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On Auslander-Reiten sequences for Borel-Schur algebras

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Abstract

We classify Borel-Schur algebras having finite representation type. We also determine Auslander-Reiten sequences for a large class of simple modules over Borel-Schur algebras. A partial information on the structure of the socles of Borel-Schur algebras is given.

1 Introduction

Consider the general linear group $GL_n(\mathbb{K})$, where $\mathbb{K}$ is an infinite field, and let B^+ be the Borel subgroup of $GL_n(\mathbb{K})$ consisting of all upper triangular matrices in $GL_n(\mathbb{K})$. The Schur algebras $S(n,r)$ and $S(B^+) := S(B^+, n, r)$ corresponding to $GL_n(\mathbb{K})$ and B^+, respectively, are powerful tools in the study of polynomial representations of $GL_n(\mathbb{K})$ and B^+. In particular, the simple modules of $S(B^+)$ labelled by partitions induce to Weyl modules for $S(n,r)$, and Weyl modules are central objects of study. In the recent paper [18], Borel-Schur algebras were crucial to construct resolutions for Weyl modules. Therefore one would like to understand better the algebra $S(B^+)$.

Given a finite-dimensional algebra over $\mathbb{K}$, we denote by A-mod the category of all finite-dimensional left A-modules. The algebra A is said to have finite representation type if there are only finitely many isomorphism classes of indecomposable modules in A-mod. Representation type of Schur algebras and of infinitesimal Schur algebras was determined in [7] and [5]. In particular, it is known when Schur algebras are of finite type. In this paper we obtain the corresponding classification for Borel-Schur algebras,

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determining the conditions on n, r and characteristic of $\mathbb{K}$ under which $S(B^+, n, r)$ is of finite representation type. This is Theorem 6.6.

One of the motivations for this classification was the construction of Auslander-Reiten sequences for Borel-Schur algebras. The class of these sequences, also known as almost split sequences, is an important invariant of the module category of a finite-dimensional algebra. It provides part of a presentation of the module category.

Taking advantage of the easy multiplication of some basis elements of $S(B^+)$, we determine Auslander-Reiten sequences for a large class of simple $S(B^+)$-modules. We are able to do this, for an arbitrary n, under some combinatorial conditions. We note that when these are satisfied, the relevant simple module does not occur in the socle of $S(B^+)$.

Several recipes were given in the 80's for the construction of Auslander-Reiten sequences (see for example [4, 10, 11]). Although we do not use any of these, we should remark that the recipe due to J.A. Green [11] was the motivation for this work.

The proofs of our results in this article are based on an explicit description of the multiplication in the Borel-Schur algebras $S(B^+, n, r)$ obtained by Green in [12]. It would be interesting if one could get similar results for the Borel-Schur algebras of other Dynkin types considered in [20].

The paper is organized as follows. Section 2 recalls the definitions of the algebras and some basic background. We also describe the quiver presentation of $S(B^+, 2, r)$.

In Section 3, we construct Auslander-Reiten sequences ending in a simple module $\mathbb{K}_\lambda$, where λ satisfies a condition given in (3.5). As a by-product we see that this condition implies that $\mathbb{K}_\lambda$ does not occur in the (left) socle of the algebra. The main result of this section is Theorem 3.6. Some observations about the middle term of the Auslander-Reiten sequences are also given.

In Section 4 we consider $n = 2$ and find Auslander-Reiten sequences ending in an arbitrary simple module, that is we deal with the cases missing in Section 3. As an easy consequence of the results in this section we can obtain a necessary and sufficient condition for a simple module to occur in the socle of $S(B^+, 2, r)$. This and other results involving the socle of the Borel-Schur algebra $S(B^+, n, r)$, for arbitrary n, are summarized in Theorem 4.5.

In Section 5 we discuss reduction of rank. This may be of more general interest, and is in fact used in Section 6, where we determine precisely which Borel-Schur algebras are of finite type.

2 Notation and basic results

In this section we establish the notation we will use and give some basic results. We will follow [17] and any undefined term may be found there. For further details on the general theory of Schur algebras see [13] and [14].

Throughout the paper $\mathbb{K}$ is an infinite field of arbitrary characteristic, n and r are arbitrary fixed positive integers and p is any prime number.

For any natural number s, we denote by $\mathbf{s}$ the set $\{1, \ldots, s\}$ and by Σ_s the symmetric group on $\mathbf{s}$. Define the sets of multi-indices $I(n,r)$ and of compositions $\Lambda(n,r)$ by

$$I(n,r) = \{i = (i_1, \dots, i_r) \,|\, i_\rho \in \mathbf{n} \text{ for all } \rho \in \mathbf{r}\}$$
$$\Lambda(n,r) = \{\, \lambda = (\lambda_1, \dots, \lambda_n) \,|\, \lambda_\nu \in \mathbb{Z},\ \lambda_\nu \geq 0\ (\nu \in \mathbf{n}),\ \sum_{\nu \in \mathbf{n}} \lambda_\nu = r\}.$$

We will often write I instead of $I(n,r)$ and Λ instead of $\Lambda(n,r)$.

Given $i \in I$ and $\lambda \in \Lambda$, we say that i has *weight* λ and write $i \in \lambda$ if, for all $\nu \in \mathbf{n}$, we have $\lambda_\nu = \#\{\, \rho \in \mathbf{r} \,|\, i_\rho = \nu\}$.

The group Σ_r acts on the right on I and on $I \times I$, respectively, by

$$i\pi = (i_{\pi 1}, \dots, i_{\pi r})$$

and $(i,j)\pi = (i\pi, j\pi)$, for all $\pi \in \Sigma_r$ and $i,\, j \in I$. If i and j are in the same Σ_r-orbit of I we write $i \sim j$. Also $(i,j) \sim (i',j')$ means these two pairs are in the same Σ_r-orbit of $I \times I$. We denote the stabilizer of i in Σ_r by Σ_i, that is $\Sigma_i = \{\, \pi \in \Sigma_r \,|\, i\pi = i\}$. We write $\Sigma_{i,j} = \Sigma_i \cap \Sigma_j$. Given $i,\, j \in I$, then $i \leq j$ means that $i_\rho \leq j_\rho$ for all $\rho \in \mathbf{r}$, and $i < j$ means that $i \leq j$ and $i \neq j$.

We use $\trianglelefteq$ for the "dominance order" on Λ, that is $\alpha \trianglelefteq \beta$ if $\sum_{\nu=1}^{\mu} \alpha_\nu \leq \sum_{\nu=1}^{\mu} \beta_\nu$ for all $\mu \in \mathbf{n}$. Obviously if $i \in \alpha$ and $j \in \beta$ (where $\alpha,\, \beta \in \Lambda$), then $i \leq j$ implies $\beta \trianglelefteq \alpha$.

Given $\lambda \in \Lambda$, we consider in I the special element

$$l = l(\lambda) = (\underbrace{1, \dots, 1}_{\lambda_1}, \underbrace{2, \dots, 2}_{\lambda_2}, \dots, \underbrace{n, \dots, n}_{\lambda_n}).$$

Clearly $\Sigma_{l(\lambda)}$ is the parabolic subgroup associated with λ

$$\Sigma_\lambda = \Sigma_{\{1,\dots,\lambda_1\}} \times \Sigma_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \times \cdots \times \Sigma_{\{\lambda_1+\cdots+\lambda_{n-1}+1,\dots,r\}}.$$

For each $\nu \in \mathbf{n-1}$, and each non-negative integer $m \leq \lambda_{\nu+1}$, we define

$$\lambda(\nu,m) = (\lambda_1, \dots, \lambda_\nu + m, \lambda_{\nu+1} - m, \dots, \lambda_n) \in \Lambda,$$

and write $l(\nu,m)$ for $l\left(\lambda\left(\nu,m\right)\right)$. We have $l(\nu,m) \leq l$.

For the notation of λ-tableaux the reader is referred to [17]. Given $\lambda \in \Lambda$, we choose the basic λ-tableau

$$T^\lambda = \begin{array}{ccccccc} 1 & 2 & \dots & \lambda_1 & & & \\ \lambda_1+1 & \lambda_1+2 & \dots & \dots & \dots & \lambda_1+\lambda_2 & \\ \dots & & & & & & \\ \lambda_1+\cdots+\lambda_{n-1}+1 & \dots & \dots & \dots & r & & \end{array}$$

The row-stabilizer of T^λ, i.e. the subgroup of Σ_r consisting of all those $\pi \in \Sigma_r$ which preserve the rows of T^λ is the parabolic subgroup Σ_λ.

Given $i \in I$, we define the λ-tableau T_i^λ as

$$T_i^\lambda = \begin{array}{cccccc} i_1 & i_2 & \dots & i_{\lambda_1} & & \\ i_{\lambda_1+1} & i_{\lambda_1+2} & \dots & \dots & \dots & i_{\lambda_1+\lambda_2} \\ \dots & & & & & \\ i_{\lambda_1+\dots\lambda_{n-1}+1} & \dots & \dots & \dots & i_r. & \end{array}$$

Then T_l^λ has only 1's in the first row, 2's in the second row, ..., n's in row n. Notice also that $T_{l(\nu,m)}^\lambda$ differs from T_l^λ only by the first m entries of row $\nu+1$: these entries are all equal to ν.

We say that a λ-tableau T_i^λ is row-semistandard if the entries in each row of T_i^λ are weakly increasing from left to right. We define

$$I^-(\lambda) := \left\{ i \in I \,\middle|\, i \leq l(\lambda) \text{ and } T_i^\lambda \text{ is row-semistandard } \right\}$$

and

$$I^+(\lambda) := \left\{ j \in I \,\middle|\, j \geq l(\lambda) \text{ and } T_j^\lambda \text{ is row-semistandard}\right\}.$$

The following obvious fact will be used later in this paper:

$$\text{If } \lambda_n \neq 0 \text{ and } m \leq \lambda_n, \text{ then } I^+(\lambda) = \{ j \in I^+(\lambda(n-1,m)) \,|\, j \geq l(\lambda)\}. \tag{2.1}$$

Next we recall the definition of Schur algebra and of Borel-Schur algebra as they were introduced in [12].

The general linear group $\mathrm{GL}_n(\mathbb{K})$ acts on $\mathbb{K}^n$ by multiplication. So $\mathrm{GL}_n(\mathbb{K})$ acts on the r-fold tensor product $(\mathbb{K}^n)^{\otimes r}$ by the rule

$$g(v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r, \text{ all } g \in \mathrm{GL}_n(\mathbb{K}),\ v_1, \ldots, v_r \in \mathbb{K}^n.$$

Extending by linearity this action to the group algebra $\mathbb{K}\mathrm{GL}_n(\mathbb{K})$, we obtain a homomorphism of algebras $T\colon \mathbb{K}\mathrm{GL}_n(\mathbb{K}) \to \mathrm{End}_{\mathbb{K}}\left((\mathbb{K}^n)^{\otimes r}\right)$. The image of T, i.e. $T(\mathbb{K}\mathrm{GL}_n(\mathbb{K}))$ is called the *Schur algebra* for $\mathbb{K}$, n, r and is denoted by $S(n,r)$. Let $B^+ = B^+_{\mathbb{K}}(n,r)$ denote the Borel subgroup of $\mathrm{GL}_n(\mathbb{K})$ consisting of all upper triangular matrices in $\mathrm{GL}_n(\mathbb{K})$. The *Borel-Schur algebra* $S(B^+) = S(B^+, n, r)$ is the subalgebra $T(\mathbb{K}B^+)$ of $S(n,r)$.

Associated with each pair $(i,j) \in I \times I$, there is a well defined element $\xi_{i,j}$ of $S(n,r)$ (see [12]). These elements have the property that $\xi_{i,j} = \xi_{k,h}$ if and only if $(i,j) \sim (k,h)$. If we eliminate repetitions in the set $\{ \xi_{i,j} \,|\, (i,j) \in I \times I\}$ then we obtain a basis of $S(n,r)$. Also $S(B^+) = \mathbb{K}\{ \xi_{i,j} \,|\, i \leq j,\ (i,j) \in I \times I\}$.

If i has weight $\alpha \in \Lambda$, we write $\xi_{i,i} = \xi_\alpha$. The set $\{ \xi_\alpha \,|\, \alpha \in \Lambda\}$ is a set of orthogonal idempotents and $1_{S(n,r)} = \sum_{\alpha \in \Lambda} \xi_\alpha$.

