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On Connectedness via Closure Operators

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Abstract. This paper describes a convenient modification of the approach presented in the paper "Closure operators and connectedness" by G. Castellini and D. Hajek, which is shown to give a suitable generalization of left- and right-constant subcategories, both at the object and the morphism levels. We show in particular that the framework we introduce here allows the simultaneous study of the classes of connected topological spaces, of concordant continuous maps and of monotone continuous maps.

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0. Introduction

The study of connectedness and disconnectedness in a topological setting goes back to the 60's, when G. Preuß and H. Herrlich (cf. [16] and [13]) introduced the idea of left- and right-constant subcategories, which capture the categorical features of the subcategories of connected and of hereditarily disconnected spaces. Since then this subject has been widely investigated (for further references see [15] and [4]).

Recently, closure operators were used in the study of connectedness by W. Tholen and the author of this paper (cf. [17, 4, 5]). They proved that subcategories of c-connected and of c-separated objects (that is, of objects with a c-dense (c-closed) diagonal $\delta_X = \langle 1_X, 1_X\rangle : X \to X \times X$) describe left-(right-)constant subcategories in convenient settings (cf. [4]). However the same authors showed in [5] that, at the morphism level (that is: in the slices of a "good" category), this no longer holds true. In fact, they proved that, although concordant (dissonant) maps form a left-(right-)constant subcategory, there is no closure operator c describing them as c-connected (c-separated) objects.

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In the meantime, G. Castellini and D. Hajek in [3] proposed a different way of describing left- and right-constant subcategories via closure operators, defining c-indiscrete and c-discrete objects in a concrete category, for a given closure operator c. However, their conditions on the category are too restrictive, preventing in particular an application to sliced categories. The reason for this failure is their notion of c-indiscrete object, which is translated directly from the topological notion of indiscrete space, but does not lead itself to a natural and general construction in an abstract category.

Although we follow the ideas of the paper just mentioned, we use a different starting point: our notions of *c-coarse* and *c-fine objects* with respect to a closure operator c (that correspond to c-indiscrete and c-discrete objects in [3]) depend on two previously defined closure operators: the discrete closure operator, that is widely used in the literature, and a newly defined indiscrete closure operator. (We prefer the use of c-fine – instead of c-discrete – object because the latter one has already been used by other authors with a different meaning.)

Guided by the paradigmatic example of the connected-component closure operator conn, which has the property that the spaces X with conn_X indiscrete are exactly the connected spaces and the spaces X with conn_X discrete are exactly the hereditarily disconnected spaces, we define, in a category $\mathcal{X}$ equipped with a closure operator c, c-coarse objects and c-fine objects, and consider the corresponding (full) subcategories $\mathcal{C}oar(c)$ and $\mathcal{F}ine(c)$.

In Section 3 we characterize the subcategories of the type $\mathcal{C}oar$ and $\mathcal{F}ine$, introducing the closure operators $\mathrm{fine}^{\mathcal{A}}$ and $\mathrm{coar}^{\mathcal{B}}$, for subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$, which generalize the constructions presented in [3]. The latter closure operator was already used in a different context by G. Brümmer and E. Giuli in [2], and the former one is mentioned in [10] (Chapter 7). It is interesting to note that the characterizations established in Theorem 3.4 are similar to those obtained in [4] for ∇- and Δ-subcategories. These subcategories are in fact closely related to subcategories of coarse and fine objects, as is shown in Section 5. Previously (Theorem 4.2) we show that left- and right-constant subcategories are particular instances of subcategories of coarse and fine objects respectively, under mild conditions on $\mathcal{X}$.

Finally, in Section 6 we present a few examples of subcategories of coarse and fine objects that show in particular that, at the morphism level, this wider approach allows the simultaneous study of interesting classes of maps, like Collins' concordant and dissonant maps, and also monotone and light maps.

1. Preliminaries

1.1. Throughout this paper we consider a finitely complete category $\mathcal{X}$ (in particular, $\mathcal{X}$ has a terminal object 1) with a proper factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms (cf. [11]). Therefore $\mathcal{E}$ is a class of epimorhisms and $\mathcal{M}$ is a class of monomorphisms, both closed under composition, such that every morphism in $\mathcal{X}$ has an $(\mathcal{E}, \mathcal{M})$-factorization and the $(\mathcal{E}, \mathcal{M})$-diagonalization property holds.

We also assume that $\mathcal{X}$ is $\mathcal{M}$-complete, so that $\mathcal{X}$ has multiple pullbacks of (arbitrary) sinks of $\mathcal{M}$-morphisms. This is equivalent to the assumption that the $(\mathcal{E}, \mathcal{M})$-factorization system can be extended to an $(\mathbb{E}, \mathcal{M})$-factorization system for sinks.

From the assumptions above several properties for $\mathcal{M}$ are derived; namely, $\mathcal{M}$ is closed under limits, it is stable under pullback and left-cancellable (so that $m \cdot n \in \mathcal{M}$ implies $n \in \mathcal{M}$). Furthermore, the class $\operatorname{sub} X$ of morphisms in $\mathcal{M}$ with codomain X can be preordered by

$$m \leq n \;\Leftrightarrow\; (\exists k)\; n \cdot k = m$$

(k will be usually denoted by $\frac{m}{n}$), so that $m \cong n$ if $m \leq n$ and $n \leq m$. The (large) lattice $\operatorname{sub} X$ has all infima and suprema, with suprema formed by $(\mathbb{E}, \mathcal{M})$-factorizations. It has in particular a largest element, $1_X : X \to X$, and a least one, $0_X : 0_X \to X$.

For every $f : X \to Y$ in $\mathcal{X}$, one has an image-preimage adjunction

$$f(-) \dashv f^{-1}(-) : \operatorname{sub} Y \to \operatorname{sub} X,$$

where $f(m)$ is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \cdot m$ and $f^{-1}(n)$ is the pullback of n along f, for every $m \in \operatorname{sub} X$ and $n \in \operatorname{sub} Y$ (cf. [10]).

1.2. We denote by $\mathcal{P}$ the (full) subcategory of preterminal objects of $\mathcal{X}$, that is, of those objects P such that for all $X \in \mathcal{X}$ and $f, g : X \to P$, $f = g$. The terminal object and the morphism from $X \in \mathcal{X}$ into it will be denoted by 1 and $!_X : X \to 1$, respectively.

DEFINITIONS (cf. [4]). (1) A *morphism* $f : X \to Y$ in $\mathcal{X}$ is said to be *constant* if its image is a preterminal object (i.e., in its $(\mathcal{E}, \mathcal{M})$-factorization $f = m \cdot e$, the domain of m is preterminal).

(2) For full subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$, the *right-* and *left-constant subcategory of* $\mathcal{A}$ and $\mathcal{B}$ are, respectively,

$$r(\mathcal{A}) = \{B \mid (\forall A \in \mathcal{A})\; A||B\},$$
$$l(\mathcal{B}) = \{A \mid (\forall B \in \mathcal{B})\; A||B\},$$

where $X||Y$ if every morphism $X \to Y$ is constant. Every subcategory of the form $r(\mathcal{A})$ ($l(\mathcal{B})$) is called a *right-*(*left-*)*constant subcategory*.

