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A converging finite volume scheme for hyperbolic conservation laws with source terms

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Abstract

In this paper we study convergence of numerical discretizations of hyperbolic nonhomogeneous scalar conservation laws. Particular attention is devoted to point source problems. Standard numerical methods, obtained by a direct discretization of the differential form, fail to converge, even in the linear case. We consider the equation in integral form in order to construct a class of convergent accurate methods. Numerical examples are included. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we study the convergence of numerical discretizations of nonhomogeneous scalar conservation laws of type

$$u_t + f_x(u) = q_x(x,t), \quad -\infty < x < \infty \text{ and } t > 0, \tag{1}$$

where f is a given function of u, depending in general nonlinearly on u. We will assume that f is a smooth convex function ($f''(u) > 0$ for all u), and q is a bounded, piecewise smooth function.

Hyperbolic conservation laws with source terms have recently been analysed by several authors (see, e.g., [1–3,8,9]). In particular, the authors in [2] study a conservation law of type (1) with a steady source term. They construct L^∞ stable Godunov-type difference schemes, which have a similar

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equilibrium structure to the continuous case. However, convergence problems are not considered. Koren in [3] suggests that by taking the source term as the derivative of a certain function, more accurate results may be achieved.

In this work we present a class of pure advection–reaction problems where the reaction term is represented by time-dependent point sources. In order to represent a point source localized at $x=0$, we consider for instance

$$q(x)=\begin{cases} b_{\mathrm{l}}, & x<0,\\ b_{\mathrm{r}}, & x>0,\end{cases}$$

thus obtaining in the distributional sense, $q_x(x)=(b_{\mathrm{l}}-b_{\mathrm{r}})\delta(x)$, where δ is the Dirac delta function.

Standard numerical methods obtained by a direct discretization of the differential form fail to converge, even in the linear case. In order to construct a class of convergent accurate methods, to approximate the solution of the point source problem, we consider the equation in its integral form. Also we discretize it with a certain finite volume approach, making use of the fact that a Dirac delta function is the distributional derivative of a Heaviside function. We prove that the class of methods thus obtained produces solutions that converge to a weak solution of the problem. Furthermore, this convergence result, for hyperbolic conservation laws with point sources, leads to an extended version of Lax–Wendroff theorem [6]. As what concerns stability we introduce a nonlinear form, which enables us to prove convergence results. An entropy condition is presented. Sufficient conditions, that guarantee the convergence of the numerical solution to the entropy solution, are also established.

The paper is structured as follows. In Section 2 we construct a weak solution of a time-dependent point source problem. In Section 3, we begin by constructing a convergent method via an integral formulation. An important part of the section is devoted to the establishment of a convergence result for a class of numerical discretizations of (1). We end the section by analysing the stability problem and the entropy conditions. In Section 4 two numerical examples are considered. Finally, in Section 5 some comments are presented.

2. Formal solutions and weak solutions

We begin by considering a linear problem with $f(u)=\alpha u$, $\alpha\in\mathbb{R}$. Let us consider the linear advection–reaction problem with a point source

$$\begin{aligned} &u_t+\alpha u_x=\delta(x-a_i), \quad x\in\mathbb{R},\ t>0,\\ &u(x,0)=u_0(x), \quad x\in\mathbb{R}. \end{aligned} \tag{2}$$

Integrating along the characteristic lines $\mathrm{d}x/\mathrm{d}t=\alpha$, and knowing that, in the distributional sense, $(\mathrm{d}H_i(x)/\mathrm{d}x)=\delta(x-a_i)$, where H_i is the Heaviside function defined by

$$H_i(x)=\begin{cases} 0, & x<a_i,\\ 1, & x>a_i,\end{cases}$$

it is easy to establish that

$$u(x,t)=\frac{H_i(x)-H_i(x-\alpha t)}{\alpha}+u_0(x-\alpha t). \tag{3}$$

Let us consider now the following advection–reaction problem

$$\begin{aligned} &u_t + \alpha u_x = g(t)\delta(x - a_i), \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{4}$$

where the reaction term is represented by a time-dependent point source. We construct, in an analogous way, its formal solution, obtaining

$$u(x,t) = \begin{cases} u_0(x - \alpha t), & x < a_i \vee x \geqslant a_i + \alpha t, \\ \frac{1}{\alpha} g\left(\frac{a_i - (x - \alpha t)}{\alpha}\right) + u_0(x - \alpha t), & a_i \leqslant x < a_i + \alpha t. \end{cases} \tag{5}$$

We observe that in the case of point sources, the density $u(x,t)$ given by (3) (resp. (5)) satisfies only formally (2) (resp. (4)). The question arises then: — Are the formal solutions, previously constructed, weak solutions?

