An action of the Hecke monoid on rational modules for the Borel subgroup of a quantised general linear group.

Ana Paula Santana*
aps@mat.uc.pt

Ivan Yudin*†
yudin@mat.uc.pt

CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal

In memory of J.A. Green, with the highest admiration.

Abstract

We construct an action of the Hecke monoid on the category of rational modules for the quantum negative Borel subgroup of the quantum general linear group. We also show that this action restricts to the category of polynomial modules for this quantum subgroup and induces an action on the category of modules for the quantised Borel-Schur algebra $S^-_{\alpha,\beta}(n,r)$.

1 Introduction

The idea of "*categorification*" originates from the joint work [3] of Crane and Frenkel, and the term was coined later in Crane's article [2]. Recently *categorification* became an intensively studied subject in several mathematical areas. A detailed account on this topic can be found in [12].

Given an action ρ of a monoid M on the Grothendieck group $\mathrm{Gr}(\mathcal{C})$ of a category $\mathcal{C}$, one can ask if it can be categorified, that is if there exists an action of M on $\mathcal{C}$ that induces ρ. Note that, to find such an action, one has: (i) to find a set of functors F_m, $m \in M$, whose action on $\mathcal{C}$ gives operators $\rho(m)$ on $\mathrm{Gr}(\mathcal{C})$; (ii) to show the existence of a coherent family of natural isomorphisms $\lambda_{m,m'}\colon F_m F_{m'} \to F_{mm'}$. Usually, (ii) is much more tricky than (i).

Let B_n be the negative Borel subgroup of the general linear group of degree n over an algebraically closed field. Denote by Gr the Grothendieck group of the category of finite dimensional polynomial B_n-modules. The tensor product of modules turns Gr into a ring. Let N be a finite dimensional polynomial B_n-module. Then we can consider the formal character ch_N of N in $\mathbb{Z}[x_1,\dots,x_n]$. In fact ch is a ring homomorphism from Gr to $\mathbb{Z}[x_1,\dots,x_n]$. In [5] Demazure showed how the characters of certain B_n-modules can be calculated by applying what is now called the *Demazure operators*, π_i, $1 \le i \le n-1$, to a monomial $x_1^{k_1}\dots x_n^{k_n}$. It turns out that the operators π_i, $1 \le i \le n$, define an action of the Hecke monoid, $\mathfrak{H}(\Sigma_n)$, on $\mathbb{Z}[x_1,\dots,x_n]$.

*This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

†The second author's work was supported by Programa Investigador FCT IF/00016/2013.

Later Magyar in [11] generalised Demazure's character formula for the class of flag Weyl modules corresponding to percentage-avoiding shapes. While researching this class we were lead to the idea that a categorification of the action of $\mathfrak{H}(\Sigma_\mathrm{n})$ on $\mathbb{Z}[x_1,\ldots,x_n]$ can be useful to prove some conjectures stated in [14].

In this article we show that the Hecke monoid acts on the category of rational modules for the quantum negative Borel subgroup of the quantum general linear group. In fact, we construct what we call a *preaction* of $\mathfrak{H}(\Sigma_\mathrm{n})$ on this category. In [17] the second author proves that the category of actions of $\mathfrak{H}(\Sigma_\mathrm{n})$ on a category $\mathcal{C}$ is equivalent to the category of preactions of $\mathfrak{H}(\Sigma_\mathrm{n})$ on $\mathcal{C}$. Therefore, via this equivalence, from the constructed preaction one can obtain an action of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the above referred category. It is then quite simple to get a preaction, and so an action, of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the category of $S^-_{\alpha,\beta}(n,r)$-modules, where $S^-_{\alpha,\beta}(n,r)$ is the quantised (negative) Borel-Schur algebra. In a forthcoming paper we will show that the action of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the category of rational modules for the quantum Borel subgroup induces an action of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the corresponding derived category. This action will provide a categorification of the action of $\mathfrak{H}(\Sigma_\mathrm{n})$ on $\mathbb{Z}[x_1,\ldots,x_n]$.

The paper is organised as follows. In Section 2 we introduce the notion of a preaction of the Hecke monoid $\mathfrak{H}(\Sigma_\mathrm{n})$ on a category $\mathcal{C}$. Section 3 contains some results, on cotensor product and induction for coalgebras, that are due to Takeuchi [15] and Donkin [7, 8]. In Section 4 we study some subgroups of the quantum general linear group (or rather their coordinate Hopf algebras), namely quantum parabolic subgroups and quantum Borel subgroups, following [16] and [10]. We also prove the exactness of the short exact sequences (7) and (8), which play a crucial role in our construction. Section 5 is dedicated to the definition of functors F_i, $1 \le i \le n-1$, that generate an action of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the category of rational modules for the quantum negative Borel subgroup. We also construct natural isomorphisms τ_{ij}, $1 \le i \le j \le n-1$, which, together with the functors F_i, give a preaction of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the above mentioned category. The proof that (F,τ) is in fact a preaction is given in Section 6. In Section 7 we show that (F,τ) induces a preaction of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the category of $S^-_{\alpha,\beta}(n,r)$-modules. In Section 8 we consider explicit examples of the application of the functors F_i to B_n-modules.

2 (Pre)actions of Hecke monoids

Let n be a positive integer and Σ_n the symmetric group of degree n.

The *Hecke monoid*, $\mathfrak{H}(\Sigma_\mathrm{n})$, is the monoid with elements T_w, $w \in \Sigma_n$, and multiplication determined by the rule

$$T_\sigma T_w = \begin{cases} T_{\sigma w}, & \text{if } l(\sigma w) = l(w)+1 \\ T_w, & \text{if } l(\sigma w) = l(w)-1, \end{cases}$$

where $w \in \Sigma_n$ and σ is an elementary transposition of the form $(k,k+1)$, $1 \le k \le n-1$.

Let $\mathcal{C}$ be a category and M a monoid with neutral element e. A *(pseudo)action* $(\mathcal{F},\lambda)$ of M on $\mathcal{C}$ is

i) a collection $\mathcal{F}$ of endofunctors $F_a\colon \mathcal{C} \to \mathcal{C}$, $a \in M$, such that F_e is the identity functor;

ii) natural isomorphisms $\lambda_{a,b}\colon F_aF_b \to F_{ab}$, such that for a, b, $c \in M$ the diagram

$$\begin{tikzcd} F_aF_bF_c \arrow[r, "\lambda_{a,b}F_c"] \arrow[d, "F_a\lambda_{b,c}"'] & F_{ab}F_c \arrow[d, "\lambda_{ab,c}"] \\ F_aF_{bc} \arrow[r, "\lambda_{a,bc}"] & F_{abc} \end{tikzcd}$$

commutes, and $\lambda_{e,a} = \lambda_{a,e}$ is the identity isomorphism of F_a (see [4]).

Suppose $(\mathcal{F}, \lambda)$ is an action of $\mathfrak{H}(\Sigma_n)$ on $\mathcal{C}$. To simplify notation we write F_a for F_{T_a} and $\lambda_{a,b}$ for λ_{T_a,T_b}. We also replace $(i, i+1)$ by i in the subscript of F and λ.

We define natural isomorphisms τ_{ij}, $1 \le i \le j \le n-1$, as follows:

$$\tau_{ii} = \lambda_{i,i} \colon F_i^2 \to F_i\,;$$

$$\tau_{ij} = \lambda_{i,j}^{-1}\lambda_{j,i} \colon F_jF_i \to F_iF_j\,, \quad \text{for } i+2 \le j\,;$$

$$\tau_{i,i+1} \colon F_{i+1}F_iF_{i+1} \to F_iF_{i+1}F_i$$

is the composition of

$$F_{i+1}F_iF_{i+1} \xrightarrow{F_{i+1}\lambda_{i,i+1}} F_{i+1}F_{(i,i+1,i+2)} \xrightarrow{\lambda_{i+1,(i,i+1,i+2)}} F_{(i,i+2)}$$

followed by the inverse of the map

$$F_iF_{i+1}F_i \xrightarrow{F_i\lambda_{i+1,i}} F_iF_{(i,i+2,i+1)} \xrightarrow{\lambda_{i,(i,i+2,i+1)}} F_{(i,i+2)}\ .$$

The natural transformations τ_{ij} fit in the following commutative diagrams:

$$\begin{tikzcd}
F_i^3 \arrow[r, "\tau_{ii}F_i"] \arrow[d, "F_i\tau_{ii}"'] & F_i^2 \arrow[d, "\tau_{ii}"] \\
F_i^2 \arrow[r, "\tau_{ii}"] & F_i
\end{tikzcd}$$

$$\begin{tikzcd}
F_{i+1}^2F_iF_{i+1} \arrow[rr, "\tau_{i+1,i+1}F_iF_{i+1}"] \arrow[d, "F_{i+1}\tau_{i,i+1}"'] & & F_{i+1}F_iF_{i+1} \arrow[d, "\tau_{i,i+1}"] \\
F_{i+1}F_iF_{i+1}F_i \arrow[r, "\tau_{i,i+1}F_i"] & F_iF_{i+1}F_i^2 \arrow[r, "F_iF_{i+1}\tau_{ii}"] & F_iF_{i+1}F_i
\end{tikzcd}$$

$$\begin{tikzcd}
F_{i+1}F_iF_{i+1}^2 \arrow[rr, "F_{i+1}F_i\tau_{i+1,i+1}"] \arrow[d, "\tau_{i,i+1}F_{i+1}"'] & & F_{i+1}F_iF_{i+1} \arrow[d, "\tau_{i,i+1}"] \\
F_iF_{i+1}F_iF_{i+1} \arrow[r, "F_i\tau_{i,i+1}"] & F_i^2F_{i+1}F_i \arrow[r, "F_iF_{i+1}\tau_{ii}"] & F_iF_{i+1}F_i
\end{tikzcd}$$

$$\begin{tikzcd}
F_jF_i^2 \arrow[rr, "F_j\tau_{ii}"] \arrow[d, "\tau_{i,j}F_i"'] & & F_jF_i \arrow[d, "\tau_{ij}"] \\
F_iF_jF_i \arrow[r, "F_i\tau_{i,j}"] & F_i^2F_j \arrow[r, "\tau_{ii}F_j"] & F_iF_j
\end{tikzcd}
\qquad
\begin{tikzcd}
F_j^2F_i \arrow[rr, "\tau_{jj}F_i"] \arrow[d, "F_j\tau_{i,j}"] & & F_jF_i \arrow[d, "\tau_{ij}"] \\
F_jF_iF_j \arrow[r, "\tau_{i,j}F_j"] & F_iF_j^2 \arrow[r, "F_i\tau_{jj}"] & F_iF_j
\end{tikzcd}$$

$$\begin{tikzcd}
F_{i+1}F_iF_{i+1}F_iF_{i+1} \arrow[rrr, "F_{i+1}F_i\tau_{i,i+1}"] \arrow[ddd, "\tau_{i,i+1}F_iF_{i+1}"'] & & & F_{i+1}F_i^2F_{i+1}F_i \arrow[d, "F_{i+1}\tau_{ii}F_{i+1}F_i"] \\
& & & F_{i+1}F_iF_{i+1}F_i \arrow[d, "\tau_{i,i+1}F_i"] \\
& & & F_iF_{i+1}F_i^2 \arrow[d, "F_iF_{i+1}\tau_{ii}"] \\
F_iF_{i+1}F_i^2F_{i+1} \arrow[r, "F_iF_{i+1}\tau_{ii}F_{i+1}"] & F_iF_{i+1}F_iF_{i+1} \arrow[r, "F_i\tau_{i,i+1}"] & F_i^2F_{i+1}F_i \arrow[r, "\tau_{ii}F_{i+1}F_i"] & F_iF_{i+1}F_i
\end{tikzcd}$$

$$\begin{tikzcd}
F_jF_iF_{i-1}F_i \arrow[r, "\tau_{ij}F_{i-1}F_i"] \arrow[d, "F_j\tau_{i-1,i}"'] & F_iF_jF_{i-1}F_i \arrow[r, "F_i\tau_{i-1,j}F_i"] & F_iF_{i-1}F_jF_i \arrow[d, "F_iF_{i-1}\tau_{ij}"'] \\
F_jF_{i-1}F_iF_{i-1} \arrow[d, "\tau_{i-1,j}F_iF_{i-1}"'] & & F_iF_{i-1}F_iF_j \arrow[d, "\tau_{i-1,i}F_j"] \\
F_{i-1}F_jF_iF_{i-1} \arrow[r, "F_{i-1}\tau_{ij}F_{i-1}"] & F_{i-1}F_iF_jF_{i-1} \arrow[r, "F_{i-1}F_i\tau_{i-1,j}"] & F_{i-1}F_iF_{i-1}F_j
\end{tikzcd}$$

$$\begin{tikzcd}
F_{j+1}F_jF_{j+1}F_i \arrow[r, "F_{j+1}F_j\tau_{i,j+1}"] \arrow[d, "\tau_{j,j+1}F_i"'] & F_{j+1}F_jF_iF_{j+1} \arrow[r, "F_{j+1}\tau_{ij}F_{j+1}"] & F_{j+1}F_iF_jF_{j+1} \arrow[d, "\tau_{i,j+1}F_jF_{j+1}"] \\
F_jF_{j+1}F_jF_i \arrow[d, "F_jF_{j+1}\tau_{ij}"] & & F_iF_{j+1}F_jF_{j+1} \arrow[d, "F_i\tau_{j,j+1}"] \\
F_jF_{j+1}F_iF_j \arrow[r, "F_j\tau_{i,j+1}F_j"] & F_jF_iF_{j+1}F_j \arrow[r, "\tau_{ij}F_{j+1}F_j"] & F_iF_jF_{j+1}F_j
\end{tikzcd}$$

$$\begin{tikzcd}
F_kF_jF_i \arrow[rr, "F_k\tau_{ij}"] \arrow[dd, "\tau_{jk}F_i"'] & & F_kF_iF_j \arrow[d, "\tau_{ik}F_j"] \\
& & F_iF_kF_j \arrow[d, "F_i\tau_{jk}"] \\
F_jF_kF_i \arrow[r, "F_j\tau_{ik}"] & F_jF_iF_k \arrow[r, "\tau_{ij}F_k"] & F_iF_jF_k
\end{tikzcd}$$

$$\begin{tikzcd}
F_{i+2}F_{i+1}F_iF_{i+2}F_{i+1}F_{i+2} \arrow[r, "F_{i+2}F_{i+1}\tau^{-1}_{i,i+2}F_{i+1}F_{i+2}"] \arrow[d, "F_{i+2}F_{i+1}F_i\tau_{i+1,i+2}"'] & F_{i+2}F_{i+1}F_{i+2}F_iF_{i+1}F_{i+2} \arrow[d, "\tau_{i+1,i+2}F_iF_{i+1}F_{i+2}"] \\
F_{i+2}F_{i+1}F_iF_{i+1}F_{i+2}F_{i+1} \arrow[d, "F_{i+2}\tau_{i,i+1}F_{i+2}F_{i+1}"'] & F_{i+1}F_{i+2}F_{i+1}F_iF_{i+1}F_{i+2} \arrow[d, "F_{i+1}F_{i+2}\tau_{i,i+1}F_{i+2}"] \\
F_{i+2}F_iF_{i+1}F_iF_{i+2}F_{i+1} \arrow[d, "\tau_{i,i+2}F_{i+1}F_iF_{i+2}F_{i+1}"'] & F_{i+1}F_{i+2}F_iF_{i+1}F_iF_{i+2} \arrow[d, "F_{i+1}\tau_{i,i+2}F_{i+1}F_iF_{i+2}"] \\
F_iF_{i+2}F_{i+1}F_iF_{i+2}F_{i+1} \arrow[d, "F_iF_{i+2}F_{i+1}\tau^{-1}_{i,i+2}F_{i+1}"'] & F_{i+1}F_iF_{i+2}F_{i+1}F_iF_{i+2} \arrow[d, "F_{i+1}F_iF_{i+2}F_{i+1}\tau^{-1}_{i,i+2}"] \\
F_iF_{i+2}F_{i+1}F_{i+2}F_iF_{i+1} \arrow[d, "F_i\tau_{i+1,i+2}F_iF_{i+1}"'] & F_{i+1}F_iF_{i+2}F_{i+1}F_{i+2}F_i \arrow[d, "F_{i+1}F_i\tau_{i+1,i+2}F_i"] \\
F_iF_{i+1}F_{i+2}F_{i+1}F_iF_{i+1} \arrow[d, "F_iF_{i+1}F_{i+2}\tau_{i,i+1}"'] & F_{i+1}F_iF_{i+1}F_{i+2}F_{i+1}F_i \arrow[d, "\tau_{i,i+1}F_{i+2}F_{i+1}F_i"] \\
F_iF_{i+1}F_{i+2}F_iF_{i+1}F_i \arrow[r, "F_iF_{i+1}\tau_{i,i+2}F_{i+1}F_i"] & F_iF_{i+1}F_iF_{i+2}F_{i+1}F_i
\end{tikzcd}$$

We will say that a collection of functors $F_1, \ldots, F_{n-1}$ and natural isomorphisms τ_{ij}, $1 \leq i \leq j \leq n-1$, satisfying the above commutative diagrams defines a *preaction* of $\mathfrak{H}(\Sigma_\mathrm{n})$ on $\mathcal{C}$. There is proved in [17] that the category of actions of $\mathfrak{H}(\Sigma_\mathrm{n})$ on $\mathcal{C}$ is equivalent to the category of preactions of $\mathfrak{H}(\Sigma_\mathrm{n})$ on $\mathcal{C}$.

