

The quantale of Galois connections

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ABSTRACT. Galois connections were originally expressed in a contravariant form with transformations that reverse (rather than preserve) order. Nowadays its covariant form (as residuated maps) is more often used since it is more convenient; namely compositions of residuated maps are handled more easily. In this paper we show that this is not a serious disadvantage of the contravariant form (at least in the natural context for uniform structures, where we need it), by introducing an operation of composition in the complete lattice $Gal(L, L)$ of all (contravariant) Galois connections in a complete lattice L , that allows us to work with Galois connections in the same way as one usually works with residuated maps. This operation endows $Gal(L, L)$ with a structure of quantale whenever L is a locale, allowing the description of uniform structures in terms of Galois connections.

1. Introduction

Originally Galois connections were expressed in a symmetric but contravariant form with transformations that reverse order (early references to this form are [3], [6] and [17]). A *Galois connection* [3] between the partially ordered sets A and B is a pair (f, f^+) of order-reversing (antitone) maps $f: A \rightarrow B$ and $f^+: B \rightarrow A$ such that

$$a \leq f^+f(a) \text{ for all } a \in A \text{ and } b \leq ff^+(b) \text{ for all } b \in B \quad (1.1)$$

or, equivalently,

$$a \leq f^+(b) \text{ iff } b \leq f(a) \text{ for every } a \in A \text{ and } b \in B. \quad (1.2)$$

$Gal(A, B)$, also denoted $A \otimes B$ [22], is the set of all *Galois maps* between A and B , that is, all antitone maps $f: A \rightarrow B$ for which there is $f^+: B \rightarrow A$ such that (f, f^+) is a Galois connection, with the pointwise partial order. Galois connections appear in many areas of mathematics (cf. the survey [5]).

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Nowadays many authors prefer to work with Galois connections in the covariant form because of its convenience (the survey [5] contains a list of references to this form). In order to avoid confusion we refer to this dualized form as a residuated pair: a *residuated pair* between the partially ordered sets A and B is a pair (f, g) of order-preserving maps $f: A \rightarrow B$ and $g: B \rightarrow A$ such that

$$a \leq gf(a) \text{ for all } a \in A \text{ and } fg(b) \leq b \text{ for all } b \in B. \quad (1.3)$$

The map $f: A \rightarrow B$ in a residuated pair (f, g) is called *residuated* and the map $g: B \rightarrow A$ is called *residual*. Thus f is residuated (resp., residual) if and only if $f \in Gal(A, B^{op})$ (resp., $f \in Gal(A^{op}, B)$).

As Blyth and Janowitz (in their monograph on Residuation Theory [4], p. 19) and Ern e, Koslowski, Melton and Strecker (in [5]) remarked, the reason for preferring residuated maps rather than Galois connections can be found in the fact that compositions of residuated pairs are handled more easily: two residuated maps may be composed to yield a new residuated map and this is not the case with antitone maps. Indeed, if $f: A \rightarrow B$ and $g: B \rightarrow C$ are residuated maps, then gf is clearly residuated with $(gf)^+ = f^+g^+$ [4]. This composition endows $Gal(A, A^{op})$ with a structure of quantale (complete residuated semigroup). We call it the *quantale of residuated maps*.

But there remains the question: how to compose Galois connections directly, in order to get a structure of quantale for $Gal(A, A)$? This is the problem which motivated us to write this paper. Its relevance comes from the fact that this quantale reveals to be an important tool in the entourage-like approaches to uniform structures (cf. [7]). In this paper we present a composition operation \circ for Galois maps. Instead of presenting a direct proof that $(Gal(A, A), \circ)$ is a quantale by proving the associativity of \circ and the distributivity of it over joins in $Gal(A, A)$, we prefer to follow an indirect but more general and elegant approach, by showing that $Gal(A, A)$ is a quotient of the set $Ant(A, A)$ of all antitone maps from A to itself, expressed in terms of a suitable quantic nucleus. This way $Gal(A, A)$ simply inherits the algebraic properties from the more general and simpler $Ant(A, A)$, and we avoid the technical difficulties inherent to proving associativity of \circ and the distributivity of it over joins.

A more detailed description of this paper is as follows. We begin by recalling, in Section 2, some general background about frames, locales, quantales and Galois connections. We also fix terminology and notation that will be used in later sections. In the third section we introduce the composition operation \circ for Galois connections and in Section 4 we introduce an auxiliary operation for antitone maps and prove that this is an associative operation when we specialize to frames. Then we show that this operation endows $Ant(A, A)$ with a structure of quantale whenever A is a frame (Proposition 4.4). In Section 5 we recall some facts about quantic

nuclei and, as a useful intermediate step, we introduce the notion of quantic prenucleus extending the definition of localic prenucleus of [1]. These facts allow us, in Section 6, to conclude from Proposition 4.4 that $(Gal(A, A), \circ)$ is also a quantale. In the last section we illustrate the importance of this quantale with a brief survey describing its application to uniform structures effected in [7]. Namely we refer that the theory of uniform structures (both classical and pointfree) may be completely described in terms of Galois connections and conclude that the equivalence between the somewhat different approaches to uniform locales of [8] and [20] is a special case of a very general phenomenon on Galois connections.