We now prove that the formal solution given by (3) is, in fact, a weak solution of (2).

A weak solution of (2) satisfies the integral equation

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\,\mathrm{d}x - \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,\mathrm{d}x &= \int_{t_n}^{t_{n+1}} [-\alpha u(x_{j+1/2},t) + H_i(x_{j+1/2})]\,\mathrm{d}t \\ &+ \int_{t_n}^{t_{n+1}} [\alpha u(x_{j-1/2},t) - H_i(x_{j-1/2})]\,\mathrm{d}t, \quad \forall t_n, t_{n+1}, x_j, x_{j+1} \end{aligned} \tag{6}$$

with $x_{j\pm 1/2} = x_j \pm h/2$, where h is a known spatial stepsize. To verify that (3) is a solution of (6), we replace in this last equation $u(x,t)$ by $(H_i(x) - H_i(x - \alpha t))/\alpha$, where, without loss of generality, we have considered $u_0(x) = 0,\ x \in \mathbb{R}$.

Using a suitable change of variables we prove that (6) is equivalent to

$$\int_{x_{j-1/2}-\alpha t_{n+1}}^{x_{j+1/2}-\alpha t_{n+1}} H_i(\tau)\,\mathrm{d}\tau - \int_{x_{j-1/2}-\alpha t_n}^{x_{j+1/2}-\alpha t_n} H_i(\tau)\,\mathrm{d}\tau = \int_{x_{j+1/2}-\alpha t_n}^{x_{j+1/2}-\alpha t_{n+1}} H_i(\tau)\,\mathrm{d}\tau - \int_{x_{j-1/2}-\alpha t_n}^{x_{j-1/2}-\alpha t_{n+1}} H_i(\tau)\,\mathrm{d}\tau. \tag{7}$$

Considering that

$$\int_{x_{j-1/2}-\alpha t_{n+1}}^{x_{j+1/2}-\alpha t_{n+1}} H_i(\tau)\,\mathrm{d}\tau = \int_{x_{j-1/2}-\alpha t_{n+1}}^{x_{j-1/2}-\alpha t_n} H_i(\tau)\,\mathrm{d}\tau + \int_{x_{j-1/2}-\alpha t_n}^{x_{j+1/2}-\alpha t_{n+1}} H_i(\tau)\,\mathrm{d}\tau,$$

$$\int_{x_{j-1/2}-\alpha t_n}^{x_{j+1/2}-\alpha t_n} H_i(\tau)\,\mathrm{d}\tau = \int_{x_{j-1/2}-\alpha t_n}^{x_{j+1/2}-\alpha t_{n+1}} H_i(\tau)\,\mathrm{d}\tau + \int_{x_{j+1/2}-\alpha t_{n+1}}^{x_{j+1/2}-\alpha t_n} H_i(\tau)\,\mathrm{d}\tau,$$

we easily conclude that (7) is verified and, consequently, that (3) is a solution of (6), that is, the formal solution is a weak solution.

We can prove in an analogous way that $u(x,t)$ given by (5) is a weak solution of (4), because it satisfies the integral equation

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\,\mathrm{d}x - \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,\mathrm{d}x &= \int_{t_n}^{t_{n+1}} [-\alpha u(x_{j+1/2},t) + g_i(t)H_i(x_{j+1/2})]\,\mathrm{d}t \\ &+ \int_{t_n}^{t_{n+1}} [\alpha u(x_{j-1/2},t) - g_i(t)H_i(x_{j-1/2})]\,\mathrm{d}t. \end{aligned} \tag{8}$$

3. Convergence of numerical discretizations

3.1. A convergent finite volume discretization

Standard numerical methods obtained by a direct discretization of the differential form fail to converge even in the linear case.

