} \wedge U_\epsilon^{d_2}$. □

A word of warning: $d = d_1 \vee d_2$ is not necessarily metric, even if d_1 and d_2 are metric. Nevertheless, we can take the associated metric diameter $\tilde{d}$ that we shall denote by $d_1 \sqcup d_2$.

We say that a non-empty collection $\mathcal{G}$ of metric diameters on L is a *gauge structure* on L if it satisfies the following conditions:

(G1) $d_1 \sqcup d_2 \in \mathcal{G}$ whenever $d_1, d_2 \in \mathcal{G}$;
(G2) if d is a metric diameter and

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \exists d' \in \mathcal{G} : U_\delta^{d'} \subseteq U_\epsilon^{\delta},$$

then $d \in \mathcal{G}$,
(G3) for every $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{G}}{\lhd} x\}$, where $y \overset{\mathcal{G}}{\lhd} x$ means that there is $d \in \mathcal{G}$ such that $y \overset{d}{\lhd} x$.

A collection $\mathcal{G}$ satisfying only (G2) and (G3) is a *basis* for the gauge obtained by taking all possible metric diameters $d_1 \sqcup \cdots \sqcup d_n$, where $d_1, \ldots, d_n \in \mathcal{G}$. Any collection of metric diameters on L is contained in a smallest possible gauge, called the *gauge generated* by $\mathcal{G}$.

PROPOSITION 2.2. *Let $\mathcal{U}$ be a uniformity on L. The family $\psi(\mathcal{U}) := \{d_\alpha \mid \alpha \in \Lambda\}$ of all metric diameters such that, for every $\alpha \in \Lambda$ and $\epsilon > 0$, $U_\epsilon^{d_\alpha} \in \mathcal{U}$, is a gauge structure on L.*

Proof. (G1) Let $\alpha, \beta \in \Lambda$ and $\epsilon > 0$. Since $U_\epsilon^{d_\alpha} \wedge U_\epsilon^{d_\beta} \le U_\epsilon^{d_\alpha \sqcup d_\beta}$, $U_\epsilon^{d_\alpha \sqcup d_\beta} \in \mathcal{U}$. Hence $d_\alpha \sqcup d_\beta \in \psi(\mathcal{U})$.

(G2) Assume d is a metric diameter such that

$$\forall\epsilon > 0 \quad \exists\delta > 0 \quad \exists d_\alpha \in \psi(\mathcal{U}) : U_\delta^{d_\alpha} \subseteq U_\epsilon^d.$$

Then $U_\epsilon^d \in \mathcal{U}$ for any $\epsilon > 0$, i.e., $d \in \psi(\mathcal{U})$.

(G3) Let $x \in L$. By hypothesis, $x = \bigvee\{y \in L \mid y \overset{\mathcal{U}}{\lhd} x\}$. It suffices to show that $y \overset{\mathcal{U}}{\lhd} x$ implies $y \overset{\psi(\mathcal{U})}{\lhd} x$. So consider $U \in \mathcal{U}$ such that $U\cdot y \leq x$ and take inductively $U_1, U_2, \ldots, U_n, \ldots$ such that $U_1 = U, U_{n+1}^* \leq U_n$. The family $\{U_1, U_2, \ldots, U_n, \ldots\}$ generates a uniformity with a countable basis. So, according to Theorem 1.3, there is a metric diameter d in $\psi(\mathcal{U})$ such that $U_\epsilon^d \subseteq U$ for some $\epsilon > 0$. Hence $U_\epsilon^d\cdot y \leq U\cdot y \leq x$. □

We call $\psi(\mathcal{U})$ the *gauge of the uniformity* $\mathcal{U}$. Note that the converse

$$y \overset{\psi(\mathcal{U})}{\lhd} x \text{ implies } y \overset{\mathcal{U}}{\lhd} x$$

is obviously true.