In the next sections we will construct a preaction of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the category of rational modules for the negative quantum Borel subgroups of the quantum general linear groups. Therefore, we obtain an action of $\mathfrak{H}(\Sigma_\mathrm{n})$ on this category, via the above referred equivalence.

3 Cotensor product and induction

In this section we collect some general definitions and results concerning coalgebras, bialgebras, and Hopf algebras. In our treatment of the cotensor product we follow [15].

We start with some notation. We will denote by $\mathbb{K}$ the ground field. By a coalgebra we will always mean a $\mathbb{K}$-coalgebra and we use Sweedler summation notation for coalgebras and for comodules.

Let C be a $\mathbb{K}$-coalgebra. By a C-comodule we mean a right C-comodule and Comod-C denotes the category of C-comodules. If $M \in$ Comod-C we write ρ_M for the structure map $M \to M \otimes C$ of M. If N is a left C-comodule, we denote the structure map $N \to C \otimes N$ by λ_N.

The cotensor product $M \otimes^C N$ of these right and left comodules is defined as the kernel of the map

$$\rho_M \otimes N - M \otimes \lambda_N \colon M \otimes N \to M \otimes C \otimes N.$$

If M is a C'-C-bicomodule and N is a C-C''-bicomodule, then $M \otimes^C N$ is a C'-C''-bicomodule with coactions given by restricting $\lambda_M \otimes N$ and $M \otimes \rho_N$ to $M \otimes^C N$.

Remark 3.1. If L is a C''-C'''-bicomodule, then $(M \otimes^C N) \otimes^{C''} L$ and $M \otimes^C (N \otimes^{C''} L)$ are isomorphic, and this isomorphism is given by restricting the natural isomorphism $(M\otimes N)\otimes L \to M \otimes (N \otimes L)$, $(m\otimes n)\otimes l \mapsto m \otimes (n \otimes l)$. Moreover, both $(M \otimes^C N) \otimes^{C''} L$ and $M \otimes^C (N \otimes^{C''} L)$ can be identified with the intersection of the kernels of the maps

$$\rho_M \otimes N \otimes L - M \otimes \lambda_N \otimes L \colon M \otimes N \otimes L \to M \otimes C \otimes N \otimes L$$
$$M \otimes \rho_N \otimes L - M \otimes N \otimes \lambda_L \colon M \otimes N \otimes L \to M \otimes N \otimes C'' \otimes L.$$

Suppose $f\colon C \to B$ is a homomorphism of coalgebras. Then we can consider every left (right) C-comodule as a left (right) B-comodule via f. We will denote the resulting left (right) B-comodule by $f_\circ M$, or simply by M if no confusion arises. In particular, we can consider C as a B-C-bicomodule. Thus for every right B-comodule M, we get that $M \otimes^B C$ is a C-comodule. The C-comodule $M \otimes^B C$ is called the induced comodule and will be denoted either by $f^\circ M$ or $\operatorname{Ind}_B^C M$.

If H is a Hopf algebra, then the category Comod-H of (right) comodules over H is endowed with a monoidal structure. Namely, if M and N are H-comodules, the coaction of H on $M \otimes N$ is defined by

$$m \otimes n \mapsto \sum m_{(0)} \otimes n_{(0)} \otimes m_{(1)} n_{(1)}.$$

The trivial H-comodule $\mathbb{K}_{\mathrm{tr}}$ is the one-dimensional comodule with underlying vector space $\mathbb{K}$ and coaction given by $1 \mapsto 1 \otimes 1$. It is clear that $\mathbb{K}_{\mathrm{tr}}$ can be chosen as the identity object for the above tensor product.

We will frequently use the tensor identity:

Theorem 3.2 ([8, Proposition 1.3])**.** *Suppose $f\colon H_1 \to H_2$ is a homomorphism of Hopf algebras and M is an H_1-comodule. Then for every H_2-comodule N there are natural isomorphisms*

$$R^k \operatorname{Ind}_{H_2}^{H_1} (M \otimes N) \cong M \otimes R^k \operatorname{Ind}_{H_2}^{H_1} N, \quad \textit{for all } k \geq 0.$$

The explicit formula for the above isomorphism when $k = 0$ is given in [7]. Namely, we have

$$\begin{aligned} \phi\colon M \otimes \operatorname{Ind}_{H_2}^{H_1} N &\to \operatorname{Ind}_{H_2}^{H_1} (M \otimes N) \\ x \otimes \sum_i y_i \otimes c_i &\mapsto \sum_i x_{(0)} \otimes y_i \otimes x_{(1)} c_i. \end{aligned} \tag{1}$$

4 Quantisation

In this section we study properties of some quantised bialgebras and of subgroups of quantum general linear groups. We start with an overview of notions introduced in [16] and some results proved in [10].

From here on, α and β are non-zero elements of $\mathbb{K}$. We will also denote by $\mathbf{n}$ the set of integers $\{1, \cdots, n\}$.

Let $F(n)$ be the free $\mathbb{K}$-algebra with n^2 generators x_{ij}, for $i, j \in \mathbf{n}$. Denote by $I_{\alpha,\beta}$ the ideal of $F(n)$ generated by elements of the form

$$\begin{array}{ll} x_{is}x_{ir} - \alpha x_{ir}x_{is}, & \text{for } 1 \leq i \leq n \text{ and } 1 \leq r < s \leq n; \\ x_{jr}x_{ir} - \beta x_{ir}x_{jr}, & \text{for } 1 \leq i < j \leq n \text{ and } 1 \leq r \leq n; \\ x_{jr}x_{is} - \alpha^{-1}\beta x_{is}x_{jr}, & \text{for } 1 \leq i < j \leq n \text{ and } 1 \leq r < s < n; \\ x_{js}x_{ir} - x_{ir}x_{js} - (\beta - \alpha^{-1}) x_{is}x_{jr}, & \text{for } 1 \leq i < j \leq n \text{ and } 1 \leq r < s \leq n. \end{array} \tag{2}$$

The algebra $F(n)/I_{\alpha,\beta}$ is denoted by $A_{\alpha,\beta}(n)$ and the canonical image $x_{ij}+I_{\alpha,\beta}$ of x_{ij} in $A_{\alpha,\beta}(n)$ by c_{ij}. In what follows, we will often skip the subscripts α, β. For a matrix $\omega \in M_n(\mathbb{N})$ we write c^ω for the product

$$c_{11}^{\omega_{11}} c_{12}^{\omega_{12}} \dots c_{1n}^{\omega_{1n}} \dots c_{nn}^{\omega_{nn}}$$

and, similarly, x^ω for the product

$$x_{11}^{\omega_{11}} x_{12}^{\omega_{12}} \dots x_{1n}^{\omega_{1n}} \dots x_{nn}^{\omega_{nn}}.$$

On the set $\{\, x_{is} \,|\, i,s \in \mathbf{n}\} \subset F(n)$ we define an ordering by $x_{js} > x_{it}$ if $j > i$, and $x_{is} > x_{ir}$ if $s > r$. We consider the corresponding lexicographical ordering on the set of monomials $\{\, x^\omega \,|\, \omega \in M_n(\mathbb{N})\}$.

The following fact is well-known but we include a short sketch of a proof.

Theorem 4.1. *The set* (2) *is a Gröbner basis of $I_{\alpha,\beta}$ with respect to the above ordering. Moreover, $\{\, c^\omega \,|\, \omega \in M_n(\mathbb{N})\}$ is a basis of $A(n)$.*

Proof. Note that every element in (2) is written so that the leading monomial is a first term. To show that (2) is a Gröbner basis one has to check that all the critical pairs are resolvable.

Suppose $m \geq 3$. Let S' be the set (2) for $n = 3$, and S the set (2) for $n = m$. It is easy to see that every critical pair of S involves at most three row and three column indices. Let us fix two triples of indices $1 \leq i_1 < i_2 < i_3 \leq m$ and $1 \leq r_1 < r_2 < r_3 \leq m$. Then we have a homomorphism of free algebras $\varphi\colon F(3) \to F(m)$ defined by $\varphi(x_{js}) = x_{i_j,r_s}$. Clearly $\varphi(S') \subset S$. Now every critical pair involving row indices i_1, i_2, i_3, and column indices r_1, r_2, r_3 lies in $\phi(S')$. This shows that it is enough to prove that S' is a Gröbner basis. Hence the claim of the theorem has to be verified for $n \leq 3$ only. The case $n = 1$ is trivial, and the cases $n = 2$, $n = 3$ can be checked using a computer algebra system, such as Magma [1].

It is easy to see that the set $\{\, x^\omega \,|\, \omega \in M_n(\mathbb{N})\}$ is the set of non-reducible monomials with respect to the Gröbner basis (2). Thus $\{\, c^\omega \,|\, \omega \in M_n(\mathbb{N})\}$ is a basis of $A(n)$. □

Given a sequence $b = (b_1, \dots, b_n) \in \mathbf{n}^n$, let $I(b) = I_{\alpha,\beta}(b)$ be the ideal of $A(n)$ generated by $\{c_{is} | s > b_i\}$. We define the quotient algebra

$$A(b) = A(n)/I(b)\ ,$$

and denote by $[c_{is}]_b$ the image of c_{is} under the canonical projection from $A(n)$ to $A(b)$.

Theorem 4.2. *On the set $\{\, x^\omega \,|\, \omega \in M_n(\mathbb{N})\} \subset F(n)$ consider the ordering used in Theorem 4.1. Let $b = (b_1, \dots, b_n) \in \mathbf{n}^n$ be a non-decreasing sequence. Denote by S'' the union of the set* (2) *and $S' := \{\, x_{is} \,|\, s > b_i\}$. Then S'' is a Gröbner basis of the ideal generated by S''. In particular,*

$$\{\, [c^\omega]_b \,|\, \omega \in M_n(\mathbb{N}),\, \omega_{is} = 0 \;\; \textit{for } s > b_i\}$$

is a basis of $A(b)$.

Proof. To prove the theorem we have to check that all ambiguities in S'' are resolvable. Let us write S for the set (2). For pairs of elements in S the ambiguities are resolvable, since S is a Gröbner basis, by Theorem 4.1. There are no ambiguities between pairs of elements in S'. Thus we only have to check that all the ambiguities between an element in S and an element in S' are resolvable. The only interesting case is when the element of S is of the form $x_{js}x_{ir} - x_{ir}x_{js} - (\beta - \alpha^{-1})x_{is}x_{jr}$, for $i < j$ and $r < s$, and the element of S' is either x_{js} or x_{ir}. In the first case we get that $s > b_j \geq b_i$ and therefore also $x_{is} \in S'$. In the second case $s > r > b_i$ and again $x_{is} \in S'$. Therefore in both cases the ambiguity is resolvable.

It is now straightforward that $\{ x^\omega \mid \omega \in M_n(\mathbb{N}),\ \omega_{is} = 0 \text{ for } s > b_i\}$ is the set of non-reducible monomials with respect to S''. Thus the set $\{ [c^\omega]_b \mid \omega \in M_n(\mathbb{N}),\ \omega_{is} = 0 \text{ for } s > b_i\}$ is a $\mathbb{K}$-basis of $A(b)$. □

Corollary 4.3. *Suppose $b = (b_1, \dots, b_n) \in \mathbf{n}^n$ is a non-decreasing sequence. Then the algebra $A(b)$ has no zero divisors.*

Proof. Consider the subset S'' of $F(n)$ defined in Theorem 4.2. We have that S'' is a Gröbner basis of the kernel of the canonical projection $F(n) \twoheadrightarrow A(b)$. One can check that the leading term of (the reduced expression for) $[c^\omega]_b[c^\tau]_b$ is $[c^{\omega+\tau}]_b$ multiplied by $\alpha^s\beta^t$, for suitable s, $t \in \mathbb{N}$. Thus given two non-zero elements in $A(b)$ with leading monomials $[c^\omega]_b$ and $[c^\tau]_b$, respectively, we get that their product has leading monomial $[c^{\omega+\tau}]_b$ and so it is non-zero. □

The algebra $A(n)$ has a unique structure of bialgebra with comultiplication $\triangle\colon A(n) \to A(n) \otimes A(n)$ and counit $\varepsilon\colon A(n) \to \mathbb{K}$, satisfying

$$\triangle(c_{ij}) = \sum_{k=1}^{n} c_{ik} \otimes c_{kj}, \quad \varepsilon(c_{ij}) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

The next theorem, proved in [10], allows us to identify the coalgebra $A(n) = A_{\alpha,\beta}(n)$ with the coalgebra $A_{1,\alpha\beta}(n)$ studied in [6, 8].

For a matrix $\omega \in M_n(\mathbb{N})$, we denote by $J(\omega)$ the number

$$\sum_{i<j,s<t} \omega_{it}\omega_{js}.$$

Theorem 4.4 ([10, Proposition 2.1])**.** *Suppose that α, β, α', β' are non-zero elements in $\mathbb{K}$ such that $\alpha'\beta' = \alpha\beta$. Then the map*

$$\begin{aligned} A_{\alpha,\beta}(n) &\to A_{\alpha',\beta'}(n) \\ c^\omega &\mapsto (\alpha/\alpha')^{J(\omega)} c^\omega \end{aligned}$$

is an isomorphism of coalgebras.

