Regularity properties for the porous medium equation

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Abstract

The theory of degenerate/singular nonlinear partial differential equations has gained a great importance over the last decades. In this work, we study a famous equation with these carachteristics, *the porous medium equation*,

$$u_t - \Delta u^m = 0, \qquad m > 1. \tag{PME}$$

We study regularity results for this equation, more especifically we prove the Hölder continuity of nonnegative weak solutions.

Our main goal is to describe the method used to achieve this result, the intrinsic scaling method. This method can be used for a great number of degenerate and singular equations, wich makes it a strong tool in this area.

Keywords: Porous medium equation, Intrinsic scaling, Hölder continuity, *a priori* estimates, Signed porous medium equation

Resumo

A teoria das equações com derivadas parciais degeneradas/singulares ganhou uma grande importância ao longo das últimas décadas. Neste trabalho, estudamos uma famosa equação com estas características, a *equação dos meios porosos*,

$$u_t - \Delta u^m = 0, \qquad m > 1. \tag{PME}$$

Estudamos resultados de regularidade para esta equação, mais concretamente, provamos a continuidade à Hölder das suas soluções fracas não negativas.

O principal objetivo deste trabalho é descrever o método usado para obter este resultado. Este método pode ser usado para tratar um vasto número de equações degeneradas e singulares, o que faz dele uma ferramenta essencial nesta área.

Palavras Chave: Equação dos meios porosos, Método da mudança intrínseca de escala, Continuidade à Hölder, Estimativas *a priori*, Equação dos meios porosos com sinal

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"Gratitude is the only treasure of the humble."

William Shakspeare

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André Martins Rodrigues

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Chapter 1

Introduction

The theory of nonlinear partial differential equations that exhibit some sort of degeneracy or singularity has been widely investigated in the last decades, both for the great number of applications it has in the fields of mathematical biology, fluid mechanics, heat propagation, and for their mathematical interest. In fact, this area is pivotal in combining the theory of nonlinear evolution PDEs with geometry, dynamical systems and other areas of applied mathematics. Despite the fact that the theory of linear PDEs experienced much progress, it was soon observed that most of equations modelling physical phenomena, without excessive simplification are nonlinear and some of them with singularities/degeneracies. Unfortunately, just after the 1960s the development of functional analysis made it possible to start building theories for these type of PDEs with full mathematical rigour.

Two of the most important degenerate/singular nonlinear PDEs are the *porous medium equation*, shortly PME,

$$u_t - \Delta u^m = 0, \qquad m > 1, \tag{PME}$$

and the parabolic p-Laplace equation, shortly PLE,

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad p > 1, \tag{PLE}$$

where $\Delta = \Delta_x$, $\nabla = \nabla_x$ represent the Laplace operator and the gradient operator respectively, acting only on the space variables. Both equations can be posed for all $x \in \mathbb{R}^d$, $d \geq 1$ and $0 < t < \infty$ (Cauchy problem), but are quite often posed in a bounded subdomain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ and $0 < t < T$ (Dirichlet problem). In the Cauchy problem the determination of a unique solution asks only for initial conditions, however in the second case boundary conditions are also needed. In this text we will work just with local solutions, leaving aside the boundary conditions. Both equations have great importance since they are both, somehow a generalization of the heat equation (notice that for $m = 1$ and $p = 2$ the equations are reduced to the heat equation $u_t = \Delta u$).

Our main goal is to describe the method of intrinsic scaling, a method for obtaining continuity results, in particular Hölder continuity for weak solutions of degenerate and singular parabolic equations. This method can be used for a great number of equations, but it is famous mostly because it can be applied to the PLE, the PME and equations with a similar structure. To avoid some technical difficulties we will use the PME as a model for the method.

1.1. The *porous medium equation*

The reader may wonder why we have choosen a simple-looking variation of the famous heat equation,

$$u_t - \Delta u = 0, \tag{HE}$$

for this text. First of all, the fact that the theory of the PME departs strongly from the heat equation was a great booster both for the pure mathematician and the applied scientist to start studying this equation. This equation appears in several useful applications and its study reveals to be very helpful on the development of interesting and sophisticate results in nonlinear analysis.

In this regard, there are a number of physical applications where the simple PME model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. The best known is probably the description of the flow of an ideal gas in a homogeneous porous medium, indeed the PME owes its name to this application, modelled by Leibenzon [8] and Muskat [11] around 1930. Another important application, essential for the historic development of the field, happens in the theory of heat propagation, more specifically Zel'dovich and Raizer in [17], propose a PME as a model to describe heat propagation by radiation occurring in plasmas (ionized gases) at very high temperatures. In fact, this subject motivated the rigorous mathematical development of the theory. Other applications have been proposed in mathematical biology, boundary layer theory, and other fields. This equation is also a good model to study hard subjects as finite propagation, free boundaries, self-similar solutions, among others. The reader can explore these areas in [16].

1.1.1. The PME as a nonlinear degenerate parabolic equation

The PME, and also the PLE, constitute the most famous examples of nonlinear degenerate and singular parabolic equation. However, before exploring this fact let us remember what this concepts are.

Linear parabolic PDEs are the most common evolution equations. These PDEs can be defined in divergence form, as

$$u_t - \sum_{i,j=1}^{n} (a_{ij}(x,t))u_{x_i})_{x_j} + \sum_{i}^{n} b_i(x,t)u_{x_i} + c(x,t)u = f,$$

with a_{ij} measurable and bounded coefficients satisfying the ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \tag{1.1}$$

for all $\xi \in \mathbb{R}^n$. However in this text we will just study diffusion equations, which means that we will suppose $b_i = 0,\ c = 0$. To simplify the calculations, we will also consider the homogeneous case, $f = 0$.

The next step was extending this definition to quasilinear equations of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = 0, \tag{1.2}$$

permitting the function $\mathbf{A}$ to be nonlinear, only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,\nabla u) \cdot \nabla u \geq \lambda(x,t,u,\nabla u)|\nabla u|^2 \\ |\mathbf{A}(x,t,u,\nabla u)| \leq |\Lambda(x,t,u,\nabla u)||\nabla u|, \end{cases} \tag{1.3}$$

where λ and Λ are given functions. This type of equations are still referred to as parabolic if there exist two positive constants $\lambda_0 \leq \Lambda_1$ such that

$$\lambda_0 \leq \lambda(x,t,u,\nabla u) \quad \text{and} \quad \Lambda(x,t,u,\nabla u) \leq \Lambda_1. \tag{1.4}$$

Lastly, a parabolic partial differential equation of the type (4.2) is termed degenerate if the modulus of ellipticity $\lambda(x,t,u,\nabla u)$ tends to zero at some point of its domain of definition; whereas it is termed singular if $\Lambda(x,t,u,\nabla u)$ tends to infinity.

Easily we can conclude that the *porous medium equation* (PME) is of the degenerate type. In fact, it can be rewritten in the divergence form as

$$u_t - \operatorname{div}(mu^{m-1}\nabla u) = 0, \quad m > 1,$$

where $\mathbf{A} = mu^{m-1}\nabla u$ and $\lambda(x,t,u,\nabla u) = mu^{m-1}$ which, evidently, tends to zero when u tends to zero.

Notice that if we choose $m < 1$, then $\Lambda = mu^{m-1}$ tends to infinity as u also goes to infinity and the equation will be singular. There is also a vast literature about this case, however we will not study it in this text.

It is also important to highlight that the p-Laplace equation (PLE) has similar properties. It is another example of this type of equation because it is degenerate for $p > 2$ and singular for $1 < p < 2$.

The fact that the PME is degenerate parabolic has deep mathematical consequences, both qualitative and quantitative. Even if, in the second half of the last century, when the tools of functional analysis were ready for it, the theory of nonlinear parabolic equations became a main area of research in PDEs not every result could be extended to degenerate parabolic equation.

On the other hand, and as a sort of compensation, the equation enjoys a number of nice properties due to its simple form, which makes the PME an interesting benchmark in the development of nonlinear analytical tools, and continues to make its way into the pure and applied sciences, and consequently into the mainstream of Mathematics.

1.2. Regularity theory and the method of intrinsic scaling

Nowadays, the regularity theory for PDEs is an important area for mathematicians. As many others recent areas, regularity theory just rose in 1900, after the well-known International Congress of Mathematicians in Paris where David Hilbert presented his list of 23 problems that would shape the mathematics of the 20th century. The way this theory started to develop, with the great mathematicians Ennio De Giorgi, John Nash and Jürgen Moser, is one of the most famous in the history in Mathematics. This subject was the main focus of my work in the first semester and the reader can learn more about it in my seminar [13]. Although the work was written in Portuguese, if the language is a restraint there is also the paper [14] and the book [15].

In the previously mentioned work, there is a detailed explanation on how De Giorgi proved that weak solutions of an elliptic PDE of the form

$$\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_j}\right)_{x_i} = 0,$$

with bounded and measurable coefficients a_{ij} satisfying an ellipticity condition (1.1) are Hölder continuous. Since the linearity does not play a role in the proofs of De Giorgi, Ladyzhenskaya and Uralt'seva, during the mid 1960's, were able to prove in [7] the same results for weak solutions of quasi-linear equations of the form

$$\text{div } \mathbf{a}(x, u, \nabla u) = 0,$$

with structure assumptions of the type

$$\begin{cases} \mathbf{a}(x,u,\nabla u)\cdot\nabla u \geq C_1|\nabla u|^p - C_2 \\ |\mathbf{a}(x,u,\nabla u)| \leq C_2(|\nabla u|^{p-1}+1), \end{cases}$$

with $p>1$, and constants $C_1>0$ and $C_2\geq 0$.

The next step is to pass to parabolic equations

$$\sum_{i,j=1}^{n}\left(a_{ij}(x)u_t - u_{x_j}\right)_{x_i} = 0,$$

with bounded and measurable coefficients a_{ij} satisfying an ellipticity condition (1.1) and

$$u_t - \text{div }\mathbf{a}(x,u,\nabla u) = 0,$$

with the same structure assumptions. Notice that this assumptions are a bit stronger than (1.3), and unfortunately these ones does not include the PME as a particular case, just the PLE. This was how the theory evolved historically and just afterwards the results were generalized, reducing the assumptions.

Moser [10] was able to prove that weak solutions of linear parabolic equations are locally Hölder continuous, however it was not possible to extend the results to quasi-linear equations as we might expect, just for special cases. In 1967, Ladyzenskaja, Solonnikov and Ural'ceva, in [12], were able to prove it for quasi-linear equations with assumptions that include the parabolic case. Some other examples appeared in the literature, however this problem remained open until mid 1980's when DiBenedetto, in [4], showed that the solutions of quasilinear equations with the previous assumptions are locally Hölder continuous. It was necessary to employ a new approach to this problem and the answer was found with the intrinsic scaling method. Surprisingly, this showed to be very versatile and could be appropriately modified to establish the same results for more general assumptions, including the *porous medium equation*.

Continuity results of a function at a certain point can be obtained by measuring its oscillation in a sequence of nested and shrinking cylinders with vertex at that point, and proving that the oscillation converges to zero as the cylinders shrink to the point. For the heat equation this idea work for the simple standard parabolic cylinders,

$$Q(R^2,R) = \{(x,t)\in\mathbb{R}^d\times\mathbb{R}_+; ||x-x_0||_\infty \leq R^2, 0\leq t_0-t<R\},$$

however for more complicated equations as the PME thus is not enough. These cylinders must be rescaled to take into account the nature of the degeneracy or singularity. A precise description of this fact is linked to the technical implementation that we will develop in this works and will hopefully clarify the idea.

Although it could seem that the proof would be similar to the heat equation a new problem arises, the nature of the scaling factors. The only way the cylinders could reflect the geometry of the equation is with an intrinsic scaling, which means that the scaling factors must depend on the solution itself.

At the basis of the intrinsic scaling method is an iteration technique, which constitutes a refinement of the technique by De Giorgi and Moser [13, 14], based on *a priori* estimates for the weak solutions. These estimates will be integral inequalities that measure the behaviour of the function near its infimum and its supremum. In the case of solutions of degenerate or singular equations, one more difficulty arises; the estimates are not homogeneous and involve integral norms corresponding to different powers. However, since our cylinders are intrinsically rescaled, we can recover the homogeneity when we rewrite the estimates in this cylinders, absorbing in this way many analysis difficulties in the geometry.

The *a priori* estimates will be the main tools in establishing the local Hölder continuity for the solutions. Actually, once these estimates are obtained we don't need the original equation any more as our problem turns to be purely analytic. We may forget the initial equation and we just need to show that functions that satisfy our integral inequalities are locally Hölder continuous.

The main goal of this work is to help the reader understand this method in full depth and to bring to light what is really essential in this powerful tool in the analysis of degenerate and singular equations. We want to highlight how the geometry and the *a priori* estimates are important through all the proof. We want to convince the reader of the strength this approach to regularity for an important and relevant class of nonlinear partial differential equations.

1.3. Outline of the text

We have chosen the *porous medium equation* as a model to present the theory for degenerate nonlinear equations. This equation highlights what is really essential in the intrinsic scaling method, leaving aside technical difficulties that come along with more general equations.

This work is based on a similar work done for the p-Laplace equation in the first part of [15]. Some proofs are presented in more detail and some ideas are also inspired on others articles and books as [6] and [5, 16]. However, we do not pretend this work to be just a rewrite of different books and articles. Although the main idea of the method is identical, we found several technical difficulties. Some of them were solved by consulting other papers, and others with original ideas.

We start in Chapter 2 by presenting the problem formally. In section 2.1 we introduce the precise definition of weak solution for the model problem. This is followed, in section 2.2, by the derivation of the building blocks of the theory: the local energy and logarithmic estimates. Then, section 2.3 will highlight the technical tools that will be essential for the establishment of some results. The 3th chapter presents in full detail the idea of intrinsic scaling, at least for the degenerate equation. Section 3.1 deals with the construction of the appropriate geometric setting, bringing in full detail the idea of intrinsic scaling. At this point, the problem is divided in two alternatives, that will be studied separately in the sections 3.2 and 3.3. Finally, chapter 4 culminates with the proof of the Hölder continuity. In section 4.2 we refer to a simpler case that is not included in the previous approach to the parabolic case. We close the chapter with remarks on possible generalizations, mainly to equations with the full quasilinear structure and to the p-Laplace equation.