PROPOSITION 2.3. *Let $\mathcal{G}$ be a gauge structure on L. The family $\mathcal{B}_\mathcal{G} := \{U_\epsilon^d \mid d \in \mathcal{G}, \epsilon > 0\}$ is a basis for a uniformity $\phi(\mathcal{G})$ on L.*

Proof. For $\epsilon, \delta > 0$ and $d_1, d_2 \in \mathcal{G}$ take $\gamma = \min(\frac{\epsilon}{2}, \frac{\delta}{2})$. Immediately,

$$U_\gamma^{d_1\sqcup d_2} \subseteq U_\epsilon^{d_1} \cap U_\delta^{d_2} = U_\epsilon^{d_1} \wedge U_\delta^{d_2}$$

and $U_\gamma^{d_1\sqcup d_2} \in \mathcal{B}_\mathcal{G}$ so $\mathcal{B}_\mathcal{G}$ is a filter basis.

Let us now show properties (U1) and (U2).

(U1) We prove (U1) by showing that, for any $d \in \mathcal{G}$ and $\epsilon > 0$, $U_{\frac{\epsilon}{3}}^{d^*} \leq U_\epsilon^d$. Consider $x \in U_{\frac{\epsilon}{3}}^d$ and choose $y_0 \in U_{\frac{\epsilon}{3}}^d$ such that $y_0 \wedge x \neq 0$. The set $S = \{y \vee y_0 \mid y \in U_{\frac{\epsilon}{3}}^d, y \wedge x \neq 0\}$ is strongly connected and $U_{\frac{\epsilon}{3}}^d\cdot x = \bigvee S$. Thus $d(U_{\frac{\epsilon}{3}}^d\cdot x) \leq 2\sup\{d(x) \mid x \in S\} < \epsilon$, so $U_{\frac{\epsilon}{3}}^d\cdot x \in U_\epsilon^d$.

(U2) It is obvious that $x \overset{\mathcal{G}}{\lhd} y$ if and only if $x \overset{\phi(\mathcal{G})}{\lhd} y$. □

In other words, $\phi(\mathcal{G}) = \{U \in L \mid \exists d \in \mathcal{G}\, \exists\epsilon > 0 : U_\epsilon^d \leq U\}$ is a uniformity on L.

THEOREM 2.4. *There is a one-to-one correspondence between the class of uniformities on L and the class of gauge structures on L.*

Proof. We first show that $\psi\phi(\mathcal{G}) = \mathcal{G}$, for any gauge structure $\mathcal{G}$. If $d \in \psi\phi(\mathcal{G})$ then $U_\epsilon^d \in \phi(\mathcal{G})$, for every $\epsilon > 0$, i.e., for every $\epsilon > 0$ there is a $\delta > 0$ and a $d' \in \mathcal{G}$ such that $U_\delta^{d'} \subseteq U_\epsilon^d$. Hence $d \in \mathcal{G}$. The reverse inclusion is trivial.

On the other hand, for any uniformity $\mathcal{U}$, the inclusion $\phi\psi(\mathcal{U}) \subseteq \mathcal{U}$ is obvious. The reverse inclusion is a consequence of Theorem 1.3: for $U \in \mathcal{U}$ take inductively

$U_1, U_2, \ldots, U_n, \ldots$ such that $U_1 = U$, $U_{n+1}^* \leq U_n$ and, according to Theorem 1.3, the metric diameter d in $\psi(\mathcal{U})$ such that $U_\epsilon^d \subseteq U$ for some $\epsilon > 0$. Hence $U \in \phi\psi(\mathcal{U})$. □

In conclusion, we can treat uniformities as gauge structures. Naming the pairs $(L, \mathcal{G})$, where L is a frame and $\mathcal{G}$ is a gauge structure on L, *as gauge frames* we have a bijective correspondence between uniform frames and gauge frames. Under this correspondence, uniform homomorphisms correspond precisely to *the gauge homomorphisms*, i.e., to the frame maps $f : L_1 \to L_2$ between gauge frames $(L_1, \mathcal{G}_1)$ and $(L_2, \mathcal{G}_2)$ such that, for every $\epsilon > 0$ and $d_1 \in \mathcal{G}_1$, there exist $\delta > 0$ and $d_2 \in \mathcal{G}_2$ satisfying $U_\delta^{d_2} \leq f[U_\epsilon^{d_1}]$. Therefore, the category `UFrm` is isomorphic to the category of gauge frames and gauge homomorphisms.