Before we proceed, we need to introduce some notation concerning sequences of natural numbers. We denote by v_l the n-tuple $(0, \cdots, 0, 1, 0, \cdots, 0)$ (1 in the lth position). Given a composition $\lambda = (\lambda_1, \dots, \lambda_m)$ of n, we write

$$(\lambda_1^{\lambda_1}, (\lambda_1 + \lambda_2)^{\lambda_2}, \dots, n^{\lambda_m}) \tag{3}$$

for

$$(\lambda_1, \dots, \lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \lambda_2, \dots, n, \dots, n),$$

where $\lambda_1 + \cdots + \lambda_k$ is repeated λ_k times. For $\lambda = (1^n)$, we obtain the sequence $\delta = (1, 2, \dots, n)$. If $\lambda = (1^{l-1}, 2, 1^{n-l-1})$, for some natural number $1 \leq l \leq n-1$, we denote the corresponding sequence by $a[l]$. Thus

$$a[l] = (1, 2, \dots, l-1, l+1, l+1, l+2, \dots, n) = \delta + v_l.$$

Remark 4.5. Suppose that $b = (\lambda_1^{\lambda_1}, (\lambda_1 + \lambda_2)^{\lambda_2}, \dots, n^{\lambda_m})$ for some composition λ of n. By [8, Proposition 2.3]. $I_{1,q}(b)$ is a biideal of $A_{1,q}(n)$, $q \in \mathbb{K}^*$. Combining this fact with Theorem 4.4, we see that $I_{\alpha,\beta}(b)$ is a coideal, for every α, $\beta \in \mathbb{K}^*$. As $I_{\alpha,\beta}(b)$ is an ideal by definition, we get that $A_{\alpha,\beta}(b)$ is a bialgebra. In particular, $A_{\alpha,\beta}[l] := A_{\alpha,\beta}(a[l])$ and $A_{\alpha,\beta}(\delta)$ are bialgebras.

Proposition 4.6. *Suppose that $b = (b_1, \dots, b_n) \in \mathbf{n}^n$ satisfies $b_k \neq l$ for all k. Then $\triangle(I(b)) \subset I(b) \otimes A(n) + A(n) \otimes I(a[l])$. In particular, $A(b)$ is an $A[l]$-comodule with the coaction given by*

$$\begin{aligned} A(b) &\to A(b) \otimes A[l] \\ [x]_b &\mapsto \sum [x_{(1)}]_b \otimes [x_{(2)}]_{a[l]}. \end{aligned}$$

Proof. Let $c_{ij} \in I(b)$. Then $j > b_i$ and

$$\triangle(c_{ij}) = \sum_{k > b_i} c_{ik} \otimes c_{kj} + \sum_{k=1}^{b_i} c_{ik} \otimes c_{kj}.$$

For $k > b_i$, we get that $c_{ik} \in I(b)$, and so the elements of the first sum are in $I(b) \otimes A(n)$. For $k \leq b_i$, we get $j > b_i \geq k$. It is easy to see that $c_{kj} \in I(a[l])$ if and only if $j > k$ and $(k, j) \neq (l, l+1)$. Suppose $(k, j) = (l, l+1)$. Then

$$j = l + 1 > b_i \geq k = l$$

implies $b_i = l$, which contradicts our assumption on b. Therefore $c_{kj} \in I(a[l])$ for all $k \leq b_i$. $\square$

With a proof similar to the above one, we obtain the following result for $A(b)$ and the bialgebra $A(\delta)$.

Proposition 4.7. *For any $b = (b_1, \dots, b_n) \in \mathbf{n}^n$, we have $\triangle(I(b)) \subset I(b) \otimes A(n) + A(n) \otimes I(\delta)$. In particular, $A(b)$ has a structure of $A(\delta)$-comodule with the coaction given by*

$$[x]_b \mapsto \sum [x_{(1)}]_b \otimes [x_{(2)}]_\delta.$$

Note that the $A(\delta)$-coaction on $A(b)$ is multiplicative in the sense of the following proposition.

Proposition 4.8. *Denote by ρ the coaction of $A(\delta)$ both on $A(b)$ and on $A(b) \otimes A(b)$, and by μ the multiplication in the algebras $A(n)$ and $A(b)$. Then the following diagram is commutative*

$$\begin{tikzcd} A(b) \otimes A(b) \arrow[r, "\rho"] \arrow[d, "\mu"'] & A(b) \otimes A(b) \otimes A(\delta) \arrow[d, "\mu \otimes A(\delta)"] \\ A(b) \arrow[r, "\rho"] & A(b) \otimes A(\delta). \end{tikzcd}$$

Proof. Let h denote the following composition of maps

$$\begin{tikzcd} A(n) \otimes A(n) \arrow[r, "\triangle \otimes \triangle"] & A(n) \otimes A(n) \otimes A(n) \otimes A(n) \arrow[dl, "\tau_{23}"] \\ A(n) \otimes A(n) \otimes A(n) \otimes A(n) \arrow[r, "A(n) \otimes A(n) \otimes \mu"] & A(n) \otimes A(n) \otimes A(n), \end{tikzcd}$$

where τ_{23} is the twist map of the second and third factors of $A(n) \otimes A(n) \otimes A(n) \otimes A(n)$. Then we have the diagram

$$
\begin{tikzcd}
A(b)\otimes A(b) \arrow[rrr, "\rho"] \arrow[ddd, "\mu"'] & & & A(b)\otimes A(b)\otimes A(\delta) \arrow[ddd, "\mu\otimes A(\delta)"] \\
& A(n)\otimes A(n) \arrow[ul, two heads] \arrow[r, "h"] \arrow[d, "\mu"'] & A(n)\otimes A(n)\otimes A(n) \arrow[ur, two heads] \arrow[d, "\mu\otimes A(n)"] & \\
& A(n) \arrow[dl] \arrow[r, "\triangle"] & A(n)\otimes A(n) \arrow[dr] & \\
A(b) \arrow[rrr, "\rho"] & & & A(b)\otimes A(\delta).
\end{tikzcd}
$$

The internal square in the above diagram commutes since $A(n)$ is a bialgebra. The trapezoids commute by the definition of the $A(\delta)$-coaction on $A(b)$ and of the multiplication on $A(b)$. Since the upper-left diagonal arrow is surjective, we conclude that the exterior square is also commutative. $\square$

Let $a = (a_1, \cdots, a_n) \in \mathbb{N}^n$. We denote by $\mathbb{K}_a$ the 1-dimensional $A(\delta)$-comodule with underlying vector space $\mathbb{K}$ and structure map

$$
\begin{aligned}
\mathbb{K}_a &\to \mathbb{K}_a \otimes A(\delta) \\
1 &\mapsto 1 \otimes [c_{11}^{a_1} \cdots c_{nn}^{a_n}]_\delta.
\end{aligned}
\tag{4}
$$

Given $b = (b_1, \ldots, b_n) \in \mathbf{n}^n$, consider the map

$$
\begin{aligned}
f\colon A(b) \otimes \mathbb{K}_{v_{b_l}} &\to A(b) \\
x \otimes 1 &\mapsto x[c_{l,b_l}]_b.
\end{aligned}
$$

Proposition 4.9. *If b is non-decreasing, then the map f defined above is an injective homomorphism of $A(\delta)$-comodules.*

Proof. Note that, since there are no zero divisors in $A(b)$, f is injective.

Denote by ρ the coaction of $A(\delta)$ on $A(b)$ and by ρ_{b_l} the coaction of $A(\delta)$ on $A(b) \otimes \mathbb{K}_{v_{b_l}}$. Then, for any $x \in A(b)$, we have

$$\rho_{b_l}(x \otimes 1) = \sum([x_{(1)}]_b \otimes 1) \otimes [x_{(2)}]_\delta [c_{b_l,b_l}]_\delta.$$

Hence

$$f \otimes A(\delta)\,(\rho_{b_l}(x \otimes 1)) = \sum [x_{(1)}]_b [c_{l,b_l}]_b \otimes [x_{(2)}]_\delta [c_{b_l,b_l}]_\delta. \tag{5}$$

Further

$$\rho([c_{l,b_l}]_b) = \sum_{k=1}^{n} [c_{lk}]_b \otimes [c_{k,b_l}]_\delta.$$

Now we have $[c_{k,b_l}]_\delta = 0$ for $k < b_l$, and $[c_{lk}]_b = 0$ for $k > b_l$. Therefore $\rho([c_{l,b_l}]_b) = [c_{l,b_l}]_b \otimes [c_{b_l,b_l}]_\delta$. Using this and Proposition 4.8, we get

$$\rho(x[c_{l,b_l}]_b) = \sum [x_{(1)}]_b [c_{l,b_l}]_b \otimes [x_{(2)}]_\delta [c_{b_l,b_l}]_\delta. \tag{6}$$

Compairing (5) and (6), we see that f is indeed a homomorphism of $A(\delta)$-comodules. $\square$

Proposition 4.10. *Suppose that b and $b - v_l \in \mathbf{n}^n$ are non-decreasing sequences. Then we have the following short exact sequence of $A(\delta)$-comodules*

$$0 \longrightarrow A(b) \otimes \mathbb{K}_{v_{b_l}} \xrightarrow{f} A(b) \xrightarrow{\pi} A(b - v_l) \longrightarrow 0\ . \tag{7}$$

Proof. Clearly $I(b) \subset I(b - v_l)$. So we can consider the canonical projection $\pi\colon A(b) \twoheadrightarrow A(b - v_l)$.

By Proposition 4.2 the sets $\{\, [c^\omega]_b \,|\, \omega_{ij} = 0,\ j > b_i\}$ and

$$\{\, [c^\omega]_{b-v_l} \,|\, \omega_{ij} = 0,\ j > b_i \text{ if } i \neq l, \text{ and } j > b_l - 1, \text{ if } i = l\}$$

are basis of $A(b)$ and $A(b - v_l)$, respectively. Therefore $\{\, [c^\omega]_b \,|\, \omega_{ij} = 0,\ j > b_i;\ \omega_{l,b_l} \neq 0\}$ is a basis of the kernel of π.

Let $\omega \in M_n(\mathbb{N})$ be such that $\omega_{ij} = 0$ for $j > b_i$. Then, in particular, ω_{l,b_l} is the last possible non-zero element in the lth row of ω. Define $\omega' \in M_n(\mathbb{N})$ to be the matrix with the same elements in the first l rows as ω and zeros elsewhere. Denote $\omega - \omega'$ by ω''. Then from the definition of c^ω, we get $[c^\omega]_b = [c^{\omega'}]_b [c^{\omega''}]_b$. Moreover, $[c^{\omega'}]_b [c_{l,b_l}]_b [c^{\omega''}]_b = [c^{\omega + e_{l,b_l}}]_b$, where e_{l,b_l} denotes the matrix with 1 in position (l, b_l) and zeros elsewhere. We claim that $[c^{\omega''}]_b [c_{l,b_l}]_b = \alpha^s \beta^t [c_{l,b_l}]_b [c^{\omega''}]_b$ for suitable integers s and t. In fact $[c^{\omega''}]_b$ is the product of the elements $[c_{ij}]_b$ with $i > l$. If $j < b_l$, we get

$$[c_{ij}]_b [c_{l,b_l}]_b = \alpha^{-1} \beta [c_{l,b_l}]_b [c_{ij}]_b.$$

If $j = b_l$, then

$$[c_{ij}]_b [c_{l,b_l}]_b = [c_{i,b_l}]_b [c_{l,b_l}]_b = \beta [c_{l,b_l}]_b [c_{ij}]_b.$$

If $j > b_l$, then

$$[c_{ij}]_b [c_{l,b_l}]_b = [c_{l,b_l}]_b [c_{ij}]_b + (\beta - \alpha^{-1}) [c_{lj}]_b [c_{i,b_l}]_b.$$

Since $j > b_l$, we get that $[c_{lj}]_b = 0$. Thus

$$[c_{ij}]_b [c_{l,b_l}]_b = [c_{l,b_l}]_b [c_{ij}]_b$$

in this last case. Therefore, we have $[c^\omega]_b [c_{l,b_l}]_b = \alpha^s \beta^t [c^{\omega + e_{l,b_l}}]_b$. This shows that the image of f and the kernel of π coincide. By Proposition 4.9, the map f is injective and so (7) is exact. □

Let $l(\sigma)$ denote the length of the permutation $\sigma \in \Sigma_n$. The quantum determinant is the element of $A(n)$ defined by

$$\begin{aligned} d = d_{\alpha,\beta} &= \sum_{\sigma \in \Sigma_n} (-\alpha)^{-l(\sigma)} c_{1,\sigma(1)} c_{2,\sigma(2)} \cdots c_{n,\sigma(n)} \\ &= \sum_{\sigma \in \Sigma_n} (-\beta)^{-l(\sigma)} c_{\sigma(1),1} c_{\sigma(2),2} \cdots c_{\sigma(n),n}. \end{aligned}$$

The determinant d is a group-like element of $A(n)$, see [10]. For every nondecreasing $b \in \mathbf{n}^n$ such that $b_i \geq i$, we get that $[d]_b$ is a non-zero element of $A(b)$ and so a non-zero divisor, by Corollary 4.3. We also have $[c_{ij}]_b [d]_b = (\alpha^{-1}\beta)^{i-j} [d]_b [c_{ij}]_b$. Hence, we can localize $A(b)$ with respect to d. We will denote the resulting localization by $A(b)_d$.

Remark 4.11. Since d is group-like this localization process preserves the coalgebra and comodule structures. Therefore, $A(n)_d$, $A(\delta)_d$, $A[l]_d$ are bialgebras, $A(b)_d$ is an $A(\delta)_d$-comodule, and for b such that $b_i \neq l$, for all i, $A(b)_d$ is an $A[l]_d$-comodule.

The bialgebra $A(n)_d$ admits a Hopf algebra structure with the antipode given by

$$S(c_{is}) = (-\beta)^{s-i} d^{-1} d_{si},$$

where d_{si} denotes the quantum determinant of the subalgebra of $A(n)$ obtained by deleting all generators c_{sk} and c_{ki} with $1 \leq k \leq n$ (see [10, (1.7)]).

Proposition 4.12. *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a composition of n, and*

$$b = (\lambda_1^{\lambda_1}, (\lambda_1 + \lambda_2)^{\lambda_2}, \ldots, n^{\lambda_m}).$$

Then, the kernel $J(\lambda)$ of the canonical projection $A(n)_d \to A(b)_d$ is a Hopf ideal generated, as an ideal, by $\{ c_{is} \mid s > b_i \}$. Therefore $A(b)_d$ admits a Hopf algebra structure with the antipode given by

$$S([c_{is}]_b) = (-\beta)^{s-i} [d]_b^{-1} [d_{si}]_b.$$

Proof. We know that, in this case, $A(b)_d$ is a bialgebra. It is obvious that the projection $A(n)_d \to A(b)_d$ is a homomorphism of bialgebras, which implies that $J(\lambda)$ is a biideal.

Suppose $d^{-k} y \in J(\lambda)$, for some $y \in A(n)$. Then $[d]_b^{-k}[y]_b = 0$ in $A(b)_d$. By the definition of localization this implies that $[y]_b = 0$, and so $y \in I(b)$. Therefore $y = \sum_{i=1}^{n} \sum_{s>b_i} y_{is} c_{is} y'_{is}$ for some elements y_{is}, $y'_{is} \in A(n)$. Since $d^{-k} y_{is} \in A(n)_d$, we get that the ideal $J(\lambda)$ is generated by the elements c_{is} with $s > b_i$.

As S is an anti-endomorphism of $A(n)$, to show that $S(J(\lambda)) \subset J(\lambda)$ it is enough to check that $S(c_{is}) \in J(\lambda)$, for every pair (i, s) such that $s > b_i$. But as $S(c_{is}) = (-\beta)^{s-i} d^{-1} d_{si}$ it is sufficient to verify that $d_{si} \in I(b)$.