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Chapter 2

Weak solutions and *a priori* estimates

The main goal of this work is to describe the intrinsic scaling method. In this chapter we will introduce some properties of the PME, definitions and results that will be necessary trough the text. This method is used to prove the Hölder continuity for the weak solutions of degenerate and singular parabolic equations. In this regard we will use the *porous medium equation*,

$$u_t - \Delta u^m = 0, \qquad m > 1, \tag{PME}$$

to explain this method in detail, leaving aside technical difficulties related to more general equations. We have chosen this equation because it is one of the simplest examples of a parabolic non-linear evolution equation, appearing however in the description of several natural phenomena. Its properties depart strongly from the heat equation,

$$u_t - \Delta u = 0,$$

one of the most important PDE. There are a number of physical applications where this simple model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion.

Most physical settings lead to the default restriction $u \geq 0$, which is mathematically convenient. As reported in the literature, in order to consider solutions without this property a variation of the PME is used,

$$u_t - \Delta(|u|^{m-1}u) = 0, \tag{2.1}$$

usually referred as the signed *porous medium equation*. Some results can be extended to this equation, the Hölder continuity of the weak solutions inclusively. For the purpose of this work and to highlight the important aspect of the intrinsic scaling method we decided that the simple PME is enough, but we will leave remarks along the text to guide the prove for the signed PME.

Additionally, we will also suppose that u is bounded and set

$$M := ||u||_{L^\infty}.$$

Although only local boundedness is necessary we assume that u is bounded for simplification. So our goal is to prove that a non-negative and bounded solution, in some sense, of the *porous medium equation*, (PME), is Hölder continuity.

If we consider Ω to be a bounded domain in $\mathbb{R}^d$, with smooth boundary $\partial\Omega$, we can define the space time domain as $\Omega_T = \Omega \times (0, T], \quad T > 0$, with parabolic boundary $\partial_p\Omega_T = (\partial\Omega \times (0, T)) \cup (\Omega \times \{0\})$. Next we will consider the degenerate intrinsic parabolic distance from a compact set $K \subset \Omega_T$ to $\partial_p\Omega_T$ as

$$\operatorname{dist}(K; \partial_p\Omega_T) := \inf_{\substack{(x,y)\in K \\ (y,s)\in\partial_p\Omega_T}} \left(|x-y| + |t-s|^{\frac{1}{2}}\right).$$

Definition 2.1 (Locally Hölder continuous). *A function u is locally Hölder continuous in Ω_T if there exist a constant C and $\beta \in (0, 1)$, depending only on the data, such that, for every compact subset K of Ω_T,*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C \left(\frac{|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}}{dist(K; \partial_p\Omega_T)}\right)^\beta,$$

for every pair of points $(x_i, t_i) \in K$, $i = 1, 2$.

2.1. Two definitions of weak solutions

Not many second-order partial differential equations have solutions in the strong sense. So we will start by defining weak solution for our problem. As usual, the definition of weak solutions is obtained by multiplying by a test function, integrate over a compact set and then integrating by parts. Formally it results in the following definition.

Definition 2.2. *A local weak solution of* (PME) *is a bounded measurable function*

$$u \in C_{loc}\left(0, T; L^2_{loc}(\Omega)\right) \cap L^2_{loc}\left(0, T; W^{1,2}_{loc}(\Omega)\right),$$

such that, for every compact $K \subset \Omega$ and for every $0 < t_1 \leq t_2 \leq T$,

$$\int_K u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \left\{-u\varphi_t + mu^{m-1}\nabla u \cdot \nabla\varphi\right\} dxdt = 0, \tag{2.2}$$

for all $\varphi \in H^1_{loc}\left(0, T; L^2(K)\right) \cap L^2_{loc}\left(0, T; W^{1,2}_0(K)\right)$.

However, this way of defining the weak solution does not involve the time derivative u_t, which would be very useful in some proofs. As u_t sometimes can only exist in the sense of distributions we will introduce the Steklov average of a function $v \in L^1(\Omega_T)$ for $0 < h < T$:

$$v_h := \begin{cases} \frac{1}{h}\int_t^{t+h} v(.,\tau)d\tau & \text{if } t \in (0, T-h], \\ 0 & \text{if } t \in (T-h, T]. \end{cases}$$

The Steklov average has nice properties: for v in the right spaces it converges to v as h goes to 0.

Lemma 2.3. *If $v \in L^r(0,T,L^q(\Omega))$ then, as $h \to 0$, the Steklov average v_h converges to $v \in L^{q,r}(\Omega_{T-\epsilon})$, for every $\epsilon \in (0,T)$. If $v \in C(0,T;L^q(\Omega))$ then, as $h \to 0$, the Steklov average $v_h(\cdot,t)$ converges to $v(\cdot,t)$ in $L^q(\Omega)$, for every $t \in (0,T-\epsilon)$ and every $\epsilon \in (0,T)$.*

Using the Steklov average it is possible to obtain another definition that will be more useful through the calculations.

Definition 2.4. *A local weak solution of* (PME) *is a bounded measurable function*

$$u \in C_{loc}\left(0,T;L^2_{loc}(\Omega)\right) \cap L^2_{loc}\left(0,T;W^{1,2}_{loc}(\Omega)\right)$$

such that, for every compact $K \subset \Omega$ and for every $0 < t < T-h$,

$$\int_{K\times\{t\}} (u_h)_t\,\varphi + m\left(u^{m-1}\nabla u\right)_h \cdot \nabla\varphi dx = 0, \tag{2.3}$$

for all $\varphi \in W^{1,2}_0(K)$.

In fact, let us prove that these definitions are equivalent.

Theorem 2.5. *The definitions 2.2 and 2.4 of weak solutions are equivalent.*

Proof. ($\Rightarrow$) For all $t \in]0,T-h[$, let us choose $t_1 = t$ and $t_2 = t+h$, and the test function φ independent of the time and belonging to $W^{1,2}_0(K)$. In this way the equation (2.2) can be simplified to read

$$\int_K u\Big|_t^{t+h}\varphi dx + \int_K \left\{m\int_t^{t+h} u^{m-1}\nabla u dt \cdot \nabla\varphi\right\} dx = 0.$$

To finish this implication we just need to multiply both sides for $\frac{1}{h}$ and notice that $(u_h)_t = \frac{1}{h}\left(\int_t^{t+h} u(\cdot,\tau)d\tau\right)_t = \frac{1}{h}u\Big|_t^{t+h}$.

($\Leftarrow$) For this part let us start to fix $0 < t_1 < t_2 \leq T$ and $\varphi \in H^1_{loc}\left(0, T; L^2(K)\right) \cap L^2_{loc}\left(0, T; W_0^{1,2}(K)\right)$. Then we can choose for every $t \in [t_1, t_2 - h]$ the function $\varphi(\cdot, t) \in W_0^{1,2}(K)$ and integrate the equation (2.3) between t_1 and $t_2 - h$, obtaining

$$\int_{t_1}^{t_2-h} \int_K (u_h)_t \, \varphi + m \left(u^{m-1} \nabla u\right)_h \cdot \nabla \varphi dx dt = 0.$$

Now we are in conditions to integrate by parts in time

$$\int_K u_h \varphi dx \Big|_{t_1}^{t_2-h} + \int_K \int_{t_1}^{t_2-h} -u_h \varphi_t + m \left(u^{m-1} \nabla u\right)_h \cdot \nabla \varphi dx dt = 0,$$

and then, we just need to let $h \to 0$. Using **Lemma** 2.3, the proof is over. □

Remark 2.6. For thesigned PME (2.1) the definition of weak solutions just has one small difference, it will appear u^{m-1} instead of $|u|^{m-1}$.

2.2. *A priori* estimates

Regarding the intrinsic scaling method, the first step is to derive *a priori* estimates for weak solutions. We need integral inequalities that measure the behaviour of the weak solutions near its infimum and its supremum in the interior of an appropriate cylinder. At this point we can ignore the equation and it remains to show that functions that satisfy these inequalities are locally Hölder continuous.

Before presenting these inequalities, let us first define the cylinders mentioned before. Given a point $x_0 \in \mathbb{R}^d$, we denote the n-dimensional cube with centre at x_0 and wedge 2ρ by:

$$K_\rho(x_0) := \left\{ x \in \mathbb{R}^d : ||x - x_0||_\infty < \rho \right\}.$$

We also define the cylinder of radius ρ and height $\tau > 0$, with vertex at (x_0, t_0), as:

$$C(x_0, t_0, \tau, \rho) = (x_0, t_0) + Q(\tau, \rho) = K_\rho(x_0) \times (t_0 - \tau, t_0),$$

with parabolic boundary

$$\partial_p C(x_0, t_0, \tau, \rho) = \partial K_\rho(x_0) \times (t_0 - \tau, t_0) \cup (K_\rho(x_0) \times \{t_0 - \tau\}).$$

We can, without loss of generality, restrict to cylinders with vertex at the origin $(0, 0)$. Since the *porous medium equation* is invariant under displacement of the coordinate axes, we can change to a generic point (x_0, t_0) with a simple translation. This way, although we just prove the results for cylinders with vertex at the origin, we claim that they are valid for cylinders with vertex at any point, which is necessary for the final conclusions. We will also write K_ρ to denote $K_\rho(0)$ and $Q(\tau,\rho)$ instead of $C(0,0,\tau,\rho)$.

We recall that

$$w_+ = \max(w,0), \qquad w_- = -\min(w,0)$$

and introduce the auxiliary function $u^l_\pm = \pm\min\{\pm u, \pm l\}$.

Finally, it will also be essential to use a piecewise smooth cutoff function in $Q(\tau,\rho)\subset\Omega_T$ such that

$$0\le\zeta\le 1, \qquad |\nabla\zeta|<\infty, \qquad \zeta(x,t)=0, \quad x\notin K_\rho.$$

2.2.1. Local energy estimates

The first inequality that needs to be proven is, somehow, a generalization of the famous Caccioppoli inequality that is often used for elliptic equations. We use this inequality to control, in some sense, the norm of ∇u with the norm of the function u itself. Since we will only use this estimates for ζ such that

$$\zeta = 0 \quad \text{in} \quad \partial_p Q(\tau,\rho),$$

we will suppose this now, just for simplification. The difficulty without this assumption is the same, but unnecessary terms appear.

Proposition 2.7 (Local energy estimates)**.** *Let u be a local weak solution of* (PME), $k,l\in\mathbb{R}^+$ *and ζ a cutoff function. There exists a constant $C\equiv C(m)>0$ such that, for every cylinder $Q(\tau,\rho)\subset\Omega_T$.*

$$\begin{aligned}\operatorname*{ess\,sup}_{-\tau<t<0}\int_{K_\rho\times\{t\}}(u^l_\pm-k)^2_\pm\zeta^2dx+\int_{-\tau}^0\int_{K_\rho}(u^l_\pm)^{m-1}|\nabla(u^l_\pm-k)_\pm\zeta|^2dxdt\le\\ C\int_{-\tau}^0\int_{K_\rho}(u^l_\pm)^{m-1}(u^l_\pm-k)^2_\pm|\nabla\zeta|^2dxdt+2\int_{-\tau}^0\int_{K_\rho}((u^l_\pm-k)^2_\pm+2(l-k)_\pm(u-l)_\pm)\zeta\zeta_tdxdt\\ \pm C(l-k)_\pm\int_{-\tau}^0\int_{K_\rho}\left(\int_l^u s^{m-1}ds\right)\left(|\nabla\zeta|^2+\zeta\Delta\zeta\right)\chi_{[u\gtrless l]}dxdt. \quad (2.4)\end{aligned}$$

Proof. Let $\varphi=\pm((u^l_\pm)_h-k)_\pm\zeta^2$, and $K=K_\rho$ in the weak formulation (2.3). Integrate over $(-\tau,t)$, for every $t\in(-\tau,0)$, to obtain

$$\begin{aligned}\int_{-\tau}^t\int_{K_\rho}\pm(u_h)_t\,((u^l_\pm)_h-k)_\pm\zeta^2\\ \pm m\left(u^{m-1}\nabla u\right)_h\cdot\left(\nabla((u^l_\pm)_h-k)_\pm\zeta^2+2((u^l_\pm)_h-k)_\pm\zeta\nabla\zeta\right)dxd\theta=0. \quad (2.5)\end{aligned}$$

Concerning the first term, we can divide it in two integrals

$$I_1 + I_2 := \int_{-\tau}^{t}\int_{K_\rho} \pm (u_h)_t \left((u_\pm^l)_h - k\right)_\pm \zeta^2 \chi_{[u_\pm^l = u]} dx d\theta + \int_{-\tau}^{t}\int_{K_\rho} \pm (u_h)_t \left((u_\pm^l)_h - k\right)_\pm \zeta^2 \chi_{[u_\pm^l = l]} dx d\theta.$$

At first sight it seems that we are double counting the set where $u = l$. In fact we are, but since the integrand function is 0 when $u = l$ there is no problem.

To analyse I_1 we define $S_\pm^h = \operatorname{supp}(u_h - k)_\pm$. Then it is possible to observe that in $S_\pm^h$, $((u_h - k)_\pm)_t = \pm(u_h)_t$, and outside the integrand function is 0. This way, we can conclude that I_1 can be rewritten as

$$\int_{-\tau}^{t}\int_{K_\rho} \frac{1}{2}\left(((u_\pm^l)_h - k)_\pm^2\right)_t \zeta^2 \chi_{[u_\pm^l = u]} dx d\theta = \int_{-\tau}^{t}\int_{K_\rho} \frac{1}{2}\left(((u_\pm^l)_h - k)_\pm^2\right)_t \zeta^2 dx d\theta.$$

After this changes, we can integrate by parts and pass to the limit in $h \to 0$, using **Lemma 2.3**, to obtain that

$$\int_{-\tau}^{t}\int_{K_\rho} \pm (u_h)_t \left((u_\pm^l)_h - k\right)_\pm \zeta^2 dx d\theta \to \frac{1}{2}\int_{K_\rho \times \{t\}} (u_\pm^l - k)_\pm^2 \zeta^2 dx - \int_{-\tau}^{0}\int_{K_\rho} (u_\pm^l - k)_\pm^2 \zeta \zeta_t dx dt.$$

Just one term related to the boundary appears because $\zeta = 0$ in $K_\rho \times \{-\tau\}$ by definition.