Gauge frames are the exact pointfree translation of the gauge spaces of [2].

We remark that gauge structures could be defined as collections of star-diameters satisfying axioms (G2), (G3) and

$$\text{(G1}'\text{)} \quad d_1 \vee d_2 \in \mathcal{G} \text{ whenever } d_1, d_2 \in \mathcal{G} \tag{2.1}$$

because it can be proved, in a similar way, that these collections of star diameters on L are also in one-to-one correspondence to the uniformities on L.

3. Some Consequences

Gauge structures make clear the nature of the generalization from metric frames to uniform frames: a metric frame is a uniform frame whose gauge is generated by a single diameter.

Moreover, metric frames generate all gauge frames, in the sense that each gauge frame is a quotient of a coproduct of metric frames. To conclude this we need first to show some facts about metric and gauge frames.

For any diameter d on L let L_d denote the subframe

$$\left\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{d}{\lhd} x\}\right\}$$

of L. Note that, for any $x \in L$, $\bigvee\{y \in L \mid y \overset{d}{\lhd} x\} \in L_d$. Indeed, denoting $\bigvee\{y \in L \mid y \overset{d}{\lhd} x\}$ by x_d we have that $x_d = \bigvee\{y \in L \mid y \overset{d}{\lhd} x_d\}$ because if $y \overset{d}{\lhd} x$, i.e., if there is $\epsilon > 0$ with $U_\epsilon^d \cdot y \leq x$ then $U_{\frac{\epsilon}{2}}^d \cdot (U_{\frac{\epsilon}{2}}^d \cdot y) \leq U_\epsilon^d \cdot y \leq x$, that is, $U_{\frac{\epsilon}{2}}^d \cdot y \overset{d}{\lhd} x$ and so $U_{\frac{\epsilon}{2}}^d \cdot y \leq x_d$.

Therefore we may consider the map $f_d : L \to L_d$ given by $f_d(x) = x_d$ which is a surjective frame homomorphism. By Proposition 1.3, $d(y) = \inf\{d(x) \mid y \leq x_d\}$ defines a metric diameter on L_d whenever d is a metric diameter on L. Note that, because of the particular definitions of L_d and f_d, $\overset{\circ}{d}$ is the restriction of d to L_d. Moreover, we have:

PROPOSITION 3.1. *Let d be a metric diameter on L. Then $(L_d, \overset{\circ}{d})$ is a metric frame.*

Proof. It remains to prove the compatibility of $\overset{\circ}{d}$. Let $x \in L_d$. Then $x = \bigvee\{y \in L \mid y \overset{d}{\lhd} x\}$ and, since $x = f_d(x)$, we have $x = f_d(\bigvee\{y \in L \mid y \overset{d}{\lhd} x\})$, that is,

$$\begin{aligned} x &= \bigvee\{f_d(y) \mid y \in L, y \overset{d}{\lhd} x\} \leq \bigvee\{f_d(y) \mid y \in L, f_d(y) \overset{d}{\lhd} x\} \\ &= \bigvee\{z \in L_d \mid z \overset{d}{\lhd} x\}. \end{aligned}$$

But, for any $z \in L_d$ and $\epsilon > 0$, $U_\epsilon^{\overset{\circ}{d}} \cdot z \leq U_\epsilon^d \cdot z$ so

$$z \overset{d}{\lhd} x \Rightarrow z \overset{\overset{\circ}{d}}{\lhd} x \quad (z, x \in L_d)$$

and, consequently, $x \leq \bigvee\{z \in L_d \mid z \overset{\overset{\circ}{d}}{\lhd} x\} \leq x$. □

PROPOSITION 3.2. *Let d be a metric diameter on L. For each $x \in L$ and $\epsilon > 0$, there exists $y \in L_d$ such that $x \leq y$ and $d(y) < d(x) + \epsilon$.*