A formula for the product of two basis elements is the following (see [12]): $\xi_{i,j}\xi_{k,h} = 0$, unless $j \sim k$; and

$$\xi_{i,j}\xi_{j,h} = \sum_\sigma \left[\Sigma_{i\sigma,h} : \Sigma_{i\sigma,j,h}\right] \xi_{i\sigma,h} \tag{2.2}$$

where the sum is over a transversal $\{\sigma\}$ of the set of all double cosets $\Sigma_{i,j}\sigma\Sigma_{j,h}$ in Σ_j.

Observation 2.1. *1. $\xi_\alpha \xi_{i,j} = \xi_{i,j}$ or zero, according to $i \in \alpha$ or $i \notin \alpha$. Similarly, $\xi_{i,j}\xi_\beta = \xi_{i,j}$ or zero, according to $j \in \beta$ or $j \notin \beta$.*

2. Suppose in the formula (2.2) *there is only one double coset. Then one can take the only representative to be $\sigma = 1$. We see from* (2.2) *that the product $\xi_{i,j}\xi_{j,h}$ is a scalar multiple of $\xi_{i,h}$. The scalar is an element of the prime field.*

We are particularly interested in products of the type $\xi_{l(\nu,m),l}\xi_{l,j}$, for $l = l(\lambda)$, and $j \in I^+(\lambda)$, for some $\lambda \in \Lambda$.

Lemma 2.2. *Let $\lambda \in \Lambda$, $\nu \in \mathbf{n-1}$, $0 \leq m \leq \lambda_{\nu+1}$, and $j \in I^+(\lambda)$. If the $\nu+1$-st row of T_j^λ is constant then the double coset $\Sigma_{l(\nu,m),l}\, 1\, \Sigma_{l,j}$ coincides with Σ_l.*

Proof. We have $\Sigma_l = \Sigma_\lambda$. Now we know that $\Sigma_{l(\nu,m)}$ differs from Σ_λ only in factors ν and $\nu+1$ and in these it is

$$\Sigma_{\{t+1,\ldots,t+\lambda_\nu+m\}} \times \Sigma_{\{t+\lambda_\nu+m+1,\ldots,t+\lambda_\nu+\lambda_{\nu+1}\}},$$

where $t = \lambda_1 + \cdots + \lambda_{\nu-1}$. It follows that the intersection $\Sigma_{l(\nu,m),l}$ differs from Σ_λ only in factor $\nu+1$ and this is

$$\Sigma_{\{t+\lambda_\nu+1,\ldots,t+\lambda_\nu+m\}} \times \Sigma_{\{t+\lambda_\nu+m+1,\ldots,t+\lambda_\nu+\lambda_{\nu+1}\}}.$$

We can write $\Sigma_{l,j} = U_1 \times \cdots \times U_n$, where U_s is a subgroup of Σ_{λ_s}. Therefore the double coset $\Sigma_{l(\nu,m),l}\, 1\, \Sigma_{l,j}$ coincides with Σ_l as soon as the product of the two $(\nu+1)$-st factors is $\Sigma_{\lambda_{\nu+1}}$. This holds if $U_{\nu+1} = \Sigma_{\lambda_{\nu+1}}$, i.e., if the $(\nu+1)$-st row of T_j^λ is constant. □

Lemma 2.3. *Let $\lambda \in \Lambda$, $\nu \in \mathbf{n-1}$, $0 \leq m \leq \lambda_{\nu+1}$. Given $j \in I^+(\lambda)$, suppose that the $(\nu+1)$-st row of T_j^λ is constant with all entries equal to c, and that c occurs exactly a times in row ν. Then*

$$\xi_{l(\nu,m),l}\xi_{l,j} = \binom{a+m}{m}\xi_{l(\nu,m),j}.$$

If $\nu = n-1$ then the hypothesis holds for all $j \in I^+(\lambda)$.

Proof. From Lemma 2.2 and Observation 2.1, we know that

$$\xi_{l(\nu,m),l}\xi_{l,j} = \left[\Sigma_{l(\nu,m),j} : \Sigma_{l(\nu,m),l,j}\right] \xi_{l(\nu,m),j}.$$

Now $\Sigma_{l(\nu,m),j}$ and $\Sigma_{l(\nu,m),j,l}$ differ only in factors ν and $\nu+1$. If the entries of row ν of T^λ where c occurs in T_j^λ are $t_1, \ldots, t_a$, then factors ν and $\nu+1$ of $\Sigma_{l(\nu,m),j}$ and $\Sigma_{l(\nu,m),l,j}$ are, respectively,

$$\cdots \times \Sigma_{\{t_1,\ldots,t_a,\lambda_1+\cdots+\lambda_\nu+1,\ldots,\lambda_1+\cdots+\lambda_\nu+m\}} \times \cdots$$

and

$$\cdots \times \Sigma_{\{t_1,\ldots,t_a\}} \times \Sigma_{\{\lambda_1,\ldots,\lambda_\nu+1,\ldots,\lambda_1+\cdots+\lambda_\nu+m\}} \times \ldots.$$

Therefore $\left[\Sigma_{l(\nu,m),j} : \Sigma_{l(\nu,m),l,j}\right] = \binom{a+m}{m}$. □

Given $\lambda \in \Lambda$, let $\mathbb{K}_\lambda$ denote the one-dimensional $S(B^+)$-module $\mathbb{K}$, where ξ_λ acts as identity and all the other basis elements, $\xi_{i,j}$, where $i \leq j$ and $(i,j) \not\sim (l,l)$, act as zero. The following results were proved in [17].

Theorem 2.4. *1. $\{\mathbb{K}_\lambda \mid \lambda \in \Lambda\}$ is a full set of irreducible $S(B^+)$-modules.*

2. The module $S(B^+)\,\xi_\lambda$ is a projective cover of $\mathbb{K}_\lambda$.

3. The module $S(B^+)\,\xi_\lambda$ has a $\mathbb{K}$-basis $\{\xi_{i,l} \mid i \in I^-(\lambda)\}$.

4. The module $\mathbb{K}_\lambda$ is projective if and only if $\lambda = (r, 0, \ldots, 0)$. This is a consequence of $\#I^-(\lambda) = 1$ if and only if $\lambda = (r, 0, \ldots, 0)$.

As we see the simple $S(B^+)$-modules are one-dimensional. Hence $S(B^+)$ is a basic algebra. This makes many calculations easy. The projective module $S(B^+)\xi_\lambda$ has a composition series with one-dimensional quotients and hence the dimension is the same as the composition length. The composition factors are therefore completely described by $I^-(\lambda)$. The radical series of the indecomposable projective modules can be very complicated if n is large. However [17, Theorem 4.5] has determined minimal generators for the radical of $S(B^+)\xi_\lambda$ in general. These then allow the description of the first two steps of a minimal projective resolution of $\mathbb{K}_\lambda$. Define

$$P_0 := S\left(B^+\right)\xi_\lambda; \qquad P_1 := \begin{cases} \bigoplus_{\nu\in\mathbf{n-1}} S\left(B^+\right)\xi_{\lambda(\nu,1)}, & \text{if } \operatorname{char}\mathbb{K}=0; \\ \bigoplus_{\nu\in\mathbf{n-1}} \bigoplus_{1\le p^{d_\nu}\le\lambda_{\nu+1}} S\left(B^+\right)\xi_{\lambda\left(\nu,p^{d_\nu}\right)}, & \text{if } \operatorname{char}\mathbb{K}=p. \end{cases}$$

Then by [17, Theorem 5.4] the first two steps of a minimal projective resolution of $\mathbb{K}_\lambda$ are

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} \mathbb{K}_\lambda \to 0. \tag{2.3}$$

Here the $S(B^+)$-homomorphism p_0 is defined on the generator by $p_0(\xi_\lambda)=1$. The $S(B^+)$-homomorphism p_1 is defined on generators by $p_1(\xi_{\lambda(\nu,1)}) = \xi_{l(\nu,1),l}$, when $\operatorname{char}(\mathbb{K})=0$, and $p_1(\xi_{\lambda(\nu,p^{d_\nu})}) = \xi_{l(\nu,p^{d_\nu}),l}$, when $\operatorname{char}(\mathbb{K})=p$.

Notice that this determines the quiver $\mathcal{Q}$ of the algebra $S(B^+)$. The vertices of $\mathcal{Q}$ are given by the compositions $\lambda\in\Lambda(n,r)$. There is an arrow from the vertex λ to the vertex μ if and only if $S(B^+)\xi_\mu$ occurs as a summand of P_1. Let $\mathbb{K}\mathcal{Q}$ be the path algebra of the quiver $\mathcal{Q}$. We write e_λ for the idempotent of $\mathbb{K}\mathcal{Q}$ that corresponds to the empty path based at the vertex λ. Since the algebra $S(B^+)$ is basic, there is a canonical epimorphims $\pi\colon \mathbb{K}\mathcal{Q}\to S(B^+)$ defined by $\pi(e_\lambda)=\xi_\lambda$ and by sending the arrow starting at λ and ending at $\lambda(\nu,p^d)$ to $\xi_{l(\lambda(\nu,p^d)),l(\lambda)}$.

Next we will describe the quiver presentation of $S(B^+,2,r)$. This will be used in Section 6.

To simplify the notation we will identify the composition (λ_1,λ_2) with λ_2. We have an arrow from the vertex t to vertex s if and only if $t-s=p^d$ for some natural number d if $\operatorname{char}\mathbb{K}=p$, and $t-s=1$ if $\operatorname{char}\mathbb{K}=0$. We denote this arrow by $\alpha_{t,d}$ in characteristic p and α_t in characteristic 0. Our aim is to describe the kernel of the canonical epimorphim $\pi\colon\mathbb{K}\mathcal{Q}\to S(B^+,2,r)$ defined above.

Lemma 2.5. *The dimension of* $\xi_\mu S(B^+,2,r)\xi_\lambda$ *is* 1 *if* $\lambda\trianglelefteq\mu$ *and* 0 *otherwise.*

Proof. By Theorem 2.4 the set $\left\{\xi_{i,l(\lambda)}\middle| i\in I^-(\lambda),\ i\in\mu\right\}$ is a basis of $\xi_\mu S(B^+,2,r)\xi_\lambda$. This is obviously the empty set unless $\lambda\trianglelefteq\mu$ and contains only the element $\xi_{l(\mu),l(\lambda)}$ if $\lambda\trianglelefteq\mu$. □

Proposition 2.6. *(a) If* $\operatorname{char}\mathbb{K}=0$ *the epimorphism* $\pi\colon\mathbb{K}\mathcal{Q}\to S(B^+,2,r)$ *is an isomorphism of algebras.*

(b) If $\operatorname{char}\mathbb{K}=p$ *the kernel of* $\pi\colon\mathbb{K}\mathcal{Q}\to S(B^+,2,r)$ *is generated as an ideal by*

$$\alpha_{s-(p-1)p^d,d}\,\alpha_{s-(p-2)p^d,d}\cdots\alpha_{s,d},\ \textit{where } d\in\mathbb{N},\ s-p^{d+1}\ge 0, \tag{2.4}$$

$$\alpha_{s-p^{d_1},d_2}\,\alpha_{s,d_1}-\alpha_{s-p^{d_2},d_1}\,\alpha_{s,d_2},\ \textit{where } d_1<d_2 \textit{ and } s-p^{d_1}-p^{d_2}\ge 0. \tag{2.5}$$

Proof. (a) If $\operatorname{char}\mathbb{K} = 0$ the quiver $\mathcal{Q}$ is of type A_{r+1}

$$0 \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} 2 \cdots\cdots r-2 \xleftarrow{\alpha_{r-1}} r-1 \xleftarrow{\alpha_r} r .$$

It is clear that $\dim_{\mathbb{K}} e_s\mathbb{K}\mathcal{Q}e_t$ is 1 or 0 according to $s \leq t$ or $s > t$. Now the result follows from the definition of π and Lemma 2.5.