For I_2, we just need to notice that

$$u_h \chi_{[u_\pm^l = l]} = u_h \chi_{[u_h \gtrless l]} = \pm(u_h - l)_\pm + l\chi_{[u_\pm^l = l]},$$

which implies that

$$\int_{-\tau}^{t}\int_{K_\rho} \pm (u_h)_t \left((u_\pm^l)_h - k\right)_\pm \zeta^2 \chi_{[u_\pm^l = l]} dx d\theta = (l - k)_\pm \int_{-\tau}^{t}\int_{K_\rho} \left((u_h - l)_\pm\right)_t \zeta^2 dx d\theta.$$

If we again integrate by parts and pass to the limit $h \to 0$, we arrive at

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} \pm (u_h)_t \left((u_\pm^l)_h - k\right)_\pm \zeta^2 \chi_{[u_\pm^l = l]} dx d\theta \\
&\quad \to (l - k)_\pm \int_{K_\rho \times \{t\}} (u - l)_\pm \zeta^2 dx - 2(l - k)_\pm \int_{-\tau}^{0}\int_{K_\rho} (u - l)_\pm \zeta \zeta_t dx dt \\
&\quad \geq -2(l - k)_\pm \int_{-\tau}^{0}\int_{K_\rho} (u - l)_\pm \zeta \zeta_t dx dt.
\end{aligned}$$

In the second term of (2.5), we first let $h \to 0$ and then we divide it in two integrals as before,

$$\begin{aligned}
mI_3 + mI_4 = m &\int_{-\tau}^{t}\int_{K_\rho} \pm u^{m-1} \nabla u \cdot \left(\nabla (u_\pm^l - k)_\pm \zeta^2 + 2(u_\pm^l - k)_\pm \zeta \nabla \zeta\right) \chi_{[u_\pm^l = u]} dx d\theta \\
+ m &\int_{-\tau}^{t}\int_{K_\rho} \pm u^{m-1} \nabla u \cdot \left(\nabla (u_\pm^l - k)_\pm \zeta^2 + 2(u_\pm^l - k)_\pm \zeta \nabla \zeta\right) \chi_{[u_\pm^l = l]} dx d\theta.
\end{aligned}$$

Concerning I_3, use Cauchy-Schwarz inequality, $|a \cdot b| \leq |a||b|$, observing that $\nabla u = \nabla(u-k) = \pm\nabla(u-k)_\pm$, in $S_\pm = \text{supp}(u-k)_\pm$. At this point we have that

$$\begin{aligned}
I_3 &= \int_{-\tau}^{t}\int_{K_\rho} u^{m-1}\nabla(u-k)_\pm \cdot \left(\nabla(u_\pm^l - k)_\pm\zeta^2 + 2(u_\pm^l-k)_\pm\zeta\nabla\zeta\right)\chi_{[u_\pm^l=u]}dxd\theta \\
&\geq \int_{-\tau}^{t}\int_{K_\rho}(u_\pm^l)^{m-1}|\nabla(u_\pm^l-k)_\pm|^2\zeta^2 \\
&\qquad - 2\int_{-\tau}^{t}\int_{K_\rho}(u_\pm^l)^{m-1}|\nabla(u_\pm^l-k)_\pm|(u_\pm^l-k)_\pm\zeta|\nabla\zeta|\chi_{[u_\pm^l=u]}dxd\theta \\
&\geq \int_{-\tau}^{t}\int_{K_\rho}(u_\pm^l)^{m-1}|\nabla(u_\pm^l-k)_\pm\zeta|^2 \\
&\qquad - 2\int_{-\tau}^{t}\int_{K_\rho}(u_\pm^l)^{m-1}\left(\frac{|\nabla(u_\pm^l-k)_\pm\zeta|^2}{4} + (u_\pm^l-k)_\pm^2|\nabla\zeta|^2\right)dxd\theta \\
&= \frac{1}{2}\int_{-\tau}^{t}\int_{K_\rho}(u_\pm^l)^{m-1}|\nabla(u_\pm^l-k)_\pm\zeta|^2 - 2\int_{-\tau}^{t}\int_{K_\rho}(u_\pm^l)^{m-1}(u_\pm^l-k)_\pm^2|\nabla\zeta|^2dxd\theta.
\end{aligned}$$

Regarding the last inequality, the Young's inequality was used,

$$ab \leq \frac{a^2}{4} + b^2,$$

with $a = |\nabla(u_\pm^l-k)_\pm|\zeta,\ b = (u_\pm^l-k)_\pm|\nabla\zeta|$.

Additionally, for I_4, since

$$\nabla\left(\int_l^u s^{m-1}ds\right)\chi_{[u\geq l]} = u^{m-1}\nabla u\chi_{[u\geq l]},$$

$$\nabla\left(\int_l^u s^{m-1}ds\right)\chi_{[u\leq l]} = -\nabla\left(\int_u^l s^{m-1}ds\right)\chi_{[u\leq l]} = u^{m-1}\nabla u\chi_{[u\leq l]},$$

we have that

$$\begin{aligned}
I_4 &= \pm\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}\nabla u\cdot(\nabla(u_\pm^l-k)_\pm\zeta^2 + 2(u_\pm^l-k)_\pm\zeta\nabla\zeta)\chi_{[u_\pm^l=l]}dxd\theta \\
&\qquad = \pm 2(l-k)_\pm\int_{-\tau}^{t}\int_{K_\rho}\nabla\left(\int_l^u s^{m-1}ds\right)\cdot\zeta\nabla\zeta\chi_{[u\gtreqless l]}dxd\theta.
\end{aligned}$$

Next we just need to integrate by parts, obtaining that

$$I_4 = \mp 2(l-k)_\pm\int_{-\tau}^{t}\int_{K_\rho}\left(\int_l^u s^{m-1}ds\right)\left(|\nabla\zeta|^2 + \zeta\Delta\zeta\right)\chi_{[u\gtreqless l]}dxd\theta.$$

This way, we have already proved that for all $t \in (-\tau, 0)$,

$$\begin{aligned}
&\frac{1}{2}\int_{K_\rho\times\{t\}}(u_\pm^l-k)_\pm^2\zeta^2dx - \int_{-\tau}^{0}\int_{K_\rho}(u_\pm^l-k)_\pm^2\zeta\zeta_tdxdt - 2(l-k)_\pm\int_{-\tau}^{0}\int_{K_\rho}(u-l)_\pm\zeta\zeta_tdxdt \\
&+\frac{m}{2}\int_{-\tau}^{t}\int_{K_\rho}(u_\pm^l)^{m-1}|\nabla(u_\pm^l-k)_\pm\zeta|^2dxd\theta - 2m\int_{-\tau}^{t}\int_{K_\rho}(u_\pm^l)^{m-1}(u_\pm^l-k)_\pm^2|\nabla\zeta|^2dxd\theta \\
&\qquad \mp 2m(l-k)_\pm\int_{-\tau}^{t}\int_{K_\rho}\left(\int_l^u s^{m-1}ds\right)\left(|\nabla\zeta|^2+\zeta\Delta\zeta\right)\chi_{[u\gtreqless l]}dxd\theta \leq 0,
\end{aligned}$$

and from here we can easily obtain (2.4). □

Remark 2.8. This estimate could also be extended for the signed PME (2.1) by considering the constants l and k possibly negative and by changing $(u^l_\pm)^{m-1}$ to $|(u^l_\pm)|^{m-1}$.

2.2.2. Local logarithmic estimates

The second estimate also plays a crucial role in the proof. Before proving this estimate we need to introduce the logarithmic function

$$\psi^\pm(s) \equiv \psi^\pm_{k,c}(s) := \left(\ln\left(\frac{H^\pm_{s,k}}{\left(H^\pm_{s,k}+c\right)-(s-k)_\pm}\right)\right)_+, \qquad 0<c<H^\pm_{s,k},$$

with

$$H^\pm_{s,k} := \operatorname*{ess\,sup}_{x\in Q(\tau,\rho)} |(s-k)_\pm|$$

and s a bounded function defined in $Q(\tau,\rho)$. This function was first introduced in [1], and since then has been used to prove results concerning the local behaviour of solutions of degenerate and singular equations. This function has two important properties that can be proven with simple calculations:

$$\left(\psi^\pm(s)\right)' \gtreqless 0,$$

and the second derivative , for $s \neq k \pm c$, is

$$\left(\psi^\pm(s)\right)'' = \left(\left(\psi^\pm(s)\right)'\right)^2.$$

If we also consider the previous cutoff function ζ to be independent of time in K_ρ, then we can obtain the following estimate.

Proposition 2.9. *Let u be a nonnegative local weak solution of* (PME) *and $k\in\mathbb{R}$. There exists a constant $C(m)>0$ such that, for every cylinder $Q(\tau,\rho)\subset\Omega_T$,*

$$\sup_{-\tau\le t\le 0}\int_{K_\rho\times\{t\}}[\psi^\pm(u)]^2\zeta^2dx \le \int_{K_\rho\times\{-\tau\}}[\psi^\pm(u)]^2\zeta^2dx + C(m)\int_{-\tau}^0\int_{K_\rho}u^{m-1}\psi^\pm(u)|\nabla\zeta|^2dxdt. \quad (2.6)$$

Proof. The first step is to consider $K=K_\rho$, $\varphi = 2\psi^\pm(u_h)\left[(\psi^\pm)'(u_h)\right]\zeta^2$ as a testing function in (2.3), and then integrate in time over $(-\tau,t)$ for $t\in(-\tau,0)$, as in the previous proposition, obtaining

$$\int_{-\tau}^t\int_{K_\rho} \pm 2\left(u_h\right)_t\psi^\pm(u_h)\left[(\psi^\pm)'(u_h)\right]\zeta^2 \pm m\left(u^{m-1}\nabla u\right)_h\cdot\left(\nabla\left(2\psi^\pm(u_h)\left[(\psi^\pm)'(u_h)\right]\zeta^2\right)dxd\theta = 0. \quad (2.7)$$

Since ζ is time-independent, $\zeta_t = 0$, we can deduce from the first term of (2.7) that

$$\begin{aligned}\int_{-\tau}^{t}\int_{K_\rho} &(u_h)_t 2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u_h)\right]\zeta^2 dxd\theta \\ &= \int_{-\tau}^{t}\int_{K_\rho}\left([\psi^{\pm}(u_h)]^2\zeta^2\right)_t dxd\theta \\ &= \int_{K_\rho\times\{t\}}[\psi^{\pm}(u_h)]^2\zeta^2 dx - \int_{K_\rho\times\{-\tau\}}[\psi^{\pm}(u)]^2\zeta^2 dx.\end{aligned}$$

Finally we can let $h \to 0$ to obtain

$$\int_{-\tau}^{t}\int_{K_\rho}(u_h)_t 2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u_h)\right]\zeta^2 dxd\theta \to \int_{K_\rho\times\{t\}}[\psi^{\pm}(u)]^2\zeta^2 dx - \int_{K_\rho\times\{-\tau\}}[\psi^{\pm}(u)]^2\zeta^2 dx.$$

For the remaining term we start by letting $h \to 0$ and then we use Cauchy-Schwarz,

$$\begin{aligned} m\int_{-\tau}^{t}\int_{K_\rho} & u^{m-1}\nabla u\cdot\nabla\left(2\psi^{\pm}(u)[(\psi^{\pm})'(u)]\zeta^2\right)dxd\theta \\ &= m\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}\nabla u\cdot\nabla u 2\left([(\psi^{\pm})'(u)]^2+\psi^{\pm}(u)[(\psi^{\pm})'(u)]^2\right)\zeta^2 dxd\theta \\ &\quad + m\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}\nabla u\cdot 4\psi^{\pm}(u)[(\psi^{\pm})'(u)]\zeta\nabla\zeta dxd\theta \\ &\geq m\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}|\nabla u|^2\left(2(1+\psi^{\pm}(u))\left[(\psi^{\pm})'(u)\right]^2\zeta^2\right)dxd\theta \\ &\quad - 2m\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}|\nabla u||\nabla\zeta| 2\psi^{\pm}(u)\left|(\psi^{\pm})'(u)\right|\zeta dxd\theta \\ &\geq m\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}|\nabla u|^2\left(2(1+\psi^{\pm}(u)-\psi^{\pm}(u))\left[(\psi^{\pm})'(u)\right]^2\zeta^2\right)dxd\theta \\ &\quad - m\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}\psi^{\pm}(u)|\nabla\zeta|^2 dxd\theta.\end{aligned}$$

In the last inequality, the Young's inequality was used,

$$ab \leq \frac{a^2}{2}+\frac{b^2}{2},$$

with $a = |\nabla\zeta|\,|(\psi^{\pm})'(u)|^{-1}$, and $b = |\nabla u\zeta|$.

Since $u^{m-1} \geq 0$ we have already proved that for all $t \in (-\tau, 0)$

$$\int_{K_\rho\times\{t\}}[\psi^{\pm}(u)]^2\zeta^2 dx - \int_{K_\rho\times\{-\tau\}}[\psi^{\pm}(u)]^2\zeta^2 dx - C\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}\psi^{\pm}(u)|\nabla\zeta|^2 dxd\theta \leq 0$$

and from here we can easily obtain (2.6). □

Remark 2.10. In this estimate, the change for the signed equation is even simplier, the only difference is again replacing u^{m-1} with $|u|^{m-1}$.

2.3. Some technical tools

Throughout the remaining of the proofs, some technical facts will be needed. They will be essential to establish some results, however they are not directly related to this theory. In this chapter we will present these important facts without any proof, so the reader will not loose the focus on the main goal of this text, the intrinsic scaling method.