2. Background

In this section we introduce terminology and notation along with an overview of the construction of tensor products of complete join-semilattices, presented in the way which is appropriate for its application in the later sections.

2.1. Frames and locales A *frame* is a complete lattice L in which the infinite distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. Frames are exactly complete Heyting algebras with the Heyting implication given by $x \rightarrow y := \bigvee \{s \in L \mid x \wedge s \leq y\}$.

Frame homomorphisms are defined as maps between frames which preserve finitary meets (including the top element $1 := \bigwedge \emptyset$) and arbitrary joins (including the bottom element $0 := \bigvee \emptyset$). The resulting category is called the category of frames. The study of frames was motivated by the following observation. Let X be a topological space and let $\mathcal{O}(X)$ be the complete lattice of open subsets of X . Since finite meets and arbitrary joins correspond to the set-theoretic operations of intersection and union, $\mathcal{O}(X)$ is a frame. Moreover, if $f: X \rightarrow Y$ is a continuous map of spaces, the induced map $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is a frame homomorphism. The functor $X \mapsto \mathcal{O}(X)$, $f \mapsto f^{-1}$ is contravariant and therefore the category of “generalized spaces” [12] should be the opposite of the category of frames. The objects of this category are called *locales* and morphisms *locale maps*. Thus locales form an order-theoretic counterpart of topological spaces. Locales were introduced by J. Isbell [12] to provide a better categorical context for topology.

For more information on frames and locales we refer the reader to [13].

2.2. Quantales In the definition of a frame, by replacing the commutative operation \wedge with a general associative binary operation, one obtains a quantale (a

“non-commutative generalized space”). Thus a *quantale* is a complete lattice Q equipped with an associative multiplication \cdot which distributes over arbitrary joins:

$$x \cdot \bigvee S = \bigvee \{x \cdot s \mid s \in S\} \text{ and } (\bigvee S) \cdot x = \bigvee \{s \cdot x \mid s \in S\}$$

for all $x \in Q$ and $S \subseteq Q$. A *quantale homomorphism* is a map between quantales which preserves arbitrary joins and the multiplication.

The term quantale was suggested by C. J. Mulvey [15] as a “quantization” of the term locale. Its definition provides a non-commutative extension of the concept of locale. The intention was to develop the concept of non-commutative topology, introduced by R. Giles and H. Kummer [9], while providing constructive foundations for quantum mechanics and non-commutative logic. Further details about quantales can be found in [18] and [21].

2.3. Galois connections Let A and B be complete lattices. In this case, as it is well-known (and very easy to prove), maps f and f^+ in a Galois connection uniquely determine each other; indeed

$$f(a) = \bigvee \{b \in B \mid f^+(b) \geq a\} \text{ and } f^+(b) = \bigvee \{a \in A \mid f(a) \geq b\}.$$

Furthermore, $f \in \text{Gal}(A, B)$ if and only if

$$f(\bigvee S) = \bigwedge f(S) \text{ for every } S \subseteq A \text{ (in particular, } f(0) = 1),$$

that is, if and only if f is a complete join-morphism $A \rightarrow B^{op}$. The set $\text{Gal}(A, B)$ is also a complete lattice (meets are just defined pointwise). On the other hand, $f: A \rightarrow B$ is residuated if and only if it is a join-homomorphism.

For a more categorical approach to Galois connections we refer to [11].

2.4. Tensor products Let A and B be complete join-semilattices. As Shmuely proved in [22], $\text{Gal}(A, B)$ is the tensor product of A and B in the category of complete join-semilattices (whose objects are complete join-semilattices, that is, complete lattices, and morphisms are maps preserving arbitrary joins). This is the analogue for complete join-semilattices of the construction of tensor products of modules over a commutative ring (cf. [14]). There is an equivalent way of describing the tensor product of complete join-semilattices, which will be very useful in the next section. As Shmuely proved ([22], Theorem 1.3; see also [10]), the elements of $\text{Gal}(A, B)$ stand in a one-to-one correspondence with certain semi-ideals of $A \times B$. Let us say that $E \subseteq A \times B$ is a *G-ideal* of $A \times B$ when

(a) $E \in \mathcal{D}(A \times B)$ (the set of all *down-sets* of $A \times B$), that is,

$$\downarrow E := \{(x, y) \mid (x, y) \leq (a, b) \text{ for some } (a, b) \in E\}$$

coincides with E ;

(b) $(\{x\} \times S \subseteq E \Rightarrow (x, \bigvee S) \in E)$ and $(S \times \{y\} \subseteq E \Rightarrow (\bigvee S, y) \in E)$.