Let us consider the following discretization of (4):

$$\frac{u_j^{n+1}-u_j^n}{k}+\alpha\frac{u_j^n-u_{j-1}^n}{h}=g_i(t_n)\delta(x_j-a_i), \quad n=0,1,\ldots,\ j=0,\pm 1,\ldots\ . \tag{9}$$

If we assume that $h=\alpha k$ we obtain $u_j^n=u_{j-1}^n+kg_i(t_n)\delta(x_j-a_i)$, $n=0,1,\ldots,\ j=0,\pm 1,\ldots,$ and if we consider $u_j^0=0,\ j=0,\pm 1,\pm 2,\ldots,$ we obtain $u_j^n=k\sum_{l=0}^{n-1}g_i(t_l)\delta(x_{j+l-(n-1)}-a_i)$. We have $u_j^n \underset{h,k\to 0}{\longrightarrow} 0$ and the numerical solution does not converge to the weak solution (5).

We consider now the integral formulation (6) of (2). Let in (6)

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\,\mathrm{d}x \approx hu_j^{n+1}, \qquad \int_{t_n}^{t_{n+1}} u(x_{j+1/2},t)\,\mathrm{d}t\approx ku_j^n, \qquad \int_{t_n}^{t_{n+1}} H_i(x_{j+1/2})\,\mathrm{d}t\approx kH_i(x_j).$$

We then have

$$\frac{u_j^{n+1}-u_j^n}{k}+\alpha\frac{u_j^n-u_{j-1}^n}{h}-\frac{H_i(x_j)-H_i(x_{j-1})}{h}=0. \tag{10}$$

Solving (10) with $h=\alpha k$ we obtain (assuming that $u_j^0=0,\ j=0,\pm 1,\ldots$)

$$u_j^n=\frac{H_i(x_j)-H_i(x_{j-n})}{\alpha}. \tag{11}$$

Comparing (11) with (3) (with $u_0(x)=0$), we conclude that (11) gives the exact solution. We recall that, considering $h=\alpha k$, we are integrating along the characteristics lines. It is well known that, in the case of a pure advection equation with no source, backward differentiation gives the exact solution if the Courant number is equal to one.

Let us now consider the integral form (8) of (4). In the case of a time-dependent point source, we obtain (proceeding as before)

$$\frac{u_j^{n+1}-u_j^n}{k}+\alpha\frac{u_j^n-u_{j-1}^n}{h}=g(t_n)\frac{H_i(x_j)-H_i(x_{j-1})}{h}.$$

Assuming again that, $u_j^0=0,\ j=0,\pm 1,\ldots,$ and $h=\alpha k$. We obtain the numerical solution, $u_j^n=\frac{1}{\alpha}\sum_{l=0}^{n-1}g_i(t_l)[H_i(x_{j-n+1+l})-H_i(x_{j-n+l})]$ or,

$$u_j^n=\begin{cases}0, & x_j<a_i \vee x_{j-n}\geqslant a_i,\\ \frac{1}{\alpha}g(t_\gamma), & x_{j-n}<a_i\leqslant x_j,\end{cases} \tag{12}$$

where, $0\leqslant\gamma\leqslant n-1$ $(\gamma\in\mathbb{N})$ (γ depends on the position of a_i in the interval $]x_{j-n},x_j]$).

In this case the weak solution (5) takes the form

$$u(x_j,t_n)=\begin{cases}0, & x_j<a_i \vee x_{j-n}\geqslant a_i,\\ \frac{1}{\alpha}g(t_\xi), & x_{j-n}<a_i\leqslant x_j,\end{cases} \tag{13}$$

where $\xi = (a_i - x_{j-n})/h$. For $x_{j-n} < a_i \leqslant x_j$, $\xi \in]0, n]$. We note that ξ depends on the position of a_i in the interval $]x_{j-n}, x_j]$, like γ in the numerical solution.

Comparing the numerical solution u_j^n and the exact solution $u(x_j, t_n)$, we conclude that if $x_{j-n} < a_i \leqslant x_j$, the argument of g_i presents a delay of $\mathrm{O}(k)$ in the numerical solution. However, it is obvious that u_j^n converges to a weak solution of (4).