An Embedding Theorem

Let $V_0^p(\Omega_T)$ denote the space

$$V_0^p(\Omega_T) = L^\infty(0,T;L^p(\Omega)) \cap L^p\left(0,T;W_0^{1,p}(\Omega)\right)$$

endowed with the norm

$$||v||^p_{V^p(\Omega_T)} = \operatorname*{ess\,sup}_{o\leq t\leq T} ||v(\cdot,t)||^p_{p,\Omega} + ||\nabla v||^p_{p,\Omega}.$$

The following embedding theorem holds [5, page 9].

Theorem 2.11. *For p>1, there exists a constant* $C \equiv C(d,p)$*, such that for every* $v \in V_0^p(\Omega_t)$*,*

$$||v||^p_{p,\Omega_T} \leq C\big||v|>0\big|^{\frac{p}{d+p}}||v||^p_{V^p(\Omega_T)}.$$

Geometric Convergence of Sequences

The following lemma is a fundamental instrument for the iterative schemes that will be the principal argument in some important proofs.

Lemma 2.12. *Let* X_n*,* $n = 0,1,2,\ldots$*, be a sequence of positive real numbers satisfying the recurrence relation*

$$X_{n+1} \leq Cb^n X_n^{1+a},$$

where C, b>1 and a>0 are given. If

$$X_0 \leq C^{-\frac{1}{a}} b^{-\frac{1}{a^2}},$$

then $X_n \to 0$ *as* $0 \to \infty$*.*

A Lemma of De Giorgi

Given a continuous function $v : \Omega \to \mathbb{R}$ and two real numbers $k_1 < k_2$, we define

$$[v \lessgtr k_i] := \{x \in \Omega \ : \ v(x) \lessgtr k_i\}, \qquad i = 1,2,$$

$$[k_1 < v < k_2] := \{x \in \Omega \ : \ k_1 < v(x) < k_2\}.$$

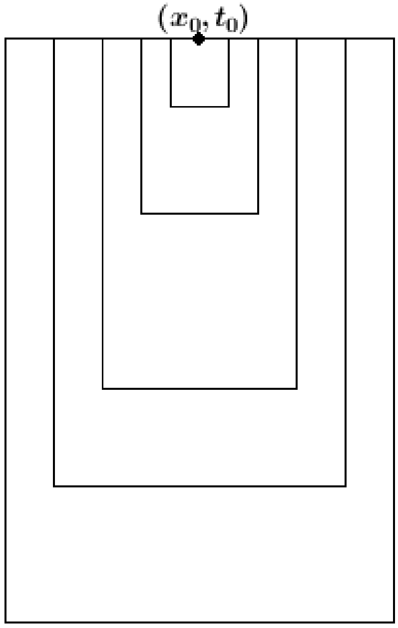
Lemma 2.13 (De Giorgi, [3]). *Let* $v \in W^{1,1}(B_\rho(x_0)) \cap C(B_\rho(x_0))$, *with* $\rho > 0$ *and* $x_0 \in \mathbb{R}^d$, *and let* $k_1 < k_2 \in \mathbb{R}$. *There exists a constant* $C \equiv C(d)$, *such that*

$$(k_2 - k_1)|[v > k_2]| \leq C \frac{\rho^{d+1}}{|[v < k_1]|} \int_{[k_1 < v < k_2]} |\nabla v| dx.$$

Chapter 3

The method of intrinsic scaling

Continuity results, at a generic point (x_0, t_0), can be obtained by measuring the oscillation of the solution in a sequence of nested and shrinking cylinders with vertex at that point. If we can show that the oscillation converges to zero as the cylinders shrink to the point, we can derive the continuity of the solution. In this regard, it is usual to prove that the oscillation is reduced by a factor $\sigma \in (0,1)$ in each iteration, *i.e.* $\omega_{n+1} \leq \sigma\omega_n$, where ω is the oscillation in the nth cylinder. This method is named Reduction of Oscillation. We will define and prove rigorously this result in the following sections.



This iterative procedure can be done using the *a priori* estimates that we obtained before. As is explained in my previous work [13], De Giorgi proved the Hölder continuity for weak solutions of elliptic equations, and the idea now is to try to generalize his work to parabolic equations.

For some simple equations, as the Heat equation,

$$u_t - \Delta u = 0, \tag{HE}$$

the standard parabolic cylinders

$$Q(R^2, R)$$

reflect the natural homogeneity between the space and time variables. Indeed, the heat equation involves one derivative with respect to the time variable, but two derivatives with respect to the space variables. Consequently, we see that if, u solves the heat equation, (HE), then so does $u(\lambda x, \lambda^2 t)$ for $\lambda \in \mathbb{R}$. This scaling indicates that the equation remains invariant through a transformation of the space-time variables that leaves constant the ratio $\frac{|x|^2}{t}$.

For more complicated equations like our degenerate parabolic equation (PME), we need different cylinders.

3.1. A geometry for the equation

The idea behind the method of intrinsic scaling is to perform the iterative process in cylinders that reflect the structure/geometry of the equation. These cylinders need to be constructed taking into account the nature of the degeneracy or singularity, and the oscillation of the solution itself (thus, the term intrinsic). They should trivially reduce to the standard parabolic cylinders, reflecting the natural homogeneity between the space and time variables for the heat equation.

For our equation, (PME), and for most degenerate or singular equations, the energy and logarithmic estimates are not homogeneous since they involve integral norms corresponding to different powers. In our case, some undesirable factors, as u^{m-1}, appear. However, if we analyse the estimates in a geometry dictated by its own degenerate structure, using the rescaled cylinders, we can accommodate the degeneracy and restore the homogeneity in the energy and logarithmic estimates.

In this regard, we can say that the equation behaves, in its own geometry, like the heat equation.

If we suppose for a moment that $u > 0$, we can rewrite equation (PME) in the following form

$$\frac{u^{1-m}}{m} u_t - \Delta u = 0.$$

Regardless of the unrealistic assumption, this equation allows us to predict that it is possible to recover the homogeneity of the function using a scaling factor. Despite this scaling factor being dependent on the solution itself, it gives us an idea of how to construct the rescaled cylinders.

The main objective of this chapter is to implement, in a rigorous way, this heuristic reasoning. For that, let us see how to choose the cylinders that better reflect the geometry of our equation.

From now on, for brevity, where no confusion is to be feared, we will always assume that u is a nonnegative local weak solution of (PME).

First let us choose $0 < R < 1$, such that

$$Q(4R^{2-\epsilon}, 2R) \subset \Omega_T, \qquad 0 < \epsilon < 1,$$

and define

$$\mu^+ := \operatorname*{ess\,sup}_{Q(4R^{2-\epsilon},2R)} u \quad \text{and} \quad \mu^- := \operatorname*{ess\,inf}_{Q(4R^{2-\epsilon},2R)} u.$$

At this point, we can define the ω as the essential oscillation of the solution u within this cylinder,*i.e.*,

$$\omega = \operatorname*{ess\,osc}_{Q(4R^{2-\epsilon},2R)} u = \mu^+ - \mu^-.$$

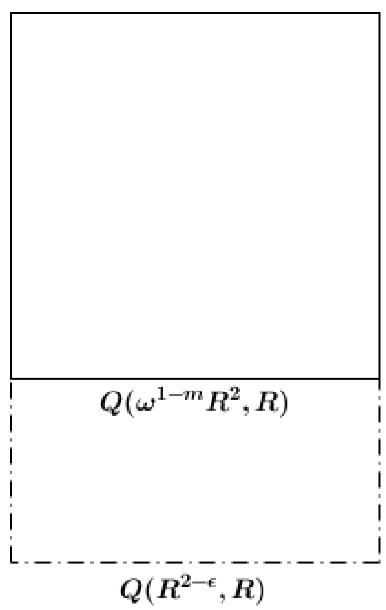
Finally, let us construct the rescaled cylinder

$$Q(\omega^{1-m}R^2, R) = K_R(0) \times (-\omega^{1-m}R^2, 0),$$

which, if we assume

$$\omega^{m-1} > R^{\epsilon}, \tag{3.1}$$

is contained in $Q(4R^{2-\epsilon}, 2R)$ and consequently in Ω_T.



Indeed, if (3.1) does not hold, we have that the oscillation is comparable to the radius, $\omega \leq R^{\epsilon(1-m)}$, and the Hölder continuity follows immediately.

This inclusion implies that

$$\operatorname*{ess\,osc}_{Q(\omega^{1-m}R^2,R)} u \leq \omega,$$

which will be the starting point of the recursive scheme.

We will also assume, without loss of generality, that

$$\mu^- < \frac{\omega}{4}. \tag{3.2}$$

Indeed, if (3.2) is not verified then the equation will not have any degeneracy, but this case will be studied better in section 4.2 anyway.

3.2. Reduction of the oscillation

At this point, to prove the highly-anticipated reduction of oscillation, we have to consider two alternatives. The proof will be divided in two complementary cases: either u is essentially away from its infimum in $Q(\omega^{1-m}R^2, R)$, or u is essentially away from its supremum. However, in both cases, we can reach identical conclusions. Let us state both alternatives in a precise way.

Consider a constant, $\nu_0 \in (0, 1)$, that will be determined later. Then either, **the first alternative**,

$$\frac{|\{(x,t) \in Q(\omega^{1-m}R^2, R) \ : \ u(x,t) < u^- + \frac{\omega}{2}\}|}{|Q(\omega^{1-m}R^2, R)|} \leq \nu_0, \tag{3.3}$$

holds, or

$$\frac{|\{(x,t) \in Q(\omega^{1-m}R^2, R) \ : \ u(x,t) \geq \mu^- + \frac{\omega}{2}\}|}{|Q(\omega^{1-m}R^2, R)|} < 1 - \nu_0,$$

which, since $\mu^+ - \frac{\omega}{2} = \mu^- + \frac{\omega}{2}$, implies the **second alternative**

$$\frac{|\{(x,t) \in Q(\omega^{1-m}R^2, R) \ : \ u(x,t) > \mu^+ - \frac{\omega}{2}\}|}{|Q(\omega^{1-m}R^2, R)|} < 1 - \nu_0. \tag{3.4}$$

3.2.1. The first alternative

Let us start to analyse our problem assuming that the first alternative holds.

Proposition 3.1. *Assume* (3.2) *is in force. If the first alternative* (3.3) *holds , then*

$$u(x,t) > \mu^- + \frac{\omega}{4}, \quad a.e. \ in \ Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right).$$

Proof. Let us start by defining the sequence

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \qquad n = 0, 1...., $$

that converges to $\frac{R}{2}$. Then we can construct the family of nested and shrinking cylinders $Q(\omega^{1-m}R_n^2, R_n)$ and, for each cylinder, let us consider a piecewise smooth cutoff function $0 \le \zeta_n \le 1$ satisfying the following assumptions:

$$\zeta_n = 1 \text{ in } Q\left(\omega^{1-m}R_{n+1}^2, R_{n+1}\right); \qquad \zeta_n = 0 \text{ on } \partial Q\left(\omega^{1-m}R_n^2, R_n\right);$$
$$|\nabla \zeta_n| \le \frac{2^{n-1}}{R}; \qquad 0 \le (\zeta_n)_t \le \frac{2^{2n-2}}{R^2}\omega^{m-1}; \qquad \Delta\zeta_n \le \frac{2^{2n-2}}{R^2}.$$

Then, we will use the energy estimates (2.4) for $(u_-^l - k_n)_-$, with $l = \mu^- + \frac{\omega}{4}$,

$$k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+1}}$$

and $\zeta = \zeta_n$.

For the left hand side of the inequality, since $u_-^l = \max\{u, \mu^- + \frac{\omega}{4}\} \ge \mu^- + \frac{\omega}{4} \ge \frac{\omega}{4}$, the following lower bound can be found:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_-^l - k_n)_-^2 \zeta^2 dx + \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} (u_-^l)^{m-1} |\nabla (u_-^l - k_n)_- \zeta|^2 dxdt \\
&\qquad \ge \operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_-^l - k_n)_-^2 \zeta^2 dx \\
&\qquad\qquad + \left(\frac{\omega}{4}\right)^{m-1} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} |\nabla (u_-^l - k_n)_- \zeta|^2 dxdt \\
&\qquad \ge 4^{1-m} \Bigg(\operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_-^l - k_n)_-^2 \zeta^2 dx \\
&\qquad\qquad + \omega^{m-1} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} |\nabla (u_-^l - k_n)_- \zeta|^2 dxdt \Bigg).
\end{aligned}$$

The next step is to look carefully at all terms in the right hand side and try to get some upper bound for each. For that purpose, we just need to notice the following four properties of our functions and constants, with χ symbolizing the characteristic function:

1. $0 \le \mu^- \le \frac{\omega}{4}$, so $u \le \frac{5\omega}{4}$ and $l \le \frac{\omega}{2}$, which implies that $u_-^l \le \frac{5\omega}{4}$;

2. $l = \mu^- + \frac{\omega}{4} < k_n$, so $\chi_{[u \le l]} \le \chi_{[u<k_n]} = \chi_{[(u-k_n)_- > 0]}$;

3. Where $u_-^l = u$, we have $\chi_{[(u-k_n)_->0]} = \chi_{[(u_-^l - k_n)_- > 0]}$, but even when $u_-^l = l$, *i.e.*, $u \le l < k_n$, the same result holds: $\chi_{[(u-k_n)_->0]} = 0 = \chi_{[(l-k_n)_->0]} = \chi_{[(u_-^l-k_n)_->0]}$. So, we have that $\chi_{[(u-k_n)_->0]} = \chi_{[(u_-^l-k_n)_->0]}$ for all (x,t);

4. $(l-k_n)_- = \frac{\omega}{2^{n+1}} \leq \frac{\omega}{2}$, $(u^l_- - k_n)_- \leq \frac{\omega}{2^{n+1}} \leq \frac{\omega}{2}$ and $(u-l)_- \leq \frac{\omega}{4}$.