Let us consider the embedding $\varphi\colon A(n-1) \to A(n)$ determined by

$$c_{jt} \mapsto \begin{cases} c_{jt}, & j < s,\ t < i \\ c_{j+1,t} & j \geq s,\ t < i \\ c_{j,t+1}, & j < s,\ t \geq i \\ c_{j+1,t+1}, & j \geq s,\ t \geq i. \end{cases}$$

Then, by the definition of d_{si}, we get that d_{si} is the image of the determinant $d \in A(n-1)$ under φ, see [10]. Now consider the ideal $\varphi^{-1}(I(b))$. Suppose k and $w \in \{1, \ldots, m\}$ are such that

$$\begin{aligned} \lambda_1 + \cdots + \lambda_{k-1} < i \leq \lambda_1 + \cdots + \lambda_k \\ \lambda_1 + \cdots + \lambda_{w-1} < s \leq \lambda_1 + \cdots + \lambda_w. \end{aligned}$$

In other words (i, s) lies in the (k, w) block determined by the composition λ. Note that $s > b_i \geq i$ implies that $w > k$. One can show that $\varphi^{-1}(I(b)) = I(b')$ where

$$b' = (\lambda_1^{\lambda_1}, \ldots, (\lambda_1 + \cdots + \lambda_k - 1)^{\lambda_k}, \ldots, (\lambda_1 + \cdots + \lambda_w - 1)^{\lambda_w - 1}, \ldots, (n-1)^{\lambda_m}).$$

Suppose $d \notin I(b')$. Then there is $\sigma \in \Sigma_{n-1}$ such that $c_{j,\sigma(j)} \notin I(b')$, for all $1 \leq j \leq n-1$. This implies that, for all $1 \leq j \leq \lambda_1$, we must have $1 \leq \sigma(j) \leq \lambda_1$. In other words, σ maps bijectively $[1, \lambda_1] \cap \mathbb{N}$ into itself. Now, for $\lambda_1 + 1 \leq j \leq \lambda_1 + \lambda_2$, we must have $1 \leq \sigma(j) \leq \lambda_1 + \lambda_2$. But since σ maps $[1, \lambda_1] \cap \mathbb{N}$ maps bijectively into itself, this implies that σ also maps $[\lambda_1 + 1, \lambda_1 + \lambda_2] \cap \mathbb{N}$ into itself. Proceeding this way, we get that σ must map $[\lambda_1 + \cdots + \lambda_{k-1} + 1, \lambda_1 + \cdots + \lambda_k] \cap \mathbb{N}$ bijectively into $[\lambda_1 + \cdots + \lambda_{k-1} + 1, \lambda_1 + \cdots + \lambda_k - 1] \cap \mathbb{N}$, which is impossible. Therefore, we get that $d \in I(b')$ and thus $d_{si} = \varphi(d) \in I(b)$. $\square$

The Hopf algebra $A(n)_d = (A(n)_{\alpha,\beta})_d$ is the coordinate algebra of the quantum general linear group $\mathrm{GL}_{\alpha,\beta}(n,\mathbb{K})$, defined by Takeuchi in [16] and also studied in [10]. The quantum groups $\mathrm{GL}_{1,\beta}(n,\mathbb{K})$ and $\mathrm{GL}_{\beta,\beta}(n,\mathbb{K})$ are, respectively, the quantum general linear groups studied by Dipper and Donkin in [6] and Parshall and Wang in [13].

If $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a composition of n, and $b = (\lambda_1^{\lambda_1}, (\lambda_1+\lambda_2)^{\lambda_2}, \ldots, n^{\lambda_m})$, then the Hopf algebra $A(b)_d$ can be considered as the coordinate algebra of a *quantum parabolic subgroup* of $\mathrm{GL}_{\alpha,\beta}(n,\mathbb{K})$. Taking $b = \delta$, we obtain the coordinate algebra $A(\delta)_d$ of the *quantum negative Borel subgroup* of $\mathrm{GL}_{\alpha,\beta}(n,\mathbb{K})$.

Quantum parabolic and Borel subgroups were extensively studied by Donkin in [8] (see also [9]), for the case $\alpha = 1$.

Consider $a = (a_1, \cdots, a_n) \in \mathbf{n}^n$. Then we also denote by $\mathbb{K}_a$ the 1-dimensional $A(\delta)_d$-comodule which is the restriction of the $A(\delta)$-comodule $\mathbb{K}_a$ defined in (4).

Given $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_n) \in \mathbf{n}^n$, we write $b \geq a$ if $b_i \geq a_i$, for $i \in \mathbf{n}$. Then we have the following extension of Proposition 4.10

Proposition 4.13. *Suppose $b \in \mathbf{n}^n$ is such that $b \geq a[l]$, and b, $b - v_l$ are non-decreasing sequences. Then we have an exact sequence of $A(\delta)_d$-comodules*

$$0 \longrightarrow A(b)_d \otimes \mathbb{K}_{v_{b_l}} \xrightarrow{f} A(b)_d \xrightarrow{\pi} A(b-v_l)_d \longrightarrow 0\ , \tag{8}$$

where f is the comodule homomorphism defined by $z \otimes 1 \mapsto z[c_{l,b_l}]_b$, for any $z \in A(b)_d$, and π is the canonical projection.

Proof. It is obvious that π is surjective. Let $[d]_b^{-k} x \in \ker(\pi)$, with $x \in A(b)$ and $k \in \mathbb{N}$. Then $[d]_{b-v_l}^{-k} \pi(x) = 0$ in $A(b-v_l)_d$. Since $[d]_{b-v_l}$ is not a zero divisor in $A(b-v_l)$, we get that $\pi(x) = 0$ in $A(b-v_l)$. This shows that x is in the kernel of the projection $A(b) \to A(b-v_l)$. Since (7) is exact, we get that there is $y \otimes 1 \in A(b) \otimes \mathbb{K}_{v_l}$ such that $f(y \otimes 1) = x$. Therefore $f([d]_b^{-k} y \otimes 1) = [d]_b^{-k} x$. This shows that (8) is exact at the second term.

Now, suppose $[d]_b^{-k} y \otimes 1 \in \ker(f)$, with $y \in A(b)$. Then $[d]_b^{-k} y c_{l,v_l} = 0$ in $A(b)_d$. Thus $y c_{l,v_l} = 0$ in $A(b)$. Since c_{l,v_l} is not a zero-divisor in $A(b)$, we get $y = 0$ in $A(b)$. Therefore f is injective. □

5 The construction of the preaction

Our next step will be to define a preaction of $\mathfrak{H}(\Sigma_\mathrm{n})$ on the category Comod-$A(\delta)_d$.

For any $1 \leq i \leq n-1$, let $\pi_i \colon A[i]_d \to A(\delta)_d$ be the canonical projection. We denote the corresponding induction functor $\mathrm{Ind}_{A(\delta)_d}^{A[i]_d}$ by π_i°. For any $A[i]_d$-comodule M, we also write M for the restricted $A(\delta)_d$-comodule $\pi_{i_\circ}(M)$.

Define F_i as the functor π_i° followed by the restriction to $A(\delta)_d$, i.e,

$$F_i = \pi_{i_\circ}\pi_i^\circ \colon\ \text{Comod-}A(\delta)_d \ \to\ \text{Comod-}A(\delta)_d.$$

Thus every F_i is an endofunctor of Comod-$A(\delta)_d$. Next we will define natural isomorphisms τ_{ij}, $1 \leq i \leq j \leq n-1$, and, in Section 6, we will prove that they satisfy all the necessary commutation relations to define a preaction of $\mathfrak{H}(\Sigma_\mathrm{n})$ on Comod-$A(\delta)_d$.

To proceed we will need the following proposition describing the behaviour of the $A(\delta)_d$-comodules $\mathbb{K}_{\mathrm{tr}} = \mathbb{K}_0$ and of $\mathbb{K}_{v_{i+1}}$ under π_i°.

Theorem 5.1. *Suppose* $1 \le i \le n-1$ *then*

$$R^k\pi_i^\circ \mathbb{K}_0 \cong \begin{cases} \mathbb{K}_{\mathrm{tr}}, & k=0 \\ 0, & k \neq 0 \end{cases} \tag{9}$$

and

$$R^k\pi_i^\circ \mathbb{K}_{v_{i+1}} = 0,\ k \ge 0.$$

Proof. First we reduce the claim of the theorem to the case $(\alpha,\beta)=(1,q)$. Then we will apply results of [8].

Let $q := \alpha\beta$. Consider the isomorphism of coalgebras $\varphi\colon A(n) \to A_{1,q}(n)$, $\varphi(c^\omega) = \alpha^{J(\omega)}c^\omega$, defined in Theorem 4.4 By Lemma 2.3 in [10], $\varphi(d_{\alpha,\beta}) = d_{1,q}$. Therefore, φ can be extended to a map $\varphi'\colon A(n)_d \to A_{1,q}(n)_d$ by $\varphi'(d^k_{\alpha,\beta}x) = d^k_{1,q}\varphi(x)$. It is shown in Theorem 2.4 of [10], that φ' is an isomorphism of coalgebras.

The isomorphism φ' induces an isomorphism of coalgebras $[\varphi']_\delta\colon A_{\alpha,\beta}(\delta)_d \to A_{1,q}(\delta)_d$, $[\varphi']_\delta([x]_\delta) = [\varphi'(x)]_\delta$ and $[\varphi']_i\colon A_{\alpha,\beta}[i]_d \to A_{1,q}[i]_d$, $[\varphi']_i([x]_{a[i]}) = [\varphi'(x)]_{a[i]}$.

Therefore we get the following commutative diagram of coalgebras

$$\begin{tikzcd} A_{\alpha,\beta}[i]_d \arrow[r, "{[\varphi']_i}", "\cong"'] \arrow[d, "\pi_i"'] & A_{1,q}[i]_d \arrow[d, "\pi_i"'] \\ A_{\alpha,\beta}(\delta)_d \arrow[r, "{[\varphi']_\delta}", "\cong"'] & A_{1,q}(\delta)_d. \end{tikzcd}$$

From this diagram it follows that we have to prove the theorem only for the case $(\alpha,\beta)=(1,q)$, since the induction of comodules involves only the coalgebra and the comodule structures. The case $(\alpha,\beta)=(1,q)$ was thoroughly studied in [8], and both claims of the theorem now follow from Lemma 3.1 and Lemma 2.12 therein. $\square$

Corollary 5.2. *The map* $\eta\colon \mathbb{K}_{\mathrm{tr}} \to \pi_i^\circ\mathbb{K}_0$, $1 \mapsto 1\otimes 1$, *is an isomorphism of* $A[i]_d$*-comodules.*

Proof. Consider the injective map $\eta'\colon \mathbb{K}_{\mathrm{tr}} \to \mathbb{K}_0 \otimes A[i]_d$ defined by $1 \mapsto 1\otimes 1$. It is easy to see that the image of η' lies in $\mathbb{K}_0 \otimes^{A(\delta)_d} A[i]_d = \pi_i^\circ\mathbb{K}_0$. Therefore we have the monomorphism of $A[i]_d$-comodules $\eta\colon \mathbb{K}_{\mathrm{tr}} \to \pi_i^\circ\mathbb{K}_0$. Since $\dim \pi_i^\circ\mathbb{K}_0$ is 1, by Therorem 5.1, we get that η is an isomorphism. $\square$

Notice that for any Hopf algebra H and any H-comodule N, $n \mapsto n\otimes 1$ defines an isomorphism between N and $N\otimes\mathbb{K}_{\mathrm{tr}}$. Now let N be an $A[i]_d$-comodule. We consider the following chain of isomorphisms of $A[i]_d$-comodules

$$N \overset{\cong}{\to} N\otimes\mathbb{K}_{\mathrm{tr}} \xrightarrow{1\otimes\eta} N\otimes\pi_i^\circ\mathbb{K}_0 \overset{\phi}{\to} \pi_i^\circ(N\otimes\mathbb{K}_0) \overset{\cong}{\to} \pi_i^\circ N,$$

where ϕ is the isomorphism (1) and η is defined in Corollary 5.2. Under this isomorphism we have for every $z \in N$

$$z \mapsto z\otimes 1 \mapsto z\otimes 1\otimes 1 \mapsto \sum z_{(0)}\otimes 1\otimes z_{(1)} \mapsto \sum z_{(0)}\otimes z_{(1)}.$$

Hence $\rho_N\colon N \to \pi_i^\circ N$, $z \mapsto \sum z_{(0)}\otimes z_{(1)}$ gives an isomorphism of $A[i]_d$-comodules.

We are now ready to define the natural isomorphism

$$\tau_{ii}\colon F_i^2 \to F_i,\ \text{ all } 1 \le i \le n-1.$$

Let $M \in \text{Comod-}A(\delta)_d$. Then $\pi_i^\circ M$ is an $A[i]_d$-comodule with the comodule structure given by

$$\sum_j z_j \otimes [x_j]_{a[i]} \mapsto \sum z_j \otimes [x_{j,(1)}]_{a[i]} \otimes [x_{j,(2)}]_{a[i]},$$

where $z_j \in M$, and $x_j \in A(n)_d$. Therefore, we get for every $M \in \text{Comod-}A(\delta)_d$ the isomorphism

$$\begin{aligned} \rho_{\pi_i^\circ M} \colon \pi_i^\circ M &\to \pi_i^\circ F_i M \\ \sum_j z_j \otimes [x_j]_{a[i]} &\mapsto \sum z_j \otimes [x_{j,(1)}]_{a[i]} \otimes [x_{j,(2)}]_{a[i]}. \end{aligned} \tag{10}$$

By restricting, we can consider $\rho_{\pi_i^\circ M}$ as a homomorphism of $A(\delta)_d$-comodules. From the explicit expression of $\rho_{\pi_i^\circ M}$ it is obvious that the class of isomorphisms $(\rho_{\pi_i^\circ M})$ is a natural transformation of functors $F_i \to F_i^2$. We define the natural isomorphism $\tau_{ii} \colon F_i^2 \to F_i$ as the inverse of $(\rho_{\pi_i^\circ M})$.

Before defining the isomorphisms τ_{ij} for $i < j$, we need to prove the following theorem.

Theorem 5.3. *Suppose $b \in \mathbf{n}^n$ satisfies $b \geq \delta$, the sequences b and $b + v_l$ are non-decreasing, and $b_{l-1} < b_l$. Then the map*

$$\begin{aligned} A(b+v_l)_d &\to \pi_{b_l}^\circ A(b)_d \\ [x]_{b+v_l} &\mapsto \sum [x_{(1)}]_b \otimes [x_{(2)}]_{a[b_l]} \end{aligned} \tag{11}$$

is a well defined isomorphism of $A[b_l]_d$-comodules, and therefore an isomorphism of $A(\delta)_d$-comodules.