3.2. A convergence result

We now return to the nonhomogeneous scalar conservation law (1), with the initial data

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \tag{14}$$

where weak solution, $u(x,t)$, satisfies the integral form

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\,\mathrm{d}x = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,\mathrm{d}x - \left\{ \left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,\mathrm{d}t - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))\,\mathrm{d}t \right] \right.$$
$$\left. - \left[\int_{t_n}^{t_{n+1}} q(x_{j+1/2},t)\,\mathrm{d}t - \int_{t_n}^{t_{n+1}} q(x_{j-1/2},t)\,\mathrm{d}t \right] \right\}. \tag{15}$$

Let us consider *conservative numerical methods* in the general form

$$u_j^{n+1} = u_j^n - \frac{k}{h}\{[F(u_{j-p}^n,\dots,u_{j+l}^n) - F(u_{j-p-1}^n,\dots,u_{j+l-1}^n)]$$
$$- [S(q_{j-p}^n,\dots,q_{j+l}^n) - S(q_{j-p-1}^n,\dots,q_{j+l-1}^n)]\}, \tag{16}$$

where $q_{j+b}^n = q((j+b)h, nk)$, $b = -(p+1),\dots,0,\dots,l$. In particular we assume, for the moment and for simplicity, that $p = 0$ and $l = 1$, obtaining

$$u_j^{n+1} = u_j^n - \frac{k}{h}[(F_{j+1/2}^n - F_{j-1/2}^n) - (S_{j+1/2}^n - S_{j-1/2}^n)], \tag{17}$$

where we have used the notation: $F_{j+1/2}^n := F(u_j^n, u_{j+1}^n)$; $S_{j+1/2}^n := S(q_j^n, q_{j+1}^n)$.

It is interesting to compare (17) and (15). We can see that

$$u_j^n \approx \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,\mathrm{d}x, \quad F_{j+1/2}^n \approx \frac{1}{k}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,\mathrm{d}t, \quad S_{j+1/2}^n \approx \frac{1}{k}\int_{t_n}^{t_{n+1}} q(x_{j+1/2},t)\,\mathrm{d}t.$$

This means that u_j^n can be considered as an approximation to a cell average of $u(x,t_n)$; that the numerical homogeneous flux function $F_{j+1/2}^n$ plays the usual role of an average homogeneous flux through $x_{j+1/2}$ over the time interval $[t_n, t_{n+1}]$ and also that $S_{j+1/2}^n$ represents the numerical source flux function playing the role of an average source flux through $x_{j+1/2}$ over the time interval $[t_n, t_{n+1}]$. $F + S$ can be considered as an extended numerical flux function.

Method (16) is *consistent* with the nonhomogeneous conservation law (1), if the numerical homogeneous flux function reduces to the true homogeneous flux for the case of constant flow and the numerical source flux satisfies $S(q,\dots,q) = q$, $\forall q \in \mathbb{R}$.

In order to extend the grid function u_j^n, we define for $n \geqslant 0$ a piecewise constant function for all x and t,

$$u_m(x,t) := u_j^n \quad \text{for} \begin{cases} nk_m \leqslant t < (n+1)k_m, \\ (j-\frac{1}{2})h_m < x \leqslant (j+\frac{1}{2})h_m, \end{cases} \tag{18}$$

where $(k_m/h_m) = \lambda$ for a fixed λ.

For homogeneous conservation laws a convergence result is provided by Lax–Wendroff theorem [6]. The particular convergence results analysed in Section 3.1, using a conservative method, lead us to an extended version of this last theorem. We shall show that a convergent sequence that is uniformly bounded and that is defined by a conservative numerical method, will converge to a solution of (1), (14) in the distributional sense.

Theorem 3.1. *Consider a sequence of grids indexed by* $m = 1, 2, \ldots,$ *with mesh parameters* $h_m, k_m \to 0$ *as* $m \to \infty$.

Let $(u_m)_m$ *be a sequence of numerical solutions as defined in* (16) *and* (18) *with respect to* h_m, k_m *and the initial values* u_j^0.