1.3. DEFINITION (cf. [9, 10]). A *closure operator* c on $\mathcal{X}$ with respect to $\mathcal{M}$ is a family of maps $(c_X : \operatorname{sub} X \to \operatorname{sub} X)_{X \in \mathcal{X}}$ satisfying the conditions

(1) $m \leq c_X(m)$,
(2) $m \leq n \;\Rightarrow\; c_X(m) \leq c_X(n)$,
(3) $f(c_X(m)) \leq c_Y(f(m))$,

for all $f : X \to Y$ and $m, n \in \operatorname{sub} X$.

A *subobject* $m : M \to X$ is *c-closed* if $m \cong c_X(m)$ and it is *c-dense* if $c_X(m) \cong 1_X$. (When it is clear from the context, the subscripts are omitted; we often denote the domain of $c_X(m)$ by $c_X(M)$.) The class of c-closed subobjects is closed under the formation of limits, hence, in particular, it is stable under (multiple) pullback in $\mathcal{X}$.

A *closure operator* c is *idempotent* if $c(m)$ is c-closed for every $m \in \mathcal{M}$, and c is *weakly hereditary* if $\frac{m}{c(m)}$ is c-dense for every $m \in \mathcal{M}$. Idempotency of c guarantees that c-dense subobjects are closed under composition, while weak heredity guarantees that c-closed subobjects are closed under composition.

The closure operator c is *hereditary* if, for every $n : N \to M$, $m : M \to X$ in $\mathcal{M}$, $c_M(n) \cong m^{-1}(c_X(m \cdot n))$.

Given two closure operators c and d in $\mathcal{X}$ (w.r.t. $\mathcal{M}$), one says that *c is finer than d* – or *d is coarser than c* – and write $c \leq d$ if $c(m) \leq d(m)$ for every $m \in \mathcal{M}$.

DEFINITIONS. (1) A *morphism* $f : X \to Y$ is *c-closed* if $f(c_X(m)) \cong c_Y(f(m))$ for every $m \in \operatorname{sub} X$ and it is *c-open* if $f^{-1}(c_Y(n)) \cong c_X(f^{-1}(n))$ for every $n \in \operatorname{sub} Y$.

(2) A *morphism* $f : X \to Y$ is *c-initial* if, for every $m \in \operatorname{sub} X$, $c_X(m) \cong f^{-1}(c_Y(f(m)))$ and it is *c-final* if $c_Y(n) \cong n \vee f(c_X(f^{-1}(n)))$ for every $n \in \operatorname{sub} Y$.

(3) More generally, a *source* $(f_i : X \to X_i)_{i \in I}$ is *c-initial* if

$$c_X(m) \cong \bigwedge_{i \in I} f_i^{-1}(c_{X_i}(f_i(m)))$$

for every $m \in \operatorname{sub} X$; a *sink* $(g_i : Y_i \to Y)_{i \in I}$ is *c-final* if

$$c_Y(n) \cong n \vee \bigvee_{i \in I} g_i(c_{Y_i}(g_i^{-1}(n)))$$

for every $n \in \operatorname{sub} Y$.

We point out that the notion of c-final morphism we use here is weaker than the notion introduced in [6]; they coincide exactly when the morphism belongs to $\mathcal{E}$.

2. The (In)Discrete Closure Operators

2.1. Every category admits the following (idempotent) closure operators w.r.t. $(\mathcal{E}, \mathcal{M})$: the *discrete* (or *fine*) *closure operator* fine, given by $\mathrm{fine}_X(m) := m$, the *indiscrete* (or *coarse*) *closure operator* coar, with $\mathrm{coar}_X(m) := \bigwedge\{f^{-1}(f(m)) \mid f : X \to P,\ P \in \mathcal{P}\}$, and the *trivial closure operator* t, where $t_X(m) = 1_X$, for every $X \in \mathcal{X}$ and $m \in \operatorname{sub} X$.

We point out that the discrete closure can be described analogously to the indiscrete closure, since $\mathrm{fine}_X(m) = m \vee \bigvee\{h(h^{-1}(m)) \mid h : P \to X,\ P \in \mathcal{P}\}$. In fact, they are related to each other in the following way:

$$\mathrm{fine}_X(m) = m \vee \bigvee\{h(\mathrm{coar}_P(h^{-1}(m))) \mid h : P \to X,\ P \in \mathcal{P}\},$$

$$\mathrm{coar}_X(m) = \bigwedge\{f^{-1}(\mathrm{fine}_P(f(m))) \mid f : X \to P,\ P \in \mathcal{P}\}.$$

2.2. In general coar and t are distinct. However, if $\mathcal{X}$ is a pointed category, then coar $= t$, and, conversely, if coar $= t$ then every preterminal object is pre-initial. In fact, if $\mathcal{X}$ is pointed, or, equivalently, if $\mathcal{P} = \{X \mid X \cong 0\}$, then, for every $f : X \to 0$ and every $m : M \to X$ in $\mathcal{M}$, $f^{-1}(f(m)) \cong 1_X$, hence coar $= t$. Moreover, if coar $= t$, then, for every preterminal object P, the morphism $0_P : 0 \to P$ must belong to $\mathcal{E}$ because, if $0_P = m \cdot e$ is its $(\mathcal{E}, \mathcal{M})$-factorization, $m \cong \mathrm{coar}_P(m) \cong t_P(m) \cong 1_P$. Therefore P is pre-initial, since $\mathcal{E} \subseteq \mathrm{Epi}\,\mathcal{X}$.

2.3. PROPOSITION. *Let $\mathcal{E}$ be stable under pullback along monomorphisms. Then* $\mathrm{coar}_X(m) = !_X^{-1}(!_X(m))$ *for every $X \in \mathcal{X}$ and $m \in \mathrm{sub}\,X$. Moreover,* coar *is hereditary whenever it may be described this way.*

Proof. Let $m : M \to X$ belong to $\mathcal{M}$. For every $f : X \to P$ with $P \in \mathcal{P}$, $!_X^{-1}(!_X(m)) \cong f^{-1}(!_P^{-1}(!_P(f(m))))$. Since $!_P$ is a monomorphism and $\mathcal{E}$ is stable under pullback along monomorphisms, $!_P^{-1}(!_P(f(m))) \cong f(m)$, by Lemma 2.1 of [12]. Hence $!_X^{-1}(!_X(m)) \cong f^{-1}(f(m))$ for every $f : X \to P$ with P preterminal, and therefore $\mathrm{coar}_X(m) \cong !_X^{-1}(!_X(m))$.

To check that coar is hereditary, let $m : M \to X$ and $n : N \to M$ be subobjects. Then

$$\mathrm{coar}_M(n) \cong !_M^{-1}(!_M(n)) \cong m^{-1}(!_X^{-1}(!_X(m \cdot n))) \cong m^{-1}(\mathrm{coar}_X(m \cdot n)). \qquad \square$$

2.4. Every preterminal object P in $\mathcal{X}$ has the property that $\mathrm{coar}_P = \mathrm{fine}_P$. We say that *the coarse closure operator determines preterminal objects* if $\mathrm{coar}_X = \mathrm{fine}_X$ only if X is preterminal, that is, $\{X \in \mathcal{X} \mid \mathrm{coar}_X = \mathrm{fine}_X\} = \mathcal{P}$.