Applying the previous properties and the definition of our cutoff functions, we can obtain the following inequalities for each term on the right hand side of the energy estimate (2.4):

1st term:

$$\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} (u^l_-)^{m-1}(u^l_- - k_n)_-^2 |\nabla \zeta_n|^2 dxdt \\ \leq C\omega^{m-1}\left(\frac{\omega}{2}\right)^2 \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_- - k_n)_- > 0]} dxdt;$$

2nd term:

$$\begin{aligned} 2\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} & \left((u^l_- - k_n)_-^2 + 2(l-k_n)_-(u-l)_-\right) \zeta_n(\zeta_n)_t dxdt \\ & \leq C\left(\left(\frac{\omega}{2}\right)^2 + \left(\frac{\omega}{2}\right)^2\right)\omega^{m-1}\frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_- - k_n)_- > 0]} dxdt \\ & \leq C\omega^{m-1}\left(\frac{\omega}{2}\right)^2 \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_- - k_n)_- > 0]} dxdt; \end{aligned}$$

3rd term:

$$\begin{aligned} -C(l-k_n)_- & \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \left(\int_l^u s^{m-1} ds\right)\left(|\nabla\zeta_n|^2 + \zeta_n \Delta \zeta_n\right) \chi_{[u \leq l]} dxdt \\ & \leq C\frac{\omega}{2}\omega^{m-1}\frac{\omega}{4}\left(\frac{2^{2n-2}}{R^2} + \frac{2^{2n-2}}{R^2}\right) \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_- - k_n)_- > 0]} dxdt \\ & \leq C\omega^{m-1}\left(\frac{\omega}{2}\right)^2 \frac{2^{2n-1}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_- - k_n)_- > 0]} dxdt. \end{aligned}$$

Joining these four inequalities, we conclude that, for every $n > 0$, the following is verified:

$$\begin{aligned} \operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}} < t < 0} & \int_{K_{R_n} \times \{t\}} (u^l_- - k_n)_-^2 \zeta_n^2 dx + \omega^{m-1} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} |\nabla (u^l_- - k)_- \zeta_n|^2 dxdt \\ & \leq \omega^{m-1}\left(\frac{\omega}{2}\right)^2 \left(C\frac{2^{2n-2}}{R^2} + \frac{2^{2n-2}}{R^2} + C\frac{2^{2n-1}}{R^2}\right) \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_- - k_n)_- > 0]} dxdt \\ & \leq C\omega^{m-1}\left(\frac{\omega}{2}\right)^2 \frac{2^{2n}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_- - k_n)_- > 0]} dxdt. \end{aligned}$$

The next step, in which the intrinsic geometric framework is essential, consists in modifying the time variable, putting $\bar{t} = \omega^{m-1} t$, and defining

$$\bar{u}^l_-(\cdot, \bar{t}) := u^l_-(\cdot, t), \qquad \bar{\zeta_n}(\cdot, \bar{t}) := \zeta_n(\cdot, t).$$

In the new time variable, the previous inequality is simplified to

$$||(\bar{u}^l_- - k_n)_-\bar{\zeta_n}||^2_{V^2(Q(R_n^2,R_n))} \leq C\frac{2^{2n}}{R^2}\left(\frac{\omega}{2}\right)^2 A_n, \tag{3.5}$$

where

$$A_n = \int_{-R_n^2}^0 \int_{K_{R_n}} \chi_{\{(\bar{u}^l_- - k_n)_- > 0\}} dxdt.$$

From the definition of A_n, the fact that $k_n > k_{n+1}$, and because $\bar{\zeta_n} = 1$ in $Q(R_{n+1}^2, R_{n+1})$, we also have that

$$\begin{aligned}
\frac{1}{2^{2(n+2)}}\left(\frac{\omega}{2}\right)^2 A_{n+1} &= |k_n - k_{n+1}|^2 A_{n+1} \\
&= \int_{-R_n^2}^0 \int_{K_{R_n}} (k_n - k_{n+1})^2 \chi_{\{(\bar{u}^l_- - k_{n+1})_- > 0\}} dxdt \\
&\leq \int_{-R_n^2}^0 \int_{K_{R_n}} (k_n - \bar{u}^l_-)^2 \chi_{\{(\bar{u}^l_- - k_{n+1})_- > 0\}} dxdt \\
&\leq ||(\bar{u}^l_- - k_n)_-||^2_{2,Q(R_{n+1}^2,R_{n+1})} \\
&\leq ||(\bar{u}^l_- - k_n)_-\bar{\zeta_n}||^2_{2,Q(R_n^2,R_n)}.
\end{aligned}$$

Moreover, if we use **Theorem** (2.11) for $p = 2$ and inequality (3.5), we can carry on these calculations,

$$\begin{aligned}
\frac{1}{2^{2(n+2)}}\left(\frac{\omega}{2}\right)^2 A_{n+1} &\leq C||(\bar{u}_l - k_n)_-\bar{\zeta_n}||^2_{V^2(Q(R_n^2,R_n))} A_n^{\frac{2}{d+2}} \\
&\leq C\frac{2^{2n}}{R^2}\left(\frac{\omega}{2}\right)^2 A_n^{1+\frac{2}{d+2}}.
\end{aligned}$$

Next, dividing the previous inequality by $|Q(R_{n+1}^2, R_{n+1})| = R_{n+1}^{d+2} < R^{d+2}$ we obtain

$$\begin{aligned}
\frac{A_{n+1}}{|Q(R_{n+1}^2,R_{n+1})|} &\leq C\frac{2^{4n+4}}{R^2}|Q(R_{n+1}^2,R_{n+1})|^{\frac{2}{d+2}}\left(\frac{A_n}{|Q(R_{n+1}^2,R_{n+1})|}\right)^{1+\frac{2}{d+2}} \\
&\leq C\frac{2^{4n+4}}{R^2}R^2\left(\frac{|Q(R_n^2,R_n)|}{|Q(R_{n+1}^2,R_{n+1})|}\frac{A_n}{|Q(R_n^2,R_n|)}\right)^{1+\frac{2}{d+2}} \\
&\leq C4^{2n}\left(\frac{A_n}{|Q(R_n^2,R_n|)}\right)^{1+\frac{2}{d+2}}.
\end{aligned}$$

Here we used the inequality $\frac{R_n}{R_{n+1}} \leq 2$ to deduce the last inequality. If we also define

$$X_n = \frac{A_n}{|Q(R_n^2,R_n)|},$$

we can rewrite the last inequality as the following recursive relation

$$X_{n+1} \leq C4^{2n}X_n^{1+\frac{2}{d+2}},$$

with C a constant depending only on d and m. Finally, supposing without loss of generality that $C > 1$, let us define $\nu_0 := C^{-\frac{d+2}{2}} 4^{-\frac{(d+2)^2}{2}} < 1$. Since $k_0 = \frac{\omega}{2}$ and $u \leq u^l_-$, our hypothesis (3.3) is equivalent to the initial condition $X_0 \leq \nu_0$. At this point, we are in conditions to apply the **Lemma 2.12** on fast geometric convergence, which implies that

$$X_n \to 0.$$

Then, we just need to return to the original variables and, noticing that $R_n \to \frac{R}{2}$, we conclude that

$$\left| \left\{ (x,t) \in Q\left(\omega^{1-m} \left(\frac{R}{2}\right)^2, \frac{R}{2} \right) \ : \ u^l_-(x,t) \leq \mu^- + \frac{\omega}{4} \right\} \right| = 0,$$

which is equivalent to

$$u^l_-(x,t) > \mu^- + \frac{\omega}{4}, \quad \text{a.e in } Q\left(\omega^{1-m} \left(\frac{R}{2}\right)^2, \frac{R}{2} \right).$$

To finish, just notice that if $u^l_-(x,t) > \mu^- + \frac{\omega}{4}$ then $u^l_- = u$. □

After this proposition, we can easily prove reduction of the oscillation.

Corollary 3.2 (Reduction of the Oscillation)**.** *Assume* (3.2) *is in force. If the first alternative* (3.3) *holds, then there exists a constant* $\sigma_1 \in (0,1)$*, depending only on the data, such that*

$$\operatorname*{ess\,sup}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)} u \leq \sigma_1 \omega. \tag{3.6}$$

Proof. By the previous result, **Proposition 3.1**,

$$\operatorname*{ess\,inf}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)} u \geq \mu^- + \frac{\omega}{4},$$

and thus

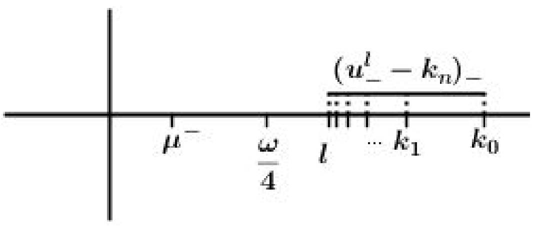
$$\begin{aligned}
\operatorname*{ess\,osc}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)} u &= \operatorname*{ess\,sup}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)} u - \operatorname*{ess\,inf}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)} u \\
&\leq \mu^+ - \mu^- - \frac{\omega}{4} \\
&= \frac{3}{4}\omega.
\end{aligned}$$

This way, the corollary follows with $\sigma_1 = \frac{3}{4}$. □

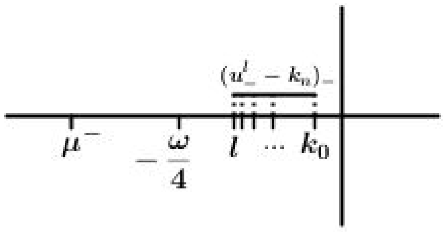
Remark 3.3. This is where more adjustments are needed to extend the result to the signed PME (2.1). We will divide the problem in 4 cases depending of μ^-, and see what changes in each one.

The first two are very similar, if $\mu^- \geq 0$ then, independently of the alternative, the proof is the same. If $\mu^- \leq -\omega$, then $-u$ is a nonnegative solution and we apply the method to $-u$. Notice that $-u$ will verify the second alternative instead, but the result obtained is equivalent. Contrarily of what the reader could think, the change of equation does not arise any issue because the essential oscillation is the same for both.

Next we will explore the case where $-\omega < \mu^- < -\frac{\omega}{4}$. In fact, for this assumption, there is no degeneracy on $\{(u^l_- - k_n)_- > 0\}$, so everything should work without significant modifications.



(a) Nonnegative solutions



(b) Solutions with $-\omega < \mu^- < -\frac{\omega}{4}$

Figure 3.1: In these two graphics we want to represent the bounds of $(u^l_- - k_n)_-$.

However, let us study which adjustments are needed d so the proof remains formally correct. We start to choose n_* such that $\mu^- + -\frac{\omega}{4} \geq \frac{\omega}{2^{n_*+1}}$ and define $k_n = \mu_- + \frac{\omega}{4} + \frac{\omega}{2^{n+n_*}}$.

At this point we can obtain the following lower bound to $|u^l_-|$ on $\{(u^l_- - k_n)_- > 0\}$, $|u^l_-| \geq -k_n \geq \frac{\omega}{2^{n_*}}$. For the upper bound, just notice that $|u^l_-| \leq \max\{|l|, \mu^+\} \leq \frac{3\omega}{4}$ and the rest remains the same. Some minor things would also need to be changed because of n_*, but since it is constant it does not bring major difficulties. Just one more problem arises, in this case $X_0 \leq \nu_0$ is not equivalent to the first alternative. However, since $k_0 < \frac{\omega}{2}$, the first alternative implies the initial condition for the convergence.

For $-\frac{\omega}{4} < \mu_- < 0$, again there is no degeneracy on $\{(u^l_- - k_n)_- > 0\}$ as we can observe in Figure 3.2.

This is a simpler case, since $u^l_- > 0$ and because there exists a n_* such that $l > \frac{\omega}{2^{n_*}}$; the proof follows identically.

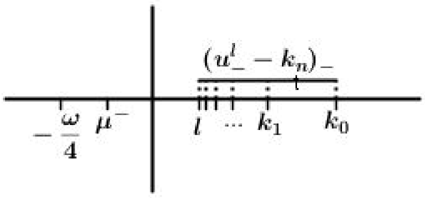


Figure 3.2: Bounds of $(u^l_- - k_n)_-$ for solutions with $-\frac{\omega}{4} < \mu_- < 0$.

In the last case, $\mu^- = \frac{\omega}{4}$, we will prove the same result but for a different σ. We will choose $l = \mu^- + \frac{\omega}{8}$ and $k_n = \mu^- + \frac{\omega}{8} + \frac{\omega}{2^{n+4}}$ and use the same techniques of the third case, $-\omega < \mu^- < -\frac{\omega}{4}$, to prove the result for $\sigma = \frac{7}{8}\omega$.

3.3. The second alternative

Remember now the second alternative

$$\frac{|\{(x,t) \in Q(\omega^{1-m}R^2, R) \ : \ u(x,t) > \mu^+ - \frac{\omega}{2}\}|}{|Q(\omega^{1-m}R^2, R)|} < 1 - \nu_0, \tag{3.7}$$

and assume that it holds. We will need to prove the reduction of the oscillation again. Recall that the constant ν_0 has already been determined, and we are still assuming that (3.2) is in force.

Start to notice that, if (3.7) is verified, then there exists a time level

$$t_0 \in \left[-\omega^{1-m}R^2, -\frac{\nu_0}{2}\omega^{1-m}R^2\right],$$

such that

$$\left|\left\{x \in K_R : u(x,t_0) > \mu^+ - \frac{\omega}{2}\right\}\right| \leq \left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R|. \tag{3.8}$$

We can easily prove this by contradiction. Suppose that (3.8) does not hold for all $t \in \left[-\omega^{1-m}R^2, -\frac{\nu_0}{2}\omega^{1-m}R^2\right]$, then

$$\begin{aligned}
&|\{(x,t) \in Q(\omega^{1-m}R^2, R) \ : \ u(x,t) > \mu^+ - \frac{\omega}{2}\}| \\
&\geq \int_{-\frac{R^2}{\omega^{m-1}}}^{-\frac{\nu_0}{2}\frac{R^2}{\omega^{m-1}}} |\{x \in K_R \ : \ u(x,\tau) > \mu^+ - \frac{\omega}{2}\}| d\tau \\
&> \left(-\frac{\nu_0}{2}\frac{R^2}{\omega^{m-1}} + \frac{R^2}{\omega^{m-1}}\right)\left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R| \\
&= (1-\nu_0)\left|Q(\omega^{1-m}R^2, R)\right|,
\end{aligned}$$

which contradicts (3.7). Although we can prove that the set where $u(\cdot,t)$ is close to its supremum, it is small not just for t_0, but in $\left[-\frac{\nu_0}{2}\omega^{1-m}R^2, 0\right]$. In detail, we mean that:

Lemma 3.4. *Assume* (3.2) *is in force. If the second alternative* (3.7) *holds, then there exists* $q \in \mathbb{N}$*, depending only on the data, such that*

$$\left|\left\{x \in K_R : u(x,t) > \mu^+ - \frac{\omega}{2^q}\right\}\right| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right) |K_R|,$$

for all $t \in \left[-\frac{\nu_0}{2}\omega^{1-m}R^2, 0\right]$

Proof. To prove this lemma we will apply the logarithmic inequality (2.6) to the function $(u-k)_+$ in the cylinder $Q(t_0, R)$, with

$$k = \mu^+ - \frac{\omega}{2} \qquad \text{and} \qquad c = \frac{\omega}{2^{n+1}},$$

for some $n \in \mathbb{N}$ greater than one that will be defined later. It is possible to apply the logarithmic estimate with this constants because we can assume that

$$H^+_{u,k} := \operatorname*{ess\,sup}_{x \in Q(R,t_0)} \left|(u - \mu^+ + \frac{\omega}{2})_+\right| > \frac{\omega}{4} \geq \frac{\omega}{2^{n+1}};$$

otherwise the lemma would be trivial for the choice of $q = 2$.