Assume that

(i) *the method is consistent with the nonhomogeneous conservation law* (1);

(ii) *the numerical homogeneous flux function F and the numerical source flux function S are Lipschitz continuous functions*;

(iii) *there exists a constant K such that*

$$\sup_m \sup_{\mathbb{R}\times[0,\infty[} |u_m(x,t)| \leqslant K.$$

Suppose that u_m *converges to a function u a.e. in* $\mathbb{R}\times[0,\infty[$, *for* $m \to \infty$. *Then, u is a solution of* (1), (14) *in the distributional sense.*

Proof. We show that the limit function $u(x,t)$ satisfies the weak form of the conservation law with source term (1), i.e., $\forall \phi \in C_0^\infty(\mathbb{R}\times[0,\infty[)$

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x(f(u) - q(x,t))]\, \mathrm{d}x\, \mathrm{d}t = -\int_{-\infty}^\infty \phi(x,0)u(x,0)\, \mathrm{d}x. \tag{19}$$

Let $\phi(x,t) \in C_0^\infty(\mathbb{R}\times[0,\infty[)$ and define ϕ_j^n

$$\phi_j^n = \frac{1}{h_m}\int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t_n)\, \mathrm{d}x. \tag{20}$$

We multiply Eq. (17) by ϕ_j^n, sum for all j and $n \geqslant 0$ and apply "summation by parts" to shift the differences to the test function ϕ. We obtain

$$\begin{aligned} &-h_m \sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}(\phi_j^n - \phi_j^{n-1})u_j^n - h_m \sum_{j=-\infty}^{\infty}\phi_j^0 u_j^0 \\ &\quad = k_m \sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}(\phi_{j+1}^n - \phi_j^n)F_{j+1/2}^n - k_m \sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}(\phi_{j+1}^n - \phi_j^n)S_{j+1/2}^n. \end{aligned} \tag{21}$$

The proof for homogeneous conservation laws is well known, see for example [4]. We will only analyse the second term in the right-hand side of (21).

We have

$$- k_m \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} (\phi_{j+1}^n - \phi_j^n) S_{j+1/2}^n \underset{k_m, h_m \to 0}{\longrightarrow} - \int_0^{\infty} \int_{-\infty}^{\infty} S_m(x,t) \phi_x(x,t) \, \mathrm{d}x \, \mathrm{d}t, \tag{22}$$

where, $S_m(x,t) := S_{j+1/2}^n$ if $x_j \leqslant x < x_{j+1}$ and $t_n \leqslant t < t_{n+1}$ for $m \geqslant 0$.

Let us consider an auxiliary function,

$$q_m(x,t) := q_j^n \quad \text{for} \begin{cases} nk_m \leqslant t < (n+1)k_m, \\ (j - \frac{1}{2})h_m < x \leqslant (j + \frac{1}{2})h_m. \end{cases} \tag{23}$$

We have $q_m(x,t) \underset{m \to \infty}{\longrightarrow} q(x,t)$. From (23),

$$S_m(x,t) := S_{j+1/2}^n = S(q_j^n, q_{j+1}^n) = S(q_m(x - h_m/2, t), q_m(x + h_m/2, t)). \tag{24}$$

To show

$$S_m(x,t) \underset{h_m \to 0}{\longrightarrow} S(q(x,t), q(x,t)), \tag{25}$$

we proceed as before. In fact, since S is a Lipschitz continuous function, we have that

$$\begin{aligned} &|S(q_m(x - h_m/2, t), q_m(x + h_m/2, t)) - S(q(x,t), q(x,t))| \\ &\quad \leqslant k_1 |q_m(x - h_m/2, t) - q(x,t)| + k_2 |q_m(x + h_m/2, t) - q(x,t)| \end{aligned}$$

and then we can easily establish (25). □

Remark 3.2. The numerical source flux function $S(q_{j-p}^n, \ldots, q_{j+l}^n)$ is a function of $(p+l+1)$ variables and is constructed satisfying the condition that S is a Lipschitz continuous function of each variable, even in the case that $q(x,t)$ is not a Lipschitz continuous function.

3.3. Nonlinear stability

For linear problems the Lax Equivalence theorem guarantees convergence. When nonlinear problems are considered we cannot use the same approach. We need some form of nonlinear stability that allows us to prove convergence results.