LEMMA. *If $\mathcal{X}$ is a pointed category then the coarse closure operator determines preterminal objects.*

Proof. Let 0 be a zero object of $\mathcal{X}$. For every $X \in \mathcal{X}$, the morphism $0_X : 0 \to X$ belongs to $\mathcal{M}$. If $\mathrm{coar}_X = \mathrm{fine}_X$, then $1_X \cong \mathrm{coar}_X(0_X) \cong \mathrm{fine}_X(0_X) \cong 0_X$, hence $X \cong 0 \in \mathcal{P}$. $\square$

This behaviour is not only expected in pointed categories, as we shall see next.

We say that *pairs of subobjects detect monosources* (*monomorphisms*) if any source (morphism) is monic provided that it distinguishes pairs of distinct subobjects; that is, $(f_i : X \to Y_i)_{i \in I}$ is monic if, for any pair $x, y : Y \to X \in \mathrm{sub}\,X$, $f_i \cdot x = f_i \cdot y$ for every $i \in I$ implies $x = y$.

PROPOSITION. *If in $\mathcal{X}$ pairs of subobjects detect monomorphisms, then the coarse closure operator determines preterminal objects.*

Proof. For every pair of subobjects $x, y : Y \to X$, if $f : X \to P$ with $P \in \mathcal{P}$, then $f \cdot x = f \cdot y$. Hence $\mathrm{coar}_X(x) \cong \mathrm{coar}_X(y)$, from which it follows that whenever $\mathrm{coar}_X = \mathrm{fine}_X$, $x \cong \mathrm{fine}_X(x) \cong \mathrm{coar}_X(x) \cong \mathrm{fine}_X(y) \cong y$. Therefore, in particular, $!_X$ is monic, and, consequently, $X \in \mathcal{P}$. □

2.5. It is obvious that any morphism is fine-open, fine-closed and fine-final. Concerning the indiscrete closure coar, one has:

PROPOSITION. *Let $\mathcal{E}$ be stable under pullback along monomorphisms.*

(1) *Every morphism in $\mathcal{E}$ is* coar*-initial and* coar*-closed.*
(2) *Consider the following conditions:*

(i) *$\mathcal{P}$ is closed under images;*

(ii) *every morphism in $\mathcal{E}$ is* coar*-open;*

(iii) *every morphism in $\mathcal{E}$ is* coar*-final.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and they are all equivalent if the coarse closure operator determines preterminal objects.*

Proof. (1) Let $f : X \to Y$ belong to $\mathcal{E}$ and $m : M \to X$ belong to $\mathcal{M}$. By Proposition 2.3,

$$\mathrm{coar}_X(m) \cong !_X^{-1}(!_X(m)) \cong f^{-1}(!_Y^{-1}(!_Y(f(m)))) \cong f^{-1}(\mathrm{coar}_Y(f(m))),$$

hence f is coar-initial. By the stability of $\mathcal{E}$ under pullback along subobjects it follows that f is also coar-closed:

$$f(\mathrm{coar}_X(m)) \cong f(f^{-1}(\mathrm{coar}_Y(f(m)))) \cong \mathrm{coar}_Y(f(m)).$$

(2) (i) $\Rightarrow$ (ii) Let $h : X \to Y$ belong to $\mathcal{E}$. For any $m : M \to Y$ in $\mathcal{M}$, since $!_Y \cdot h = !_X$, by Proposition 2.3,

$$\begin{aligned}\mathrm{coar}_X(h^{-1}(m)) &\cong !_X^{-1}(!_X(h^{-1}(m))) \cong h^{-1}(!_Y^{-1}(!_Y(h(h^{-1}(m)))))\\ &\cong h^{-1}(!_Y^{-1}(!_Y(m)))\\ &\cong h^{-1}(\mathrm{coar}_Y(m)).\end{aligned}$$

(ii) $\Rightarrow$ (iii) since the stability of $\mathcal{E}$ under pullback along subobjects assures that every coar-open $\mathcal{E}$-morphism is coar-final.

To prove that (iii) $\Rightarrow$ (i) whenever $\mathcal{P} = \{X \mid \mathrm{coar}_X = \mathrm{fine}_X\}$, consider an $\mathcal{E}$-morphism $e : P \to X$ with $P \in \mathcal{P}$. For each $m : M \to X$,

$$\mathrm{coar}_X(m) \cong e(\mathrm{coar}_P(e^{-1}(m))) \cong e(e^{-1}(m)) \cong m.$$

Hence $\mathrm{coar}_X = \mathrm{fine}_X$, and therefore $X \in \mathcal{P}$. □

3. Relative Coarse and Fine Objects and Closure Operators

3.1. For a closure operator c in $\mathcal{X}$, we say that an *object* X of $\mathcal{X}$ is *c-coarse* if $c_X \geq \mathrm{coar}_X$ and X is *c-fine* if $c_X \leq \mathrm{fine}_X$. This way we define the (full) subcategories of $\mathcal{X}$

$$\mathcal{C}oar(c) := \{A \in \mathcal{X} \mid c_A \geq \mathrm{coar}_A\}$$

and

$$\mathcal{F}ine(c) := \{B \in \mathcal{X} \mid c_B \leq \mathrm{fine}_B\}.$$

PROPOSITION. *Let c be a closure operator.*

(1) *The subcategory $\mathcal{C}oar(c)$ is closed under* coar-*final morphisms. Hence it is closed under images provided that $\mathcal{E}$ is stable under pullback along monomorphisms and $\mathcal{P}$ is closed under images.*
(2) *The subcategory $\mathcal{F}ine(c)$ is closed under* fine-*initial morphisms. In particular, it is closed under subobjects.*

Proof. (1) To prove that $\mathcal{C}oar(c)$ is closed under coar-final morphisms, let $h : X \to Y$ be coar-final and X belong to $\mathcal{C}oar(c)$. Then, for each $m : M \to Y$ in $\mathcal{M}$, $\mathrm{coar}_Y(m) \cong h(\mathrm{coar}_X(h^{-1}(m)))$ because h is coar-final, hence

$$\mathrm{coar}_Y(m) \leq h(c_X(h^{-1}(m))) \leq c_Y(h(h^{-1}(m))) \leq c_Y(m),$$

which means that $Y \in \mathcal{C}oar(c)$ as claimed.

Now, to prove the second assertion, we make use of Proposition 2.5. Whenever $\mathcal{E}$ is stable under pullback along monomorphisms and $\mathcal{P}$ is closed under images, $\mathcal{E}$-morphisms are coar-final. Therefore, we may conclude that $\mathcal{C}oar(c)$ is closed under images.

(2) If $f : X \to Y$ is a fine-initial morphism, then, for every subobject $m : M \to X$, if $Y \in \mathcal{F}ine(c)$, then

$$\begin{aligned} c_X(m) &\leq c_X(f^{-1}(f(m))) \leq f^{-1}(c_Y(f(m))) \\ &\leq f^{-1}(\mathrm{fine}_Y(f(m))) \cong \mathrm{fine}_X(m), \end{aligned}$$

hence also $X \in \mathcal{F}ine(c)$.

Since fine is obviously hereditary, every $\mathcal{M}$-morphism is fine-initial. □

3.2. To characterize subcategories of coarse and fine objects we will use suitable closure operators we introduce in the sequel.