Note that the case $S = \emptyset$ in (b) implies that every G-ideal contains the set $\downarrow(1, 0) \cup \downarrow(0, 1)$. We denote by $A \otimes B$ the set of all G-ideals of $A \times B$, partially ordered by inclusion. This is clearly a complete lattice, because the intersection of G-ideals is always a G-ideal. In summary, $A \otimes B \cong Gal(A, B)$ is the tensor product in the category of complete join-semilattices.

For every $a \in A$ and $b \in B$,

$$p_{a,b}: A \longrightarrow B$$

$$x \longmapsto \begin{cases} 1 & \text{if } x = 0 \\ b & \text{if } 0 < x \leq a \\ 0 & \text{if } x \not\leq a. \end{cases}$$

is a Galois map. These maps generate $Gal(A, B)$ by joins, that is, every $f \in Gal(A, B)$ is a join of some family $\{p_{a_i, b_i}\}_{i \in I} \subseteq Gal(A, B)$ [22]. Under the isomorphism $Gal(A, B) \cong A \otimes B$, the Galois map $p_{a,b}$ corresponds to the least G-ideal containing (a, b) , that is,

$$a \otimes b := \downarrow(a, b) \cup \downarrow(1, 0) \cup \downarrow(0, 1).$$

This is called a *pure tensor*. The tensor product construction above gives us the construction of products in the category of locales. Indeed, $A \otimes B$ is a frame whenever A and B are frames and $(A \otimes B, u_1, u_2)$ is the coproduct of A and B in the category of frames, where

$$u_1: A \longrightarrow A \otimes B$$

$$a \longmapsto a \otimes 1$$

and

$$u_2: B \longrightarrow A \otimes B$$

$$b \longmapsto 1 \otimes b.$$

3. The Galois composition

From now on, for complete lattices A and B , let $Ant(A, B)$ denote the set of all antitone maps from A to B . This is, clearly, a complete lattice (meets and joins are just defined pointwise).

For $f \in Ant(A, B)$ and $g \in Ant(B, C)$ let $E_{f,g}$ denote the G-ideal

$$\bigvee \{a \otimes c \in A \otimes C \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } g(b) \geq c\}.$$

The *Galois composition* $g \circ f: A \rightarrow C$ is defined by

$$(g \circ f)(a) := \bigvee \{c \in C \mid (a, c) \in E_{f,g}\}.$$

Proposition 3.1. *For every $f \in Ant(A, B)$ and $g \in Ant(B, C)$, $g \circ f \in Gal(A, C)$.*

Proof. Let us show that $(g \circ f)(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} (g \circ f)(a_i)$. Since G-ideals are downwards closed, $g \circ f$ is clearly order-reversing. Therefore $(g \circ f)(\bigvee_{i \in I} a_i) \leq (g \circ f)(a_i)$ for every $i \in I$. On the other hand, let $c \leq (g \circ f)(a_i)$ for every $i \in I$. By the definition of G-ideal, we have, for each $i \in I$,

$$(a_i, \bigvee \{d \in C \mid (a_i, d) \in E_{f,g}\}) \in E_{f,g},$$

that is, $(a_i, g \circ f(a_i)) \in E_{f,g}$. Thus $(a_i, c) \in E_{f,g}$ and, consequently, $(\bigvee_{i \in I} a_i, c) \in E_{f,g}$, which implies $c \leq (g \circ f)(\bigvee_{i \in I} a_i)$. Hence $(g \circ f)(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} (g \circ f)(a_i)$, which means that $g \circ f \in \text{Gal}(A, C)$. \square

For a categorical description of the Galois composition, that justifies the “correctness” of the chosen definition, see [7].

The following fact about $\text{Gal}(A, B)$ will be decisive in the sequel:

Lemma 3.2. *Let $f \in \text{Gal}(A, B)$ and $E \in \mathcal{D}(A \times B)$ satisfy*

$$(a, b) \in E \Rightarrow b \leq f(a).$$

Then

$$(a, b) \in \langle E \rangle \Rightarrow b \leq f(a),$$

where $\langle E \rangle$ denotes the G-ideal generated by E .

Proof. Consider the non-empty set

$$\mathcal{F} := \{F \in \mathcal{D}(A \times B) \mid E \subseteq F \subseteq \langle E \rangle, (a, b) \in F \Rightarrow b \leq f(a)\}.$$

If $F \in \mathcal{F}$ then also

$$F_0 := \left\{ (a, \bigvee S) \mid \{a\} \times S \subseteq F \right\} \cup \left\{ (\bigvee S, b) \mid S \times \{b\} \subseteq F \right\}$$

is in \mathcal{F} :

Let $(a, b) \in F_0$. If $(a, b) = (a, \bigvee S)$ for some S with $\{a\} \times S \subseteq F$, then $s \leq f(a)$ for every $s \in S$. Thus $\bigvee S \leq f(a)$. On the other hand, if $(a, b) = (\bigvee S, b)$ for some S with $S \times \{b\} \subseteq F$, then $b \leq f(s)$ for every $s \in S$. This implies $b \leq \bigwedge f(S) = f(\bigvee S) = f(a)$.