We will say that a numerical method is *total variation stable* [7], or simply TV-*stable*, if the approximations u_m for $m < m_0$ lies in some fixed set of the form

$$K = \{u \in L_{1,T}\colon \ \mathrm{TV}_T(u) \leqslant R \text{ and } \mathrm{Supp}(u(.,t)) \subset [-M, M] \ \forall t \in [0,T]\},$$

where $\mathrm{TV}_T(u)$ represents the total variation of u over $[0,T]$. In this definition R and M may depend on the initial data u_0, the flux function $f(u)$ and the source term q_x, but not on m. We observe that the space $L_{1,T}$ consists of all functions of x and t for which the norm

$$||u||_{1,T} = \int_0^T \int_{-\infty}^{\infty} |u(x,t)| \, \mathrm{d}x \, \mathrm{d}t$$

is finite. Following the ideas in [7], we can easily prove the following theorems:

Theorem 3.3. *Consider a conservative numerical method in the general form* (16), *where the numerical homogeneous flux function F and the numerical source flux function S are Lipschitz continuous functions. Suppose that for each initial data u_0 and source q_x with compact support, there exists some $k_0, R_1, R_2 > 0$ such that*

$$\mathrm{TV}(u^n) \leqslant R_1 \quad \textit{and} \quad \mathrm{TV}(q^n) \leqslant R_2 \quad \forall n, k \text{ with } k < k_0,\ nk < T.$$

Then the method is TV-*stable.*

Remark 3.4. In the discrete case we define for a sequence $v^n = (v_j^n)_{j \in \mathbb{Z}}$ of discrete values v_j^n, $\mathrm{TV}(v^n) := \sum_{j=-\infty}^{\infty} |v_{j+1}^n - v_j^n|$.

Theorem 3.5. *Suppose u_m is generated by a numerical method in conservation form, with Lipschitz continuous numerical homogeneous flux function and numerical source flux function, consistent with the scalar nonhomogeneous conservation law* (1). *If the method is* TV-*stable then the method is convergent.*

Remark 3.6. Like LeVeque in [7], we assume here that we have convergence of u_m to u in the following sense:

1. Over every bounded set $\Omega = [a, b] \times [0, T]$ in $x - t$ space,

$$||u_m - u||_{1,\Omega} \to 0 \quad \text{as } m \to \infty \tag{26}$$

2. For each T there is an $R > 0$ such that

$$\mathrm{TV}(u_m(.,t)) \leqslant R \quad \text{for all } 0 \leqslant t \leqslant T,\ m = 1, 2, \ldots . \tag{27}$$

For Theorem 3.1 we assumed a different form of convergence, namely that u_m converges to u almost everywhere in a uniformly bounded manner. However, using the fact that each u_m is a piecewise constant function, it can be shown [7] that this requirement is essentially equivalent to (26) and (27).

3.4. *An entropy condition*

In some cases the weak solution is not unique. An additional condition (*entropy condition*) is then needed to select the physically relevant solution — *entropy solution.*

Let $\eta, \psi \in C^2(\mathbb{R})$ be such that

$$\psi'(u) = \eta'(u) f'(u) + \eta''(u) q, \tag{28}$$

where η is strictly convex. The pair (η, ψ) is called a *pair of entropy functions* for Eq. (1).

Theorem 3.7 (Entropy condition)**.** *The function $u(x,t)$ is the entropy solution of* (1) *if, for all strictly convex functions $\eta(u) \in C^2(\mathbb{R})$ and corresponding entropy fluxes $\psi(u)$ satisfying* (28), *the* *entropy inequality*

$$\frac{\partial}{\partial t}\eta(u(x,t)) + \frac{\partial}{\partial x}\psi(u(x,t)) \leqslant \frac{\partial}{\partial x}(\eta'(u(x,t))q(x,t)), \tag{29}$$

is satisfied in the distributional sense.

Proof. The function $\eta(u)$ is conserved for smooth flows. For discontinuous solutions we are particularly interested in how the entropy behaves for the vanishing viscosity weak solution. In this sense we look at the related viscous problem and will then let the viscosity tend to zero.