(b) Suppose $\operatorname{char}\mathbb{K} = p$. Denote by $\mathcal{I}$ the ideal generated by (2.4) and (2.5). Let $A := \mathbb{K}\mathcal{Q}/\mathcal{I}$. We show first that

$$\dim_{\mathbb{K}}(e_sAe_t) = 0 \text{ if } s > t; \quad \dim_{\mathbb{K}}(e_sAe_t) \leq 1 \text{ if } s \leq t. \tag{2.6}$$

The first equality is obvious since the labels decrease along every path in $\mathcal{Q}$. Define the *degree* of the arrow $\alpha_{k,d}$ as p^d. Then we can define the *degree* of a path in $\mathcal{Q}$ as the sum of the degrees of the arrows in the path. It is immediate that all paths from vertex t to vertex s have the same degree $t - s$. Using (2.5), every element in A can be written as a linear combination of paths

$$\alpha_{t_1,d_1}\alpha_{t_2,d_2}\dots\alpha_{t_k,d_k} \tag{2.7}$$

with $d_1 \geq d_2 \geq \cdots \geq d_k$ and $t_j = t_{j+1} - p^{d_{j+1}}$ for all $1 \leq j \leq k-1$. Denote by m_d the number of occurrences of the natural number d among $d_1, \dots, d_k$. In view of (2.4) we can assume that $m_d \leq p-1$ for every d. Then the degree of the path (2.7) is $\sum_{d\geq 0} m_dp^d$. But $\sum_{d\geq 0} m_dp^d$ is the p-adic expansion of the degree of the path (2.7). This shows that the multiplicities m_d are determined by the starting $t = t_k$ and endindg $s = t_1 - p^{d_1}$ points of the path (2.7). Therefore, given vertices s and t with $s \leq t$, there is exactly one path from t to s of the form (2.7) such that $d_1 \geq \cdots \geq d_k$ and each multiplicity m_d of d among $d_1, \dots, d_k$ does not exceed $p-1$. Hence $\dim_{\mathbb{K}} e_sAe_t \leq 1$.

Next we show that $\mathcal{I} \subset \ker\pi$. It follows from the definition of π that

$$\pi(e_s) = \xi_{(r-s,s)}, \quad \pi(\alpha_{s,d}) = \xi_{l(r-s+p^d,s-p^d),l(r-s,s)}.$$

Using Lemma 2.3, when we apply π to each summand in (2.5) we obtain

$$\binom{p^{d_1}+p^{d_2}}{p^{d_1}}\xi_{l(r-s+p^{d_1}+p^{d_2},s-p^{d_1}-p^{d_2}),l(r-s,s)}.$$

Now we apply π to (2.4). The result is

$$\binom{2p^d}{p^d}\binom{3p^d}{p^d}\cdots\binom{p\cdot p^d}{p^d}\xi_{l(r-s+p^{d+1},s-p^{d+1}),l(r-s,s)}. \tag{2.8}$$

Since p divides $\binom{p^{d+1}}{p^d}$, we get that (2.8) is 0. This shows that $\mathcal{I} \subset \ker\pi$. Now, comparing the dimensions of e_sAe_t and $\pi(e_s)S(B^+,2,r)\pi(e_t)$, we see that π induces an injective map from A to $S(B^+,2,r)$. Since π is surjective, we get that this is indeed an isomorphism, that is $\mathbb{K}\mathcal{Q}/\mathcal{I} \cong S(B^+,2,r)$. □

3 Auslander-Reiten sequences

In this section we give an overview of some results and definitions connected to the notion of Auslander-Reiten sequences. Let A be a finite-dimensional algebra over $\mathbb{K}$.

A short exact sequence

$$(E) \quad 0 \to N \xrightarrow{f} E \xrightarrow{g} S \to 0$$

is said to be *Auslander-Reiten* if

(i) (E) is not split;

(ii) the modules S and N are indecomposable;

(iii) if X is an indecomposable A-module and $h\colon X \to S$ is a non-invertible homomorphism of A-modules, then h factors through g.

Theorem 3.1 ([1]). *Given any non-projective indecomposable A-module S, there is an Auslander-Reiten sequence (E) ending with S. Moreover, (E) is determined by S, uniquely up to isomorphism of short exact sequences.*

In this paper we will construct an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$, for a large number of $\lambda \in \Lambda(n,r)$. We will use two contravariant functors

$$D,\ (\cdot)^t\colon A{-}\mathrm{mod} \to A^{op}{-}\mathrm{mod}$$

where for every $X \in A{-}\mathrm{mod}$

$$X^t := \mathrm{Hom}_A(X, A), \qquad\qquad DX := \mathrm{Hom}_{\mathbb{K}}(X, \mathbb{K}).$$

Recall that A acts on the right of X^t and DX, respectively, by $(\phi\xi)(x) = \phi(x)\xi$ and $(\psi\xi)(x) = \psi(\xi x)$, where $\phi \in X^t$, $\psi \in DX$, $\xi \in A$, and $x \in X$.

Consider the Nakayama functor [9, p.10]

$$D(\cdot)^t\colon A{-}\mathrm{mod} \to A{-}\mathrm{mod}.$$

This is a covariant right exact functor which turns projectives into injectives. Let X be an indecomposable non-projective A-module. Consider the first two steps of a minimal projective resolution of X

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X \to 0.$$

Applying the Nakayama functor we get from this the exact sequence

$$0 \to \tau X \to DP_1^t \xrightarrow{Dp_1^t} DP_0^t \xrightarrow{Dp_0^t} DX^t \to 0. \tag{3.1}$$

The kernel τX of Dp_1^t is called the *Auslander-Reiten translation* of X. It is shown on pages 5-6 of [9] that it is possible to select an A-homomorphism $\theta\colon X \to DP_0^t$ such that the short exact sequence obtained from (3.1) by pullback along θ

$$0 \to \tau X \xrightarrow{f} E(\theta) \xrightarrow{g} X \to 0 \tag{3.2}$$

is an Auslander-Reiten sequence. Note that

$$E(\theta) = \left\{ (z, c) \in DP_1^t \oplus X \,\middle|\, Dp_1^t(z) = \theta(c) \right\}$$

is an A-submodule of $DP_1^t \oplus X$, and f, g are the A-homomorphisms defined by $g(z, c) = c$, $f(v) = (v, 0)$, for all $z \in DP_1^t$, $v \in \tau X$, and $c \in X$. Several recipes were given in the 80's for the construction of θ with the above property, see [4, 9, 11], but we do not need to use them in this article.

Now we consider $A = S(B^+)$ and $X = \mathbb{K}_\lambda$ with $\lambda \in \Lambda(n, r)$, $\lambda \neq (r, 0, \ldots, 0)$. It is clear that $\mathbb{K}_\lambda$ is indecomposable and non-projective. We will use the minimal projective resolution (2.3) of $\mathbb{K}_\lambda$. Since

$$\operatorname{soc}(DP_0^t) \cong D(\operatorname{hd} P_0^t) \cong D(\operatorname{hd}(\xi_\lambda S(B^+))) \cong \mathbb{K}_\lambda,$$

we have

$$\dim_{\mathbb{K}} \operatorname{Hom}_{S(B^+)}(\mathbb{K}_\lambda, DP_0^t) = 1. \tag{3.3}$$

Therefore we can take for $\theta \colon \mathbb{K}_\lambda \to DP_0^t$ any non-zero $S(B^+)$-homomorphism.

Before constructing Auslander-Reiten sequences, we will determine $\mathbb{K}$-bases of P_0 and P_1 adapted to our calculations. As $(S(B^+)\xi_\alpha)^t$ and $\xi_\alpha S(B^+)$ are isomorphic right $S(B^+)$-modules for every $\alpha \in \Lambda$, we will identify these two modules. We will also identify $\left(\bigoplus_{\alpha \in \Lambda'} S(B^+)\xi_\alpha\right)^t$ with $\bigoplus_{\alpha \in \Lambda'} \xi_\alpha S(B^+)$, for every family Λ' of elements in Λ.

Lemma 3.2. *Let $\alpha \in \Lambda$. Then $\left\{ \xi_{l(\alpha),j} \,\middle|\, j \in I^+(\alpha) \right\}$ is a $\mathbb{K}$-basis of $\xi_\alpha S(B^+)$.*

Proof. We know that $\xi_\alpha S(B^+)$ is spanned by $\left\{ \xi_{l(\alpha),j} \,\middle|\, j \in I(n, r),\ j \geq l(\alpha) \right\}$. As $\xi_{l(\alpha),j} = \xi_{l(\alpha),i}$ if and only if $i\pi = j$, for some π in the stabilizer of $l(\alpha)$ in Σ_r and this stabilizer coincides with the row stabilizer of T^α, the result follows. □

Fix $\lambda \in \Lambda(n, r)$ and consider the result of the application of $(\cdot)^t$ to (2.3). Then $P_1^t \cong \bigoplus_{\nu=1}^{n-1} \xi_{\lambda(\nu,1)} S(B^+)$, or $P_1^t \cong \bigoplus_{\nu=1}^{n-1} \bigoplus_{1 \leq p^{d_\nu} \leq \lambda_{\nu+1}} \xi_{\lambda\left(\nu, p^{d_\nu}\right)} S(B^+)$, according as $\operatorname{char} \mathbb{K} = 0$, or $\operatorname{char} \mathbb{K} = p$. Thus a $\mathbb{K}$-basis of P_1^t is given by

$$B_1 := \left\{ \xi_{l(\nu,1),j} \,\middle|\, j \in I^+(\lambda(\nu, 1)),\ \nu \in \mathbf{n-1} \right\}, \qquad \text{if } \operatorname{char} \mathbb{K} = 0,$$

$$B_2 := \left\{ \xi_{l(\nu,p^{d'}),j} \,\middle|\, \begin{array}{l} j \in I^+(\lambda(\nu, p^{d'})), \\ 1 \leq p^{d'} \leq \lambda_{\nu+1},\ \nu \in \mathbf{n-1} \end{array} \right\}, \quad \text{if } \operatorname{char} \mathbb{K} = p. \tag{3.4}$$

With the above identifications of the projective modules, the map $p_1^t \colon P_0^t \to P_1^t$ becomes $p_1^t(\eta) = \sum_{\nu=1}^{n-1} \xi_{l(\nu,1),l}\eta$, or $p_1^t(\eta) = \sum_{\nu=1}^{n-1} \sum_{1 \leq p^{d'} \leq \lambda_{\nu+1}} \xi_{l(\nu,p^{d'}),l}\eta$, according as $\operatorname{char} \mathbb{K} = 0$, or $\operatorname{char} \mathbb{K} = p$.

To construct an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$, it is convenient to obtain, from B_1 and B_2, new bases for P_1^t containing $p_1^t\left(\xi_{l(\lambda),j}\right)$, $j \in I^+(\lambda)$. Suppose λ satisfies conditions

$$\begin{cases} \lambda_n \neq 0, & \text{if } \operatorname{char} \mathbb{K} = 0, \\ \lambda_n \neq 0,\ \lambda_{n-1} < p^{d+1} - 1, & \text{if } \operatorname{char} \mathbb{K} = p \text{ and } p^d \leq \lambda_n < p^{d+1}. \end{cases} \tag{3.5}$$

Given $j \in I^+(\lambda)$, as $j \geq l = l(\lambda)$, the nth row of T_j^λ is constant with all entries equal to n, and its $(n-1)$st row has a entries equal to n and $\lambda_{n-1} - a$ entries equal to $n-1$, for some $0 \leq a \leq \lambda_{n-1}$.