Besides, for any non-empty $\mathcal{H} \subseteq \mathcal{F}$, $\bigcup_{H \in \mathcal{H}} H \in \mathcal{F}$. Therefore $T := \bigcup_{F \in \mathcal{F}} F$ belongs to \mathcal{F} , that is, \mathcal{F} has a largest element T . But then, as we proved above, T_0 also belongs to \mathcal{F} so $T = T_0$, that is, T is a G-ideal. Hence $\langle E \rangle = T \in \mathcal{F}$ which means that $b \leq f(a)$ whenever $(a, b) \in \langle E \rangle$. \square

4. The quantale of antitone maps

In order to prove that \circ is a quantic multiplication on $Gal(L, L)$ we introduce the following auxiliary operation on antitone maps:

Let $f \in Ant(A, B)$ and $g \in Ant(B, C)$; the *antitone composition* $g \cdot f: A \rightarrow C$ is defined by

$$(g \cdot f)(a) := \bigvee \{c \in C \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } g(b) \geq c\}.$$

Remarks 4.1. (1) It is obvious that $(g \cdot f)(a) = \bigvee \{g(b) \mid b \in B \setminus \{0\}, b \leq f(a)\}$ and $g \cdot f \in Ant(A, C)$.

(2) It is also easy to see that $(g \cdot f)(a)$ coincides with

$$\bigvee \left\{ c \in C \mid (a, c) \in \{(x, z) \in A \times C \mid \exists b \in B \setminus \{0\} : f(x) \geq b \text{ and } g(b) \geq z\} \right\}.$$

Thus $g \cdot f \leq g \circ f$. As we shall see in Proposition 6.4, $g \circ f$ is the least Galois map containing $g \cdot f$.

Note that $g \cdot f_1 \leq g \cdot f_2$ whenever $f_1 \leq f_2$.

Lemma 4.2. *Let $f \in Ant(A, B)$ and $g \in Ant(B, C)$. If C is a frame then $c \leq (g \cdot f)(a)$ if and only if $c = \bigvee S$ for some $S \subseteq C$ such that for each $s \in S$ there exists $b \in B \setminus \{0\}$ satisfying $f(a) \geq b$ and $g(b) \geq s$.*

Proof. Let C be a frame. If $c \leq (g \cdot f)(a)$ then, using Remark 4.1(1), we have

$$\begin{aligned} c &= c \wedge (g \cdot f)(a) \\ &= c \wedge \bigvee \{g(b) \mid b \in B \setminus \{0\}, b \leq f(a)\} \\ &= \bigvee \{c \wedge g(b) \mid b \in B \setminus \{0\}, b \leq f(a)\}, \end{aligned}$$

where, for each such b , $c \wedge g(b) \leq g(b)$ and $b \leq f(a)$.

Conversely, let $c = \bigvee_{i \in I} s_i$ where, for each $i \in I$, there exists $b_i \in B \setminus \{0\}$ such that $s_i \leq g(b_i)$ and $b_i \leq f(a)$. Then, obviously, $s_i \leq (g \cdot f)(a)$ and, consequently, $c \leq (g \cdot f)(a)$. □

In order to conclude the associativity of the antitone composition we have to restrict ourselves to frames.

Proposition 4.3. *Let $f \in Ant(A, B)$, $g \in Ant(B, C)$ and $h \in Ant(C, D)$. If C and D are frames then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.*

Proof. First, let us show that $((h \cdot g) \cdot f)(a) \leq (h \cdot (g \cdot f))(a)$, for every $a \in A$, that is, $\bigvee \{d \in D \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } (h \cdot g)(b) \geq d\} \leq (h \cdot (g \cdot f))(a)$. Of course, it suffices to ensure that $d \leq (h \cdot (g \cdot f))(a)$ whenever $f(a) \geq b \neq 0$ and $(h \cdot g)(b) \geq d$. So let $f(a) \geq b \neq 0$ and $(h \cdot g)(b) \geq d$. Then, since D is a frame, we may apply Lemma 4.2 and write d as a join $\bigvee_{i \in I} d_i$ of a family $\{d_i \mid i \in I\} \subseteq D$ for which there

exist $c_i \in C \setminus \{0\}$, $i \in I$, satisfying $g(b) \geq c_i$ and $h(c_i) \geq d_i$. Then $(g \cdot f)(a) \geq c_i \neq 0$ and $h(c_i) \geq d_i$ for every $i \in I$, and we may conclude that $(h \cdot (g \cdot f))(a) \geq d_i$ for every $i \in I$; consequently, $(h \cdot (g \cdot f))(a) \geq \bigvee_{i \in I} d_i = d$.