Let us assume that (η, ψ) is a pair of entropy functions and let $u_\varepsilon \in C^2(\mathbb{R}\times\mathbb{R}^+)$ be the solution of

$$(u_\varepsilon)_t + f_x(u_\varepsilon) = \varepsilon(u_\varepsilon)_{xx} + q_x \quad \text{in } \mathbb{R}\times\mathbb{R}^+.$$

Then

$$\eta_t(u_\varepsilon) + \psi_x(u_\varepsilon) - \varepsilon\eta_{xx}(u_\varepsilon) - (\eta'(u_\varepsilon)q)_x = -\varepsilon\eta''(u_\varepsilon)(u_\varepsilon)_x^2 \leqslant 0,$$

since η is convex. That is

$$\eta_t(u_\varepsilon) + \psi_x(u_\varepsilon) - \varepsilon\eta_{xx}(u_\varepsilon) - (\eta'(u_\varepsilon)q)_x \leqslant 0.$$

For all test functions $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$ and $\phi(x,t)\geqslant 0$ we get

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \{\eta(u_\varepsilon)\phi_t + [\psi(u_\varepsilon) - \eta'(u_\varepsilon)q]\phi_x + \varepsilon\eta(u_\varepsilon)\phi_{xx}\}\,\mathrm{d}x\,\mathrm{d}t \geqslant 0. \tag{30}$$

As it can be proved that (see [5])

$$\left.\begin{array}{l} u_\varepsilon \to u \\ \eta(u_\varepsilon) \to \eta(u) \\ \psi(u_\varepsilon) \to \psi(u) \\ \eta'(u_\varepsilon) \to \eta'(u) \end{array}\right\} \quad \text{in } L^1$$

$$|\eta(u_\varepsilon)| \leqslant \text{const} \quad \text{as } \varepsilon \to 0,$$

taking limits in (30), when $\varepsilon \to 0$, we obtain the weak form of the entropy inequality,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \{\eta(u)\phi_t + [\psi(u) - \eta'(u)q]\phi_x\}\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.$$

We finally conclude that the limit u has to satisfy the entropy inequality

$$\eta_t(u) + \psi_x(u) \leqslant (\eta'(u)q)_x,$$

in the distributional sense. □

When we develop numerical algorithms for the nonhomogeneous conservation law (1) it is important to guarantee that the corresponding scheme will select the entropy solution.

Definition 3.8 (Discrete entropy condition). Assume that u is the entropy solution of (1). Let (η, ψ) be a pair of entropy functions and Ψ, S, Θ be Lipschitz continuous functions of each variable, such that $\Psi(u,\ldots,u) = \psi(u)$, $\Theta(u,\ldots,u) = \eta'(u)$ $\forall u \in \mathbb{R}$ and $S(q,\ldots,q) = q$ $\forall q \in \mathbb{R}$. Let (u_j^n) be a solution generated by a numerical method in conservation form. Then, we shall say that (u_j^n) satisfies a discrete entropy condition if

$$\begin{aligned}\eta(u_j^{n+1}) \leqslant \eta(u_j^n) - \frac{k}{h}\{&[\Psi(u_{j-p}^n,\dots,u_{j+l}^n) - \Psi(u_{j-p-1}^n,\dots,u_{j+l-1}^n)] \\ &- [\Theta(u_{j-p}^n,\dots,u_{j+l}^n)S(q_{j-p}^n,\dots,q_{j+l}^n) - \Theta(u_{j-p-1}^n,\dots,u_{j+l-1}^n)S(q_{j-p-1}^n,\dots,q_{j+l-1}^n)]\}.\end{aligned} \tag{31}$$

Furthermore, if this inequality holds uniformly for $k,h \to 0$ for any η,Ψ,Θ and S, we say that the numerical scheme is consistent with the entropy condition.

Theorem 3.9 (Convergence to the entropy solution)**.** *Let us assume that the conditions of Theorem* 3.1 *are satisfied and that the corresponding numerical scheme is consistent with the entropy condition. Then the numerical solution will converge to a weak solution of* (1), (14) *that satisfies the entropy condition* (29).

Proof. In Theorem 3.1 we have already proved that the limit function u is a solution of (1), (14) in the distributional sense. In an analogous way we can show that with the conditions of the present theorem, u satisfies the entropy inequality (29), or, more precisely, the weak form of the entropy inequality, i.e., $\forall\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$ and $\phi(x,t)\geqslant 0$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\{\eta(u)\phi_t + [\psi(u) - \eta'(u)q]\phi_x\}\,\mathrm{d}x\,\mathrm{d}t \geqslant 0. \qquad \square$$

4. Numerical examples

In Section 4 we present two problems solved with different conservative numerical methods. The first problem is a linear one and convergence is obtained from Lax Equivalence theorem. The second problem is nonlinear. From a theoretical point of view, we can prove that the numerical scheme used to solve it, is TV-*stable* and, consequently, using Theorem 3.8, convergent. Unfortunately, we have not succeeded in proving that the numerical methods used satisfy the entropy condition. However, we observe that the numerical solutions obtained are, in fact, the entropy solutions.