Since $u \leq \frac{5}{4}\omega$, the logarithmic inequality (2.6) implies that

$$\sup_{t_0 \leq t \leq 0} \int_{K_R \times \{t\}} [\psi^+(u)]^2 \zeta^2 dx \leq \int_{K_R \times \{t_0\}} [\psi^+(u)]^2 \zeta^2 dx + C \int_{t_0}^0 \int_{K_R} \omega^{m-1} \psi^+(u) |\nabla\zeta|^2 dx dt. \tag{3.9}$$

As in the previous proposition, the next step is to bound the terms of the previous inequality. For that, we will need some initial calculations again. Let us start by remembering that $\psi^+(u)$ is defined in the whole cylinder $Q(t_0, R)$, and it is given by

$$\psi^+_{\{H^+_{u,k}, k, \frac{\omega}{2^{n+1}}\}}(u) = \begin{cases} \ln\left(\frac{H^+_{u,k}}{H^+_{u,k} - u + k + \frac{\omega}{2^{n+1}}}\right) & \text{if } u > k + \frac{\omega}{2^{n+1}}, \\ 0 & \text{if } u \leq k + \frac{\omega}{2^{n+1}}. \end{cases}$$

In addition, in this cylinder, we have that

$$u - k \leq H^+_{u,k} \leq \frac{\omega}{2}, \tag{3.10}$$

which implies that

$$\psi^+ \leq \ln\left(\frac{H^+_{u,k}}{H^+_{u,k} - u + k + \frac{\omega}{2^{n+1}}}\right) \leq \ln\left(\frac{\frac{\omega}{2}}{\frac{\omega}{2^{n+1}}}\right) = n\ln(2). \tag{3.11}$$

Lastly, we will choose our cutoff function $0 \leq \zeta(x) \leq 1$ defined in K_R such that, for some $\delta \in (0,1)$,

$$\zeta(x) = 1, \text{ for } x \in K_{(1-\delta)R} \quad \text{and} \quad |\nabla\zeta| \leq (\delta R)^{-1}.$$

We are now ready to bound the right hand side terms in (3.9). As $\psi^+(u) = 0$ on the set $\{x \in K_R \;:\; u(x,\cdot) \le \mu^+ - \frac{\omega}{2}\}$, by (3.8) and (3.11), the first term of the right hand side of (3.9) is bounded by

$$\int_{K_R\times\{t_0\}} [\psi^+]^2\zeta^2 dx \le n^2 \ln(2)^2 \left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R|.$$

Using (3.11) again and the fact that $-t_0 \le \frac{R^2}{\omega^{m-1}}$, we can also bound the second term as follows:

$$\begin{aligned} C\int_{t_0}^{0}\int_{K_R} \omega^{m-1}\psi^+(u)|\nabla\zeta|^2 dxdt &\le Cn\ln(2)\omega^{m-1}(\delta R)^{-2}(-t_0)|K_R| \\ &\le Cn\omega^{m-1}\frac{1}{\delta^2R^2}\frac{R^2}{\omega^{m-1}}|K_R| \\ &\le Cn\frac{1}{\delta^2}|K_R|. \end{aligned}$$

For the left hand side of the inequality we start to integrate in a smaller set,

$$S_t = \left\{x \in K_{(1-\delta)R} : u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\right\} \subset K_R, \qquad t\in(t_0,0).$$

In this set, we have that $\zeta = 1$ and, since $-u + k + \frac{\omega}{2^{n+1}} < \frac{\omega}{2^{n+1}} - \frac{\omega}{2} + \frac{\omega}{2^{n+1}} < 0$,

$$\frac{H^+_{u,k}}{H^+_{u,k} - u + k + \frac{\omega}{2^{n+1}}}$$

is a decreasing function of $H^+_{u,k}$. Thus, from (3.10),

$$\frac{H^+_{u,k}}{H^+_{u,k} - u + k + \frac{\omega}{2^{n+1}}} \ge \frac{\frac{\omega}{2}}{\frac{\omega}{2} - u + k + \frac{\omega}{2^{n+1}}} > \frac{\frac{\omega}{2}}{\frac{\omega}{2^{n+1}} + \frac{\omega}{2^{n+1}}} = 2^{n-1}.$$

Therefore, in S_t,

$$[\psi^+(u)]^2 \ge [\ln(2^{n-1})]^2 = (n-1)^2(\ln(2))^2,$$

and consequently

$$\sup_{t_0\le t\le 0}\int_{K_R\times\{t\}} [\psi^+]^2\zeta^2 dx \ge (n-1)^2(\ln(2))^2|S_t|.$$

Combining this three estimates, we arrive at

$$\begin{aligned} |S_t| &\le \left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R| + C\frac{n}{(n-1)^2}\frac{1}{\delta^2}|K_R| \\ &\le \left(\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\nu_0/2}\right) + C\frac{1}{n\delta^2}\right)|K_R|. \end{aligned}$$

Additionally,

$$\begin{aligned} &\left|\left\{x\in K_R : u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\right\}\right| \\ &\le |S_t| + |K_R\backslash K_{(1-\delta)R}| = |S_t| + d\delta|K_R| \end{aligned}$$

is also true and, from this, it holds that

$$\left|\left\{x \in K_R : u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\right\}\right| \leq \left(\left(\frac{n}{n-1}\right)^2 \left(\frac{1-\nu_0}{1-\nu_0/2}\right) + \frac{C}{n\delta^2} + d\delta\right) |K_R|,$$

for all $t \in (t_0, 0) \supset \left[-\frac{\nu_0}{2}\omega^{1-m}R^2, 0\right]$. Finally, if we choose $d\delta \leq \frac{3}{8}\nu_0^2$ and n so large that

$$\left(\frac{n}{n-1}\right)^2 \leq \left(1 - \frac{\nu_0}{2}\right)(1+\nu_0) := \beta \qquad \text{and} \qquad \frac{C}{n\delta^2} \leq \frac{3}{8}\nu_0^2,$$

our lemma follows with $q = n+1$. □

In fact, we could also state that u is strictly below its supremum in a smaller cylinder, $Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)$.

Proposition 3.5. *Assume* (3.2) *is in force. If the second alternative* (3.7) *holds, then there exists a number* $s_0 > 1$*, independent of* ω*, such that*

$$u(x,t) \leq \mu^+ - \frac{\omega}{2^{s_0}} \quad \forall (x,t) \in Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right).$$

Proof. Similarly to the proof of the first alternative, we start by defining the sequence

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \qquad n = 0, 1, 2\ldots,$$

and to construct the family of nested and shrinking cylinders $Q(\omega^{1-m}R_n^2, R_n)$. We will also define the cutoff function $0 \leq \zeta \leq 1$ with resemblant properties:

$$\zeta_n = 1 \text{ in } Q\left(\nu_0\frac{R_{n+1}^2}{2\omega^{m-1}}, R_{n+1}\right); \qquad \zeta_n = 0 \text{ on } \partial Q\left(\nu_0\frac{R_n^2}{2\omega^{m-1}}, R_n\right);$$

$$|\nabla\zeta_n| \leq \frac{2^{n-1}}{R}; \qquad 0 \leq (\zeta_n)_t \leq \frac{2^{2n-2}}{R^2}\omega^{m-1}; \qquad \Delta\zeta_n \leq \frac{2^{2n-2}}{R^2}.$$

This time, we will use the energy estimates in the cylinder $Q\left(\nu_0\frac{R_n^2}{2\omega^{m-1}}, R_n\right)$ for $(u_+^l - k_n)_+$, with $l = \mu^+ - \frac{\omega}{2^{s_0}}$,

$$k_n = \mu^+ - \frac{\omega}{2^{s_0}} - \frac{\omega}{2^{s_0+n}},$$

and $\zeta = \zeta_n$.

The next step is similar to the proof of **Proposition 3.1**. For the left hand side of the energy inequality, we can deduce again the following lower bound,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_+^l - k_n)_+^2 \zeta_n^2 dx + \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} (u_+^l)^{m-1} |\nabla (u_+^l - k_n)_+ \zeta_n|^2 dxdt \\
&\quad \geq \operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_+^l - k_n)_+^2 \zeta_n^2 dx \\
&\qquad\qquad + \left(\frac{\omega}{2}\right)^{m-1} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} |\nabla (u_+^l - k_n)_+ \zeta_n|^2 dxdt \\
&\quad \geq 2^{1-m} \Bigg(\operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_+^l - k_n)_+^2 \zeta_n^2 dx \\
&\qquad\qquad + \omega^{m-1} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} |\nabla (u_+^l - k_n)_+ \zeta_n|^2 dxdt \Bigg),
\end{aligned}$$

since when $|\nabla (u_+^l - k_n)_+ \zeta_n|$ is not zero,

$$u_+^l > k_n \geq \mu^+ - \frac{\omega}{2} \geq \omega - \frac{\omega}{2} = \frac{\omega}{2}.$$

Let us, one more time, deduce upper bounds for each term on the right hand side. The following properties will be used:

1. $0 \leq \mu^- \leq \frac{\omega}{4}$, so $u \leq \frac{5\omega}{4}$ and consequently $u_+^l = \min\{u, l\} \geq \frac{5\omega}{4}$;
2. $l = \mu^+ - \frac{\omega}{2^{s_0}} > k_n$, so $\chi_{[u \geq l]} \leq \chi_{[u > k_n]} = \chi_{[(u-k_n)_+ > 0]}$;
3. $\chi_{[(u-k_n)_+ > 0]} = \chi_{[(u_+^l - k_n)_+ > 0]}$ for all (x, t);
4. $(l - k_n)_+ = \frac{\omega}{2^{s_0+n}} \leq \frac{\omega}{2^{s_0-1}}$, $(u_+^l - k_n)_+ \leq \frac{\omega}{2^{s_0+n}} \leq \frac{\omega}{2^{s_0-1}}$ and $(u - l)_+ \leq \frac{\omega}{2^{s_0-1}}$.

Applying the previous properties and definitions of our cutoff functions, we can obtain the following inequalities for each term on the right hand side of the energy estimate, (2.4):

1st term:

$$\begin{aligned}
&\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} (u_+^l)^{m-1} (u_+^l - k_n)_+^2 |\nabla \zeta_n|^2 dxdt \\
&\qquad\qquad \leq C\omega^{m-1} \left(\frac{\omega}{2^{s_0-1}}\right)^2 \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u_+^l - k_n)_+ > 0]} dxdt;
\end{aligned}$$

2nd term:

$$\begin{aligned}
&2 \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \left((u_+^l - k_n)_+^2 + 2(l - k_n)_+ (u - l)_+ \right) \zeta_n (\zeta_n)_t dxdt \\
&\quad \leq C \left(\left(\frac{\omega}{2^{s_0-1}}\right)^2 + 2\left(\frac{\omega}{2^{s_0-1}}\right)^2 \right) \omega^{m-1} \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u_+^l - k_n)_+ > 0]} dxdt \\
&\quad \leq C\omega^{m-1} \left(\frac{\omega}{2^{s_0-1}}\right)^2 \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u_+^l - k_n)_+ > 0]} dxdt;
\end{aligned}$$

3rd term:

$$\begin{aligned}
&C(l-k_n)_+\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0}\int_{K_{R_n}}\left(\int_l^u s^{m-1}ds\right)\left(|\nabla\zeta_n|^2+\zeta_n\Delta\zeta_n\right)\chi_{[u\geq l]}dxdt\\
&\quad\leq C(l-k_n)_+\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0}\int_{K_{R_n}}(u-l)_+u^{m-1}\left(|\nabla\zeta_n|^2+\zeta_n\Delta\zeta_n\right)\chi_{[u\geq l]}dxdt\\
&\quad\leq C\left(\frac{\omega}{2^{s_0-1}}\right)\left(\frac{5\omega}{4}\right)^{m-1}\left(\frac{2^{2n-2}}{R^2}+\frac{2^{2n-2}}{R^2}\right)\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0}\int_{K_{R_n}}\chi_{[(u_+^l-k_n)_+>0]}dxdt\\
&\quad\leq C\omega^{m-1}\left(\frac{\omega}{2^{s_0-1}}\right)^2\frac{2^{2n-1}}{R^2}\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0}\int_{K_{R_n}}\chi_{[(u_+^l-k_n)_+>0]}dxdt.
\end{aligned}$$

Joining these four inequalities we conclude that, for every $n\geq 0$, the following is verified:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0}\int_{K_{R_n}\times\{t\}}(u_+^l-k_n)_+^2\zeta_n^2dx+\omega^{m-1}\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0}\int_{K_{R_n}}|\nabla(u_+^l-k)_+\zeta_n|^2dxdt\\
&\qquad\qquad\leq C\omega^{m-1}\left(\frac{\omega}{2^{s_0-1}}\right)^2\frac{2^{2n}}{R^2}\int_{-\frac{R_n^2}{\omega^{m-1}}}^{0}\int_{K_{R_n}}\chi_{[(u_+^l-k_n)_+>0]}dxdt.
\end{aligned}$$