Finally, let us prove the reverse inequality $((h \cdot g) \cdot f)(a) \geq (h \cdot (g \cdot f))(a)$. We need to show that $d \leq ((h \cdot g) \cdot f)(a)$ whenever $(g \cdot f)(a) \geq c \neq 0$ and $h(c) \geq d$. Again by Lemma 4.2, since C is a frame, we may write $c = \bigvee_{i \in I} c_i$ for $c_i \in C \setminus \{0\}$ satisfying $f(a) \geq b_i$ and $g(b_i) \geq c_i$ for some $b_i \in B \setminus \{0\}$. But $d \leq h(c) \leq h(c_i)$ so $d \leq (h \cdot g)(b_i)$ for each $i \in I$. Hence $d \leq ((h \cdot g) \cdot f)(a)$. \square

Theorem 4.4. *If L is a frame then $(\text{Ant}(L, L), \cdot)$ is a quantale.*

Proof. By Remark 4.1(3), we have immediately that $g \cdot (\bigvee_{i \in I} f_i) \geq \bigvee_{i \in I} (g \cdot f_i)$. Let us check that $(g \cdot (\bigvee_{i \in I} f_i))(a) \leq (\bigvee_{i \in I} (g \cdot f_i))(a)$, for every $a \in L$, that is,

$$\bigvee \left\{ c \in L \mid \exists b \in L \setminus \{0\} : \bigvee_{i \in I} f_i(a) \geq b \text{ and } g(b) \geq c \right\}$$

is equal or less than

$$\bigvee_{i \in I} \bigvee \{ c \in L \mid \exists b \in L \setminus \{0\} : f_i(a) \geq b \text{ and } g(b) \geq c \}.$$

So, consider $c \in L$ for which there exists $b \in L \setminus \{0\}$ satisfying $b \leq \bigvee_{i \in I} f_i(a)$ and $c \leq g(b)$. Then $b = b \wedge \bigvee_{i \in I} f_i(a) = \bigvee_{i \in I} (b \wedge f_i(a))$. This implies the existence of $i \in I$ for which $b \wedge f_i(a) \neq 0$. Denoting this element $b \wedge f_i(a)$ by b' we have $c \leq g(b) \leq g(b')$ and $0 \neq b' \leq f_i(a)$. This proves $(g \cdot (\bigvee_{i \in I} f_i))(a) \leq (\bigvee_{i \in I} (g \cdot f_i))(a)$.

Similarly,

$$\left(\bigvee_{i \in I} g_i \right) \cdot f = \bigvee_{i \in I} (g_i \cdot f).$$

This, together with Proposition 4.3, shows that $(\text{Ant}(L, L), \cdot)$ is a quantale. \square

5. Quantic nuclei and prenuclei

We need some facts about quotients of quantales, most of which can be found in [16] or [21].

A *quantic nucleus* [16] on a quantale Q is a closure operator $j: Q \rightarrow Q$ such that $j(x) \cdot j(y) \leq j(x \cdot y)$, for all $x, y \in Q$. Note that if j is a quantic nucleus, then

$$j(x \cdot y) = j(x \cdot j(y)) = j(j(x) \cdot y) = j(j(x) \cdot j(y)),$$

for all $x, y \in Q$. The set of j -closed elements

$$Q_j := \text{Fix}(j) = \{x \in Q \mid j(x) = x\}$$

is closed under \wedge and hence is complete (of course, this is true for any closure operator j on a complete lattice Q). In this case, j being a quantic nucleus on a

quantale Q , Q_j is a quantale with joins $\bigvee^j x_i = j(\bigvee x_i)$ and multiplication $x \cdot_j y = j(x \cdot y)$. Indeed,

$$(x \cdot_j y) \cdot_j z = j(j(x \cdot y) \cdot z) = j((x \cdot y) \cdot z) = j(x \cdot (y \cdot z)) = j(x \cdot j(y \cdot z)) = x \cdot_j (y \cdot_j z),$$

$$x \cdot_j \left(\bigvee^j y_i\right) = j(x \cdot j(\bigvee y_i)) = j(x \cdot \bigvee y_i) = j(\bigvee (x \cdot y_i)) \leq j(\bigvee (j(x \cdot y_i))) = \bigvee^j (x \cdot_j y_i)$$

and, consequently, $x \cdot_j \left(\bigvee^j y_i\right) = \bigvee^j (x \cdot_j y_i)$; similarly, $\left(\bigvee^j x_i\right) \cdot_j y = \bigvee^j (x_i \cdot_j y)$.

Furthermore, the map $j: Q \rightarrow Q_j$ is a homomorphism of quantales and every surjective quantale homomorphism arises in this manner, up to isomorphism [16].