We shall look first at the case $\operatorname{char} \mathbb{K} = 0$. Then by Lemma 2.3

$$\xi_{l(n-1,1),l}\xi_{l,j} = (a+1)\,\xi_{l(n-1,1),j},$$

and $a+1 \neq 0$. Using this, we shall show that we can replace $\xi_{l(n-1,1),j}$ by $p_1^t(\xi_{l,j})$ in B_1 and obtain a new basis for P_1^t. Notice that $I^+(\lambda) \subset I^+(\lambda(n-1,1))$, and so $\xi_{l(n-1,1),j} \in B_1$. On the other hand, $\xi_{l,j} \in P_0^t = \xi_\lambda S(B^+)$, and

$$p_1^t(\xi_{l,j}) = \xi_{l(n-1,1),l}\xi_{l,j} + \sum_{\nu=1}^{n-2} \xi_{l(\nu,1),l}\xi_{l,j} = (a+1)\,\xi_{l(n-1,1),j} + \sum_{\nu=1}^{n-2} \xi_{l(\nu,1),l}\xi_{l,j}.$$

Now $\xi_{l(\nu,1),l}\xi_{l,j}$ is a linear combination of basis elements of the type $\xi_{l(\nu,1),j\pi}$, for some $\pi \in \Sigma_r$. As, for $\nu = 1, \ldots, n-2$, we have that $l(\nu,1) \not\sim l(n-1,1)$, we get that $\xi_{l(n-1,1),j}$ is always different from $\xi_{l(\nu,1),i}$, for any $i \in I(n,r)$. Therefore, we can replace $\xi_{l(n-1,1),j}$ in B_1 by $p_1^t(\xi_{l,j})$ and still get a basis for P_1^t. We have proved the following:

Proposition 3.3. *If* $\operatorname{char} \mathbb{K} = 0$ *and* $\lambda_n \neq 0$, *then*

$$\begin{aligned}\overline{B}_1 = &\left\{ \xi_{l(\nu,1),j} \,\middle|\, j \in I^+(\lambda(\nu,1)),\ \nu = 1, \ldots, n-2 \right\} \\ &\cup \left\{ \xi_{l(n-1,1),j} \,\middle|\, j \in I^+(\lambda(n-1,1)) \setminus I^+(\lambda) \right\} \cup \left\{ p_1^t(\xi_{l,j}) \,\middle|\, j \in I^+(\lambda) \right\}\end{aligned}$$

is a basis of P_1^t. *In particular,* p_1^t *is a monomorphism.*

Suppose now that $\operatorname{char} \mathbb{K} = p$ and that λ satisfies condition (3.5). We will apply Lemma 2.3, together with the following well known consequence of Lucas' Theorem:

Proposition 3.4. *Assume* m *and* q *are positive integers and that*

$$m = m_0 + m_1 p + \cdots + m_t p^t \quad \text{and} \quad q = q_0 + q_1 p + \cdots + q_s p^s$$

are the p*-adic expansions of* m *and* q*. Then* p *divides* $\binom{m}{q}$ *if and only if* $m_\nu < q_\nu$ *for some* ν*.*

Given $j \in I^+(\lambda)$, we denote by $a = a(j)$ the number of entries equal to n in the $(n-1)$st row of T_j^λ. Let $a_0, \ldots, a_d$ be the coefficients in the p-adic expansion of a, with a_d possibly equal to zero. As $a \leq \lambda_{n-1} < p^{d+1} - 1$, and all the coefficients in the p-adic expansion of $p^{d+1} - 1$ are equal to $p-1$, we get that there is some $a_t \neq p-1$. Let

$$m(j) := \min\{\, t \mid a_t < p-1 \,\} \tag{3.6}$$

and define

$$I^+(\lambda, d') = \left\{ j \in I^+(\lambda) \,\middle|\, m(j) = d' \right\}.$$

Obviously $I^+(\lambda) = \dot{\bigcup}_{0 \leq d' \leq d} I^+(\lambda, d')$.

Proposition 3.5. *Suppose that* $\operatorname{char}\mathbb{K} = p$ *and that* $\lambda \in \Lambda(n,r)$ *satisfies condition* (3.5). *Then*

$$\begin{aligned}\overline{B}_2 = &\left\{ \xi_{l\left(\nu,p^{d'}\right),j} \,\middle|\, j \in I^+(\lambda(\nu,p^{d'})),\ 1 \le p^{d'} \le \lambda_{\nu+1},\ \nu = 1,\dots,n-2 \right\} \\ &\cup \left\{ \xi_{l\left(n-1,p^{d'}\right),j} \,\middle|\, j \in I^+(\lambda(n-1,p^{d'})) \setminus I^+(\lambda,d'),\ 0 \le d' \le d \right\} \\ &\cup \left\{ p_1^t(\xi_{l,j}) \,\middle|\, j \in I^+(\lambda) \right\}\end{aligned}$$

is a $\mathbb{K}$*-basis for* P_1^t*. In particular,* p_1^t *is a monomorphism.*

Proof. Let $j \in I^+(\lambda)$. Just like in the characteristic zero case, we consider the basis element $\xi_{l,j}$ of P_0^t and look at

$$p_1^t(\xi_{l,j}) = \sum_{\nu=1}^{n-1} \sum_{1 \le p^{d'} \le \lambda_{\nu+1}} \xi_{l\left(\nu,p^{d'}\right),l}\xi_{l,j}.$$

By Lemma 2.3 for any d' such that $0 \le p^{d'} \le \lambda_n$ we have

$$\xi_{l\left(n-1,p^{d'}\right),l}\xi_{l,j} = \binom{a+p^{d'}}{p^{d'}} \xi_{l\left(n-1,p^{d'}\right),j}.$$

Here a is the number of times n occurs in row $n-1$ of T_j^λ. It follows from Proposition 3.4 and the definition of $m(j)$ (see (3.6)), that p does not divide $\binom{a+p^{m(j)}}{p^{m(j)}}$ and divides all $\binom{a+p^{d'}}{p^{d'}}$ for $d' < m(j)$. Therefore

$$p_1^t(\xi_{l,j}) = \sum_{\nu=1}^{n-2} \sum_{1 \le p^{d'} \le \lambda_{\nu+1}} \xi_{l\left(\nu,p^{d'}\right),l}\xi_{l,j} + \sum_{p^{m(j)} \le p^{d'} \le \lambda_n} \binom{a+p^{d'}}{p^{d'}} \xi_{l\left(n-1,p^{d'}\right),j} \tag{3.7}$$

and the coefficient of $\xi_{l\left(n-1,p^{m(j)}\right),j}$ in this sum is non-zero. As, for $\nu \ne n-1$, we have $l\left(n-1,p^{m(j)}\right) \not\sim l\left(\nu,p^{d'}\right)$ it follows that $\xi_{l(n-1,p^{m(j)}),j}$ does not appear in the basis expansion of $\xi_{l\left(\nu,p^{d'}\right),l}\xi_{l,h}$, for any $h \in I^+(\lambda)$. Also, if $d' \ne m(j)$, then $l\left(n-1,p^{d'}\right) \not\sim l\left(n-1,p^{m(j)}\right)$ and so $\xi_{l\left(n-1,p^{d'}\right),h} \ne \xi_{l\left(n-1,p^{m(j)}\right),j}$ for all $h \in I^+(\lambda)$. Finally, suppose $h \in I^+(\lambda)$ satisfies $(\,l(n-1,p^{m(j)}),j\,) \sim (\,l(n-1,p^{m(h)}),h\,)$. Then $h = j\pi$ for some $\pi \in \Sigma_{\lambda\left(n-1,p^{m(j)}\right)}$. But, since both $h, j \ge l$, we cannot move any entry $n-1$ in row $(n-1)$ of T_j^λ to row n to obtain T_h^λ. This implies that π belongs to the row stabilizer of T^λ. As both T_j^λ and T_h^λ are row semistandard we get $h = j$. Therefore $\xi_{l\left(n-1,p^{m(j)}\right),j}$ appears only once in $\overline{B}_2$: in the expression (3.7) of $p_1^t(\xi_{l,j})$ with the coefficient $\binom{a+p^{m(j)}}{p^{m(j)}}$. Hence, we can replace $\xi_{l\left(n-1,p^{m(j)}\right),j}$ by $p_1^t(\xi_{l,j})$ in B_2 for all $j \in I^+(\lambda)$ and still have a basis for P_1^t. □

It is now easy to obtain an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$ for λ satisfying (3.5). Denote, respectively, by B_1^* and B_2^* the $\mathbb{K}$-basis of DP_1^t dual to $\overline{B}_1$ and $\overline{B}_2$. For $j \in I^+(\lambda)$, we denote by $z_{l,j}$ the element in B_1^* (respectively, in B_2^*) that is dual to

$p_1^t(\xi_{l,j})$. Let U_λ be the subspace of DP_1^t with $\mathbb{K}$-basis $B_1^* \setminus \{ z_{l,j} \mid j \in I^+(\lambda)\}$ if $\operatorname{char}\mathbb{K} = 0$ or $B_2^* \setminus \{ z_{l,j} \mid j \in I^+(\lambda)\}$ if $\operatorname{char}\mathbb{K} = p$. Then U_λ is in fact a $S(B^+)$-submodule. Define

$$E(\lambda) = \left\{ (z, c) \in DP_1^t \oplus \mathbb{K}_\lambda \,\middle|\, z \in (U_\lambda + c z_{l,l}) \right\}.$$

Then we have the following result.

Theorem 3.6. *Suppose that $\lambda \in \Lambda(n, r)$ satisfies* (3.5). *Then the sequence*

$$0 \to U_\lambda \xrightarrow{f} E(\lambda) \xrightarrow{g} \mathbb{K}_\lambda \to 0, \tag{3.8}$$

where f and g are defined by $f(z) = (z, 0)$ and $g(z', c) = c$, for all $z \in U_\lambda$, $(z', c) \in E(\lambda)$, is an Auslander-Reiten sequence.

Proof. Notice first that $\mathbb{K}_\lambda$ is not projective, since $\lambda \neq (r, 0, \ldots, 0)$. Hence an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$ exists.

By (3.3) the dimension of $\operatorname{Hom}_{S(B^+)}(\mathbb{K}_\lambda, DP_0^t)$ is one. Therefore, for any non-zero $\theta \in \operatorname{Hom}_{S(B^+)}(\mathbb{K}_\lambda, DP_0^t)$, the sequence

$$0 \to \tau\mathbb{K}_\lambda \xrightarrow{f} E(\theta) \xrightarrow{g} \mathbb{K}_\lambda \to 0 \tag{3.9}$$

is an Auslander-Reiten sequence. We will consider θ defined by

$$\theta(c)(\eta) = \eta c, \text{ for all } \eta \in P_0^t = \xi_\lambda S(B^+) \text{ and all } c \in \mathbb{K}_\lambda.$$

Note that as P_0^t has $\mathbb{K}$-basis $\{ \xi_{l,j} \mid j \in I^+(\lambda)\}$ and, for $j \in I^+(\lambda)$ and $c \in \mathbb{K}_\lambda$, $\xi_{l,j}c = c$ or 0, according as $j = l$ or $j \neq l$, we have that θ is completely determined by saying that $\theta(c)(\xi_{l,j}) = c$ if $j = l$, and 0 otherwise. Given $z \in DP_1^t$ we can write z as a linear combination of the elements of B_1^* or B_2^*, according to $\operatorname{char}\mathbb{K} = 0$ or $\operatorname{char}\mathbb{K} = p$. Then, for any $c \in \mathbb{K}_\lambda$, we have $Dp_1^t(z) = \theta(c)$ if and only if $zp_1^t = \theta(c)$, which in turn holds if and only if for all $j \in I^+(\lambda)$ there holds $zp_1^t(\xi_{l,j}) = c$ if $j = l$, and 0 otherwise. Thus $z = cz_{l,l} + u$ for some $u \in U_\lambda$. Hence

$$E(\theta) = \left\{ (z, c) \in DP_1^t \oplus \mathbb{K}_\lambda \,\middle|\, Dp_1^t(z) = \theta(c) \right\} = E(\lambda).$$

In a similar way, we see that $z \in \tau\mathbb{K}_\lambda = \ker Dp_1^t$ if and only if $zp_1^t = 0$, that is if and only if $z \in U_\lambda$. Therefore $\tau\mathbb{K}_\lambda = U_\lambda$. □

Remark 3.7. We have explained that any non-zero homomorphism from the simple module $\mathbb{K}_\lambda$ into DP_0^t gives an Auslander-Reiten sequence. In particular if we replace θ by $c\theta$, where c is a non-zero scalar, then this gives the same Auslander-Reiten sequence. In fact we can say more. Recall the Auslander-Reiten formula. For any modules X, Y of some algebra, we have (see, for example, Theorem 2.20 in [11])

$$\operatorname{Ext}^1(X, \tau Y) \cong D\underline{\operatorname{Hom}}(Y, X).$$

Here $\underline{\operatorname{Hom}}(U, V)$ is the quotient space of $\operatorname{Hom}(U, V)$ modulo homomorphisms which factor through a projective module. We apply this with $X = Y = \mathbb{K}_\lambda$. Then the right hand side is trivially one-dimensional. Hence $\operatorname{Ext}^1(\mathbb{K}_\lambda, \tau\mathbb{K}_\lambda) \cong \mathbb{K}$. Therefore, by the previous observation, if we have a non-split exact sequence with end terms $\mathbb{K}_\lambda$ and $\tau\mathbb{K}_\lambda$ this must be an Auslander-Reiten sequence.