In order to do that, we first point out that the indiscrete closure operator coar is the coarsest closure operator that is discrete in $\mathcal{P}$ and the discrete closure is the finest closure operator that is indiscrete (= discrete) in $\mathcal{P}$. In an analogous way, for subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$, we may consider *the coarsest closure operator that is discrete in* $\mathcal{B}$

$$\mathrm{coar}_X^{\mathcal{B}}(m) := \bigwedge\{f^{-1}(f(m)) \mid f : X \to B,\, B \in \mathcal{B}\},$$

and *the finest closure operator that is indiscrete in* $\mathcal{A}$

$$\mathrm{fine}_X^{\mathcal{A}}(m) := m \vee \bigvee\{h(\mathrm{coar}_A(h^{-1}(m))) \mid h : A \to X,\, A \in \mathcal{A}\}.$$

We remark that $\mathrm{coar} = \mathrm{coar}^{\mathcal{P}} = \mathrm{fine}^{\mathcal{X}}$ and $\mathrm{fine} = \mathrm{fine}^{\mathcal{P}} = \mathrm{coar}^{\mathcal{X}}$, and that $\mathrm{coar}^{\mathcal{B}}$ is called the *splitting closure operator defined by* $\mathcal{B}$ in [2].

3.3. The closure $\mathrm{coar}^{\mathcal{B}}$ is always idempotent, but in general it is not weakly hereditary (see Example 6.1(b)). By contrast, in general $\mathrm{fine}^{\mathcal{A}}$ is not even idempotent, and not weakly hereditary either (see Examples 6.1(e) and 6.2(b)). Next we present a necessary and sufficient condition for weak heredity of $\mathrm{fine}^{\mathcal{A}}$.

PROPOSITION. *Let the indiscrete closure operator be weakly hereditary. Then* $\mathrm{fine}^{\mathcal{A}}$ *is weakly hereditary if and only if the subcategory* $\mathcal{C}oar(\mathrm{fine}^{\mathcal{A}})$ *is closed under* coar*-closed subobjects.*

Proof. Let $\mathrm{fine}^{\mathcal{A}}$ be weakly hereditary, and let $m : M \to A$ be a coar-closed subobject of an object A of $\mathcal{A}$. By the weak heredity of both coar and $\mathrm{fine}^{\mathcal{A}}$ and the equality $\mathrm{coar}_A = \mathrm{fine}_A^{\mathcal{A}}$, it follows that $\mathrm{coar}_M = \mathrm{fine}_M^{\mathcal{A}}$, and therefore $M \in \mathcal{C}oar(\mathrm{fine}^{\mathcal{A}})$.

Conversely, let $m : M \to X$ be a morphism in $\mathcal{M}$, and let $\tilde{m}$ be the morphism $\frac{m}{\mathrm{fine}_X^{\mathcal{A}}(m)}$; that is, $m = \mathrm{fine}_X^{\mathcal{A}}(m) \cdot \tilde{m}$. For each $h : A \to X$ with $A \in \mathcal{A}$, since $h(\mathrm{coar}_A(h^{-1}(m))) \le \mathrm{fine}_X^{\mathcal{A}}(m)$, there is a morphism $k : A' \to \mathrm{fine}_X^{\mathcal{A}}(M)$ ($A' \in \mathcal{A}$) such that $h \cdot \mathrm{coar}_A(h^{-1}(m)) = \mathrm{fine}_X^{\mathcal{A}}(m) \cdot k$.

Hence

$$\begin{aligned}
\mathrm{fine}_X^{\mathcal{A}}(m) \cdot k(\mathrm{coar}_{A'}(k^{-1}(\tilde{m}))) &\cong (h \cdot \mathrm{coar}_A(h^{-1}(m)) \cdot \mathrm{coar}_{A'}(k^{-1}(\tilde{m})))\\
&\cong h(\mathrm{coar}_A(h^{-1}(m)) \cdot \mathrm{coar}_{A'}(k^{-1}(\tilde{m})))\\
&\ge h(\mathrm{coar}_A(h^{-1}(m))),
\end{aligned}$$

since $\mathrm{coar}_A(h^{-1}(m))\cdot\mathrm{coar}_{A'}(k^{-1}(\tilde{m}))$ is $\mathrm{fine}^{\mathcal{A}}$-closed and greater or equal to $h^{-1}(m)$. Therefore $\mathrm{fine}_X^{\mathcal{A}}(m) \cdot \mathrm{coar}_{\mathrm{fine}_X^{\mathcal{A}}(M)}(\tilde{m}) \ge \mathrm{fine}_X^{\mathcal{A}}(m)$, whence it follows that $\tilde{m}$ is $\mathrm{fine}^{\mathcal{A}}$-dense, as claimed. $\square$

Using Proposition 2.3, we can conclude that:

COROLLARY. *If* $\mathcal{E}$ *is stable under pullback along monomorphisms, then, for every subcategory* $\mathcal{A}$ *of* $\mathcal{X}$*,* $\mathrm{fine}^{\mathcal{A}}$ *is weakly hereditary if and only if* $\mathcal{C}oar(\mathrm{fine}^{\mathcal{A}})$ *is closed under* coar*-closed subobjects.*

Since in every topological category over $\mathcal{S}et$ (in the sense of [14]) – for the class $\mathcal{M}$ of embeddings – coar is the usual indiscrete closure, the condition of

$\mathcal{C}oar(\text{fine}^{\mathcal{A}})$ being closed under coar-closed subobjects trivializes. This condition is also trivially satisfied when $\text{coar} = t$. Hence:

COROLLARY. *If $\mathcal{X}$ is a topological category over $\mathcal{S}et$ or if $\mathcal{X}$ is pointed, then* $\text{fine}^{\mathcal{A}}$ *is weakly hereditary for every subcategory $\mathcal{A}$ of $\mathcal{X}$.*

3.4. The consideration of the closure operators $\text{fine}^{\mathcal{A}}$ and $\text{coar}^{\mathcal{B}}$ leads to straightforward characterizations of subcategories of the form $\mathcal{C}oar(c)$ and $\mathcal{F}ine(c)$ for some closure operator c, respectively.

THEOREM. (1) *For a subcategory $\mathcal{A}$ of $\mathcal{X}$ the following conditions are equivalent:*

(i) $\mathcal{A} = \mathcal{C}oar(c)$ *for some closure operator c;*
(ii) $\mathcal{A} = \mathcal{C}oar(\text{fine}^{\mathcal{A}})$*;*
(iii) $\mathcal{A}$ *is closed under* coar-*final sinks.*

(2) *For a subcategory $\mathcal{B}$ of $\mathcal{X}$ the following conditions are equivalent:*

(i) $\mathcal{B} = \mathcal{F}ine(c)$ *for some closure operator c;*
(ii) $\mathcal{B} = \mathcal{F}ine(\text{coar}^{\mathcal{B}})$*;*
(iii) $\mathcal{B}$ *is closed under* fine-*initial sources.*

Proof. The proof of (ii) $\Leftrightarrow$ (iii) is technically routine and (ii) $\Leftrightarrow$ (i) is trivially true.