Proof. By Proposition 4.13, we have an exact sequence of $A(\delta)_d$-comodules

$$0 \to A(b+v_l)_d \otimes \mathbb{K}_{v_{b_l+1}} \to A(b+v_l)_d \to A(b)_d \to 0.$$

Applying $\pi_{b_l}^\circ$, we get the exact sequence

$$\pi_{b_l}^\circ(A(b+v_l)_d \otimes \mathbb{K}_{v_{b_l+1}}) \to \pi_{b_l}^\circ A(b+v_l)_d \to \pi_{b_l}^\circ A(b)_d \to R^1\pi_{b_l}^\circ(A(b+v_l)_d \otimes \mathbb{K}_{v_{b_l+1}})$$

with the middle arrow given by

$$\begin{aligned} \pi_{b_l}^\circ A(b+v_l)_d &\to \pi_{b_l}^\circ A(b)_d \\ \sum_k [x_k]_{b+v_l} \otimes [y_k]_{a[b_l]} &\mapsto \sum_k [x_k]_b \otimes [y_k]_{a[b_l]}. \end{aligned} \tag{12}$$

Note that since $b_{l-1} < b_l$ and $b + v_l$ is non-decreasing, the vector $b + v_l$ does not have any component equal to b_l. Therefore, by Remark 4.11, $A(b + v_l)_d$ is an $A[b_l]_d$-comodule. Hence, by the tensor identity (see Theorem 3.2), we have

$$R^i\pi_{b_l}^\circ(A(b+v_l)_d \otimes \mathbb{K}_{v_{b_l+1}}) \cong A(b+v_l)_d \otimes R^i\pi_{b_l}^\circ \mathbb{K}_{v_{b_l+1}}.$$

But, by Theorem 5.1,

$$R^i\pi_{b_l}^\circ \mathbb{K}_{v_{b_l+1}} = 0,\ i \geq 0.$$

Therefore, (12) is an isomorphism. Now, using (1) for $H_1 = A[b_l]_d$, $H_2 = A(\delta)_d$, $M = A(b+v_l)_d$, and $N = \mathbb{K}_0$, we get the isomorphism

$$\begin{aligned} A(b+v_l)_d \otimes \pi_{b_l}^\circ \mathbb{K}_0 &\to \pi_{b_l}^\circ A(b+v_l)_d \\ [x]_{b+v_l} \otimes 1 \otimes [y]_{a[b_l]} &\mapsto \sum [x_{(1)}]_{b+v_l} \otimes [x_{(2)}]_{a[b_l]}[y]_{a[b_l]}. \end{aligned} \tag{13}$$

Recall that, in Corollary 5.2, we defined the isomorphism of $A[b_l]_d$-comodules $\eta\colon \mathbb{K}_{\mathrm{tr}} \to \pi^\circ_{b_l}\mathbb{K}_0$.

Composing $A(b+v_l)_d \otimes \eta$ with (13) and (12), we get the isomorphism

$$\begin{aligned} A(b+v_l)_d \otimes \mathbb{K}_0 &\to \pi^\circ_{b_l} A(b)_d \\ [x]_{b+v_l} \otimes 1 &\mapsto \sum [x_{(1)}]_b \otimes [x_{(2)}]_{a[b_l]}. \end{aligned} \tag{14}$$

Composing this with the natural isomorphism $A(b+v_l)_d \to A(b+v_l)_d \otimes \mathbb{K}_0$, we see that (11) is indeed a well-defined isomorphism of $A[b_l]_d$-comodules. □

To define the isomorphisms τ_{ij}, $i+2 \leq j$, we proceed as follows. Applying Theorem 5.3 with $l = j$ and $l = i$, respectively, we get the isomorphisms

$$\begin{aligned} A(\delta + v_i + v_j)_d &\to \pi^\circ_j A(\delta + v_i)_d \\ A(\delta + v_i + v_j)_d &\to \pi^\circ_i A(\delta + v_j)_d. \end{aligned} \tag{15}$$

As $\delta + v_i = a[i]$ and $\delta + v_j = a[j]$, composing the inverse of the first of these isomorphisms with the second one, we obtain the isomorphism

$$t_{ij}\colon \pi^\circ_j A[i]_d \to \pi^\circ_i A[j]_d.$$

For $i+2 \leq j$, we define $\tau_{ij}\colon F_jF_i \to F_iF_j$, by $(\tau_{ij})_M = M \otimes^{A(\delta)_d} t_{ij}$. Clearly, the family (τ_{ij}) is a natural transformation of functors.

Finally, we will define now the natural transformations $\tau_{i,i+1}$. Applying Theorem 5.3 with $l = i$, we get the isomorphisms

$$\begin{aligned} A(\delta + 2v_i + v_{i+1})_d &\to \pi^\circ_{i+1} A(\delta + v_i + v_{i+1})_d \\ A(\delta + v_i + v_{i+1})_d &\to \pi^\circ_i A(\delta + v_{i+1})_d. \end{aligned} \tag{16}$$

Therefore, we have the isomorphism of $A(\delta)_d$-comodules

$$A(\delta + 2v_i + v_{i+1})_d \to A[i+1]_d \otimes^{A(\delta)_d} A[i]_d \otimes^{A(\delta)_d} A[i+1]_d. \tag{17}$$

Since comultiplication is coassoative on $A(n)_d$, the explicit formula for (17) is given by

$$[x]_{\delta+2v_i+v_{i+1}} \mapsto \sum [x_{(1)}]_{a[i+1]} \otimes [x_{(2)}]_{a[i]} \otimes [x_{(3)}]_{a[i+1]}.$$

Proposition 5.4. *The map*

$$\begin{aligned} \rho_{\alpha,\beta}\colon A(\delta + 2v_i + v_{i+1})_d &\to A[i]_d \otimes^{A(\delta)_d} A[i+1]_d \otimes^{A(\delta)_d} A[i]_d \\ [x]_{\delta+2v_i+v_{i+1}} &\mapsto \sum [x_{(1)}]_{a[i]} \otimes [x_{(2)}]_{a[i+1]} \otimes [x_{(3)}]_{a[i]} \end{aligned} \tag{18}$$

is a well defined isomorphism of $A(\delta)_d$-comodules.

Proof. The idea of the proof is to exhibit an isomorphism that identifies (18) with (17).

Without loss of generality we can assume that $q := (\alpha\beta)^{\frac{1}{2}} \in \mathbb{K}$. In fact, if (18) is not an isomorphism, then it will not be an isomorphism upon field extension either.

By Theorem 4.4 the map $\varphi\colon A_{\alpha,\beta}(n) \to A_{q,q}(n)$, defined by $\varphi(c^\omega) = (\alpha\beta^{-1})^{\frac{1}{2}J(\omega)}c^\omega$, is an isomorphism of coalgebras. Using Proposition 4.2, we see that φ induces an isomorphism of vector spaces $\varphi_a\colon A_{\alpha,\beta}(a) \to A_{q,q}(a)$ for every non-decreasing sequence $a \in \mathbf{n}^n$. If a is of the form $(\lambda_1^{\lambda_1}, (\lambda_1+\lambda_2)^{\lambda_2}, \ldots, n^{\lambda_r})$, then $A_{\alpha,\beta}(a)$ and $A_{q,q}(a)$ are coalgebras and we see that φ_a is an isomorphism of coalgebras. This is the case of the sequences δ, $a[i]$, $a[i+1]$ and $b = \delta + 2v_i + v_{i+1}$. So we get the following commutative diagram

$$\begin{tikzcd}
& A_{\alpha,\beta}(b) \arrow[rrr, "\varphi_b"] \arrow[ddl] \arrow[ddd] \arrow[dr] & & & A_{q,q}(b) \arrow[ddl] \arrow[ddd] \arrow[dr] & \\
& & A_{\alpha,\beta}[i] \arrow[rrr, "\varphi_{a[i]}"] \arrow[ddl] & & & A_{q,q}[i] \arrow[ddl] \\
A_{\alpha,\beta}[i+1] \arrow[rrr, "\varphi_{a[i+1]}"'] \arrow[dr] & & & A_{q,q}[i+1] \arrow[dr] & & \\
& A_{\alpha,\beta}(\delta) \arrow[rrr, "\varphi_\delta"] & & & A_{q,q}(\delta) &
\end{tikzcd}$$

where all the maps are homomorphisms of coalgebras and the horizontal arrows are isomorphisms.

From [10, Lemma 2.3], we get that $\varphi_a[d_{\alpha,\beta}]_a = [d_{q,q}]_a$. Thus the above diagram remains commutative upon localization. This shows that we have the following commutative diagram, whose vertical arrows are isomorphisms

$$\begin{tikzcd}
A_{\alpha,\beta}(b)_d \arrow[r, "\rho_{\alpha,\beta}"] \arrow[d, "\varphi_b"'] & A_{\alpha,\beta}[i]_d \otimes^{A_{\alpha,\beta}(\delta)_d} A_{\alpha,\beta}[i+1]_d \otimes^{A_{\alpha,\beta}(\delta)_d} A_{\alpha,\beta}[i]_d \arrow[d, "\varphi_{a[i]} \otimes \varphi_{a[i+1]} \otimes \varphi_{a[i]}"] \\
A_{q,q}(b)_d \arrow[r, "\rho_{q,q}"] & A_{q,q}[i]_d \otimes^{A_{q,q}(\delta)_d} A_{q,q}[i+1]_d \otimes^{A_{q,q}(\delta)_d} A_{q,q}[i]_d .
\end{tikzcd}$$

Therefore it is enough to prove the proposition in the case $(\alpha, \beta) = (q, q)$. In this case we can use the results of Parshall and Wang in [13]. Proposition 3.7.1(3) of that work says that the map h sending c_{is} to $c_{n+1-s,n+1-i}$ extends to an anti-automorphsims of $A_{q,q}(n)$ considered both as a coalgebra and an algebra. It is not difficult to check that

$$\begin{aligned}
&h(I(\delta)) = I(\delta) \\
&h(I(a[i])) = I(a[n-i]) \\
&h(I(a[i+1])) = I(a[n-i-1]) \\
&h(I(b)) = I(\delta + 2v_{n-i-1} + v_{n-i}).
\end{aligned}$$

Thus if $b' = \delta + 2v_{n-i-1} + v_{n-i}$, we get the commutative diagram

$$\begin{tikzcd}
& A_{q,q}(b) \arrow[rrr, "h_b"] \arrow[ddl] \arrow[ddd] \arrow[dr] & & & A_{q,q}(b')^{op} \arrow[ddl] \arrow[ddd] \arrow[dr] & \\
& & A_{q,q}[i] \arrow[rrr, "h_{a[i]}"] \arrow[ddl] & & & A_{q,q}[n-i]^{op} \arrow[ddl] \\
A_{q,q}[i+1] \arrow[rrr, "h_{a[i+1]}"'] \arrow[dr] & & & A_{q,q}[n-i-1]^{op} \arrow[dr] & & \\
& A_{q,q}(\delta) \arrow[rrr, "h_\delta"] & & & A_{q,q}(\delta)^{op} &
\end{tikzcd}$$

where all the horizontal arrows are isomorphisms of coalgebras, and all slanted arrows are natural projections preserving comultiplication. It is shown in [13, Lemma 4.2.3], that $h(d_{q,q}) = d_{q,q}$. Therefore, we have a similar diagram with all the bialgebras replaced by their localizations with respect to $[d]_a$, for a suitable a. We get then the commutative diagram

$$\begin{array}{ccc} A_{q,q}(b)_d & \xrightarrow{\rho_{q,q}} & A_{q,q}[i]_d \otimes^{A_{q,q}(\delta)_d} A_{q,q}[i+1]_d \otimes^{A_{q,q}(\delta)_d} A_{q,q}[i]_d \\ \downarrow{\scriptstyle h_b} & & \downarrow{\scriptstyle h_{a[i]}\otimes h_{a[i+1]}\otimes h_{a[i]}} \\ A_{q,q}(b')^{op}_d & \xrightarrow{\rho'_{q,q}} & A_{q,q}[n-i]^{op}_d \otimes^{A_{q,q}(\delta)^{op}_d} A_{q,q}[n-i-1]^{op}_d \otimes^{A_{q,q}(\delta)^{op}_d} A_{q,q}[n-i]^{op}_d. \end{array}$$

whose vertical arrows are isomorphisms and the map $\rho'_{q,q}$ is given by

$$\rho'_{q,q}\colon [x]_{b'} \mapsto \sum [x_{(3)}]_{a[n-i]} \otimes [x_{(2)}]_{a[n-i-1]} \otimes [x_{(1)}]_{a[n-i]}.$$

Thus it is enough to prove that $\rho'_{q,q}$ is an isomorphism. It follows, from Remark 3.1, that the linear isomorphism $A_{q,q}[n-i]^{op}_d \otimes A_{q,q}[n-i-1]^{op}_d \otimes A_{q,q}[n-i]^{op}_d \to A_{q,q}[n-i]_d \otimes A_{q,q}[n-i-1]_d \otimes A_{q,q}[n-i]_d$ given by

$$a_1 \otimes a_2 \otimes a_3 \mapsto a_3 \otimes a_2 \otimes a_1$$

induces a linear isomorphism ν between $A_{q,q}[n-i]^{op}_d \otimes^{A_{q,q}(\delta)^{op}_d} A_{q,q}[n-i-1]^{op}_d \otimes^{A_{q,q}(\delta)^{op}_d} A_{q,q}[n-i]^{op}_d$ and $A_{q,q}[n-i]_d \otimes^{A_{q,q}(\delta)_d} A_{q,q}[n-i-1]_d \otimes^{A_{q,q}(\delta)_d} A_{q,q}[n-i]_d$

Therefore we get the commutative diagram

$$\begin{array}{ccc} A_{q,q}(b')^{op}_d & \xrightarrow{\rho'_{q,q}} & A_{q,q}[n-i]^{op}_d \otimes^{A_{q,q}(\delta)^{op}_d} A_{q,q}[n-i-1]^{op}_d \otimes^{A_{q,q}(\delta)^{op}_d} A_{q,q}[n-i]^{op}_d \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \nu} \\ A_{q,q}(b')_d & \xrightarrow{\rho''_{q,q}} & A_{q,q}[n-i]_d \otimes^{A_{q,q}(\delta)_d} A_{q,q}[n-i-1]_d \otimes^{A_{q,q}(\delta)_d} A_{q,q}[n-i]_d, \end{array}$$

where $\rho''_{q,q}$ is the isomorphism (17). This shows that $\rho'_{q,q}$ is an isomorphism, and the result follows.

□

We define the map $t_{i,i+1}$ as the composition of the inverse of (17) followed by (18). Therefore

$$t_{i,i+1}\colon A[i+1]_d \otimes^{A(\delta)_d} A[i]_d \otimes^{A(\delta)_d} A[i+1]_d \to A[i]_d \otimes^{A(\delta)_d} A[i+1]_d \otimes^{A(\delta)_d} A[i]_d.$$

Now the natural transformations $\tau_{i,i+1}$ are defined by $(\tau_{i,i+1})_M = M \otimes^{A(\delta)_d} t_{i,i+1}$, i.e.,

$$\begin{aligned} F_{i+1}F_iF_{i+1}M &\to F_iF_{i+1}F_iM \\ \sum_k m_k \otimes w_k &\mapsto \sum_k m_k \otimes t_{i,i+1}(w_k), \end{aligned}$$

all $m_k \in M,\ w_k \in A[i+1]_d \otimes^{A(\delta)_d} A[i]_d \otimes^{A(\delta)_d} A[i+1]_d$.

6 The commutativity of the preaction diagrams

We will show now that the natural isomorphisms τ_{ij}, defined in the previous section, satisfy all the necessary relations so that $(F_i, 1 \le i \le n-1; \tau_{ij}, 1 \le i \le j \le n-1)$ is a preaction (in the sense of Section 2) of $\mathfrak{H}(\Sigma_n)$ on the category Comod-$A(\delta)_d$.