We have constructed Auslander-Reiten sequences ending with $\mathbb{K}_\lambda$, for every $\lambda \in \Lambda(n,r)$ satisfying conditions (3.5). For this we only need to deal with the multiplication of basis elements of $S(B^+)$ where the formula in Lemma 2.3 can be used. For λ not satisfying (3.5), the calculations for multiplication of basis elements get very tortuous, with many particular cases to consider, and the method we use does not work well in the construction of the desired sequences. We will give throughout treatment of the case $n = 2$ in Section 4. For $n = 3$, we know some examples, but the calculation get quickly out of control, so we do not list them.

Now we will look at the following problem. Given the Auslander-Reiten sequence (3.8), one would like to know when the module $E(\lambda)$ is indecomposable. This seems to be a difficult question in general, as one can see for the Borel-Schur algebras of finite type (see Section 6). In fact one of the motivations for our classification was to get a better understanding of this question. We have two easy observations, which deal with most of the cases when the algebra has finite type. The first one involves the indecomposability of the module P_1. Notice that P_1 is indecomposable if and only if $\lambda = (\lambda_1, 0, \cdots, 0, \lambda_\nu, 0, \cdots, 0)$, for some $2 \le \nu \le n$, $\lambda_\nu \ge 1$, if $\operatorname{char}\mathbb{K} = 0$, and $1 \le \lambda_\nu < p$, if $\operatorname{char}\mathbb{K} = p$.

Proposition 3.8. *Given $\lambda \in \Lambda(n,r)$, assume the module P_1 is indecomposable. Then the middle term $E(\lambda)$ is indecomposable.*

Proof. We construct $E(\lambda)$ as a pullback, and hence we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau\mathbb{K}_\lambda & \longrightarrow & E(\lambda) & \longrightarrow & \mathbb{K}_\lambda & \longrightarrow & 0 \\ & & {\scriptstyle 1}\downarrow & & {\scriptstyle \tilde{\theta}}\downarrow & & {\scriptstyle \theta}\downarrow & & \\ 0 & \longrightarrow & \tau\mathbb{K}_\lambda & \longrightarrow & DP_1^t & \xrightarrow{Dp_1^t} & DP_0^t & & \end{array} \tag{3.10}$$

By the Snake Lemma, the map $\tilde{\theta}$ is injective. Since P_1 is indecomposable, the module DP_1^t is indecomposable injective and hence has a simple socle. Therefore the socle of $E(\lambda)$ is simple, and the module is indecomposable. □

Note that if $n = 2$ and $\operatorname{char}\mathbb{K} = 0$, then P_1 is always indecomposable. Therefore we have the following result.

Corollary 3.9. *If $n = 2$ and $\operatorname{char}\mathbb{K} = 0$, then the middle term $E(\lambda)$ is always indecomposable.*

We can also identify from (3.10) the Auslander-Reiten sequence for $\mathbb{K}_\lambda$ when $\lambda = (0, 0, \ldots, r)$. This is the unique simple module which is injective. It follows that $DP_0^t \cong \mathbb{K}_\lambda$ and the map θ is an isomorphism. In this case the map Dp_1^t must be onto, and then the Auslander-Reiten sequence is equivalent to the exact sequence which is the bottom row of the diagram (3.10).

4 Auslander-Reiten sequences for $n = 2$

In this section we study the construction of an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$ in the particular case of $n = 2$. We will show that it is very easy to obtain such sequences with no restriction on λ or the characteristic of $\mathbb{K}$. We will also collect some facts on the socle of $S(B^+, n, r)$, for general n, which are easy consequences of the construction of Auslander-Reiten sequence we have done.

Let $\lambda = (\lambda_1, \lambda_2)$. Since $\mathbb{K}_\lambda$ is non-projective if and only if $\lambda_2 \neq 0$, all compositions we are interested in satisfy this condition. So we assume that $\lambda_2 \neq 0$. In particular, the construction of Auslander-Reiten sequences in the characteristic zero and $n = 2$ case is completely answered in Theorem 3.6.

Suppose now that $\operatorname{char}\mathbb{K} = p$ and d is such that $p^d \leq \lambda_2 < p^{d+1}$. Given $j \in I^+(\lambda)$, recall that $a(j)$ is the number of 2's in the first row of T_j^λ. If

$$a = a(j) = (p-1) + (p-1)p + \cdots + (p-1)p^d + \ldots \tag{4.1}$$

is the p-adic expansion of a then, by Proposition 3.4, for all $0 \leq d' \leq d$ the binomial coefficient $\binom{a+p^{d'}}{p^{d'}}$ is divisible by p. Hence

$$p_1^t(\xi_{l,j}) = \sum_{d'=0}^{d} \xi_{l\left(1,p^{d'}\right),l}\xi_{l,j} = \sum_{d'=0}^{d} \binom{a+p^{d'}}{p^{d'}} \xi_{l\left(1,p^{d'}\right),j} = 0.$$

Next we suppose that $a = a(j)$ has p-adic expansion $a = a_0 + a_1 p + \cdots + a_s p^s$, with $a_t \neq p-1$ for some $0 \leq t \leq d$. Define $m(j) = \min\{\, t \,|\, t \leq d \text{ and } a_t < p-1\}$ and

$$\widehat{I}(\lambda) = \left\{\, j \in I^+(\lambda) \,\middle|\, a(j) \neq (p-1) + (p-1)p + \cdots + (p-1)p^d + \ldots \right\}.$$

For $0 \leq d' \leq d$ we denote by $\widehat{I}(\lambda, d')$ the subset of those $j \in \widehat{I}(\lambda)$ such that $m(j) = d'$. Then $\widehat{I}(\lambda) = \dot{\bigcup}_{0\leq d'\leq d}\widehat{I}(\lambda, d')$ and $\widehat{I}(\lambda, d') \subset I^+(\lambda(1, p^{d'}))$. Now with a proof completely analogous to the proof of Proposition 3.5, we see that, for $j \in \widehat{I}(\lambda, d')$, the element $\xi_{l(1,p^{m(j)}),j}$ in B_2 can be replaced by $p_1^t(\xi_{l,j})$ and the resulting set $\overline{B}_2$ is a new basis for P_1^t. This proves the following result.

Proposition 4.1. *Suppose that* $\operatorname{char}\mathbb{K} = p$ *and* $\lambda = (\lambda_1, \lambda_2)$*, with* $\lambda_2 \neq 0$*. Then*

$$\begin{aligned}\overline{B}_2 = &\left\{\, \xi_{l\left(1,p^{d'}\right),j} \,\middle|\, j \in I^+(\lambda(1,p^{d'})) \setminus \widehat{I}(\lambda, d'),\ 0 \leq d' \leq d \right\}\\ &\cup \left\{\, p_1^t(\xi_{l,j}) \,\middle|\, j \in \widehat{I}(\lambda) \right\}\end{aligned}$$

is a $\mathbb{K}$*-basis for* P_1^t*.*

We also have that $\left\{\, \xi_{l,j} \,\middle|\, j \in I^+(\lambda) \setminus \widehat{I}(\lambda) \right\}$ *is a* $\mathbb{K}$*-basis for* $\ker(p_1^t)$*. In particular,* p_1^t *is injective if and only if* $p^d \leq \lambda_2 < p^{d+1}$ *and* $\lambda_1 < p^{d+1} - 1$*, i.e., if and only if* λ *satisfies condition 3.5.*

Denote by B_2^* the basis of DP_1^t dual to $\overline{B}_2$. We write $z_{l,j}$ for the element dual to $p_1^t(\xi_{l,j})$, where $j \in \widehat{I}(\lambda)$. Let U_λ be the $S(B^+)$-submodule of DP_1^t with $\mathbb{K}$-basis $B_2^* \setminus \left\{\, z_{l,j} \,\middle|\, j \in \widehat{I}(\lambda) \right\}$ and

$$E(\lambda) = \left\{\, (z, c) \in DP_1^t \oplus \mathbb{K}_\lambda \,\middle|\, z \in (U_\lambda + c z_{l,l}) \right\}.$$

Then, adapting the proof of Theorem 3.6, we can conclude the following result.

Theorem 4.2. *Suppose that* $\operatorname{char}\mathbb{K} = p$ *and* $\lambda = (\lambda_1, \lambda_2)$, *with* $\lambda_2 \neq 0$. *Then the sequence*

$$0 \to U_\lambda \xrightarrow{f} E(\lambda) \xrightarrow{g} \mathbb{K}_\lambda \to 0,$$

where f *and* g *are defined by* $f(z) = (z, 0)$ *and* $g(z', c) = c$, *for all* $z \in U_\lambda$, $(z', c) \in E(\lambda)$, *is an Auslander-Reiten sequence.*

Now we return to the general setting of an arbitrary n and $\lambda \in \Lambda(n,r)$. In what follows, we will use the usual notation $\Lambda^+(n,r)$ for the subset of partitions in $\Lambda(n,r)$.

While constructing Auslander-Reiten sequences, we studied the kernel of the map p_1^t. Since this kernel can be identified with $\operatorname{Hom}_{S(B^+)}(\mathbb{K}_\lambda, S(B^+))$, it provides information on the socle of the Borel-Schur algebra $S(B^+)$. Namely, p_1^t is non-injective if and only if $\mathbb{K}_\lambda$ is in the socle of $S(B^+)$. We end the present section with a compilation of this information. We start with the following auxiliary result.

Lemma 4.3. *Suppose* $\nu \in \Lambda(n,r) \setminus \Lambda^+(n,r)$ *and let* M *be an* $S(n,r)$*-module. Then* $\operatorname{Hom}_{S(B^+)}(\mathbb{K}_\nu, M) = 0$, *where we consider* M *as an* $S(B^+)$*-module by restriction.*

Proof. Let $f\colon \mathbb{K}_\nu \to M$ be an $S(B^+)$-homomorphism and $c \in \mathbb{K}_\nu$. Then $\xi_{ij} f(c) = f(\xi_{ij} c) = 0$ for all $\xi_{ij} \in S(B^+)$ different from ξ_ν. By [12, Theorem 5.2] we have $S(n,r) = \sum_{\lambda \in \Lambda^+(n,r)} S(B^-)\xi_\lambda S(B^+)$, where $S(B^-)$ denotes the lower Borel subalgebra of the Schur algebra $S(n,r)$. Since $\nu \notin \Lambda^+(n,r)$, we get that $S(n,r)f(c) = 0$. This shows that $f(c) = 0$ for all $c \in \mathbb{K}_\nu$ and thus f is the zero map. □

As a simple consequence we get:

Proposition 4.4. *Let* $\nu \in \Lambda(n,r) \setminus \Lambda^+(n,r)$. *Then* $\operatorname{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(B^+)) = 0$.

Proof. The embedding $S(B^+) \hookrightarrow S(n,r)$ induces the injective map

$$\operatorname{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(B^+)) \to \operatorname{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(n,r)).$$

Now Lemma 4.3 implies that the vector space $\operatorname{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(n,r))$ is trivial. Thus also $\operatorname{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(B^+))$ vanishes. □

Combining the results on p_1^t obtained previously with Proposition 4.4, we get the following theorem.

Theorem 4.5.

1. *The module* $\mathbb{K}_{(r,0,\ldots,0)}$ *is a direct summand of the socle of* $S(B^+)$ *independently of* $\operatorname{char}\mathbb{K}$.

2. *Suppose* $\operatorname{char}\mathbb{K} = 0$ *and* $n = 2$. *Then the socle of* $S(B^+)$ *is a direct sum of several copies of* $\mathbb{K}_{(r,0)}$.