To show that (1)(i) $\Rightarrow$ (ii), notice first that, for any subcategory $\mathcal{A}$, if $A \in \mathcal{A}$, then $\text{fine}^{\mathcal{A}}_A = \text{coar}_A$, and so $\mathcal{A} \subseteq \mathcal{C}oar(\text{fine}^{\mathcal{A}})$. Now, assume that $\mathcal{A} = \mathcal{C}oar(c)$ for some closure operator c. Since $c_A(m) \geq \text{coar}_A(m)$ for every object A of $\mathcal{A}$, and $\text{fine}^{\mathcal{A}}$ is the least closure operator that coincides with coar in $\mathcal{A}$, $\text{fine}^{\mathcal{A}} \leq c \wedge \text{coar} \leq c$. Therefore, if $X \in \mathcal{C}oar(\text{fine}^{\mathcal{A}})$, that is, if $\text{fine}^{\mathcal{A}}_X \geq \text{coar}_X$, then $c_X \geq \text{coar}_X$ and so $X \in \mathcal{A}$.

The proof of (2)(i) $\Rightarrow$ (ii) is similar. $\square$

4. Subcategories of Coarse and Fine Objects versus Connectednesses and Disconnectednesses

4.1. As we observed above, for any subcategory $\mathcal{A}$, if $A \in \mathcal{A}$ then $\text{fine}^{\mathcal{A}}_A = \text{coar}_A$, and so $\mathcal{A} \subseteq \mathcal{C}oar(\text{fine}^{\mathcal{A}})$; on the other hand, if c is a closure operator, then

$$\begin{aligned}\text{fine}^{\mathcal{C}oar(c)}_X(m) &= m \vee \bigvee\{h(\text{coar}_A(h^{-1}(m))) \mid h : A \to X,\ A \in \mathcal{C}oar(c)\}\\ &\leq m \vee \bigvee\{h(c_A(h^{-1}(m))) \mid h : A \to X,\ A \in \mathcal{C}oar(c)\}\\ &\leq c_X(m);\end{aligned}$$

that is, $\text{fine}^{\mathcal{C}oar(c)} \leq c$.

Moreover, if $\mathcal{B}$ is a subcategory of $\mathcal{X}$ then, for every $B \in \mathcal{B}$, $\mathrm{coar}_B^{\mathcal{B}} = \mathrm{fine}_B$; hence $\mathcal{B} \subseteq \mathcal{F}ine(\mathrm{coar}^{\mathcal{B}})$. On the other hand, if c is a closure operator, then, for any $g : X \to B$ with $B \in \mathcal{F}ine(c)$,

$$c_X(m) \leq c_X(g^{-1}(g(m))) \leq g^{-1}(c_B(g(m))) \leq g^{-1}(g(m)),$$

therefore $c \leq \mathrm{coar}^{\mathcal{F}ine(c)}$.

In conclusion, we have defined the Galois correspondences

$$\begin{tikzcd} \mathrm{SUB}(\mathcal{X})^{\mathrm{op}} \arrow[r, shift right, "\mathrm{coar}"'] \arrow[r, phantom, "\perp"] & \mathrm{CL}(\mathcal{X}) \arrow[l, shift right, "\mathcal{F}ine"'] \arrow[r, shift right, "\mathcal{C}oar"'] \arrow[r, phantom, "\perp"] & \mathrm{SUB}(\mathcal{X}); \arrow[l, shift right, "\mathrm{fine}"'] \end{tikzcd}$$

here $\mathrm{SUB}(\mathcal{X})$ denotes the conglomerate of all full subcategories of $\mathcal{X}$, ordered by inclusion, and $\mathrm{CL}(\mathcal{X}, \mathcal{M})$ the conglomerate of all closure operators of $\mathcal{X}$ (w.r.t. $\mathcal{M}$), preordered as indicated in 1.3.

4.2. For any closure operator c, $\mathcal{F}ine(c) \cap \mathcal{C}oar(c) \subseteq \{X \mid \mathrm{coar}_X = \mathrm{fine}_X\}$. Hence, if $\mathcal{E}$ is stable under pullback along monomorphisms and the coarse closure operator determines preterminal objects, then any morphism $f : A \to B$ with $A \in \mathcal{C}oar(c)$ and $B \in \mathcal{F}ine(c)$ is constant, since, under these assumptions, its image belongs to $\mathcal{C}oar(c) \cap \mathcal{F}ine(c) = \mathcal{P}$.

Indeed, there is a close relation between the functors defined above and the formation of left- and right-subcategories, as we show in the sequel.

THEOREM. *If $\mathcal{E}$ is stable under pullback along monomorphisms and $\mathcal{P}$ is closed under images, then the following diagram*

$$\begin{tikzcd} & \mathrm{SUB}(\mathcal{X}) \arrow[dl, shift right, "\mathrm{fine}"'] \arrow[dd, shift left, "r"] \arrow[dd, phantom, "\vdash"] \\ \mathrm{CL}(\mathcal{X}) \arrow[ur, shift right, "\mathcal{C}oar"'] \arrow[dr, shift right, "\mathcal{F}ine"'] & \\ & \mathrm{SUB}(\mathcal{X})^{\mathrm{op}} \arrow[ul, shift right, "\mathrm{coar}"'] \arrow[uu, shift left, "l"] \end{tikzcd} \tag{1}$$

commutes if and only if $\mathcal{P} = \{X \mid \mathrm{coar}_X = \mathrm{fine}_X\}$.

Proof. The necessity of the condition $\mathcal{P} = \{X \mid \mathrm{coar}_X = \mathrm{fine}_X\}$ follows from the fact that $r\mathcal{X} = \mathcal{P}$ and $\mathcal{F}ine(\mathrm{fine}^{\mathcal{X}}) = \mathcal{F}ine(\mathrm{coar}) = \{X \mid \mathrm{coar}_X = \mathrm{fine}_X\}$.

To prove the sufficiency it is enough to show that, for every subcategory $\mathcal{A}$, $\mathcal{F}ine(\mathrm{fine}^{\mathcal{A}}) = r\mathcal{A}$, since the other equality follows from this one by adjointness.

If $X \in r\mathcal{A}$ and $h : A \to X$ with $A \in \mathcal{A}$, in the $(\mathcal{E}, \mathcal{M})$-factorization of h

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "e"'] & & X \\ & M \arrow[ur, "m"'] & \end{tikzcd}$$

M is preterminal. Then, for each $n : N \to X$ in $\mathcal{M}$,

$$\begin{aligned} \mathrm{coar}_A(h^{-1}(n)) &\leq e^{-1}(e(h^{-1}(n))) && \text{(by definition of coar)} \\ &\cong e^{-1}(e(e^{-1}(m^{-1}(n)))) \\ &\cong e^{-1}(m^{-1}(n)) && \text{(because } \mathcal{E} \text{ is stable under pullback along } \mathcal{M}) \\ &\cong h^{-1}(n). \end{aligned}$$

Hence $h(\mathrm{coar}_A(h^{-1}(n))) \leq h(h^{-1}(n)) \leq n$, and so $\mathrm{fine}^{\mathcal{A}}_X(n) \cong n$ and $X \in \mathcal{F}ine(\mathrm{fine}^{\mathcal{A}})$.