Proof. For any $x \in L$ and $\epsilon > 0$, since $\overset{\circ}{d}$ is a diameter on L_d, we have that

$$\begin{aligned} x &= x \wedge \bigvee\left\{a \in L_d \mid \overset{\circ}{d}(a) < \frac{\epsilon}{2}\right\} \\ &= \bigvee\left\{x \wedge a \mid a \in L_d, x \wedge a \neq 0, \overset{\circ}{d}(a) < \frac{\epsilon}{2}\right\} \\ &\leq \bigvee\left\{a \in L_d \mid x \wedge a \neq 0, \overset{\circ}{d}(a) < \frac{\epsilon}{2}\right\}. \end{aligned}$$

Then $\bigvee\{a \in L_d \mid x \wedge a \neq 0, \overset{\circ}{d}(a) < \frac{\epsilon}{2}\} \in L_d$ is the required element y. In fact, using property $(*')$, $d(y) = d(x \vee y) \leq d(x) + \sup\{d(a_1) + d(a_2) \mid a_1, a_2 \in L_d, a_1 \neq a_2, a_1 \wedge x \neq 0, a_2 \wedge x \neq 0, \overset{\circ}{d}(a_1), \overset{\circ}{d}(a_2) < \frac{\epsilon}{2}\} < d(x) + \epsilon$. □

It follows from Proposition 3.2 that the relation $\overset{\overset{\circ}{d}}{\lhd}$ is the restriction of $\overset{d}{\lhd}$ to L_d. On the other hand, we also have:

PROPOSITION 3.3. *Let M be a subframe of L. If d is a star diameter on M then*

$$\bar{d}(x) = \inf\{d(y) \mid y \in M, x \leq y\}$$

defines a star diameter on L.

Proof. The proof that $\bar{d}$ is a diameter is straightforward. Let us verify the star property. Let $S \subseteq L$ be strongly connected. For any $\epsilon > 0$ and $s \in S$ there is an $y_s \in M$ with $d(y_s) < \bar{d}(s) + \epsilon$ and $s \leq y_s$. Clearly $S_M = \{y_s \mid s \in S\}$ is strongly connected thus

$$\bar{d}\Big(\bigvee S\Big) \leq d\Big(\bigvee S_M\Big) \leq 2\sup\{d(y_s) \mid s \in S\} < 2\sup\{\bar{d}(s) \mid s \in S\} + \epsilon.$$

Hence $\bar{d}(\bigvee S) \leq 2\sup\{\bar{d}(s) \mid s \in S\}$. □

However, $\bar{d}$ is not necessarily metric, even if d is. Nevertheless, there is, as before, a unique metric diameter $\tilde{d}$ on L such that $\frac{1}{2}\bar{d} \leq \tilde{d} \leq \bar{d}$ and $U_\epsilon^{\tilde{d}} \cdot x = U_\epsilon^{\bar{d}} \cdot x$ for every $x \in L$ and $\epsilon > 0$. Also $L_{\bar{d}} = L_{\tilde{d}}$.

PROPOSITION 3.4. *Let M be a subframe of L. For any compatible metric diameter d on M, $L_{\bar{d}} = L_{\tilde{d}} = M$.*

Proof. Consider $x \in L_{\tilde{d}}$. Let us start by proving that, for any $y \in L$ such that $y \overset{\tilde{d}}{\lhd} x$, there exists a $z \in M$ satisfying $y \leq z \overset{\tilde{d}}{\lhd} x$.

By hypothesis, there is some $\epsilon > 0$ with $U_\epsilon^{\tilde{d}} \cdot y \leq x$. Also

$$\begin{aligned} y &= y \wedge \bigvee\left\{z' \in M \mid d(z') < \frac{\epsilon}{2}\right\} \\ &\leq \bigvee\left\{z' \in M \mid z' \wedge y \neq 0, d(z') < \frac{\epsilon}{2}\right\} \in M. \end{aligned}$$

The element $z := \bigvee\{z' \in M \mid z' \wedge y \neq 0, d(z') \leq \frac{\epsilon}{2}\}$ is the element in M we are looking for. In fact, $U_{\frac{\epsilon}{2}}^{\tilde{d}} \leq U_\epsilon^{\tilde{d}} \cdot y$.