In the next section, in order to conclude that $(Gal(L, L), \circ)$ is a quantale, we will define a quantic nucleus generated by certain data. As a useful intermediate step we introduce the following notion, which generalizes Banaschewski's definition of a (localic) prenucleus [1, 2]:

Definition 5.1. A quantic prenucleus on a quantale L is a map $j_0: Q \rightarrow Q$ such that, for all $x, y \in Q$,

- (1) $x \leq j_0(x)$,
- (2) if $x \leq y$ then $j_0(x) \leq j_0(y)$,
- (3) $j_0(x) \cdot y \leq j_0(x \cdot y)$ and $x \cdot j_0(y) \leq j_0(x \cdot y)$.

It follows immediately from the first two conditions that $Q_{j_0} = Fix(j_0)$ is a closure system and the associated closure operator is given by

$$j(x) := \bigwedge \{y \in Q_{j_0} \mid x \leq y\}.$$

Lemma 5.2. The closure operator j is a quantic nucleus.

Proof. Let us show that $j(x) \cdot y \leq j(x \cdot y)$ for all $x, y \in Q$. For this, consider the set

$$\mathcal{E} := \{z \in Q \mid x \leq z \leq j(x), z \cdot y \leq j(x \cdot y)\}.$$

This is a non-empty set (since $x \in \mathcal{E}$) and $j_0(z) \in \mathcal{E}$ whenever $z \in \mathcal{E}$ by the properties of j_0 and j . Moreover, for any non-void $\mathcal{F} \subseteq \mathcal{E}$, $\bigvee_{z \in \mathcal{F}} z$ belongs to \mathcal{E} , by the distribution law of quantales. Therefore \mathcal{E} has a largest element $\bar{z} := \bigvee_{z \in \mathcal{E}} z$. But then $j_0(\bar{z}) \in \mathcal{E}$ thus $\bar{z} = j_0(\bar{z})$ i.e., $\bar{z} \in Q_{j_0}$. Then $j(x) = \bar{z} \in \mathcal{E}$ and, consequently, $j(x) \cdot y \leq j(x \cdot y)$.

By symmetry, $x \cdot j(y) \leq j(x \cdot y)$. Hence $j(x) \cdot j(y) \leq j(x \cdot j(y)) \leq j^2(x \cdot y) = j(x \cdot y)$ as claimed. □

Remarks 5.3. (1) It can be also proved that the quantale homomorphism $j: Q \rightarrow Q_{j_0}$ is universal among all quantale homomorphisms $h: Q \rightarrow Q'$ for which $h(x) = h(j_0(x))$ for all $x \in Q$.

(2) The quantic nucleus j may alternatively be described as the stable transfinite iterate of j_0 , and then the Lemma can be proved by transfinite induction.

6. The quantale of Galois connections

For each $f \in \text{Ant}(A, B)$ let us define $j_0(f): A \rightarrow B$ by

$$j_0(f)(a) := \bigvee \{b \in B \mid \exists S \subseteq A : \bigvee S = a \text{ and } f(s) \geq b \text{ for every } s \in S\}.$$

Clearly, j_0 is an extensive operator, that is, $j_0(f) \geq f$. The motivation for the definition of j_0 comes from the following proposition asserting that Galois maps are precisely the fixed points of j_0 .

Proposition 6.1. *For each $f \in \text{Ant}(A, B)$ we have:*

- (1) *If A is a frame then $j_0(f) \in \text{Ant}(A, B)$.*
- (2) *$j_0(f) = f$ if and only if $f \in \text{Gal}(A, B)$.*

Proof. (1) Let $a \leq a'$ in A . We need to show that $j_0(f)(a') \leq j_0(f)(a)$, that is,

$$\begin{aligned} & \bigvee \{b \in B \mid \exists S \subseteq A : \bigvee S = a' \text{ and } f(s) \geq b \text{ for every } s \in S\} \\ & \leq \bigvee \{b \in B \mid \exists S \subseteq A : \bigvee S = a \text{ and } f(s) \geq b \text{ for every } s \in S\}. \end{aligned}$$

So, consider b in the first join, that is, $b \in B$ satisfying $b \leq f(s)$ for every s in some $S \subseteq A$ for which $\bigvee S = a'$. Then $a = a \wedge a' = a \wedge \bigvee S = \bigvee_{s \in S} (a \wedge s)$, by the frame distributive law. Thus, since $b \leq f(s) \leq f(a \wedge s)$, b is also in the second join. (2) By the definition of j_0 it is clear that $j_0(f)(\bigvee S) \geq \bigwedge f(S)$ for every $S \subseteq A$. Therefore, if $j_0(f) = f$ then $f(\bigvee S) \geq \bigwedge f(S)$. Thus, since $f(\bigvee S) \leq \bigwedge f(S)$ is always true, $f(\bigvee S) = \bigwedge f(S)$, that is, f is a Galois map.