To prove the reverse inclusion, let $X \in \mathcal{F}ine(\mathrm{fine}^{\mathcal{A}})$. For $h : A \to X$ with $A \in \mathcal{A}$, form its $(\mathcal{E}, \mathcal{M})$-factorization $h = m \cdot e$. Let $n : N \to M$ belong to $\mathcal{M}$. Then we have

$$\begin{aligned} \mathrm{coar}_M(n) &\cong e(e^{-1}(\mathrm{coar}_M(n))) && (\mathcal{E} \text{ is stable under pullback along monomorphisms}) \\ &\cong e(\mathrm{coar}_A(e^{-1}(n))) && (e \text{ is coar-open}) \\ &\cong e(\mathrm{fine}^{\mathcal{A}}_A(e^{-1}(n))) \\ &\leq \mathrm{fine}^{\mathcal{A}}_X(e(e^{-1}(n))) \\ &\cong \mathrm{fine}^{\mathcal{A}}_X(n) \cong n. \end{aligned}$$

Hence $M \in \{X \mid \mathrm{coar}_X = \mathrm{fine}_X\} = \mathcal{P}$. □

5. Subcategories of Coarse and Fine Objects versus ∇- and Δ-Subcategories

5.1. We recall that, for a given closure operator c, an *object* X of $\mathcal{X}$ is *c-connected* if the diagonal $\delta_X = \langle 1_X, 1_X \rangle : X \to X \times X$ is c-dense and it is *c-separated* if δ_X is c-closed. The (full) subcategory of c-connected (c-separated) objects is denoted by $\nabla(c)$ ($\Delta(c)$). These subcategories were thoroughly studied in [4].

5.2. Next we shall show that, under suitable conditions, every Δ-subcategory (that is, every subcategory of the form $\Delta(c)$ for some closure operator c) is a subcategory of d-fine objects for some closure operator d. From the previous section we already know that this implies in particular that the coarse closure operator detects preterminal objects, since $\mathcal{P} = \Delta(t)$ for t the trivial closure operator, and, by Theorem 3.4, $\mathcal{P} = \mathcal{F}ine(c)$ if and only if $\mathcal{P} = \mathcal{F}ine(\mathrm{coar}^{\mathcal{P}}) = \{X \mid \mathrm{coar}_X = \mathrm{fine}_X\}$. The next result shows that a sufficient condition for this to happen mentioned in 2.4 also guarantees that subcategories of fine objects are Δ-subcategories.

THEOREM. *If pairs of subobjects detect monosources, then every subcategory closed under monosources is a subcategory of c-fine objects for some closure* *operator c. In particular, every Δ-subcategory is a subcategory of c-fine objects for a suitable closure operator c.*

Proof. The latter assertion follows immediately from the former one because every Δ-subcategory is closed under monosources.

To prove the first statement, we will show that every fine-initial source is a monosource. Let $(f_i : X \to X_i)_{i\in I}$ be a fine-initial source, and let $x, y : Y \to X$ be subobjects such that $f_i \cdot x = f_i \cdot y$ for every $i \in I$. Then $x \cong \bigwedge f_i^{-1}(f_i(x)) \cong \bigwedge f_i^{-1}(f_i(y)) \cong y$, which implies that (f_i) is a monosource under the given hypotheses.

5.3. The sufficient condition that ensures that every ∇-subcategory is a subcategory of c-coarse objects for some closure operator c we state below is more restrictive. In order to guarantee that subcategories of c-coarse objects cover ∇-subcategories we assume that points in $\mathcal{X}$ are well-behaved in the sense we explain in the sequel.

We recall from [4] that a *point* of an object X of $\mathcal{X}$ is simply a morphism $x : 1 \to X$ (which is, in particular, a subobject of X).

We say that $\mathcal{X}$ *has enough points* if every subobject $m : M \to X$ is the join of its points; that is, $m \cong \bigvee\{x : 1 \to X \mid x \leq m\}$. We say that a *sink* $(g_i : Y_i \to Y)_{i\in I}$ is pt-*surjective* if for each $y : 1 \to Y$ there are $j \in I$ and $y_j : 1 \to Y_j$ such that $g_j \cdot y_j = y$.

The following characterization of ∇-subcategories was given in [4]:

PROPOSITION. *A full subcategory $\mathcal{A}$ of $\mathcal{X}$ is of the form $\nabla(c)$ for some closure operator c if and only if, for every sink $(h_i : A_i \times A_i \to Y \times Y)_{i\in I}$ with $A_i \in \mathcal{A}$, $h_i(\delta_{A_i}) \leq \delta_Y$ for all $i \in I$ and $1_{Y\times Y} \cong \delta_Y \vee \bigvee h_i(1_{A_i\times A_i})$, one has $Y \in \mathcal{A}$.*

The following result will also be useful:

LEMMA. *If $\mathcal{X}$ has enough points and morphisms in $\mathcal{E}$ are* pt-*surjective, then, for every object X with $!_X \in \mathcal{E}$ (or, equivalently, with a point) and every subobject $m : M \to X$, the following conditions are equivalent:*

(i) *X has a point x with $x \leq m$;*
(ii) $\mathrm{coar}_X(m) \cong 1_X$*;*
(iii) *X has a point y with $y \leq$* $\mathrm{coar}_X(m)$.

Proof. (i) $\Rightarrow$ (ii) If there is $x : 1 \to X$, then, for every $g : X \to P$ with $P \in \mathcal{P}$, $P \cong 1$, hence $g^{-1}(g(x)) \cong !_X^{-1}(!_X \cdot x) \cong 1_X$ and therefore $\mathrm{coar}_X(m) \geq \mathrm{coar}_X(x) \cong 1_X$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Let $!_X \cdot m = n \cdot e$ be the $(\mathcal{E}, \mathcal{M})$-factorization of $!_X \cdot m$ and (n', f) be the pullback of $(n, !_X)$. Then, since $y \leq \mathrm{coar}_X(m) \leq !_X^{-1}(!_X(m)) \cong n'$, there exists a morphism $\frac{y}{n'} : 1 \to N'$ and then $n \cdot f \cdot \frac{y}{n'} \cong 1_1$. This implies that n is an isomorphism, hence $!_M : M \to 1$ belongs to $\mathcal{E}$, which implies (i) under our assumptions. □

THEOREM. *If $\mathcal{X}$ has enough points and sinks in $\mathbb{E}$ are* pt*-surjective, then every ∇-subcategory is a subcategory of c-coarse objects for some closure operator c.*

Proof. Let $\mathcal{A}$ be a ∇-subcategory, and let $(g_i : A_i \to Y)_{i\in I}$ be a coar-final sink with $A_i \in \mathcal{A}$ for every $i \in I$. Consider the following commutative diagram

$$\begin{tikzcd} A_i \arrow[r, "\delta_{A_i}"] \arrow[d, "g_i"'] & A_i \times A_i \arrow[d, "g_i \times g_i"] \\ Y \arrow[r, "\delta_Y"] & Y \times Y \end{tikzcd}$$

If $\langle x, y\rangle : 1 \to Y \times Y$ and $y \neq x$ then

$$y \leq 1_Y \cong \mathrm{coar}_Y(x) \cong x \vee \bigvee g_i(\mathrm{coar}_{A_i}(g_i^{-1}(x))).$$