As in the first alternative, the next step is to change the time variable, putting $\bar{t}=\frac{2\omega^{m-1}}{\nu_0}t$, and defining

$$\bar{u}_+^l(\cdot,\bar{t}):=u_+^l(\cdot,t),\qquad \bar{\zeta}_n(\cdot,\bar{t}):=\zeta_n(\cdot,t).$$

In this way, the previous inequality can be rewritten as

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-R_n^2<t<0}\int_{K_{R_n}\times\{\bar{t}\}}(\bar{u}_+^l-k_n)_+^2\bar{\zeta}_n^2dx+\frac{\nu_0}{2}\int_{-R_n^2}^{0}\int_{K_{R_n}}|\nabla(\bar{u}_+^l-k)_+\bar{\zeta}_n|^2dxdt\\
&\leq C\frac{\nu_0}{2}\left(\frac{\omega}{2^{s_0-1}}\right)^2\frac{2^{2n}}{R^2}\int_{-R_n^2}^{0}\int_{K_{R_n}}\chi_{\{(\bar{u}_+^l-k_n)_+>0\}}dxdt,
\end{aligned}$$

and, if we multiply both sides by $\frac{2}{\nu_0}>1$, it can be simplified to

$$||(\bar{u}_+^l-k_n)_+\bar{\zeta}_n||^2_{V^2(Q(R_n^2,R_n))}\leq C\left(\frac{\omega}{2^{s_0-1}}\right)^2\frac{2^{2n}}{R^2}A_n, \tag{3.12}$$

where

$$A_n=\int_{-R_n^2}^{0}\int_{K_{R_n}}\chi_{\{(\bar{u}_+^l-k_n)_+>0\}}dxdt.$$

From the definition of A_n, the fact that $k_n<k_{n+1}$ and since $\bar{\zeta}_n=1$ in $Q(R_{n+1}^2,R_{n+1})$, we similarly obtain that

$$\begin{aligned}
\frac{1}{2^{2(n+2)}}\left(\frac{\omega}{2^{s_0-1}}\right)^2A_{n+1}&=|k_{n+1}-k_n|^2A_{n+1}\\
&\leq||(\bar{u}_+^l-k_n)_+||^2_{2,Q(R_{n+1}^2,R_{n+1})}\\
&\leq||(\bar{u}_+^l-k_n)_+\bar{\zeta}_n||^2_{2,Q(R_n^2,R_n)}.
\end{aligned}$$

Moreover, if we use **Theorem 2.11** for p=2 and the inequality (3.12), we can carry on these calculations

$$\frac{1}{2^{2(n+2)}}\left(\frac{\omega}{2^{s_0-1}}\right)^2 A_{n+1} \leq C||(\bar{u}^l_+ - k_n)_-\bar{\zeta}_n||^2_{V^2(Q(R_n^2,R_n))} A_n^{\frac{2}{d+2}}$$
$$\leq C\frac{2^{2n}}{R^2}\left(\frac{\omega}{2^{s_0-1}}\right)^2 A_n^{1+\frac{2}{d+2}}. \qquad (3.13)$$

Doing exactly the same calculations as in the first alternative we obtain that

$$\frac{A_{n+1}}{|Q(R_{n+1}^2, R_{n+1})|} \leq C4^{2n}\left(\frac{A_n}{|Q(R_n^2, R_n|)}\right)^{1+\frac{2}{d+2}},$$

and again, if we define

$$X_n = \frac{A_n}{|Q(R_n^2, R_n)|}$$

we can rewrite this as the following recursive relation

$$X_{n+1} \leq C4^{2n} X_n^{1+\frac{2}{d+2}}.$$

Now, if

$$X_0 \leq C^{-\frac{d+2}{2}} 4^{-\frac{(d+2)^2}{2}} := \nu_0^*, \qquad (3.14)$$

by the fast geometric convergence, we can obtain the same conclusions as before,

$$X_n \to 0$$

which implies that

$$\left|\left\{(x,t)\in Q\left(\frac{\nu_0}{2\omega^{m-1}}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right) \ : \ u^l_+(x,t) > \mu^+ - \frac{\omega}{2^{s_0}}\right\}\right| = 0.$$

One more time, since $u^l_+ \leq \mu^+ - \frac{\omega}{2^{s_0}} = l$ implies that $u^l_+ = u$, the final result follows immediately.

At this point, to finish this proof, it just remain to prove (3.14).

To simplify the notation, let us introduce the sets

$$B_\sigma(t) = \left\{x \in K_R \ : \ u(x,t) > \mu^+ - \frac{\omega}{2^\sigma}\right\}$$

and

$$B_\sigma = \left\{(x,t) \in Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right) \ : \ u(x,t) > \mu^+ - \frac{\omega}{2^\sigma}\right\}.$$

Therefore, with this notation, (3.14) can be read as

$$|B_{s_0-1}| \leq \nu_0^* \left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|.$$

This inequality means that the subset of the cylinder $Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)$ where u is close to its supremum μ^+ can be made arbitrarily small. To prove this, we will not only use **Lemma 3.4**, but also the energy estimate. Let us start by applying the energy estimate (2.4) to the function $(u_+^{\mu^+} - k)_+ = (u-k)_+$ where

$$k = \mu^+ - \frac{\omega}{2^s}, \qquad q < s < s_0,$$

and the cutoff function $0 \le \zeta \le 1$ is defined in $Q\left(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\right)$ with the following assumptions:

$$\zeta = 1 \text{ in } Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right); \qquad \zeta = 0 \text{ on } \partial Q\left(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\right);$$
$$|\nabla\zeta| \le \frac{1}{R}; \qquad 0 \le \zeta_t \le \frac{\omega^{m-1}}{R^2}.$$

This time, we will delete the first term in the left hand side, because it is nonnegative, and integrate the second one in a smaller set, $Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)$. Then, repeating the same argument, when $|\nabla(u-k)_+\zeta| \neq 0$, $u > k = \mu^+ - \frac{\omega}{2^s} > \frac{\omega}{2}$, we arrive at

$$\operatorname*{ess\,sup}_{-\frac{\nu_0 R^2}{\omega^{m-1}}<t<0} \int_{K_{2R}\times\{t\}} (u-k)_+^2\zeta^2 dx + \int_{-\frac{\nu_0 R^2}{\omega^{m-1}}}^0 \int_{K_{2R}} u^{m-1}|\nabla(u-k)_+\zeta|^2 dxdt$$
$$\ge \left(\frac{\omega}{2}\right)^{m-1} \int_{-\frac{\nu_0 R^2}{2\omega^{m-1}}}^0 \int_{K_R} |\nabla(u-k)_+\zeta|^2 dxdt.$$

In the right hand side, the third term is now 0 because $\chi_{\{u \ge \mu^+\}} = 0$. For the others terms, using similar properties, we can obtain the following upper bounds:

1st term:

$$\int_{-\frac{\nu_0 R^2}{\omega^{m-1}}}^0 \int_{K_{2R}} u^{m-1}(u-k)_+^2|\nabla\zeta|^2 dxdt \le C\omega^{m-1}\left(\frac{\omega}{2^s}\right)^2 \frac{1}{R^2}\left|Q\left(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\right)\right|;$$

2nd term:

$$2\int_{-\frac{\nu_0 R^2}{\omega^{m-1}}}^0 \int_{K_{2R}} \left((u-k)_+^2 + 2(\mu^+ - k_n)_+(u-\mu^+)_+\right)\zeta\zeta_t dxdt$$
$$\le C\omega^{m-1}\left(\frac{\omega}{2^s}\right)^2 \frac{1}{R^2}\left|Q\left(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\right)\right|.$$

Joining these three inequalities and multiplying both sides by $\left(\frac{\omega}{2}\right)^{1-m}$, we conclude that

$$\int_{-\frac{\nu_0 R^2}{2\omega^{m-1}}}^0 \int_{K_R} |\nabla(u-k)_+\zeta|^2 dxdt \le \frac{C}{R^2}\left(\frac{\omega}{2^s}\right)^2 \left|Q\left(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\right)\right|.$$

In addition, we also have that $\left|Q\left(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\right)\right| = 2^{d+1}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|$ and $\zeta = 1$ in $Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)$, therefore

$$\int_{-\frac{\nu_0 R^2}{2\omega^{m-1}}}^0 \int_{K_R} |\nabla(u-k)_+|^2 dxdt \le \frac{C}{R^2}\left(\frac{\omega}{2^s}\right)^2 \left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|.$$

To finish this first part of the proof, notice that $B_s \subset Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)$ and, that in B_s we have $|\nabla(u-k)_+| = |\nabla(u-k)| = |\nabla u|$, and consequently

$$\int\int_{B_s} |\nabla u|^2 dxdt \leq \frac{C}{R^2}\left(\frac{\omega}{2^s}\right)^2 \left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|. \tag{3.15}$$

In the next step, we will apply **Lemma 2.13**. If we choose $k_1 = \mu^+ - \frac{\omega}{2^s}$ and $k_2 = \mu^+ - \frac{\omega}{2^{s+1}}$, then the lemma gives us that, for all $t \in (-\frac{\nu_0}{2\omega^{m-1}}, 0)$,

$$\frac{\omega}{2^{s+1}}|[u(\cdot,t) > \mu^+ - \frac{\omega}{2^{s+1}}]| \leq C\frac{R^{d+1}}{|[u(\cdot,t) < \mu^+ - \frac{\omega}{2^s}]|}\int_{[-\frac{\omega}{2^s} < u(\cdot,t)-\mu^+ < -\frac{\omega}{2^{s+1}}]} |\nabla u| dx. \tag{3.16}$$

Additionally, since $q \leq s-1$, by **Lemma 3.4** we have that $|B_{s-1}(t)| \leq |B_q(t)| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right)|K_R|$ for all $t \in (-\frac{\nu_0}{2\omega^{m-1}}, 0)$. Using this inequality we deduce that

$$\begin{aligned}\left|\left\{x \in K_R : u(x,t) < \mu^+ - \frac{\omega}{2^s}\right\}\right| &\geq \left|\left\{x \in K_R : u(x,t) \leq \mu^+ - \frac{\omega}{2^{s-1}}\right\}\right| \\ &= |K_R| - |B_{s-1}(t)| \geq |K_R| - |B_q(t)| \geq \left(\frac{\nu_0}{2}\right)^2 |K_R|.\end{aligned}$$

Therefore, with our notation, (3.16) implies that

$$\frac{\omega}{2^{s+1}}|B_{s+1}(t)| \leq \frac{CR^{d+1}}{\nu_0^2 |K_R|}\int_{B_s(t)\setminus B_{s+1}(t)} |\nabla u| dx,$$

for $t \in [-\frac{\nu_0}{2\omega^{m-1}}, 0]$.

At this point, we can integrate the previous inequality over the set $[-\frac{\nu_0}{2\omega^{m-1}}, 0]$, apply Cauchy-Schwarz inequality and use (3.15), in this order, to obtain

$$\begin{aligned}\frac{\omega}{2^{s+1}}|B_{s+1}| &\leq \frac{CR}{\nu_0^2}\int\int_{B_s\setminus B_{s+1}} |\nabla u| dxdt \\ &\leq \frac{CR}{\nu_0^2}\left(\int\int_{B_s\setminus B_{s+1}} |\nabla u|^2 dxdt\right)^{\frac{1}{2}} |B_s \setminus B_{s+1}|^{\frac{1}{2}} \\ &\leq \frac{CR}{\nu_0^2}\left(\int\int_{B_s} |\nabla u|^2 dxdt\right)^{\frac{1}{2}} |B_s \setminus B_{s+1}|^{\frac{1}{2}} \\ &\leq \frac{C}{\nu_0^2}\frac{\omega}{2^s}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|^{\frac{1}{2}} |B_s \setminus B_{s+1}|^{\frac{1}{2}}.\end{aligned}$$

Simplifying and squaring both sides we have that

$$|B_{s+1}|^2 \leq \frac{C}{\nu_0^4}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right| |B_s \setminus B_{s+1}|.$$

Since this inequality is valid for $q < s < s_0$, we can add them for $s = q+1, q+2, \ldots, s_0 - 2$, *i.e.*,

$$\sum_{s=q+1}^{s_0-2} |B_{s+1}|^2 \leq \frac{C}{\nu_0^4}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right| \sum_{s=q+1}^{s_0-2} |B_s \setminus B_{s+1}|.$$

Then, as $\sum_{s=q+1}^{s_0-2}|B_s\setminus B_{s+1}|\leq \left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2,R\right)\right|$ is trivially verified, and $B_{s_0}\subset B_{s+1}$ for $s=q+1,q+2,\dots,s_0-2$, we deduce that

$$(s_0-q)|B_{s_0}|^2\leq \frac{C}{\nu_0^4}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2,R\right)\right|^2,$$

in other words,

$$|B_{s_0}|\leq \frac{C}{\nu_0^2(s_0-q)^{\frac{1}{2}}}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2,R\right)\right|.$$

If we choose s_0 so large that

$$\frac{C}{\nu_0^2(s_0-q)^{\frac{1}{2}}}<\nu_0^*$$

we prove (3.14) and consequently our lemma. □

Finally, we arrive again at the reduction of the oscillation, in this case when the second alternative holds.

Corollary 3.6 (Reduction of the Oscillation)**.** *Assume that* (3.2) *is in force. If the second alternative* (3.7) *holds, then there exists a constant* $\sigma_2\in(0,1)$*, depending only on the data, such that*

$$\operatorname*{ess\,sup}_{Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u\leq \sigma_2\omega. \tag{3.17}$$

Proof. This proof is similar to the proof of **Corollary 3.2**. By **Proposition 3.5**, there exists $s_0\in\mathbb{N}$ such that

$$\operatorname*{ess\,sup}_{Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u\leq \mu^+-\frac{\omega}{2^{s_0}}$$

and thus

$$\begin{aligned}\operatorname*{ess\,osc}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u&\leq \mu^+-\frac{\omega}{2^{s_0}}-\mu^-\\&=\left(1-\frac{1}{2^{s_0}}\right)\omega.\end{aligned}$$

This way, the corollary follows with $\sigma_2=\left(1-\frac{1}{2^{s_0}}\right)$. □

Remark 3.7. For the signed PME, there is no need to study the second alternative. As we referred before, if u verifies the second alternative, then $-u$ will verify the first alternative and this case is already studied. Notice that the essential oscillation of u and $-u$ are the same so it is equivalent to prove the previous corollary for u or

$-u$. Since the PME is a particular case of the signed PME, it could seem that the second alternative should never be necessary. However, notice that to prove the first alternative for the signed PME we used a nonnegative solution of the original PME. So both alternatives are always necessary.