We start by describing the notation used in the diagrams below. First of all, note that if M is an $A(\delta)_d$-comodule, then

$$F_{i_k} \dots F_{i_1} M = M \otimes^{A(\delta)_d} A[i_1]_d \otimes^{A(\delta)_d} \dots \otimes^{A(\delta)_d} A[i_k]_d.$$

Suppose that $\lambda = (\lambda_1, \cdots, \lambda_m)$ is a composition of n, and $b = (\lambda_1^{\lambda_1}, (\lambda_1+\lambda_2)^{\lambda_2}, \dots, n^{\lambda_m})$. Then, using the coassosiativity of the comultiplication on $A(n)_d$, we get the map

$$\begin{aligned} A(b)_d &\to \underbrace{A(b)_d \otimes^{A(\delta)_d} \dots \otimes^{A(\delta)_d} A(b)_d}_{k \text{ times}} \\ [x]_b &\mapsto \sum [x_{(1)}]_b \otimes \dots \otimes [x_{(k)}]_b. \end{aligned} \tag{19}$$

Suppose now that $b^{(1)}, \dots, b^{(k)} \in \mathbf{n}^n$ satisfy $\delta \le b^{(i)} \le b$, for $1 \le i \le k$. Composing (19) with the canonical projections $A(b)_d \to A(b^{(i)})_d$, we get the map

$$\begin{aligned} \rho_{b;b^{(1)},\dots,b^{(k)}} \colon A(b)_d &\to A(b^{(1)})_d \otimes^{A(\delta)_d} \dots \otimes^{A(\delta)_d} A(b^{(k)})_d \\ [x]_b &\mapsto \sum [x_{(1)}]_{b^{(1)}} \otimes \dots \otimes [x_{(k)}]_{b^{(k)}}. \end{aligned}$$

Remark 6.1. In the case $k=2$, $b = b^{(1)} = b^{(2)} = a[i]$, we recover $(\tau_{ii}^{-1})_M = M \otimes^{A(\delta)_d} \rho_{b;b^{(1)},b^{(2)}}$. For $k=2$, $b = \delta + v_i + v_j$, $b^{(1)} = a[i]$ ($b^{(1)} = a[j]$), and $b^{(2)} = a[j]$ ($b^{(2)} = a[i]$), we get isomorphisms, since (15) are isomorphisms. For $k=3$, $b = \delta + 2v_i + v_{i+1}$, $b^{(1)} = b^{(3)} = v_i$, $b^{(2)} = v_{i+1}$, we get that $\rho_{b;b^{(1)},b^{(2)},b^{(3)}}$ is an isomorphism by Proposition 5.4. For $k=3$, $b = \delta + 2v_i + v_{i+1}$, $b^{(1)} = b^{(3)} = v_{i+1}$, $b^{(2)} = v_i$, we get that $\rho_{b;b^{(1)},b^{(2)},b^{(3)}}$ is an isomorphism, since (17) is an isomorphism.

In the diagrams below we will skip M and write:

i) $i_1^{\alpha_1} \dots i_l^{\alpha_l}$ for $A(\delta + \sum_{k=1}^{l} \alpha_k v_{i_k})_d$, where $1 \le i_1 < \dots < i_l \le n-1$, and $1 \le \alpha_k \le n - i_k$;

ii) dot "." for $\otimes^{A(\delta)_d}$;

iii) ρ_k for $\rho_{b;b^{(1)},\dots,b^{(k)}}$, and ρ for ρ_2.

For example,

$$(i+1).i.(i+1) \xleftarrow{\rho_3} i^2(i+1) \xrightarrow{\rho_3} i.(i+1).i\,, \qquad i.j \xleftarrow{\rho} ij \xrightarrow{\rho} j.i\,,$$

$$(i+1).i.(i+1).i \xleftarrow{\rho_3.i} i^2(i+1).i \xrightarrow{\rho_3.i} i.(i+1).i.i\,, \qquad i.i.j \xleftarrow{i.\rho} i.ij \xrightarrow{i.\rho} i.j.i\,,$$

denote, respectively, $\tau_{i,i+1}$, $\tau_{i,j}$, $F_i\tau_{i,i+1}$ and $\tau_{i,j}F_i$.

Note that all the diagrams below are commutative, since comultiplication in $A(n)$ is coassociative. Moreover, the maps at the boundaries are isomorphisms by Remark 6.1.

We have to check that two paths going from the upper-left vertex to the down-right vertex produce equal maps. For this it is enough to check that all the maps which are not at the boundary are also isomorphisms.

In the diagram

$$\begin{tikzcd} i.i.i & i.i \arrow[l, "i.\rho"'] \\ i.i \arrow[u, "\rho.i"] & i \arrow[l, "\rho"] \arrow[u, "\rho"'] \end{tikzcd}$$

there is nothing to check since there are no arrows except the boundary ones.

In the diagrams

$$\begin{tikzcd}
(i+1).i.(i+1).(i+1) & (i+1).i.(i+1) \arrow[l, "(i+1).i.\rho"'] \\
i^2(i+1).(i+1) \arrow[u, "\rho_3.(i+1)"'] \arrow[d, "\rho_3.(i+1)"'] & \\
i.(i+1).i.(i+1) & i^2(i+1) \arrow[uu, "\rho_3"'] \arrow[dd, "\rho_3"] \arrow[ul, "\rho"'] \arrow[dl, "\rho"] \\
i.i^2(i+1) \arrow[u, "i.\rho_3"] \arrow[d, "i.\rho_3"] & \\
i.i.(i+1).i & i.(i+1).i \arrow[l, "\rho.(i+1).i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(i+1).(i+1).i.(i+1) & (i+1).i.(i+1) \arrow[l, "\rho.i.(i+1)"'] \\
(i+1).i^2(i+1) \arrow[u, "(i+1).\rho_3"] \arrow[d, "(i+1).\rho_3"] & \\
(i+1).i.(i+1).i & i^2(i+1) \arrow[uu, "\rho_3"'] \arrow[dd, "\rho_3"] \arrow[ul, "\rho"'] \arrow[dl, "\rho"] \\
i^2(i+1).i \arrow[u, "\rho_3.i"] \arrow[d, "\rho_3.i"] & \\
i.(i+1).i.i & i.(i+1).i \arrow[l, "i.(i+1).\rho"']
\end{tikzcd}$$

the invertibility of non-boundary maps follows from the commutativity of the upper and lower trapezoids.

In the diagrams

$$\begin{tikzcd}
i.i.j & & & i.j \arrow[lll, "\rho.j"'] \\
i.ij \arrow[u, "i.\rho"] \arrow[d, "i.\rho"] & & & ij \arrow[lll, "\rho"'] \arrow[u, "\rho"'] \arrow[d, "\rho"'] \arrow[dll, "\rho"'] \\
i.j.i & ij.i \arrow[l, "\rho.i"'] \arrow[r, "\rho.i"'] & j.i.i & j.i \arrow[l, "j.\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
i.j.j & & & i.j \arrow[lll, "i.\rho"'] \\
ij.j \arrow[u, "\rho.j"] \arrow[d, "\rho.j"] & & & ij \arrow[lll, "\rho"'] \arrow[u, "\rho"'] \arrow[d, "\rho"'] \arrow[dll, "\rho"'] \\
j.i.j & j.ij \arrow[l, "j.\rho"'] \arrow[r, "j.\rho"'] & j.j.i & j.i \arrow[l, "\rho.i"']
\end{tikzcd}$$

the invertibility of non-boundary arrows follows from the commutativity of the upper rectangles and the commutativity of the lower-down triangles.

It is not difficult to conclude, by a recursive argument, that in the next diagrams it is enough to check that one of the radial arrows is invertible to conclude that all the radial arrows are isomorphisms.

In the diagram

$$\begin{tikzcd}
(i+1).i.(i+1).i.(i+1) & i^2(i+1).i.(i+1) \arrow[l, "\rho_3.i.(i+1)"'] \arrow[r, "\rho_3.i.(i+1)"] & i.(i+1).i.i.(i+1) \\
(i+1).i.i^2(i+1) \arrow[u, "(i+1).i.\rho_3"] \arrow[d, "(i+1).i.\rho_3"'] & & i.(i+1).i.(i+1) \arrow[u, "i.(i+1).\rho.(i+1)"'] \\
(i+1).i.i.(i+1).i & i^2(i+1) \arrow[uu, "\rho_3"] \arrow[ul, "\rho_3"'] \arrow[r, "\rho"] \arrow[ddl, "\rho"'] \arrow[ddr, "\rho_3"] & i.i^2(i+1) \arrow[u, "i.\rho_3"'] \arrow[d, "i.\rho_3"] \\
(i+1).i.(i+1).i \arrow[u, "(i+1).\rho.(i+1).i"'] & & i.i.(i+1).i \\
i^2(i+1).i \arrow[u, "\rho_3.i"'] \arrow[r, "\rho_3.i"] & i.(i+1).i.i & i.(i+1).i \arrow[l, "i.(i+1).\rho"'] \arrow[u, "\rho.(i+1).i"']
\end{tikzcd}$$

the 5 o'clock map $\rho_3\colon i^2(i+1) \to i.(i+1).i$ is invertible, by Remark 6.1.

In the diagram

$$\begin{tikzcd}
i.(i-1).i.j & i.(i-1).ij \arrow[l, "i.(i-1).\rho"'] \arrow[r, "i.(i-1).\rho"] & i.(i-1).j.i \\
(i-1)^2i.j \arrow[u, "\rho_3.j"] \arrow[d, "\rho_3.j"'] & & i.(i-1)j.i \arrow[u, "i.\rho.i"'] \arrow[d, "i.\rho.i"] \\
(i-1).i.(i-1).j & & i.j.(i-1).i \\
(i-1).i.(i-1)j \arrow[u, "(i-1).i.\rho"] \arrow[d, "(i-1).i.\rho"'] & (i-1)^2ij \arrow[uuu, "\rho_3"] \arrow[uul, "\rho"'] \arrow[uur, "\rho_3"] \arrow[l, "\rho_3"'] \arrow[r, "\rho_3"] \arrow[ddl, "\rho_3"'] \arrow[ddr, "\rho"] \arrow[ddd, "\rho_3"] & ij.(i-1).i \arrow[u, "\rho.(i-1).i"'] \arrow[d, "\rho.(i-1).i"] \\
(i-1).i.j.(i-1) & & j.i.(i-1).i \\
(i-1).ij.(i-1) \arrow[u, "(i-1).\rho.(i-1)"] \arrow[d, "(i-1).\rho.(i-1)"'] & & j.(i-1)^2i \arrow[u, "j.\rho_3"'] \arrow[d, "j.\rho_3"] \\
(i-1).j.i.(i-1) & (i-1)j.i.(i-1) \arrow[l, "\rho.i.(i-1)"'] \arrow[r, "\rho.i.(i-1)"] & j.(i-1).i.(i-1)
\end{tikzcd}$$

the 10 o'clock map $\rho\colon (i-1)^2ij \to (i-1)^2i.j$ is an isomorphism, by Theorem 5.3.

Figure 1:

$$
\begin{tikzcd}
k.j.k.i.j.k & k.j.ik.j.k \arrow[l, "k.j.\rho.j.k"'] \arrow[r, "k.j.\rho.j.k"] & k.j.i.k.j.k & k.j.i.j^2k \arrow[l, "k.j.i.\rho_3"'] \arrow[d, "k.j.i.\rho_3"'] \\
j^2k.i.j.k \arrow[u, "\rho_3.i.j.k"] \arrow[d, "\rho_3.i.j.k"'] & & & k.j.i.j.k.j \\
j.k.j.i.j.k & & & k.i^2j.k.j \arrow[u, "k.\rho_3.k.j"'] \arrow[d, "k.\rho_3.k.j"] \\
j.k.i^2j.k \arrow[u, "j.k.\rho_3.k"] \arrow[d, "j.k.\rho_3.k"'] & & & k.i.j.i.k.j \\
j.k.i.j.i.k & & & k.i.j.ik.j \arrow[u, "k.i.j.\rho.j"'] \arrow[d, "k.i.j.\rho.j"] \\
j.k.i.j.ik \arrow[u, "j.k.i.j.\rho"] \arrow[d, "j.k.i.j.\rho"'] & i^3j^2k \arrow[l, "\rho_5"'] \arrow[uuuuu, "\rho_5"] \arrow[uuuuurr, bend left, "\rho_4"] \arrow[uuurr, "\rho_4"] \arrow[urr, "\rho_5"] \arrow[drr, "\rho_5"] \arrow[dddrr, "\rho_4"] \arrow[dddddrr, bend right, "\rho_4"] \arrow[ddddddr, "\rho_5"] \arrow[ddddddl, bend left, "\rho_4"] \arrow[ddddl, "\rho_4"] \arrow[ddl, "\rho_5"] \arrow[uul, "\rho_4"] \arrow[uuuul, "\rho_4"] & & k.i.j.k.i.j \\
j.k.i.j.k.i & & & ik.j.k.i.j \arrow[u, "\rho.j.k.i.j"'] \arrow[d, "\rho.j.k.i.j"] \\
j.ik.j.k.i \arrow[u, "j.\rho.j.k.i"] \arrow[d, "j.\rho.j.k.i"'] & & & i.k.j.k.i.j \\
j.i.k.j.k.i & & & i.j^2k.i.j \arrow[u, "i.\rho_3.i.j"] \arrow[d, "i.\rho_3.i.j"'] \\
j.i.j^2k.i \arrow[u, "j.i.\rho_3.i"] \arrow[d, "j.i.\rho_3.i"'] & & & i.j.k.j.i.j \\
j.i.j.k.j.i & & & i.j.k.i^2j \arrow[u, "i.j.k.\rho_3"] \arrow[d, "i.j.k.\rho_3"'] \\
i^2j.k.j.i \arrow[u, "\rho_3.k.j.i"] \arrow[r, "\rho_3.k.j.i"] & i.j.i.k.j.i & i.j.ik.j.i \arrow[l, "i.j.\rho.j.i"'] \arrow[r, "i.j.\rho.j.i"] & i.j.k.i.j.i
\end{tikzcd}
$$

In the diagram

$$\begin{tikzcd}
i.(j+1).j.(j+1) & i(j+1).j.(j+1) \arrow[l, "\rho.j.(j+1)"'] \arrow[r, "\rho.j.(j+1)"] & (j+1).i.j.(j+1) \\
i.j^2(j+1) \arrow[u, "i.\rho_3"] \arrow[d, "i.\rho_3"'] & & (j+1).ij.(j+1) \arrow[u, "(j+1).\rho.(j+1)"'] \arrow[d, "(j+1).\rho.(j+1)"] \\
i.j.(j+1).j & & (j+1).j.i.(j+1) \\
ij.(j+1).j \arrow[u, "\rho.(j+1).j"] \arrow[d, "\rho.(j+1).j"'] & ij^2(j+1) \arrow[l, "\rho_3"'] \arrow[r, "\rho_3"] \arrow[uuu, "\rho_3"] \arrow[ddd, "\rho_3"] \arrow[uul, "\rho"] \arrow[uur, "\rho_3"'] \arrow[ddl, "\rho_3"'] \arrow[ddr, "\rho"] & (j+1).j.i(j+1) \arrow[u, "(j+1).j.\rho"'] \arrow[d, "(j+1).j.\rho"] \\
j.i.(j+1).j & & (j+1).j.(j+1).i \\
j.i(j+1).j \arrow[u, "j.\rho.j"] \arrow[d, "j.\rho.j"'] & & j^2(j+1).i \arrow[u, "\rho_3.i"'] \arrow[d, "\rho_3.i"] \\
j.(j+1).i.j & j.(j+1).ij \arrow[l, "j.(j+1).\rho"'] \arrow[r, "j.(j+1).\rho"] & j.(j+1).j.i
\end{tikzcd}$$

the 4 o'clock map $\rho\colon ij^2(j+1) \to j^2(j+1).i$ is an isomorphism, by Theorem 5.3.
In the diagram

$$\begin{tikzcd}
i.j.k & ij.k \arrow[l, "\rho.k"'] \arrow[r, "\rho.k"] & j.i.k & j.ik \arrow[l, "j.\rho"'] \arrow[r, "j.\rho"] & j.k.i \\
i.jk \arrow[u, "i.\rho"] \arrow[d, "i.\rho"'] & & ijk \arrow[ll, "\rho"] \arrow[rr, "\rho"] \arrow[ul, "\rho"] \arrow[ur, "\rho"] \arrow[dl, "\rho"] \arrow[dr, "\rho"] & & jk.i \arrow[u, "\rho.i"'] \arrow[d, "\rho.i"] \\
i.k.j & ik.j \arrow[l, "\rho.j"'] \arrow[r, "\rho.j"] & k.i.j & k.ij \arrow[l, "k.\rho"'] \arrow[r, "k.\rho"] & k.j.i
\end{tikzcd}$$

for example, the map $\rho\colon ijk \to ij.k$ is an isomorphism, by Theorem 5.3.