Hence, with $g_i^{-1}(x) : A_i' \to A_i$ and $\mathrm{coar}(g_i^{-1}(x)) : \mathrm{coar}(A_i') \to A_i$, the sink

$$(x : 1 \to Y, (g_i \cdot \mathrm{coar}(g_i^{-1}(x)) : \mathrm{coar}(A_i') \to Y)_{i\in I})$$

belongs to $\mathbb{E}$. Since $\mathbb{E}$-sinks are pt-surjective, and $y \neq x$, there are $j \in I$ and $a' : 1 \to \mathrm{coar}(A_j')$ such that $g_j \cdot \mathrm{coar}(g_j^{-1}(x)) \cdot a' = y$. Hence, the point $a := \mathrm{coar}(g_j^{-1}(x)) \cdot a'$ trivially verifies $a \leq \mathrm{coar}_{A_j}(g_j^{-1}(x))$ and $g_j \cdot a = y$. From the lemma there is also $b : 1 \to A_j$ such that $b \leq g_j^{-1}(x)$, and so $g_j \cdot b = x$. Hence $(g_j \times g_j)\langle a, b\rangle = \langle x, y\rangle$. Therefore $1_{Y\times Y} \cong \delta_Y \vee \bigvee(g_i \times g_i)(1_{A_i\times A_i})$, hence $Y \in \mathcal{A}$ by the proposition above. □

5.4. From Theorems 5.2 and 5.3 we can conclude that:

COROLLARY. *In every topological category over $\mathcal{S}et$, if $\mathcal{M}$ is the class of embeddings, every Δ-(∇-)subcategory is a subcategory of c-fine (c-coarse) objects for some closure operator c.*

In the next section we will give examples of subcategories of c-fine (c-coarse) objects in the category $\mathcal{T}op$ of topological spaces which are not Δ-(∇-)subcategories.

6. Examples

6.1. In the category $\mathcal{T}op$ of topological spaces and continuous maps, we consider the class $\mathcal{M}$ of embeddings (consequently, $\mathcal{E}$ is the class of surjective continuous maps). Hence the morphisms in $\mathcal{M}$ with codomain X may be identified with the inclusions of subspaces or simply with the subsets of X. For every topological space X, $\mathrm{coar}_X(\emptyset) = \emptyset$ and $\mathrm{coar}_X(M) = X$ for every non-empty subset M of X.

This is an example of a setting that fulfils the conditions of Theorem 4.2 and also of Theorems 5.2 and 5.3, so that every right-constant subcategory – and more generally every Δ-subcategory – is a subcategory of c-fine objects for some closure operator c, and every left-constant subcategory – and even every ∇-subcategory – is a subcategory of c-coarse objects for some c. Examples (a) and (d) show that a subcategory of c-fine objects does not need to be a Δ-subcategory, and that a subcategory of c-coarse objects does not need to be a ∇-subcategory.

(a) For the Kuratowski closure k, one has obviously that a space X is k-fine (k-coarse) if it is a (in)discrete space.

(b) The *quasi-component closure* q, where

$$q_X(M) = \bigcap\{U \mid U \text{ clopen and } M \subseteq U\}$$

for every subset M of a space X, is exactly $\mathrm{coar}^{\mathcal{B}}$, for $\mathcal{B}$ the subcategory of discrete spaces. Hence, $\mathcal{F}ine(q) = \mathcal{B}$ (by Theorem 3.4) and $\mathcal{C}oar(q) = l\mathcal{B}$ (by Theorem 4.2) is the subcategory of connected spaces. This is an example of a non weakly hereditary closure operator of the form $\mathrm{coar}^{\mathcal{B}}$.

(c) The *connected-component closure* conn, defined by

$$\mathrm{conn}_X(M) = \bigcup_{x\in M} \mathrm{conn}_X(x),$$

where $\mathrm{conn}_X(x)$ is the connected component of x in X, is exactly $\mathrm{fine}^{\mathcal{A}}$ and $\mathrm{coar}^{\mathcal{B}}$ for $\mathcal{A}$ the subcategory of connected spaces and $\mathcal{B}$ the subcategory of hereditarily disconnected spaces. Hence $\mathcal{A} = \mathcal{C}oar(\mathrm{conn})$ and $\mathcal{B} = \mathcal{F}ine(\mathrm{conn})$.

(d) Consider the *path connected-component closure* path, defined by

$$\mathrm{path}_X(M) = \{x \in X \mid \text{there is a path from } x \text{ to } y \in M\},$$

for every subset M of a space X. This closure operator is precisely $\mathrm{fine}^{\mathcal{A}}$ and also $\mathrm{coar}^{\mathcal{B}}$, for $\mathcal{A}$ the subcategory of path-connected spaces and $\mathcal{B}$ the subcategory of hereditarily path-disconnected spaces. We obviously have $\mathcal{A} = \mathcal{C}oar(\mathrm{path})$ and $\mathcal{B} = \mathcal{F}ine(\mathrm{path})$.

(e) Consider the Sierpinski space $S = \{0, 1\}$ with the non-trivial open subset $\{0\}$, and the closure operator $\mathrm{fine}^{\{S\}}$. This closure operator is not idempotent: in $S \times S$ one can easily check that

$$\mathrm{fine}^{\{S\}}(0, 1) = \{(0, 0), (1, 0), (1, 1)\}$$

and

$$\mathrm{fine}^{\{S\}}(\mathrm{fine}^{\{S\}}(0, 1)) = S \times S.$$

Moreover, the subcategory $\mathcal{C}oar(\mathrm{fine}^{\{S\}})$ is not a ∇-subcategory, since, although $S \in \mathcal{C}oar(\mathrm{fine}^{\{S\}})$, $S \times S$ does not belong to $\mathcal{C}oar(\mathrm{fine}^{\{S\}})$ and every ∇-subcategory containing S must contain also $S \times S$.

6.2. For a topological space B, consider the sliced category $\mathcal{T}op/B$ whose objects are the continuous maps f with codomain B and whose morphisms $h : (f : X \to B) \to (g : Y \to B)$ are continuous maps $h : X \to Y$ such that $g \cdot h = f$. Consider the class $\mathcal{M}^B$ of embeddings over B, that is, of embeddings with codomain B. The preterminal objects in $\mathcal{T}op/B$ are precisely the monomorphisms with codomain B. Since the class $\mathcal{E}^B$ of surjective continuous maps over B (that is, with codomain B) is stable under pullback (along monomorphisms), for every $f : X \to B$ and $M \subseteq X$,

$$\mathrm{coar}_f(M) = f^{-1}(f(M)),$$

since $f = !_f : f \to 1_B$; that is, the indiscrete closure operator is the *saturated closure operator* in $\mathcal{T}op/B$.

We point out that, although there are right-(left-)subcategories that are not Δ-(∇-)subcategories in $\mathcal{T}op/B$ (cf. [5]), Theorem 4.2 assures – since in $\mathcal{T}op/B$ subobjects detect monomorphisms – that every right-(left-)constant subcategory is a subcategory of c-fine (c-coarse) objects for a suitable closure operator c.