Consider $w \in L$ with $w \wedge z \neq 0$ and $\tilde{d}(w) < \frac{\epsilon}{2}$. Then there exists $z' \in M$ such that $z' \wedge w \neq 0$, $z' \wedge y \neq 0$ and $d(z') < \frac{\epsilon}{2}$. Therefore $w \leq z' \vee w$, $(z' \vee w) \wedge y \neq 0$ and $\tilde{d}(z' \vee w) \leq \tilde{d}(z') + \tilde{d}(w) = d(z') + \tilde{d}(w) < \epsilon$. Hence $U_{\frac{\epsilon}{2}}^{\tilde{d}} \cdot z \leq U_\epsilon^{\tilde{d}} \cdot y \leq x$, i.e., $y \leq z \overset{\tilde{d}}{\lhd} x$.

Now, returning to the proof of the inclusion $L_{\tilde{d}} \subseteq M$ we have immediately that, for any $x \in L_{\tilde{d}}$,

$$x = \bigvee\{y \in L \mid y \overset{\tilde{d}}{\lhd} x\} \leq \bigvee\{z \in M \mid z \overset{\tilde{d}}{\lhd} x\} \leq x,$$

that is,

$$x = \bigvee\{z \in M \mid z \overset{\tilde{d}}{\lhd} x\} \in M.$$

Conversely, consider $x \in M$. Since $x = \bigvee\{y \in M \mid y \overset{d}{\lhd} x\}$, we only need to check that $y \overset{d}{\lhd} x$ implies $y \overset{\tilde{d}}{\lhd} x$ in order to conclude that $x \in L_{\tilde{d}}$. So, assume that $U_\epsilon^d \cdot y \leq x$ for some $\epsilon > 0$. Then $U_{\frac{\epsilon}{4}}^{\tilde{d}} \cdot y \leq x$ because $U_{\frac{\epsilon}{4}}^{\tilde{d}} \cdot y \leq U_\epsilon^d \cdot y$: for any $z \in L$ such that $z \wedge y \neq 0$ and $\tilde{d}(z) < \frac{\epsilon}{4}$ there exists, by Proposition 3.2, $w \in L_{\tilde{d}}$ such that $z \leq w$ and $\tilde{d}(w) < \tilde{d}(z) + \frac{\epsilon}{4} < \frac{\epsilon}{2}$. We have already proved that $L_{\tilde{d}} \subseteq M$ so $w \in M$. On the other hand, $w \wedge y \neq 0$ and $\frac{1}{2}d(w) = \frac{1}{2}\mathring{\tilde{d}}(w)\tilde{d}(w) < \frac{\epsilon}{2}$, i.e., $w \leq U_\epsilon^d \cdot y$. $\square$

Clearly, $\mathring{\bar{d}} = \mathring{\tilde{d}} = d$.

COROLLARY 3.5. *Let d be a metric diameter on L. Then $\bar{\mathring{d}} = \tilde{\mathring{d}} = d$.*

Proof. For each $x \in L$, $\bar{\mathring{d}}(x) = \inf\{\mathring{d}(y) \mid y \in L_d, x \leq y\}$, thus $\bar{\mathring{d}} \geq d(x)$. The equality $\bar{\mathring{d}}(x) = d(x)$ follows immediately from Proposition 3.2. Then, trivially, $\tilde{\mathring{d}} = d(x)$. $\square$

COROLLARY 3.6. *Let $\mathcal{G}$ be a gauge structure on L. For any $d \in \mathcal{G}$, the inclusion $(L_d, \mathring{d}) \hookrightarrow (L, \mathcal{G})$ is a uniform homomorphism.*

Proof. Let us show that, for any $\epsilon > 0$ and $d \in \mathcal{G}$, $U_\epsilon^{\bar{d}} \in \mathcal{G}$. For this, it suffices to check that $U_{\frac{\epsilon}{2}}^d \leq U_\epsilon^{\bar{d}}$.