In the diagram depicted in Figure 1, we write $j = i+1$ and $k = i+2$. In this diagram the 11 o'clock map $\rho_4\colon i^3(i+1)^2(i+2) \to j^2k.i.j.k$ is an isomorphism, since it is the following composition of isomorphisms defined in Theorem 5.3

$$i^3j^2k \to i^2j^2k.k \to ij^2k.j.k \to j^2k.i.j.k \quad .$$

This concludes the proof that the collection of functors F_i, $1 \le i \le n-1$, and of natural isomorphisms τ_{ij}, $1 \le i \le j \le n-1$, defines a preaction of $\mathfrak{H}(\Sigma_{\mathrm{n}})$ on Comod-$A(\delta)_d$.

7 A (pre)action of $\mathfrak{H}(\Sigma_{\mathrm{n}})$ on $S^{-}_{\alpha,\beta}(n,r)$-Mod

In this section we show that the preaction of $\mathfrak{H}(\Sigma_{\mathrm{n}})$ on Comod-$A(\delta)_d$ induces a preaction (and so an action) of $\mathfrak{H}(\Sigma_{\mathrm{n}})$ on the category of $S^{-}(n,r)$-modules, where $S^{-}(n,r) = S^{-}_{\alpha,\beta}(n,r)$ is the quantum negative Borel-Schur algebra.

We prove first that the preaction of $\mathfrak{H}(\Sigma_{\mathrm{n}})$ on Comod-$A(\delta)_d$ can be restricted to Comod-$A(\delta)$. For each $1 \le i \le n-1$, define F_i': Comod-$A(\delta) \to$ Comod-$A(\delta)$ by

$$M \mapsto M \otimes^{A(\delta)} A[i].$$

Let $\psi\colon A(\delta) \to A(\delta)_d$ be the canonical inclusion. Then we have the associated restriction functor $\psi_\circ$: Comod-$A(\delta) \to$ Comod-$A(\delta)_d$.

Proposition 7.1. *The inclusion $\psi_i\colon A[i] \to A[i]_d$ induces a natural isomorphism $\psi_\circ F_i' \to F_i\psi_\circ$.*

Proof. Let $M \in$ Comod-$A(\delta)$. Then the natural transformation in question is given by

$$\begin{aligned} M \otimes^{A(\delta)} A[i] &\to \psi_\circ M \otimes^{A(\delta)_d} A[i]_d \\ \sum_j m_j \otimes x_j &\mapsto \sum m_j \otimes \psi_i(x_j). \end{aligned}$$

Since we have $M \cong M \otimes^{A(\delta)} A(\delta)$, and the cotensor product is associative, to prove the proposition it is enough to show that

$$\begin{aligned} A(\delta) \otimes^{A(\delta)} A[i] &\to \psi_\circ A(\delta) \otimes^{A(\delta)_d} A[i]_d \\ \sum_j z_j \otimes x_j &\mapsto \sum z_j \otimes \psi_i(x_j) \end{aligned}$$

is an isomorphism. Precomposing this with the isomorphism $A[i] \to A(\delta) \otimes^{A(\delta)} A[i]$, $[x]_{a[i]} \mapsto \sum [x_{(1)}]_\delta \otimes [x_{(2)}]_{a[i]}$, we get the map

$$\begin{aligned} A[i] &\to A(\delta) \otimes^{A(\delta)_d} A[i]_d \\ [x]_{a[i]} &\mapsto \sum [x_{(1)}]_\delta \otimes [x_{(2)}]_{a[i]}. \end{aligned} \tag{20}$$

Thus all that is left to check is that (20) is an isomorphism. For this, consider the exact sequence (7), for $b = a[i]$ and $l = i$,

$$0 \to A[i] \otimes \mathbb{K}_{v_{i+1}} \to A[i] \to A(\delta) \to 0.$$

It can also be considered as a sequence of $A(\delta)_d$-comodules. Proceeding as in the proof of Theorem 5.3 (with $b = \delta$ and $l = i$), using Theorem 5.1 and Theorem 3.2, we see that (20) is an isomorphism of $A(\delta)_d$-comodules. □

Note that, since $\psi\colon A(\delta) \to A(\delta)_d$ is a monomorphism of coalgebras over a field, the functor $\psi_\circ$ is full and faithful. Therefore, for any $M \in$ Comod-$A(\delta)$, we have an isomorphism

$$\text{Comod-}A(\delta)((F_i')^2 M, F_i' M) \xrightarrow{\cong} \text{Comod-}A(\delta)_d(F_i^2\psi_\circ M, F_i\psi_\circ M),$$

for every $1 \le i \le n-1$. Hence we can define $(\tau_{ii}')_M$ as the map that corresponds to $(\tau_{ii})_M$ under this isomorphisms. It is clear that τ_{ii}' is a natural transformation from $(F_i')^2$ to F_i'. Similarly, one can define the natural transformations $\tau_{i,j}'$ for $i < j$. Since (F,τ) is a preaction on Comod-$A(\delta)_d$, we get that (F',τ') is a preaction on Comod-$A(\delta)$.

Let r be a natural number. Then the subset $A(\delta;r)$ of r-homogeneous elements in $A(\delta)$ is a finite dimensional subcoalgebra of $A(\delta)$. Similarly, the set $A(a[i];r)$ of r-homogeneous elements in $A[i]$ is a finite dimensional subcoalgebra of $A[i]$. Let M be an $A(\delta;r)$-comodule. Then from the definition of the cotensor product we get

$$M \otimes^{A(\delta)} A[i] = M \otimes^{A(\delta;r)} A(a[i];r).$$

Thus $F_i'M$ is an $A(\delta;r)$-comodule. Hence the preaction (F',τ') defines a preaction of $\mathfrak{H}(\Sigma_{\mathrm{n}})$ on Comod-$A(\delta;r)$.

As it is well known, see e.g. [8], [9], the associative algebra $S^-(n,r) = S^-_{\alpha,\beta}(n,r)$ dual to $A(\delta;r)$ is called the (negative) quantised *Borel-Schur* algebra. As usual, we have a canonical equivalence between the categories $S^-(n,r)$-Mod and Comod-$A(\delta;r)$. Therefore we get that (F',τ') induces an action of $\mathfrak{H}(\Sigma_{\mathrm{n}})$ on $S^-(n,r)$-Mod.

8 Examples

In this section we consider some explicit examples of the application of the functors F_w to $A(\delta)_d$-comodules. For simplicity, we will work within the non-quantised setting over an infinite field. In particular, the coordinate variables c_{ij} commute with each other.

We will need some additional notation. We denote by $\mathbb{K}[T_n]$ the coordinate algebra of the subgroup of diagonal matrices in $\mathrm{GL}_n(\mathbb{K})$. The canonical projection $\pi\colon A(\delta)_d \to \mathbb{K}[T_n]$ is defined by

$$\pi(c_{ij}) = \begin{cases} c_{ii}, & i=j \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward to verify that π is a homomorphism of coalgebras. Therefore, every $A(\delta)_d$-comodule (M,ρ) can be considered a $\mathbb{K}[T_n]$-comodule with the coaction given by

$$\rho_T(x) := (\mathrm{id}\otimes\pi)(\rho(x)),$$

for all $x\in M$. For every $a\in\mathbb{Z}^n$ we define the one-dimensional $\mathbb{K}[T_n]$-comodule $\mathbb{K}_a$ by

$$\rho(1) := 1\otimes c_{11}^{a_1}\dots c_{nn}^{a_n}.$$

It is well-known that every finite dimensional indecomposable comodule over $\mathbb{K}[T_n]$ is isomorphic to $\mathbb{K}_a$ for some $a\in\mathbb{Z}^n$. Given a finite dimensional $A(\delta)_d$-comodule M, we can write $M=\bigoplus_a M_a$, where each M_a is the $\mathbb{K}[T_n]$- submodule of M satisfying

$$\rho_T(x) = x\otimes c_{11}^{a_1}\dots c_{nn}^{a_n}$$

for all $x\in M_a$. The subspaces M_a of M are called *weight subspaces* of M. We will say that the elements of M_a have *weight* a.

Fix $i\in\mathbf{n}$. We will write $\mathbb{K}[G_i]$ for the coordinate algebra of the Levi subgroup

$$G_i := \mathrm{GL}_1(\mathbb{K})^{(i-1)}\times\mathrm{GL}_2(\mathbb{K})\times\mathrm{GL}_1(\mathbb{K})^{(n-i-1)}.$$

Thus $\mathbb{K}[G_i]$ is the localization of $\mathbb{K}[c_{11},c_{22},\dots,c_{nn},c_{i,i+1},c_{i+1,i}]$ with respect to

$$c_{11}\dots c_{i-1,i-1}(c_{ii}c_{i+1,i+1}-c_{i,i+1}c_{i+1,i})c_{i+2,i+2}\dots c_{nn}.$$

Note that $A[i]_d$ is the coordinate algebra of the corresponding parabolic subgroup in $\mathrm{GL}_n(\mathbb{K})$. Since the Levi subgroup G_i is a quotient of the corresponding parabolic subgroup we get a well defined homomorphism of Hopf algebras

$$\zeta_i\colon \mathbb{K}[G_i]\to A[i]_d$$

determined by

$$\zeta_i(c_{kl}) = c_{kl},$$

where $k = l \in \mathbf{n}$ or $\{k, l\} = \{i, i+1\}$. Thus every $\mathbb{K}[G_i]$-comodule can be considered as an $A[i]_d$-comodule via ζ_i.

For every composition $\mu = (\mu_1, \ldots, \mu_n)$ such that $\mu_i = \mu_{i+1}$, we denote by $\mathbb{K}_\mu$ the one-dimensional $A[i]_d$-comodule with coaction given by

$$\rho(1) = 1 \otimes c_{11}^{\mu_1} \ldots c_{i-1,i-1}^{\mu_{i-1}} (c_{ii}c_{i+1,i+1} - c_{i,i+1}c_{i+1,i})^{\mu_i} c_{i+2,i+2}^{\mu_{i+2}} \ldots c_{nn}^{\mu_n}.$$

From [8, Section 3], we know that $\mathbb{K}[c_{ii}, c_{i,i+1}]$ is a $\mathbb{K}[G_i]$-subcomodule of the regular $\mathbb{K}[G_i]$-comodule $\mathbb{K}[G_i]$. For a natural number m, we denote by $Y_{i,m}$ the mth homogeneous component of $\mathbb{K}[c_{ii}, c_{i,i+1}]$. Then $Y_{i,m}$ is a $\mathbb{K}[G_i]$-subcomodule of the $\mathbb{K}[G_i]$-comodule $\mathbb{K}[c_{ii}, c_{i,i+1}]$. We write $\widetilde{Y}_{i,m}$ for $Y_{i,m}$ considered as $A[i]_d$-comodule via ζ_i.

It follows from Lemma 3.1 and Lemma 2.12 in [8] that

1) *If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is such that $\lambda_i - \lambda_{i+1} = m \geq 0$, then $\pi_i^\circ \mathbb{K}_\lambda \cong \mathbb{K}_\mu \otimes \widetilde{Y}_{i,m}$, where*

$$\mu = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_{i+1}, \ldots, \lambda_n)$$

   and $R^k \pi_i^\circ \mathbb{K}_\lambda \cong 0$ for $k \geq 1$.

2) *If $\lambda_i - \lambda_{i+1} = -1$, then $R^k \pi_i^\circ \mathbb{K}_\lambda \cong 0$ for all $k \geq 0$.*

From now on we fix $n = 3$. Using the above facts we will give an explicit description of the $A(\delta)_d$-comodules $F_w \mathbb{K}_{(1,1,0)}$, for all $w \in \mathfrak{H}(\Sigma_3)$.

As $\widetilde{Y}_{1,0}$ is the trivial $A[1]_d$-comodule, we get that

$$\pi_1^\circ \mathbb{K}_{(1,1,0)} \cong \mathbb{K}_{(1,1,0)}.$$

Therefore, $F_1 \mathbb{K}_{(1,1,0)} \cong \mathbb{K}_{(1,1,0)}$. This implies that $F_2 F_1 \mathbb{K}_{(1,1,0)} \cong F_2 \mathbb{K}_{(1,1,0)}$ and

$$F_2 F_1 F_2 \mathbb{K}_{(1,1,0)} \cong F_1 F_2 F_1 \mathbb{K}_{(1,1,0)} \cong F_1 F_2 \mathbb{K}_{(1,1,0)}. \tag{21}$$

Thus, to know all the $A(\delta)_d$-comodules $F_w \mathbb{K}_{(1,1,0)}$, we only have to compute $F_2 \mathbb{K}_{(1,1,0)}$ and $F_1 F_2 \mathbb{K}_{(1,1,0)}$.

We start by studying $F_2 \mathbb{K}_{(1,1,0)}$. For this, consider $\widetilde{Y}_{2,1}$. It has $\mathbb{K}$-basis $\{c_{22}, c_{23}\}$ and $A[2]_d$-comodule structure given by

$$\rho(c_{22}) = c_{22} \otimes c_{22} + c_{23} \otimes c_{32}, \quad \rho(c_{23}) = c_{22} \otimes c_{23} + c_{23} \otimes c_{33}.$$

Let us compute the $A[2]_d$-comodule structure on

$$\pi_2^\circ \mathbb{K}_{(1,1,0)} \cong \mathbb{K}_{(1,0,0)} \otimes \widetilde{Y}_{2,1}.$$

Since $\rho(x) = x \otimes c_{11}$ for $x \in \mathbb{K}_{(1,0,0)}$, we get in $\mathbb{K}_{(1,0,0)} \otimes \widetilde{Y}_{2,1}$

$$\rho(1 \otimes c_{22}) = (1 \otimes c_{22}) \otimes c_{11}c_{22} + (1 \otimes c_{23}) \otimes c_{11}c_{32}, \quad \rho(1 \otimes c_{23}) = (1 \otimes c_{22}) \otimes c_{11}c_{23} + (1 \otimes c_{23}) \otimes c_{11}c_{33}.$$

Therefore, the $A(\delta)_d$-comodule $F_2 \mathbb{K}_{(1,1,0)} = \pi_{2\circ} \pi_2^\circ \mathbb{K}_{(1,0,0)}$ is two-dimensional, with basis $\{1 \otimes c_{22}, 1 \otimes c_{23}\}$ and $A(\delta)_d$-comodule structure given by

$$\rho(1 \otimes c_{22}) = (1 \otimes c_{22}) \otimes c_{11}c_{22} + (1 \otimes c_{23}) \otimes c_{11}c_{32}, \quad \rho(1 \otimes c_{23}) = (1 \otimes c_{23}) \otimes c_{11}c_{33}. \tag{22}$$

It is now easy to determine the weight subspace structure of $F_2\mathbb{K}_{(1,1,0)}$. This structure will be useful to study $F_1F_2\mathbb{K}_{(1,1,0)}$.