3. *Suppose* $\operatorname{char}\mathbb{K} = p$ *and* $n = 2$. *Then* $\mathbb{K}_\lambda$ *is a direct summand of the socle of* $S(B^+)$ *if and only if* $\lambda = (r, 0)$ *or* λ *is a partition satisfying*

$$\lambda_1 \geq p^{\lfloor \log_p \lambda_2 \rfloor + 1} - 1.$$

4. *Suppose* $n \geq 3$ *and* $\operatorname{char}\mathbb{K} = 0$. *Then all the composition factors of the socle of* $S(B^+)$ *are of the form* $\mathbb{K}_\lambda$ *with* λ *a partition such that* $\lambda_n = 0$.

5. *Suppose* $n \geq 3$ *and* $\operatorname{char}\mathbb{K} = p$. *Then all the composition factors of the socle of* $S(B^+)$ *are of the form* $\mathbb{K}_\lambda$ *with* λ *a partition such that either* $\lambda_n = 0$ *or*

$$\lambda_{n-1} \geq p^{\lfloor \log_p \lambda_n \rfloor + 1} - 1.$$

Proof. The module $\mathbb{K}_{(r,0,\dots,0)}$ is isomorphic to its projective cover $S(B^+)\xi_{(r,0,\dots,0)}$. Thus $\operatorname{Hom}_{S(B^+)}(\mathbb{K}_{(r,0,\dots,0)}, S(B^+)) \cong \xi_{(r,0,\dots,0)}S(B^+)$ is non trivial. This shows that $\mathbb{K}_{(r,0,\dots,0)}$ is a direct summand of the socle of $S(B^+)$.

Now the claims (2)-(5) follow from Propositions 3.3, 3.5, and 4.1. □

5 Functors between different algebras

A Borel-Schur algebra is *triangular*, that is, its quiver does not have oriented cycles. Furthermore, a Borel-Schur algebra $S(B^+, m, r)$ is a non-unital subalgebra of a Borel-Schur algebra $S(B^+, n, r)$ for $m < n$. This gives rise to functors between their module categories, and one would like to understand when an Auslander-Reiten sequence of a module for the smaller algebra lifts to an Auslander-Reiten sequence over the larger algebra.

We study this question in a slightly more general setting. Assume $\Lambda^* \subset \Lambda(n, r)$ is a coideal with respect to the dominance order. That is, if $\lambda \in \Lambda^*$ and $\lambda \trianglelefteq \mu$ then $\mu \in \Lambda^*$. For example, if $m < n$ then

$$\Lambda^*(m, r) := \{\alpha \in \Lambda(n, r) \,|\, \alpha = (\alpha_1, \dots, \alpha_m, 0, \dots, 0)\}$$

is a coideal in $\Lambda(n, r)$. Let

$$e = e_{\Lambda^*} := \sum_{\lambda \in \Lambda^*} \xi_\lambda.$$

This is an idempotent in $A := S(B^+, n, r)$. Then eAe can be regarded as a non-unital subalgebra of A, see also [13, § 6.5]. This algebra has simple modules precisely the $\mathbb{K}_\lambda$ with $\lambda \in \Lambda^*$. In case $\Lambda^* = \Lambda^*(m, r)$ we have

$$eS(B^+, n, r)e \cong S(B^+, m, r). \tag{5.1}$$

We have an exact functor

$$\begin{aligned} F\colon A\text{-mod} &\to eAe\text{-mod} \\ V &\mapsto eV \\ (\theta\colon V \to V') &\mapsto \theta|_{eV}. \end{aligned}$$

The functor F has a left adjoint $G = Ae\otimes_{eAe}\colon eAe\text{-mod} \to A\text{-mod}$. In our case G is the identity functor, since $(1-e)Ae = 0$. We get as an easy consequence:

Proposition 5.1. *With the above notation, the following hold:*

(i) $FG(M) = M$ *for all* $M \in eAe$*-mod;*

(ii) G *is exact;*

(iii) G preserves indecomposable modules;

(iv) G preserves simple modules.

We will use the following notation. If $\lambda \in \Lambda(n, r)$ belongs to the subset Λ^* we write $\bar{\lambda}$ if we view it as a weight of the algebra eAe, and we write $\mathbb{K}_{\bar{\lambda}}$ for the simple eAe-module labelled by λ. Then $G(\mathbb{K}_{\bar{\lambda}}) = \mathbb{K}_\lambda$. We also use τ_A and τ_{eAe} for the Auslander-Reiten translation in A-mod and eAe-mod, respectively. Consider the Auslander-Reiten sequence in eAe-mod

$$0 \to \tau_{eAe}\mathbb{K}_{\bar{\lambda}} \to E(\bar{\lambda}) \to \mathbb{K}_{\bar{\lambda}} \to 0.$$

By applying the functor G we obtain the exact sequence

$$0 \to G(\tau_{eAe}\mathbb{K}_{\bar{\lambda}}) \to G(E(\bar{\lambda})) \to \mathbb{K}_\lambda \to 0. \tag{5.2}$$

Proposition 5.2. *The sequence (5.2) is an Auslander-Reiten sequence if and only if* $G(\tau_{eAe}\mathbb{K}_{\bar{\lambda}}) \cong \tau_A\mathbb{K}_\lambda$.

Proof. One direction is clear. For the converse, assume $G(\tau_{eAe}\mathbb{K}_{\bar{\lambda}}) \cong \tau_A\mathbb{K}_\lambda$. The exact sequence (5.2) is non-split since, if we apply F, we get the Auslander-Reiten sequence in eAe-mod. By Remark 3.7, any non-split exact sequence with end terms $\tau_A\mathbb{K}_\lambda$ and $\mathbb{K}_\lambda$ is the Auslander-Reiten sequence. This proves the claim. □

We give now an example where the sequence (5.2) is not an Auslander-Reiten sequence.

Example 5.3. Suppose char($\mathbb{K}$) $\neq 2$. Assume $\Lambda^* = \Lambda(2, 3)$ viewed as a subset of $\Lambda(3, 3)$, so that $eAe \cong S(B^+, 2, 3)$ and $A = S(B^+, 3, 3)$. Consider the weight $\lambda = (1, 2, 0)$. From the construction, we have the exact sequence

$$0 \to \tau_A\mathbb{K}_\lambda \to DP_1^t \xrightarrow{Dp_1^t} DP_0^t.$$

In this case, DP_1^t is the injective $I_{(2,1,0)}$ with socle $\mathbb{K}_{(2,1,0)}$ and DP_0^t is the injective $I_{(1,2,0)}$.

There is a uniserial module U of dimension 2 with top isomorphic to $\mathbb{K}_{(2,0,1)}$ and socle isomorphic to $\mathbb{K}_{(2,1,0)}$. This is a submodule of $I_{(2,1,0)}$ and it is contained in the kernel of Dp_1^t, as the socle of $I_{(1,2,0)}$ is not a composition factor of U. That is, U is a submodule of $\tau_A\mathbb{K}_\lambda$ and we see that $\tau_A\mathbb{K}_\lambda$ is not of the form $G(M)$ for any M. So by the Proposition 5.2, (5.2) cannot be an Auslander-Reiten sequence.

6 Finite type classification

In this section we assume that the ground field $\mathbb{K}$ is algebraically closed. We will determine precisely which Borel-Schur algebras are of finite type. We start by summarising general results which can be used to identify representation type.

Assume A is some finite-dimensional algebra over $\mathbb{K}$. Recall that A has *finite type* if there are only finitely many indecomposable A-modules up to isomorphism. Otherwise, A has *infinite type*. If $A = \mathbb{K}\mathcal{Q}$, where $\mathcal{Q}$ is a quiver with no oriented cycles, then by Gabriel's Theorem [8, Theorem 1.2] A has finite type if and only if $\mathcal{Q}$ is a disjoint union of Dynkin quivers of types A_k with $k \geq 1$, D_k with $k \geq 4$, or E_k with $6 \leq k \leq 8$.

In some cases, we will consider algebras of the form $\mathbb{K}\mathcal{Q}/\mathcal{I}$ where $\mathcal{I}$ is generated by a single commutativity relation. We will use a result by Ringel (see [16]) to identify algebras of infinite type. Furthermore, the following is a general reduction method (see for example Lemma 1 in [2]).

Lemma 6.1. *Assume e is an idempotent of A. If eAe has infinite type then A has infinite type.*

We will combine this result with the idea of regular covering of quivers. Let $\mathcal{Q}$ be a quiver and G a group acting freely on the right of $\mathcal{Q}$. Then one has the quotient quiver $\mathcal{Q}' := \mathcal{Q}/G$ and the canonical projection $\phi\colon \mathcal{Q} \to \mathcal{Q}'$. In that situation one says that $\mathcal{Q}$ is a *regular covering* of $\mathcal{Q}'$.

The action of G on $\mathcal{Q}$ induces an action of G on the path category $P(\mathcal{Q})$ of the quiver $\mathcal{Q}$. It is obvious that $P(\mathcal{Q})/G$ is isomorphic to the path category $P(\mathcal{Q}')$. In particular, following Bongartz and Gabriel [3], we can define the pushdown functor $\phi_*\colon P(\mathcal{Q})\text{-mod} \to P(\mathcal{Q}')\text{-mod}$. In our case the definition takes the following form. Let M be a representation of $\mathcal{Q}$, then for any vertex $xG \in \mathcal{Q}'$

$$\phi_*(M)_{xG} := \bigoplus_{g\in G} M_{xg};$$

and for every arrow $x \xrightarrow{\alpha} y$ in $\mathcal{Q}$, we define the map

$$\phi_*(M)_{\alpha G}\colon \phi_*(M)_{xG} \to \phi_*(M)_{yG}$$

to be the matrix with zeros at positions (g_1, g_2) if $g_1 \neq g_2$ and αg at position (g, g).

Similarly, for each $g \in G$, we define a new representation g_*M of $\mathcal{Q}$ by

$$(g_*M)_x = M_{xg^{-1}}$$

and

$$(g_*M)_\alpha = M_{\alpha g^{-1}}\colon M_{xg^{-1}} \to M_{yg^{-1}}$$

for every vertex x and every arrow $x \xrightarrow{\alpha} y$ in $\mathcal{Q}$.

The following theorem is a restatement of [10, Lemma 3.5] in our notation.

Theorem 6.2. *Let $\mathcal{Q}$ be a quiver and G a group acting freely on $\mathcal{Q}$. Suppose that M is a finite-dimensional indecomposable representation of $\mathcal{Q}$ such that $g_*M \ncong M$, for every $g \in G$, $g \neq 1_G$. Then ϕ_*M is indecomposable. Moreover, if $N \ncong M$ is a representation of $\mathcal{Q}$ such that $\phi_*N \cong \phi_*M$, then there is $g \in G$, $g \neq 1_G$, such that $g_*M \cong N$.*

Returning to Borel-Schur algebras, we have already seen in (5.1) that $S(B^+, m, r)$ and $eS(B^+, n, r)e$ are isomorphic for some idempotent $e \in S(B^+, n, r)$, whenever $n > m$, and we will apply Lemma 6.1 in this case. It also applies to relate $S(B^+, 2, r)$ with $S(B^+, 2, r')$ for $r' \geq r$. Namely we have:

Lemma 6.3. *There is an idempotent e of $S(B^+, 2, r+1)$ such that*

$$eS(B^+, 2, r+1)e \cong S(B^+, 2, r).$$

Proof. The usual basis of $S(B^+,2,r)$ can be parametrized as $\xi_{\nu,\mu}$ where $\nu,\mu \in \Lambda(2,r)$ and $\mu \trianglelefteq \nu$, see [21]. If $\lambda \in \Lambda(2,r)$, then set $\hat{\lambda} = (\lambda_1+1,\lambda_2) \in \Lambda(2,r+1)$. Let

$$e := \sum_{\lambda \in \Lambda(2,r)} \xi_{\hat{\lambda}}.$$

Then we claim that $eS(B^+,2,r+1)e \cong S(B^+,2,r)$. Namely the linear map, defined on the basis by $\xi_{\mu,\lambda} \mapsto \xi_{\hat{\mu},\hat{\lambda}}$, is bijective and, by Lemma 22 of [21], it is an algebra map. □

Some of the Borel-Schur algebras $S(B^+,2,r)$ are special biserial.

Definition 6.4. An algebra of the form $A = \mathbb{K}\mathcal{Q}/\mathcal{I}$ is called *special biserial* if

(i) for each vertex i of $\mathcal{Q}$, there are at most two arrows starting at i and at most two arrows ending at i;

(ii) for each arrow α of $\mathcal{Q}$ there is at most one arrow β and one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ do not belong to $\mathcal{I}$.