(a) Consider the closure operator c defined by

$$c_f(M) = \bigcup_{x \in M} (q_X(x) \cap f^{-1}(f(x)))$$

(where q is the quasi-component closure), for every $f : X \to B$ and $M \subseteq X$. Then $\mathcal{C}oar(c)$ is the subcategory $\mathcal{A}$ of concordant maps and $\mathcal{F}ine(c)$ is the subcategory $\mathcal{B}$ of dissonant maps with codomain B, in the sense of Collins [7]. Therefore $\mathrm{fine}^{\mathcal{A}} \leq c \leq \mathrm{coar}^{\mathcal{B}}$; moreover, it is easy to check that c coincides with $\mathrm{coar}^{\mathcal{B}}$ but we do not know whether $\mathrm{fine}^{\mathcal{A}}$ and c are the same. To prove that $c = \mathrm{coar}^{\mathcal{B}}$, take any $f : X \to B$ and its (quotient-concordant)-dissonant factorization:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

Then, for any $M \subseteq X$, $\mathrm{coar}^{\mathcal{B}}_f(M) \subseteq h^{-1}(h(M))$. Hence, for any $y \in \mathrm{coar}^{\mathcal{B}}_f(M)$, there is $x \in M$ with $h(x) = h(y)$. This gives, on one hand, that $y \in f^{-1}(f(x))$, and, on the other hand, that $y \in q_X(x)$ since $y \in h^{-1}(h(x))$ and this set is contained in some quasi-component, by definition of concordant map.

(b) Consider the closure operator c defined by

$$c_f(M) = \bigcup_{x \in M} (\mathrm{conn}_X(x) \cap f^{-1}(f(x)))$$

for every $f : X \to B$ and $M \subseteq X$. If $\mathcal{C}$ is the subcategory of connected spaces over B, then $\mathcal{F}ine(c)$ is the subcategory of $\mathcal{C}$-dissonant maps with codomain

B and $\mathcal{C}oar(c)$ is the subcategory $\mathcal{A}$ of $\mathcal{C}$-concordant maps with codomain B (in the sense of [4]). Moreover, $c = \text{fine}^{\mathcal{A}} = \text{coar}^{\mathcal{B}}$, as we show next. Since $\mathcal{A} = \mathcal{C}oar(c)$ and $\mathcal{B} = \mathcal{F}ine(c)$, it is immediate that $\text{fine}^{\mathcal{A}} \leq c \leq \text{coar}^{\mathcal{B}}$. To check that $c \leq \text{fine}^{\mathcal{A}}$, let $f : X \to B$ be any continuous map, let $x \in X$, and consider the following commutative diagram

$$\begin{tikzcd} \text{conn}_X(x) \arrow[rr, "m"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

where m is the inclusion and g is the restriction of f to $\text{conn}_X(x)$. Since g is $\mathcal{C}$-concordant, $\text{fine}^{\mathcal{A}}_f(x) \geq m(\text{fine}_g(x)) \cong m(g^{-1}(g(x))) \cong \text{conn}_X(x) \cap f^{-1}(f(x))$. Hence, for any subset M of X,

$$\text{fine}^{\mathcal{A}}_f(M) \supseteq \bigcup_{x\in M} \text{fine}^{\mathcal{A}}_f(x) \cong c_f(M).$$

In order to show the remaining inequality, proceed as in (a), considering ($\mathcal{C}$-concordant-$\mathcal{C}$-dissonant)-factorizations.

Finally we remark that this closure operator is an example of a non weakly hereditary closure operator of the form $\text{fine}^{\mathcal{A}}$: take $B = S^1$ and the map $f : [0, 1] \to S^1$ with $f(x) = (\cos 2\pi x, \sin 2\pi x)$; the inclusions $m : \{0, 1\} \to [0, 1]$ and $n : \{0\} \to \{0, 1\}$ (where $m : f \cdot m \to f$ and $n : f \cdot m \cdot n \to f \cdot m$) are $\text{fine}^{\mathcal{A}}$-closed, although $m \cdot n : \{0\} \to [0, 1]$ is not.

(c) If c is defined by

$$c_f(M) = \bigcup_{x\in M} \text{conn}_{f^{-1}(f(x))}(x)$$

for every $f : X \to B$ and $M \subseteq X$, $\mathcal{F}ine(c)$ is the subcategory of light maps with codomain B and $\mathcal{C}oar(c)$ is the subcategory of monotone maps with codomain B. As in the previous example, $\text{fine}^{\mathcal{A}} \leq c \leq \text{coar}^{\mathcal{B}}$, for $\mathcal{A}$ ($\mathcal{B}$) the subcategory of monotone (light) maps with codomain B. The first inequality is in fact an identity, which can be checked following an argument analogous to the argument used in the previous example.

6.3. For a ring R with unit element, let $\mathcal{M}od_R$ be the category of (left) R-modules with its (epi, mono)-factorization system. Since $\mathcal{M}od_R$ is pointed, the indiscrete closure operator is the trivial closure operator and it determines preterminal objects. Hence, every right-constant subcategory (= torsion-free subcategory – cf. [8]) is a subcategory of c-fine objects for some closure operator c, and every left-constant subcategory (= torsion subcategory) is a subcategory of c-coarse objects for some closure operator c.

In fact, if r is a preradical and $\min^r$ is the closure operator defined by

$$\min^r_X(M) = M + rX$$

for every R-module X and submodule M of X (cf. [10], Section 3.4), then

$$\mathcal{F}ine(\min^r) = \{X \mid \min^r \leq \mathrm{fine}_X\} = \{X \mid rX = 0\},$$

that is, $\mathcal{F}ine(\min^r)$ is exactly the torsion-free subcategory $\mathcal{F}_r$ induced by r, and

$$\mathcal{C}oar(\min^r) = \{X \mid \min^r \geq \mathrm{coar}_X\} = \{X \mid \min^r \geq t_X\} = \{X \mid rX = X\},$$

which means that $\mathcal{C}oar(\min^r)$ is exactly the torsion-subcategory $\mathcal{T}_r$ defined by r. It is easy to check that, if r is idempotent, then $\mathrm{fine}^{\mathcal{T}_r} = \min^r$.

Considering now the closure operator $\max^r$ defined by

$$\max^r_X(M) = \rho^{-1}(r(X/M))$$

for every R-module X and submodule M of X, where $\rho : X \to X/M$ is the projection, it is easily seen that

$$\mathcal{C}oar(\max^r) = \{X \mid \max^r_X \geq t_X\} = \mathcal{T}_r = \mathcal{C}oar(\min^r).$$

More generally, a subcategory $\mathcal{A}$ of $\mathcal{M}od_R$ is $\mathcal{C}oar(c)$ for some closure operator c if and only if $\mathcal{A} = \mathcal{T}_r = \{X \mid rX = X\} = \mathcal{C}oar(\min^r)$, where r is the preradical defined by c (that is, $rX := c_X(0)$).

However, the corresponding argument fails for $\mathcal{F}ine(c)$: take in $\mathcal{A}b\mathcal{G}rp$ the radical r defined by the torsion subgroup $rX = \{x \in X \mid (\exists n \in \mathbb{Z})\ n > 0 \text{ and } na = 0\}$ for every abelian group X; then $\mathcal{F}ine(\min^r)$ is the subcategory of torsion-free abelian groups, while $\mathcal{F}ine(\max^r) = \{0\}$.

We do not know whether every subcategory of the form $\mathcal{F}ine(c)$ is $\mathcal{F}_r$ for a suitable preradical r.

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