From (22), we get

$$\rho_T(1\otimes c_{22}) = (1\otimes c_{22})\otimes c_{11}c_{22}, \quad \rho_T(1\otimes c_{23}) = (1\otimes c_{23})\otimes c_{11}c_{33}.$$

This implies that

$$\big(F_2\mathbb{K}_{(1,1,0)}\big)_{(1,1,0)} = \langle 1\otimes c_{22}\rangle\,, \quad \big(F_2\mathbb{K}_{(1,1,0)}\big)_{(1,0,1)} = \langle 1\otimes c_{23}\rangle\,.$$

Moreover, from (22) it also follows that $\langle 1\otimes c_{23}\rangle$ is an $A(\delta)_d$-subcomodule of $F_2\mathbb{K}_{(1,1,0)}$ isomorphic to $\mathbb{K}_{(1,0,1)}$. The corresponding quotient has the following $A(\delta)_d$-coaction

$$\rho([1\otimes c_{22}]) = [1\otimes c_{22}]\otimes c_{11}c_{22}$$

and so it is isomorphic to $\mathbb{K}_{(1,1,0)}$. Thus we get a short exact sequence of $A(\delta)_d$-comodules

$$0\to\mathbb{K}_{(1,0,1)}\to F_2\mathbb{K}_{(1,1,0)}\to\mathbb{K}_{(1,1,0)}\to 0. \tag{23}$$

Next we will study the $A(\delta)_d$-comodule structure of $F_1F_2\mathbb{K}_{(1,1,0)}$. For this we will exhibit first its weight subspaces, and then determine the $A(\delta)_d$-coaction on a weight basis of $F_1F_2\mathbb{K}_{(1,1,0)}$. We start by applying π_1° to (23). As $R^1\pi_1^\circ\mathbb{K}_{(1,0,1)}\cong 0$, we get from the long exact sequence that

$$0\to\pi_1^\circ\mathbb{K}_{(1,0,1)}\to\pi_1^\circ F_2\mathbb{K}_{(1,1,0)}\to\pi_1^\circ\mathbb{K}_{(1,1,0)}\to 0$$

is an exact sequence of $A[1]_d$-comodules. Applying $\pi_{1\circ}$ and taking into account that $F_1\mathbb{K}_{(1,1,0)}\cong\mathbb{K}_{(1,1,0)}$ we get the exact sequence

$$0\to F_1\mathbb{K}_{(1,0,1)}\to F_1F_2\mathbb{K}_{(1,1,0)}\to\mathbb{K}_{(1,1,0)}\to 0 \tag{24}$$

of $A(\delta)_d$-comodules.

By a computation similar to the case of $F_2\mathbb{K}_{(1,1,0)}$, we can see that $F_1\mathbb{K}_{(1,0,1)}$ has basis $\{1\otimes c_{11}, 1\otimes c_{12}\}$ and $A(\delta)_d$-comodule structure

$$\rho(1\otimes c_{11}) = (1\otimes c_{11})\otimes c_{11}c_{33} + (1\otimes c_{12})\otimes c_{21}c_{33}, \quad \rho(1\otimes c_{12}) = (1\otimes c_{12})\otimes c_{22}c_{33}. \tag{25}$$

Hence

$$\rho_T(1\otimes c_{11}) = (1\otimes c_{11})\otimes c_{11}c_{33}, \quad \rho_T(1\otimes c_{12}) = (1\otimes c_{12})\otimes c_{22}c_{33},$$

and therefore

$$\big(F_1\mathbb{K}_{(1,0,1)}\big)_{(1,0,1)} = \langle 1\otimes c_{11}\rangle\,, \quad \big(F_1\mathbb{K}_{(1,0,1)}\big)_{(0,1,1)} = \langle 1\otimes c_{12}\rangle\,.$$

Denote by u the image of $1\otimes c_{12}$ in $F_1F_2\mathbb{K}_{(1,1,0)}$ under the monomorphism in (24), and by v the image of $1\otimes c_{11}$ under the same map. Since the monomorphism in (24) is a homomorphism of $A(\delta)_d$-comodules, from (25), we get

$$\rho(u) = u\otimes c_{22}c_{33}, \quad \rho(v) = v\otimes c_{11}c_{33} + u\otimes c_{21}c_{33}.$$

Note that u has weight $(0,1,1)$ and v has weight $(1,0,1)$

The sequence (24) splits if considered as a sequence of $\mathbb{K}[T_3]$-comodules. Thus

$$F_1F_2\mathbb{K}_{(1,1,0)}\cong\mathbb{K}_{(0,1,1)}\oplus\mathbb{K}_{(1,0,1)}\oplus\mathbb{K}_{(1,1,0)}$$

as $\mathbb{K}[T_3]$-comodules. In particular, every weight subspace of $F_1F_2\mathbb{K}_{(1,1,0)}$ is one-dimensional and

$$(F_1F_2\mathbb{K}_{(1,1,0)})_{(0,1,1)} = \langle u\rangle\,, \quad (F_1F_2\mathbb{K}_{(1,1,0)})_{(1,0,1)} = \langle v\rangle\,.$$

From the explicit description (25) of the $A(\delta)_d$-coaction on $F_1\mathbb{K}_{(1,0,1)}$, we get the short exact sequence

$$0 \to \mathbb{K}_{(0,1,1)} \to F_1\mathbb{K}_{(1,0,1)} \to \mathbb{K}_{(1,0,1)} \to 0. \tag{26}$$

As $R^k\pi_2^\circ\mathbb{K}_{(1,0,1)} \cong 0$ for all $k \geq 0$, applying π_2° to (26), we get that

$$\pi_2^\circ F_1\mathbb{K}_{(1,0,1)} \cong \pi_2^\circ\mathbb{K}_{(0,1,1)} \cong \mathbb{K}_{(0,1,1)}; \quad R^k\pi_2^\circ F_1\mathbb{K}_{(1,0,1)} \cong R^k\pi_2^\circ\mathbb{K}_{(0,1,1)} \cong 0,\ k \geq 1.$$

So applying π_2° to (24), we get the short exact sequence of $A[2]_d$-comodules

$$0 \to \mathbb{K}_{(0,1,1)} \to \pi_2^\circ F_1F_2\mathbb{K}_{(1,1,0)} \to \pi_2^\circ\mathbb{K}_{(1,1,0)} \to 0.$$

As $\pi_{2\circ}$ is exact and $F_2 = \pi_{2\circ}\pi_2^\circ$ we obtain the short exact sequence

$$0 \to \mathbb{K}_{(0,1,1)} \to F_2F_1F_2\mathbb{K}_{(1,1,0)} \to F_2\mathbb{K}_{(1,1,0)} \to 0 \tag{27}$$

of $A(\delta)_d$-comodules. In view of (21), the short exact sequence (27) becomes

$$0 \to \mathbb{K}_{(0,1,1)} \to F_1F_2\mathbb{K}_{(1,1,0)} \xrightarrow{\theta} F_2\mathbb{K}_{(1,1,0)} \to 0. \tag{28}$$

Let $\bar{v}$ be the image of v in $F_2\mathbb{K}_{(1,1,0)}$ under the epimorphimsm θ in (28). Since v has weight $(1,0,1)$, the same is true for $\bar{v}$. As the weight subspace $(F_2\mathbb{K}_{(1,1,0)})_{(1,0,1)}$ is one-dimensional and is spanned by $1 \otimes c_{23}$, there is a non-zero $\gamma \in \mathbb{K}$ such that $\bar{v} = \gamma \otimes c_{23}$. The epimorphism θ in (28) induces an isomorphism between the weight subspaces $(F_1F_2\mathbb{K}_{(1,1,0)})_{(1,1,0)}$ and $(F_2\mathbb{K}_{(1,1,0)})_{(1,1,0)} = \langle 1 \otimes c_{22}\rangle$. Let us denote by w the element in $(F_1F_2\mathbb{K}_{(1,1,0)})_{(1,1,0)}$ that corresponds to $\gamma \otimes c_{22}$ under this isomorphism. Then by (22)

$$\rho(\theta(w)) = \gamma\rho(1 \otimes c_{22}) = (\gamma \otimes c_{22}) \otimes c_{11}c_{22} + (\gamma \otimes c_{23}) \otimes c_{11}c_{32}. \tag{29}$$

We have that $\big(F_1F_2\mathbb{K}_{(1,1,0)}\big)_{(1,1,0)} = \langle w\rangle$ and that $\{u, v, w\}$ is a basis of $F_1F_2\mathbb{K}_{(1,1,0)}$. Therefore, there are unique h and f in $A(\delta)_d$ such that

$$\rho(w) = w \otimes c_{11}c_{22} + v \otimes h + u \otimes f \tag{30}$$

and $\pi(h) = \pi(f) = 0$. Thus

$$(\theta \otimes \mathrm{id})\rho(w) = (\gamma \otimes c_{22}) \otimes c_{11}c_{22} + (\gamma \otimes c_{23}) \otimes h. \tag{31}$$

Since $(\theta \otimes \mathrm{id})\rho = \rho\theta$, compairing (29) and (31), we get that $h = c_{11}c_{32}$. Hence it is left to determine f.

As $\mathbb{K}_{(1,1,0)}$ is an $A(\delta;2)$-comodule, we get from the considerations at the end of Section 7, that $F_1F_2\mathbb{K}_{(1,1,0)}$ is an $A(\delta;2)$-comodule. Therefore f is an element of degree two in $A(\delta) \cong \mathbb{K}[c_{11}, c_{22}, c_{33}, c_{21}, c_{31}, c_{32}]$.

From (30), we get

$$\begin{aligned}(\rho \otimes \mathrm{id})\rho(w) &= w \otimes c_{11}c_{22} \otimes c_{11}c_{22} + v \otimes (c_{11}c_{32} \otimes c_{11}c_{22} + c_{11}c_{33} \otimes c_{11}c_{32})\\ &\quad + u \otimes (f \otimes c_{11}c_{22} + c_{21}c_{33} \otimes c_{11}c_{32} + c_{22}c_{33} \otimes f);\\ (\mathrm{id} \otimes \Delta)\rho(w) &= w \otimes c_{11}c_{22} \otimes c_{11}c_{22} + v \otimes (c_{11}c_{32} \otimes c_{11}c_{22} + c_{11}c_{33} \otimes c_{11}c_{32})\\ &\quad + u \otimes \Delta(f).\end{aligned}$$

As $(\rho \otimes \mathrm{id})\rho = (\mathrm{id} \otimes \Delta)\rho$ we obtain that f satisfies the equation

$$\Delta(f) = f \otimes c_{11}c_{22} + c_{21}c_{33} \otimes c_{11}c_{32} + c_{22}c_{33} \otimes f. \tag{32}$$

Denote by V the subspace of $A(\delta; 2) \otimes A(\delta; 2)$ spanned by

$$\{c_{ij}c_{kl} \otimes c_{js}c_{lt} \mid i \geq j \geq s,\ k \geq l \geq t\}. \tag{33}$$

From the definition of the comultiplication in $A(\delta)$, we get that $\Delta(A(\delta; 2)) \subset V$.

Suppose $c_{ij}c_{kl}$ has non-zero coefficient in the expansion of f with respect to the monomial basis of $A(\delta; 2)$. Note that $c_{ij}c_{kl} \neq c_{11}c_{22}$, $c_{ij}c_{kl} \neq c_{22}c_{33}$ as $\pi(f) = 0$. Then from (32), we see that $c_{ij}c_{kl} \otimes c_{11}c_{22}$ and $c_{22}c_{33} \otimes c_{ij}c_{kl}$ have non-zero coefficients in the expansion of $\Delta(f) \in \Delta(A(\delta; 2)) \subset V$ with respect to the basis (33) of V. Thus $\{j, l\} = \{1, 2\}$ and $\{i, k\} = \{2, 3\}$. Therefore the only basis elements of $A(\delta; 2)$ that can have non-zero coefficients in the expansion of f are $c_{21}c_{32}$ and $c_{22}c_{31}$. Direct computation now shows that the only linear combination of $c_{21}c_{32}$ and $c_{22}c_{31}$ that satisfy (32) is

$$f = c_{21}c_{32} - c_{22}c_{31}.$$

Hence we get a full description of $A(\delta)_d$-comodule structure on $F_1F_2\mathbb{K}_{(1,1,0)}$:

$$\rho(u) = u \otimes c_{22}c_{33},\ \rho(v) = v \otimes c_{11}c_{33} + u \otimes c_{21}c_{33},\ \rho(w) = w \otimes c_{11}c_{22} + v \otimes c_{11}c_{32} + u \otimes (c_{21}c_{32} - c_{22}c_{31}).$$

References

[1] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR MR1484478

[2] Louis Crane, *Clock and category: is quantum gravity algebraic?*, J. Math. Phys. **36** (1995), no. 11, 6180–6193. MR 1355904 (96j:83036)

[3] Louis Crane and Igor B. Frenkel, *Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases*, J. Math. Phys. **35** (1994), no. 10, 5136–5154, Topology and physics. MR 1295461 (96d:57019)

[4] P. Deligne, *Action du groupe des tresses sur une catégorie*, Invent. Math. **128** (1997), no. 1, 159–175. MR 1437497 (98b:20061)

[5] Michel Demazure, *Une nouvelle formule des caractères*, Bull. Sci. Math. (2) **98** (1974), no. 3, 163–172. MR 0430001 (55 #3009)

[6] Richard Dipper and Stephen Donkin, *Quantum* GL_n, Proc. London Math. Soc. (3) **63** (1991), no. 1, 165–211. MR 1105721 (92g:16055)

[7] Stephen Donkin, *Hopf complements and injective comodules for algebraic groups*, Proc. London Math. Soc. (3) **40** (1980), no. 2, 298–319. MR 566493 (81f:20053)

[8] ———, *Standard homological properties for quantum* GL_n, J. Algebra **181** (1996), no. 1, 235–266. MR 1382035 (97b:20065)

[9] Stephen Donkin, Ana Paula Santana, and Ivan Yudin, *Homological properties of quantised Borel-Schur algebras and resolutions of quantised Weyl modules*, J. Algebra **402** (2014), 120–157. MR 3160417

[10] Jie Du, Brian Parshall, and Jian Pan Wang, *Two-parameter quantum linear groups and the hyperbolic invariance of q-Schur algebras*, J. London Math. Soc. (2) **44** (1991), no. 3, 420–436. MR 1149005 (93d:20084)

[11] Peter Magyar, *Schubert polynomials and Bott-Samelson varieties*, Comment. Math. Helv. **73** (1998), no. 4, 603–636. MR 1639896 (2000e:14085)

[12] Volodymyr Mazorchuk, *Lectures on algebraic categorification*, QGM Master Class Series, European Mathematical Society (EMS), Zürich, 2012. MR 2918217

[13] Brian Parshall and Jian Pan Wang, *Quantum linear groups*, Mem. Amer. Math. Soc. **89** (1991), no. 439, vi+157. MR 1048073 (91g:16028)

[14] Victor Reiner and Mark Shimozono, *Percentage-avoiding, northwest shapes and peelable tableaux*, J. Combin. Theory Ser. A **82** (1998), no. 1, 1–73. MR 1616579 (2000a:05220)

[15] Mitsuhiro Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 3, 629–644. MR 0472967 (57 #12646)

[16] ———, *A two-parameter quantization of* $\mathrm{GL}(n)$ *(summary)*, Proc. Japan Acad. Ser. A Math. Sci. **66** (1990), no. 5, 112–114. MR 1065785 (92f:16049)

[17] Ivan Yudin, *Decreasing diagrams and coherent presentations*, arXiv preprint (2015).