Chapter 4

The Hölder continuity

At this point, using **Corollaries 3.2** and **3.6** we can easily prove the general reduction of oscillation. This result will be the basis of our final iterative scheme that will prove the Hölder Continuity of weak solutions of the PME.

Proposition 4.1. *Assume* (3.2) *is in force; then there exists a constant* $\sigma \in (0,1)$, *depending only on the data, such that*

$$\operatorname*{ess\,sup}_{Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)} u \leq \sigma\omega. \tag{4.1}$$

Proof. Since $\frac{\nu_0}{2} < 1$,

$$Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right) \subset Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right),$$

and by **Corollaries 3.2** and **3.6**, we just need to choose $\sigma = \max\{\sigma_1, \sigma_2\}$ to finish the proof . □

4.1. The Recursive Argument

We finally prove the Hölder continuity for weak solutions through an iterative scheme based on all previous results.

Lemma 4.2. *There exist constants* $\gamma > 1$ *and* $\beta \in (0,1)$, *that can be determined a priori in terms of the data, such that for all* $0 < r \leq R$,

$$\operatorname*{ess\,osc}_{Q(\omega^{1-m}r^2, r)} u \leq \gamma\omega\left(\frac{r}{R}\right)^{\beta}.$$

Proof. Let us start by defining

$$R_k = c_0^k R, \qquad c_0 = \frac{1}{2}\sigma^{\frac{m-1}{2}}\left(\frac{\nu_0}{2}\right) < \frac{1}{2},$$

for $k = 0, 1, 2 \ldots$, where σ is given by **Proposition 4.1**. Since we are supposing that

$$\omega^{m-1} > R^{\epsilon},$$

for $k = 0$, we have that our initial condition

$$\operatorname*{ess\,osc}_{Q(\omega^{1-m}R_0^2, R_0)} u \le \operatorname*{ess\,osc}_{Q(R_0^{2-\epsilon}, R_0)} u \le \omega$$

is verified.

For $k = 1$, by (4.1), we also have that

$$\operatorname*{ess\,osc}_{Q\left(\omega^{1-m}R_1^2, R_1\right)} u \le \operatorname*{ess\,osc}_{Q\left(\omega^{1-m}R_1^2, \frac{R}{2}\right)} u \le \sigma\omega = \omega_1.$$

Repeating all the process again starting in $Q\left(\omega^{1-m}R_1^2, R_1\right)$, with $\omega_1 = \sigma\omega$, we can deduce inductively that, for all $k = 0, 1, 2 \ldots$,

$$\operatorname*{ess\,osc}_{Q\left(\omega^{1-m}R_k^2, R_k\right)} u \le \omega_k = \sigma^k \omega.$$

In addition, for every $0 < r \le R$, there exists some k such that

$$Rc_0^{(k+1)} \le r \le Rc_0^k,$$

so, if we choose $\beta = \frac{\log \sigma}{\log c_0} > 0$, we arrive at

$$\sigma^{k+1} \le \left(\frac{r}{R}\right)^{\beta},$$

which means that

$$\operatorname*{ess\,osc}_{Q(\omega^{1-m}r^2, r)} u \le \operatorname*{ess\,osc}_{Q\left(\omega^{1-m}R_k^2, R_k\right)} u \le \sigma^k \omega \le \gamma\omega \left(\frac{r}{R}\right)^{\beta},$$

with $\gamma = \sigma^{-1}$.

Without loss of generality, we can also suppose that $\sigma > \frac{1}{2}$, which implies that $\beta \in (0, 1)$. □

Finally, we can easily prove the first part of the main theorem rigorously.

Theorem 4.3. *Assume* (3.2) *is in force. Let u be a nonnegative bounded local weak solution of* (PME) *in Ω_T and $M = ||u||_{\infty, \Omega_T}$. Then u is locally Hölder continuous in Ω_T.*

Proof. Let us start by fixing $(x, t_i) \in K,\ i = 1, 2$, with $t_2 > t_1$ and constructing the cylinder

$$S = (x, t_2) + Q(R^2, R).$$

Since Ω_T there is a constant l such that $\operatorname{dist}(K;\partial_p\Omega_T) < l$ for every compact set K. This fact implies that if we choose

$$R = \frac{1}{2l}\operatorname{dist}(K;\partial_p\Omega_T),$$

then $S \subset \Omega_T$ and $R < 1$.

Moreover, supposing that

$$t_2 - t_1 < R^2,$$

it is possible to choose

$$r = |t_2 - t_1|^{\frac{1}{2}} \in (0, R).$$

We can then apply **Lemma 4.2** to

$$(x, t_2) + Q(\omega^{1-m}r^2, r)$$

and conclude that

$$|u(x,t_1) - u(x,t_2)| \le \gamma\omega(2l)^\beta \left(\frac{|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p\Omega_T)}\right)^\beta = C\left(\frac{|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p\Omega_T)}\right)^\beta.$$

On the other hand, if $t_2 - t_1 \ge R^2$, we have that

$$\frac{2l|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p\Omega_T)} \ge 1,$$

which implies that

$$|u(x,t_1) - u(x,t_2)| \le 2M \le 2M(2l)^\beta \left(\frac{|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p\Omega_T)}\right)^\beta = C\left(\frac{|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p\Omega_T)}\right)^\beta.$$

Identically, we can prove the Hölder continuity in the space variables,*i.e.*, for all $(x_i, t) \in K,\ i = 1, 2$

$$|u(x_1,t) - u(x_2,t)| \le C\left(\frac{|x_2 - x_1|}{\operatorname{dist}(K;\partial_p\Omega_T)}\right)^\beta.$$

To finish the proof we can use both inequality to arrive at

$$\begin{aligned}
|u(x_1,t_1) - u(x_2,t_2)| &= |u(x_1,t_1) - u(x_2,t_1) + u(x_2,t_1) - u(x_2,t_2)| \\
&\le C\left(\left(\frac{|x_2 - x_1|}{\operatorname{dist}(K;\partial_p\Omega_T)}\right)^\beta + \left(\frac{|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p\Omega_T)}\right)^\beta\right) \\
&\le C\left(\frac{|x_2 - x_1| + |t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p\Omega_T)}\right)^\beta,
\end{aligned}$$

for all $(x_i, t) \in K,\ i = 1, 2.$ □

We thus complete the application of the intrinsic scaling method, remaining just to prove this result when (3.2) does not hold.

Remark 4.4. These last proofs just depend on the Reduction of the Oscillation, and not on the equation itself. Since in the previous remarks we described how to extend the Reduction of the Oscillation result for the signed PME, at this point we can also state that weak solutions of the signed PME, (2.1), are locally Hölder continuous.

4.2. Uniformly parabolic equation

Let us consider the case when (3.2) does not hold and the infimum is not comparatively small, i.e.,

$$\mu^- > \frac{\omega}{4} > \frac{R^\epsilon}{4},$$

and consequently $u \geq C > 0$ in $Q(4R^{2-\epsilon}, 2R)$.

In this case, the *porous media equation* does not have any degeneracy. In fact, it is simply a parabolic equation in $Q(4R^{2-\epsilon}, 2R)$, and this type of equations had already been studied in detail [12]. Remember that, in the introduction, we introduced the evolution equation,

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) = 0,$$

with $\mathbf{A}$ measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, \nabla u) \cdot \nabla u \geq \lambda(x, t, u, \nabla u)|\nabla u|^2 \\ |\mathbf{A}(x, t, u, \nabla u)| \leq |\Lambda(x, t, u, \nabla u)||\nabla u|, \end{cases}$$

where λ and Λ are given functions. We also highlighted that, if there exist two positive constants $\lambda_0 \leq \Lambda_1$, such that

$$\lambda_0 \leq \lambda(x, t, u, \nabla u) \quad \text{and} \quad \Lambda(x, t, u, \nabla u) \leq \Lambda_1,$$

then the equation is considered to be parabolic.

The (PME) can be rewritten in the divergence form as

$$u_t - \operatorname{div}(mu^{m-1}\nabla u) = 0 \quad m > 1,$$

where $\mathbf{A} = mu^{m-1}\nabla u$. This means that $\lambda(x, t, u, \nabla u) \geq mC^{m-1} > 0$. Since u is considered to be bounded, we also have that $\Lambda(x, t, u, \nabla u) \leq mM^{m-1}$, which implies that, in fact, the PME, in this particular case, is uniformly parabolic.

The theory of this type of equation is already well studied, and the proof of the Hölder continuity of weak solutions can be consulted in [12, chapter 5].

4.3. Generalizations

The intrinsic scaling method is very versatile and we can adjust it to a great number of singular and degenerate parabolic differential equations. As indicated earlier, the Hölder continuity is solely a consequence of the *a priori* estimates and, for this reason, the techniques just presented are rather flexible and adjustable to a variety of other equations.

To highlight this versatility and the strength of this method, we will state two of the most important possible generalizations.

Along the text we already left remarks to demonstrate how we can generalize the method to the signed PME, although this results may also be generalized to all quasilinear equations of *porous medium* type. This type of equations can be cast in the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = \mathrm{b}(x,t,u,\nabla u), \tag{4.2}$$

where $\mathbf{A}_i$ and b are measurable functions satisfying the following structure assumptions:

$$\begin{cases} \mathbf{A}(x,t,u,\nabla u)\cdot\nabla u \geq C_0|u|^m - 1|\nabla u|^2 - \varphi_0(x,t), \quad m>0; \\ |\mathbf{A}(x,t,u,\nabla u)| \leq C_1|u|^{m-1}|\nabla u| + \varphi_1(x,t); \\ |b(x,t,u,\nabla u)| \leq C_2|\nabla|u|^m|^2 + \varphi_2(x,t), \end{cases}$$

for a.e. $(x,t)\in\Omega_T$. The constants C_i are given and positive and the functions φ_i are nonnegative subject to the integrability conditions $\varphi_0,\ \varphi_1^2,\ \varphi_2 \in L^{q,r}(\Omega_T)$ with $q,r\geq 1$ satisfying, for $d\geq 2$,

$$\frac{1}{r} + \frac{d}{2q} \in (0,1).$$

As we already mentioned, another equation to which the theory applies is the p-Laplace equation (PLE). In fact, it can be extended to equations with the full p-Laplacian type quasilinear structure, *i.e.*, with the form (4.2) verifying the following structure assumptions:

$$\begin{cases} \mathbf{A}(x,t,u,\nabla u)\cdot\nabla u \geq C_0|\nabla u|^p - \varphi_0(x,t), \quad m>0; \\ |\mathbf{A}(x,t,u,\nabla u)| \leq C_1|\nabla u u|^{p-1} + \varphi_1(x,t); \\ |b(x,t,u,\nabla u)| \leq C_2|\nabla u|^p + \varphi_2(x,t), \end{cases}$$

with similar conditions for C_i and φ_i.

It is possible to find in the literature even more equations, and for that case we refer the reader to [5] and [6].

Chapter 5

Conclusion

The goal initially proposed was successfully achieved. Although some ideas were not presented or even studied we were able to show the powerful method of intrinsic scaling.

Along this project we went trough different functional analysis techniques, trying to achieve a balance between explaining them thoroughly without losing the focus of the general idea. Overall, we believe this balance was successfully achieved, making this work not hard to follow and at same time self-contained.

Considering the complexity of the method and to ease comprehension we divided this work in several parts. We initially define weak solutions and with them we reach to two integral inequalities, named *a priori* estimates. At this point, the method is composed by two similar alternatives. Despite of some slight differences in the last part the arguments are essentially the same for both alternatives. We highlight the fact that the *a priori* estimates were the building blocks of the theory, all proofs start by its application. Then, we took advantage of the geometry of the equation changing our time variable and we start working on the "geometry and time" of the equation. Finally, after the Reduction of Oscillation has been proved for both alternatives, we generalized the recursive argument used by De Giorgi in [3], proving the Hölder continuity of the solutions.

Applying the intrinsic scaling method to the PME not only allowed learning about the method itself in full depth but also increased our knowledge on the equation and its geometry.

We would like to emphasize the fact that that the general idea of supporting this work has been published by J. Urbano in 2008 [15]. In this work the Hölder continuity of the p-Laplace equation has been proved, in this regard we adapted the method used to the PME and the signed PME. For that reason, some proofs are identical but explained in more detail to ease comprehension, while others went through significant adjustments. Several articles [2, 6, 8, 10, 14] were read and some ideas were adapted, however for some small steps original adjustments were needed.

Unfortunately the method applied does not answer all the questions risen by this theory. There is still much to discover and many problems to solve. The most famous unsolved issue related with this theory is the determination of optimal regularity. A wide number of mathematicians are still trying to understand what is the best Hölder regularity for general non-negative solutions, in other words, searching what is the worst possible case. For one-dimension this problem has been already answered, and optimal regularity was found [16, chapter 9]. In the case of greater dimensions other results were discovered. The application of the maximum principle made it possible to determine Hölder estimates with explicit Hölder exponents. Some others results were discovered [9], however the optimal exponent and regularity remains unknown.

Several hours were also spent around the better understanding of this open problem, although no considerable advances were made.

Additionally this work contributed to improvement of our knowledge on several topics on nonlinear PDEs. Studying the regularity of weak solutions, made us travel trough fascinating themes, including self-similar solutions, free-boundaries, finite propagation, radial equivalence for the PME and PLE amongst other themes. This last subject was of special interest along the year although it has not been included in this work. As it would deviate from the main topic of this dissertation and the final result was not completed we decided not to include in this work

In the end remains the feeling that it was impossible to transmit all the knowledge acquired about all the different areas studied. However, overall we hope with this manuscript that valuable ideas were clearly transmitted, in an easy and enjoyable way.

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