Conversely, let $f \in \text{Gal}(A, B)$ and $a \in A$. For each b in

$$\{b \in B \mid \exists S \subseteq A : \bigvee S = a \text{ and } f(s) \geq b \text{ for every } s \in S\}$$

we have $b \leq \bigwedge f(S) = f(\bigvee S) = f(a)$, which shows that $j_0(f)(a) \leq f(a)$. Hence $j_0(f) \leq f$. Then, immediately, $j_0(f) = f$ since j_0 is extensive. \square

Proposition 6.2. *If L is a frame then $j_0: \text{Ant}(L, L) \rightarrow \text{Ant}(L, L)$ is a quantic prenucleus.*

Proof. Obviously j_0 is extensive and monotone. Let us prove that $(g \cdot j_0(f))(x) \leq (j_0(g \cdot f))(x)$ for all $x \in L$ and $f, g \in \text{Ant}(L, L)$.

So, consider $y \in L$ satisfying $j_0(f)(x) \geq z$ and $g(z) \geq y$ for some nonzero $z \in L$. Then

$$\begin{aligned} z &= z \wedge j_0(f)(x) \\ &= z \wedge \bigvee \{a \in L \mid \exists S \subseteq L : \bigvee S = x \text{ and } f(s) \geq a \text{ for every } s \in S\} \\ &= \bigvee \{z \wedge a \mid a \in L, \exists S \subseteq L : \bigvee S = x \text{ and } f(s) \geq a \text{ for every } s \in S\}. \end{aligned}$$

The fact $z \neq 0$ implies the existence of $a \in L$ and $S \subseteq L$ such that $x = \bigvee S$, $f(s) \geq a$ for all $s \in S$ and $z \wedge a \neq 0$. Then $g(z \wedge a) \geq g(z) \geq y$ and, for each $s \in S$, $f(s) \geq a \geq z \wedge a \neq 0$. This means $(g \cdot f)(s) \geq y$ for every $s \in S$. Therefore $(j_0(g \cdot f))(x) \geq y$.

The condition $j_0(g) \cdot f \leq j_0(g \cdot f)$ may be proved in a similar manner. \square

Then $Ant(L, L)_{j_0}$ (which, by Proposition 6.1(2), coincides with $Gal(L, L)$) is a closure system and, by Lemma 5.2, the associated closure operator $j: Ant(L, L) \rightarrow Ant(L, L)$ given by

$$j(f) := \bigwedge \{g \in Gal(L, L) \mid f \leq g\}$$

is a quantic nucleus. Hence, by Theorem 4.4 and the results on quantic nuclei of [16] recalled at the beginning of Section 5, we have:

Corollary 6.3. *If L is a frame then $(Gal(L, L), \cdot_j)$ is a quantale.*

Now the conclusion that the operation \cdot_j coincides with the Galois composition \circ introduced in Section 3 follows from the following general result:

Proposition 6.4. *Let A , B and C be complete lattices. For every $f \in Ant(A, B)$ and $g \in Ant(B, C)$, $j(g \cdot f) = g \circ f$.*

Proof. By Remark 4.1(2), $g \cdot f \leq g \circ f$. Now let $h \in Gal(A, C)$ satisfying $h \geq g \cdot f$. We need to show that $h \geq g \circ f$ in order to conclude the proof. Let $a \in A$. Since $(g \circ f)(a) = \bigvee \{c \in C \mid (a, c) \in E_{f, g}\}$, by Lemma 3.2 it suffices to check that $c \leq h(a)$ whenever $(a, c) \in \{(x, z) \in A \times C \mid \exists y \in B \setminus \{0\} : f(x) \geq y \text{ and } g(y) \geq z\}$, which is easy: if $(a, c) = (x, z)$ for such a pair (x, z) then $c = z \leq g(y)$ and $0 \neq y \leq f(x) = f(a)$, which implies $c \leq (g \cdot f)(a) \leq h(a)$. \square

Corollary 6.5. *If L is a frame then $(Gal(L, L), \circ)$ is a quantale.*

7. Applications

For any set X , the power set $\mathcal{P}(X \times X)$ equipped with the usual composite relation $S \circ R$ is a quantale. This quantale of binary relations on X is the basic structure on which the theory of uniform spaces of Weil [23] relies (the calculus of relations has also played an important rôle in the interaction between algebra and logic since the middle of the nineteenth century; a first adequate development of such algebras was given by de Morgan and Peirce).