A special biserial algebra $A = \mathbb{K}\mathcal{Q}/\mathcal{I}$ is called a *string* algebra if $\mathcal{I}$ is generated by monomials.

Remark 6.5. By [6, II.1.3], for every special biserial algebra $A = \mathbb{K}\mathcal{Q}/\mathcal{I}$, there is a string algebra A_{str} such that A and A_{str} have the same representation type. To construct A_{str} one proceeds as follows. Let X be the set of vertices $x \in \mathcal{Q}$ such that the modules Ae_x are injective-projective and non-uniserial. Then $\mathcal{J} := \bigoplus_{x\in X} \operatorname{soc}(Ae_x)$ is an ideal of A. The algebra A_{str} is defined to be $A/\mathcal{J}$. Moreover, every indecomposable A-module is either annihilated by $\mathcal{J}$, in which case it can be considered as an indecomposable A_{str}-module, or it is isomorphic to Ae_x for some $x \in X$. Therefore the number of isomorphism classes of indecomposable A-modules differs from the number of isomorphism classes of indecomposable A_{str}-modules by the number of elements in X. Hence A and A_{str} have the same representation type.

The indecomposable modules over a string algebra were classified in [19]. Let V be the set of vertices of $\mathcal{Q}$ and E the set of arrows of $\mathcal{Q}$. Denote by $\bar{\mathcal{Q}}$ the quiver with the set of vertices

$$\bar{E} = E \cup \left\{ \alpha^t \,\middle|\, \alpha \in E \right\},$$

where α^t denotes the arrow going in the opposite direction of α. We define the involution $\cdot^t$ on $\bar{E}$ by $(\alpha)^t = \alpha^t$, $(\alpha^t)^t = \alpha$ for every $\alpha \in E$. A *walk* in $\mathcal{Q}$ is a path in $\bar{\mathcal{Q}}$. A cycle in $\bar{\mathcal{Q}}$ is called a *tour* in $\mathcal{Q}$. We say that a walk $w = \alpha_1 \dots \alpha_k$ is *admissible* if neither w nor w^t contain a subpath in $\mathcal{I}$ or a subpath of the form $\alpha\alpha^t$ with $\alpha \in E$. A tour $\tau = \alpha_1 \dots \alpha_k$ is called *admissible* if

(i) neither τ nor τ^t is a cycle;

(ii) for any $1 < j \le k$ the *cyclic shift* $\alpha_j \dots \alpha_k\alpha_1 \dots \alpha_{j-1}$ of τ is different from τ;

(iii) there are neither subpaths from $\mathcal{I}$ nor of the form $\alpha\alpha^t$ either in cyclic shifts of τ or in cyclic shifts of τ^t.

Given a walk w in $\mathcal{Q}$, Wald and Waschbüsch construct on page 487 of [19] an A-module $M(w)$. Further, given a tour τ, a natural number m and $\kappa \in \mathbb{K}^*$, they construct an A-module $M(\tau, m, \kappa)$. They show in [19, Proposition 2.3] that the modules $M(w)$ and $M(\tau, m, k)$ are indecomposable and

(i) if M is an indecomposable A-module, then either $M \cong M(w)$ for some admissible walk or $M \cong M(\tau, m, \kappa)$ for some admissible tour, $m \in \mathbb{N}$ and $\kappa \in \mathbb{K}$;

(ii) $M(w) \ncong M(\tau, m, k)$ for any admissible walk w, admissible tour τ, $m \in \mathbb{N}$, $\kappa \in \mathbb{K}^*$;

(iii) if $M(w_1) \cong M(w_2)$ for two different admissible walks w_1 and w_2, then $w_1 = w_2^t$;

(iv) if $M(\tau_1, m_1, \kappa_1) \cong M(\tau_2, m_2, \kappa_2)$ for $(\tau_1, m_1, \kappa_1) \neq (\tau_2, m_2, \kappa_2)$, then $m_1 = m_2$, $\kappa_1 = \kappa_2$ and τ_2 is a cyclic shift of either τ_1 or τ_1^t.

We can now state the classification of finite type for Borel-Schur algebras.

Theorem 6.6. *1. The algebra $S(B^+, 2, r)$ has finite type if and only if one of the following holds:*

(i) $\mathrm{char}(K) = 0$*;*

(ii) $\mathrm{char}(K) = p \geq 5$ *and* $r \leq p$*;*

(iii) $\mathrm{char}(K) = 3$ *and* $r \leq 4$*;*

(iv) $\mathrm{char}(K) = 2$ *and* $r \leq 3$.

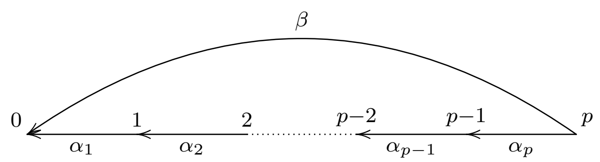
2. For $n \geq 3$, the algebra $S(B^+, n, r)$ has finite type if and only if $r = 1$.

Proof. (1) We show first that the algebras listed above have finite type.

(a) If $\mathrm{char}(\mathbb{K}) = 0$, or if $\mathrm{char}(\mathbb{K}) = p$ and $r < p$, then by Proposition 2.6 $S(B^+, 2, r)$ is isomorphic to $\mathbb{K}\mathcal{Q}$, where $\mathcal{Q}$ is the Dynkin quiver of type A_{r+1} with linear orientation. By Gabriel's Theorem [8, Theorem 1.2], the algebra has finite type.

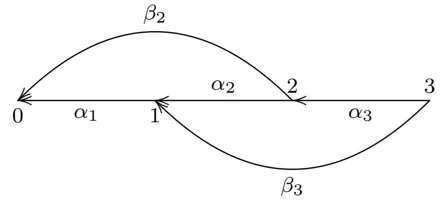
From now we assume that $\mathbb{K}$ has characteristic p.

(b) If $r = p$ then $S(B^+, 2, p)$ has finite type: its quiver described in Proposition 2.6 has the form



with $\alpha_i : i \to i-1$ and $\beta : p \to 0$, where the only relation is that the product of all the α_i is zero. This is (trivially) special biserial. Any admissible walk is a subwalk of $\alpha_2\alpha_3 \ldots \alpha_p \beta^t \alpha_1 \alpha_2 \ldots \alpha_{p-1}$ (or its transpose) and there are no admissible tours. Hence there are only finitely many admissible words and the algebra has finite type.

(c) The algebra $S(B^+, 2, 3)$ for $p = 2$ is also special biserial. By Proposition 2.6, its quiver is



(6.1)

and the relations are $\alpha_1\alpha_2 = 0$, $\alpha_2\alpha_3 = 0$, and $\alpha_1\beta_3 = \beta_2\alpha_3$. The only injective-projective indecomposable module of $S(B^+, 2, 3)$ is $S(B^+, 2, 3)\xi_{(0,3)}$. Its socle is the one-dimensional subspace of $S(B^+, 2, 3)$ generated by $\alpha_1\beta_3$. Thus the corresponding string algebra A_{str} has the same quiver (6.1) and the relations $\alpha_1\alpha_2 = 0$, $\alpha_2\alpha_3 = 0$, $\alpha_1\beta_3 = 0$, and $\beta_2\alpha_3 = 0$. Now every admissible walk for A_{str} is either a subwalk of $w = \alpha_3\beta_3^t\alpha_2\beta_2^t\alpha_1$ or of w^t. In particular, there are no admissible tours and therefore, by Remark 6.5, both A_{str} and A have finite representation type.

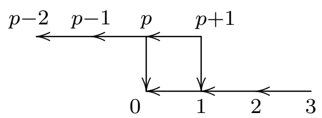
(d) The algebra $S(B^+, 2, 4)$ has finite type for $p = 3$. To prove this, we calculate Auslander-Reiten sequences, and we find that the Auslander-Reiten quiver has a finite component, a drawing of which we include (see Figure 1). Auslander's Theorem (see for example Proposition on page 116 of [15]) states that if the Auslander-Reiten quiver of an indecomposable algebra has a connected component whose modules have lengths bounded by some natural number, then this component is finite and it contains precisely all indecomposable modules. In particular the algebra is of finite type. It is clear that the modules in the component of the Auslander-Reiten quiver of $S(B^+, 2, 4)$ on Figure 1 have bounded length since there are finitely many of them. Hence by Auslander's Theorem the algebra $S(B^+, 2, 4)$ has finite type.

Now we find four classes of algebras and show they have infinite type.

(e) Consider $S(B^+, 2, p + 1)$ for $p \geq 7$. Define

$$e = \xi_{(p+1,0)} + \xi_{(p,1)} + \xi_{(p-1,2)} + \xi_{(p-2,3)} + \xi_{(3,p-2)} + \xi_{(2,p-1)} + \xi_{(1,p)} + \xi_{(0,p+1)}.$$

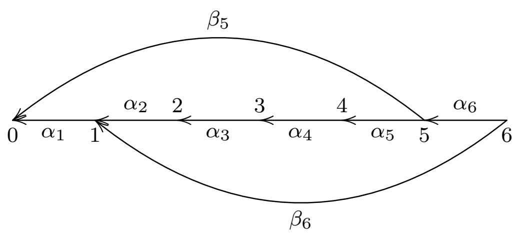
From Proposition 2.6 it follows that the algebra $eS(B^+, 2, p+1)e$ is a basic algebra whose quiver is



(6.2)

with commuting square and no other relation. The quiver (6.2) is number 32 in Ringel's list [16] and thus the algebra $S(B^+, 2, p + 1)$ has infinite type.

(f) Consider $S(B^+, 2, 6)$ for $p = 5$. By Proposition 2.6, it has the quiver presentation

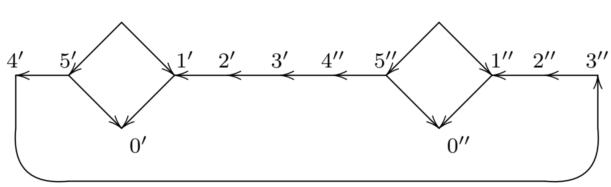


(6.3)

with the relations

$$\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = 0, \quad \alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 = 0, \quad \alpha_1\beta_6 = \beta_5\alpha_6. \tag{6.4}$$

Let us consider the quiver



(6.5)