Assume $x \in L$ is such that $d(x) < \frac{\epsilon}{2}$. By Proposition 3.2 there exists $y \in L_d$ satisfying $\bar{d}(y) = d(y) < d(x) + \frac{\epsilon}{2} < \epsilon$ and $x \leq y$. Hence $x \leq y \in U_\epsilon^{\bar{d}}$. $\square$

Notice that, for any gauge structure $\mathcal{G}$ on L, $(L, \bigsqcup_{d\in\mathcal{G}} d)$ is a metric frame and the identity $(L, \mathcal{G}) \to (L, \bigsqcup_{d\in\mathcal{G}} d)$ is uniform. Thus the inclusion $(L_d, \mathring{d}) \hookrightarrow (L, \bigsqcup_{d\in\mathcal{G}} d)$ is a uniform homomorphisrn of metric frames.

Now let $(L, \mathcal{G})$ be a gauge frame. For each $d \in \mathcal{G}$ consider the metric frame $(L_d, \mathring{d})$ of 3.1 and the coproduct $\bigoplus_{d\in\mathcal{G}}(L_d, \mathring{d})$. Then the sink

$$\sigma_{\mathcal{G}} := ((L_d, \mathring{d}) \hookrightarrow (L, \mathcal{G}))_{d\in\mathcal{G}}$$

given by 3.6, induces a unique f making the following diagram commutative:

$$\begin{tikzcd}
(L_d, \mathring{d}) \arrow[d] \arrow[dr] & \\
\bigoplus_{d\in\mathcal{G}}(L_d, \mathring{d}) \arrow[r, dashed, "f"'] & (L, \mathcal{G})
\end{tikzcd}$$

Since condition (G2) is clearly equivalent to $L = \bigvee_{d\in\mathcal{G}} L_d$, the sink $\sigma_{\mathcal{G}}$ is extremal epimorphic, which implies that f is a uniform surjection. Thus each gauge frame is a quotient of a coproduct of metric frames.

By the results above we may also describe gauge structures in a more categorical way, as sinks

$$\sigma = (m_i : (L_i, d_i) \hookrightarrow (L, d))_{i \in I}$$

of subobjects m_i of a metric frame (L, d), satisfying the following conditions (we use the usual lattice-theoretic notation in the partially ordered set $Sub(L, d)$ of all subobjects of (L, d)):

(S1) for every $m_i, m_j \in \sigma, m_i \vee m_j \in \sigma$;
(S2) if d is a metric diameter on L and

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \exists i \in I : U_\delta^{d_i} \subseteq U_\epsilon^d,$$

then $(L_d, \overset{\circ}{d}) \hookrightarrow (L, d)$ belongs to σ;
(S3) every σ is an extremal epi-sink.

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References

1. Bourbaki, N.: *Topologie Générale*, Ch. 9, Hermann, Paris, 1948.
2. Dugundji, J.: *Topology*, Brown Publishers, Dubuque (Iowa), 1989.
3. Gillman, L. and Jerison, M.: *Rings of Continuous Functions*, Graduate Texts in Math., Vol. 43, Springer-Verlag, Berlin etc., 1976.
4. Isbell, J.: Atomless parts of spaces, *Math. Scand.* **31** (1972), 5–32.
5. Johnstone, P. T.: *Stone Spaces*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1982.
6. Picado, J.: Weil uniformities for frames, *Comment. Math. Univ. Carolin.* **36** (1995), 357–370.
7. Pultr, A.: Pointless uniformities I. Complete regularity, *Comment. Math. Univ. Carolin.* **25** (1984), 91–104.
8. Pultr, A.: Pointless uniformities II. (Dia)metrization, *Comment. Math. Univ. Carolin.* **25** (1984), 105–120.
9. Pultr, A.: Diameters in locales: how bad they can be, *Comment. Math. Univ. Carolin.* **29** (1988), 731–742.
10. Pultr, A.: Categories of diametric frames, *Math. Proc. Cambridge Philos. Soc.* **105** (1989), 285–297.
11. Tukey, J. W.: *Convergence and Uniformity in Topology*, Ann. of Math. Studies 2, Princeton University Press, 1940.
12. Weil, A.: *Sur les Espaces à Structure Uniforms et sur la Topologie Générale*, Publications de l'Institute Mathématique de l'Université de Strasbourg, Hermann, Paris, 1938.