Binary relations can be described by *polarities* (i.e., Galois connections between power sets) [5]. In fact, there is a one-to-one correspondence between relations $R \subseteq X \times X$ and Galois connections on $\mathcal{P}(X)$, that assigns to each relation R the

Galois connection $(R_\forall, R^\forall): \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by

$$R_\forall(U) := \bigcap_{a \in U} \{b \in X \mid (a, b) \in R\} = \{b \in X \mid \forall a \in U (a, b) \in R\} \quad \text{for } U \subseteq X$$

and

$$R^\forall(V) := \bigcap_{b \in V} \{a \in X \mid (a, b) \in R\} = \{a \in X \mid \forall b \in V (a, b) \in R\} \quad \text{for } V \subseteq X.$$

In particular, Weil entourages E of X (that is, reflexive binary relations on X) correspond to Galois connections (E_\forall, E^\forall) satisfying $\{x\} \subseteq E_\forall(\{x\})$ for all $x \in X$ or, equivalently, $\bigcup\{A \subseteq X \mid A \subseteq E_\forall(A)\} = X$.

Similarly, binary relations may also be described by *axialities*, that is, residuated pairs between power sets (see [5] for the details).

Therefore the classical description of uniform spaces, introduced by Weil in terms of entourages, may be immediately formulated in terms of polarities and axialities. In all these entourage-like approaches to uniform structures the main tool is a certain quantic structure. The theory of uniform spaces of Weil can be generalized to the pointfree context of frames and locales (see [20] for a survey on this generalization). Of course, it seems desirable to have characterizations of uniform frames that are analogous to those of uniformities which are given by polarities and axialities. In order to pursue this goal one needs “good” quantic structures for $Gal(L, L)$ and $Gal(L, L^{op})$. The former is presented in this paper; we get it precisely in the context that we need it: when L is a frame. In [7] it is shown how to apply it to get a new (equivalent) approach to uniform frames. The latter is the well-known quantale of residuated maps ([5], [21]). By applying it to the study of uniform frames, as it is shown in [7], one gets precisely Fletcher-Hunsaker-Lindgren’s approach [8]. This puts some light on the nature of this approach: Galois connections rather than entourages are at its root.

We can say more; the equivalence between the approaches to frame uniformities presented in [8] and [19] is a special case of the following very general phenomenon on Galois connections, relating the elements of the quantales $Gal(L, L)$ and $Gal(L, L^{op})$. For each $f \in Gal(L, L)$ let $f^\uparrow: L \rightarrow L$ be defined by

$$f^\uparrow(x) := \bigvee \{y \in L \mid y \wedge x \neq 0, y \leq f(y)\}.$$

This is a residuated map on L and $f \leq g$ implies $f^\uparrow \leq g^\uparrow$. Conversely, given $f \in Gal(L, L^{op})$, define $f^\downarrow: L \rightarrow L$ by putting

$$f^\downarrow(x) := \bigvee \left\{ y \in L \mid (x, y) \in \bigvee \{a \otimes a \mid a \text{ is } f\text{-small}\} \right\},$$

where $a \in L$ is said to be *f-small* whenever $a \wedge b \neq 0$ implies $a \leq f(b)$. This way we get a Galois map f^\downarrow on L and $f \leq g$ implies $f^\downarrow \leq g^\downarrow$. Since $f^{\uparrow\downarrow}$ and $f^{\downarrow\uparrow}$ may differ from f , these correspondences do not establish a one-to-one correspondence

between $Gal(L, L)$ and $Gal(L, L^{op})$. Nevertheless we have the following (cf. [7] for the details):

Let us say that a Galois map f is *symmetric* [7] whenever $f^+ = f$. It is straightforward to check that f^\downarrow , for every residuated f , is symmetric. Similarly, a *symmetric residuated map* [7] is a residuated map f for which

$$f(x) \wedge y \neq 0 \Leftrightarrow x \wedge f(y) \neq 0 \text{ for all } x, y \in L.$$

The map f^\uparrow , for every Galois f , is symmetric. Let us also extend the concepts of polarity and axiality to any frame L by saying that $f \in Gal(L, L)$ is a *polarity* of L whenever $\bigvee\{x \in L \mid x \leq f(x)\} = 1$, and that $f \in Gal(L, L^{op})$ is an *axiality* of L if $\bigvee\{x \in L \mid x \text{ is } f\text{-small}\} = 1$. Then, for every symmetric polarity f of L ,

$$f \leq (f \circ f)^{\uparrow\downarrow} \text{ and } f^{\uparrow\downarrow} \leq f \circ f$$

and, for every symmetric axiality f of L ,

$$f \leq (f \circ f \circ f)^{\downarrow\uparrow} \text{ and } f^{\downarrow\uparrow} \leq f \text{ [7].}$$

This means that, in the presence of the square-refinement axiom (which asserts that for every f in the structure there exists g in the structure satisfying $g \circ g \leq f$) and the axiom of symmetry, we get a bijection. More precisely, the correspondences $f \mapsto f^\uparrow$ and $f \mapsto f^\downarrow$ establish an isomorphism between the filters of polarities of L satisfying the symmetry and square-refinement axioms and the filters of axialities of L satisfying the symmetry and square-refinement axioms. This fact about Galois connections explains why the somewhat different approaches to uniform frames presented in [8] and [19] are equivalent. The details can be found in [7].

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