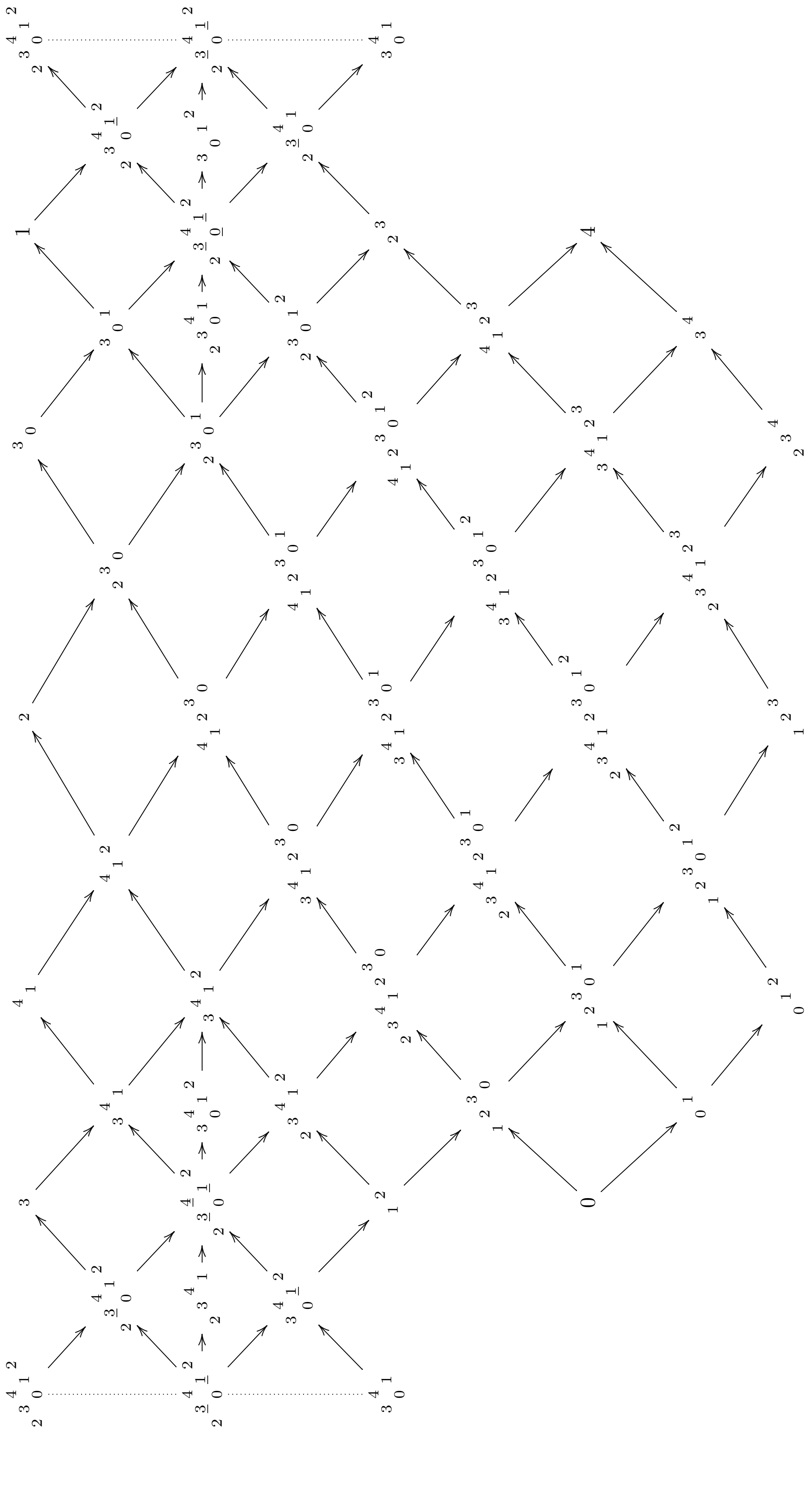
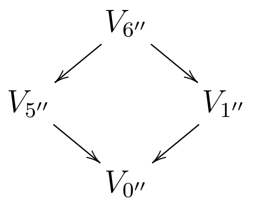


Figure 1: Auslander-Reiten quiver for $S(B^+, 2, 4)$, $p = 3$.

with an action of the symmetric group Σ_2 given by interchanging $'$ with $''$. Then the quiver (6.3) is a quotient of (6.5) under this action. The full subquiver of (6.5) spanned by the vertices $3'$, and i'' with $0 \le i \le 6$ is isomorphic to the quiver (6.2). Therefore there is an infinite set $\mathcal{J}$ of pairwise non-isomorphic indecomposable representations V of the quiver (6.5) such that $V_{i'} = 0$ for $i \neq 3$, and the square

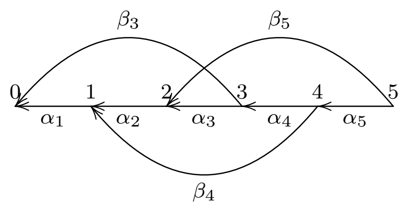


commutes. It is easy to check that for every such V the representation $\phi_*(V)$ of (6.3) satisfies relations (6.4).

Now we show that $\phi_* V$ is indecomposable. Suppose that $\phi_* V$ is not indecomposable. Then by Theorem 6.2 there is a non-unit element $\sigma \in \Sigma_2$ such that $\sigma_* V \cong V$. Since Σ_2 contains a unique non-unit element and its action on (6.5) is given by swapping i' and i'' for every $1 \le i \le 6$, we get $V_{i''} \cong V_{i'}$ for all i. In particular $V_{i''} = 0$ for $i \neq 3$. Thus the support of V is $\{3', 3''\}$. Since there are no arrows connecting $3'$ and $3''$ in (6.5), V is a direct sum of several copies of the simple modules $S_{3'}$ and $S_{3''}$. As V is indecomposable we get that either $V \cong S_{3'}$ or $V \cong S_{3''}$. But in the fist case $\sigma_* V \cong S_{3''} \ncong S_{3'} \cong V$, and in the second case $\sigma_* V \cong S_{3'} \ncong S_{3''} \cong V$. Thus we get a contradiction to the existence of $\sigma \neq 1$ such that $\sigma_* V \cong V$. This shows that $\phi_* V$ is indecomposable.

Further, if V and $W \in \mathcal{J}$ are two different representations of (6.5) such that $\phi_*(V) \cong \phi_*(W)$, then by Theorem 6.2, we have $V \cong \sigma^* W$, where $\sigma = (12) \in \Sigma_2$. This implies that $V_{i''} \cong W_{i'} = 0$ if $i \neq 3$. Therefore either V is isomorphic to the simple module $S_{3'}$ or to the simple module $S_{3''}$. Let $\mathcal{J}' = \mathcal{J} \setminus \{S_{3'}, S_{3''}\}$. Then $\mathcal{J}'$ is infinite and $\{\phi_*(V) | V \in \mathcal{J}'\}$ is the set of pairwise non-isomorphic indecomposable representations of $S(B^+, 2, 6)$ over a field of characteristic 5. This shows that $S(B^+, 2, 6)$ is of infinite type for $p = 5$.

(g) Consider $S(B^+, 2, 5)$ for $p = 3$. By Proposition 2.6 it is isomorphic to the quiver algebra of the quiver

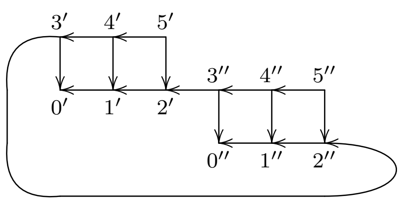


(6.6)

with relations

$$\alpha_1\alpha_2\alpha_3 = 0, \qquad \alpha_2\alpha_3\alpha_4 = 0, \qquad \alpha_3\alpha_4\alpha_5 = 0, \qquad \alpha_1\beta_4 = \beta_3\alpha_4, \qquad \alpha_2\beta_5 = \beta_4\alpha_5. \tag{6.7}$$

Let us consider the quiver



(6.8)

with an action of Σ_2 given by interchanging $'$ and $''$. Then (6.6) is the quotient of (6.8) under this action. Let $\mathcal{J}$ be the set of isomorphism classes of finite-dimensional indecomposable representations V of (6.8), such that

$$V_{0'} = 0, \qquad V_{1'} = 0, \qquad V_{5''} = 0,$$

the map from $V_{3'}$ to $V_{2''}$ is zero, and the square

$$\begin{array}{ccc} V_{3''} & \longleftarrow & V_{4''} \\ \downarrow & & \downarrow \\ V_{0''} & \longleftarrow & V_{1''} \end{array}$$

commutes. Then the elements of $\mathcal{J}$ can be identified with the isomorphism classes of finite-dimensional indecomposable representations of the quiver 33 in Ringel's list [16]. Thus $\mathcal{J}$ is infinite.

Now, for every isomorphism class $[V]$ in $\mathcal{J}$, one can check that ϕ_*V satisfies relations (6.7) and therefore can be considered as a representation of $S(B^+, 2, 5)$ over a field of characteristic 3. Theorem 6.2 implies that ϕ_*V is indecomposable. In fact, if this is not the case, then $\sigma_*V \cong V$ and thus $V_{0''} = V_{1''} = V_{5'} = 0$. Moreover, the map from $V_{3''}$ to $V_{2'}$ is zero. Hence V is the direct sum of subrepresentations V' and V'' defined by

$$V'_{i'} = V_{i'}, \qquad V'_{i''} = 0; \qquad V''_{i'} = 0, \qquad V''_{i''} = V_{i''}, \qquad (6.9)$$

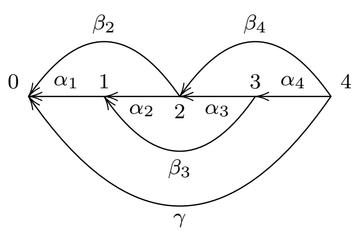
where $0 \le i \le 5$. Since V is indecomposable, we get that either $V' = 0$ or $V'' = 0$. But $\sigma_*V \cong V$ implies that $\sigma_*V' \cong V''$ and $\sigma_*V'' \cong V'$. Thus in both cases, we get that $V = 0$.

Further, for every $[V] \in \mathcal{J}$ there is at most one $[W] \in \mathcal{J}$ different from $[V]$ such that $[\phi_*V] = [\phi_*W]$. In fact, by Theorem 6.2, we get that $V \cong \sigma_*W$, where $\sigma = (12) \in \Sigma_2$. Therefore the set

$$\{\, [\phi_*V] \mid [V] \in \mathcal{J} \}$$

is infinite, and we get that $S(B^+, 2, 5)$ for $p = 3$ has infinitely many pairwise non-isomorphic indecomposable representations.

(h) Consider $S(B^+, 2, 4)$ for $p = 2$. By Proposition 2.6 it has the quiver

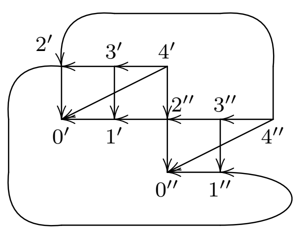


(6.10)

and its ideal of relations is generated by

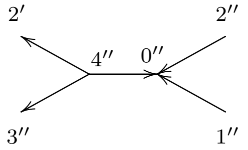
$$\alpha_1\alpha_2 = 0, \quad \alpha_2\alpha_3 = 0, \quad \alpha_3\alpha_4 = 0, \quad \alpha_1\beta_3 = \beta_2\alpha_3, \quad \alpha_2\beta_4 = \beta_3\alpha_4, \quad \beta_2\beta_4 = 0. \quad (6.11)$$

Let us consider the quiver



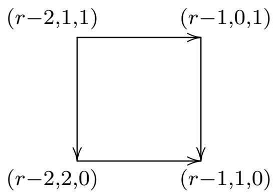
(6.12)

with an action of Σ_2 given by interchanging $'$ and $''$. Then (6.10) is the quotient of (6.12) under this action. Let $\mathcal{J}$ be the set of isomorphism classes of indecomposable representations V of (6.12) such that $V_{i'} = 0$ for $i \neq 2$ and the maps from $V_{3''}$ to $V_{1''}$, from $V_{3''}$ to $V_{2''}$ and from $V_{2'}$ to $V_{1''}$ are zero maps. Then $\mathcal{J}$ can be identified with the set of isomorphism classes of indecomposable representations of the quiver



of type $\widetilde{D}_5$. Thus $\mathcal{J}$ is infinite. One can check that for every $[V] \in \mathcal{J}$ the representation $\phi_* V$ of (6.10) satisfies the relations (6.11) and thus can be considered as a representation of $S(B^+, 2, 4)$ over a field of characteristic 2. Using Theorem 6.2 one can verify as above that $\phi_* V$ is indecomposable. Moreover, if $[V]$ and $[W] \in \mathcal{J}$ are different and $[\phi_* V] = [\phi_* W]$ then $[V] = [\sigma_* W]$. Therefore the set $\{ [\phi_* V] \mid [V] \in \mathcal{J} \}$ is infinite. Hence $S(B^+, 2, 4)$ defined over a field of characteristic 2 has infinitely many pairwise non-isomorphic indecomposable representations. This completes the proof of part *(1)* of Theorem 6.6.

(2) Now we study $S(B^+, 3, r)$ for $r \geq 2$. The quiver of $S(B^+, 3, r)$ contains a full subquiver $\mathcal{Q}'$ given by



Let e be the idempotent $e = \xi_{(r-2,1,1)} + \xi_{(r-1,0,1)} + \xi_{(r-2,2,0)} + \xi_{(r-1,1,0)}$. Then $eS(B^+, 3, r)e$ is isomorphic to $K\mathcal{Q}'$ and this is of infinite type, by Gabriel's Theorem. Hence $S(B^+, 3, r)$ is of infinite type, by Lemma 6.1. Now it follows, from (5.1) and Lemma 6.1, that $S(B^+, n, r)$ is of infinite type for any $n \geq 3$ and $r \geq 2$.

Finally, the algebra $S(B^+, n, 1)$ is isomorphic to $K\mathcal{Q}$ where $\mathcal{Q}$ is a quiver of type A_n with linear orientation, hence it has finite type. □

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