Problem 1. We consider the initial boundary value problem

$$u_t + u_x = \sin(\pi t)\delta(x-0.1), \quad x \in (0,1),\ t>0,$$
$$u(x,0) = 0, \quad x \in (0,1),$$
$$u(0,t) = 0, \quad t\geqslant 0.$$

The exact solution can be obtained from (5) with $\alpha = 1$, $g(t) = \sin(\pi t)$ and $a_i = 0.1$. We solve this problem using a generalization of Lax–Friedrichs method. This method can be written in the conservation form (17) by taking $F_{j+1/2}^n = \frac{1}{2}(u_j^n + u_{j+1}^n) + (h/2k)(u_j^n - u_{j+1}^n)$ and $S_{j+1/2}^n = \frac{1}{2}(q_j^n + q_{j+1}^n)$,

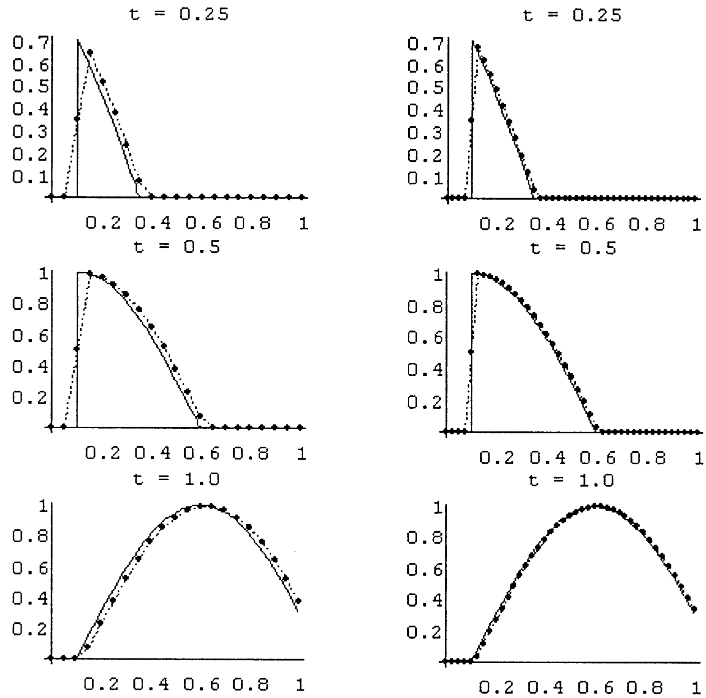

Fig. 1. Numerical solutions at $t = 0.25$, 0.5 and 1 for $k = h = \frac{1}{20}$ (in the left side) and for $k = h = \frac{1}{40}$ (in the right side).

where $q_j^n = \sin(\pi t_n) H(x_j - 0.1)$. Assuming $k = h$ (we are integrating along the characteristic lines), two situations are analysed, $h = \frac{1}{20}$ and $\frac{1}{40}$, in three moments: $t = 0.25$, 0.5, 1.0 (Fig. 1). The numerical solution is represented with a dashed line and the exact solution is represented with a continuous line.

Problem 2. We consider the inviscid Burger's equation with a point source,

$$\begin{aligned}
&u_t + (u^2/2)_x = \delta(x - 0.2), \quad x \in (0,1), \ t > 0,\\
&u(x,0) = 0, \quad x \in (0,1),\\
&u(0,t) = 0, \quad t \geqslant 0.
\end{aligned}$$

We solve this problem using the conservation form (16) with $p = l = 1$, by taking

$$F(u_{j-1}^n, u_j^n, u_{j+1}^n) = \frac{(u_j^n)^2}{2} \quad \text{and} \quad S(q_{j-1}^n, q_j^n, q_{j+1}^n) = -\frac{\rho}{3} q_{j-1}^n + \frac{3 + 4\rho}{6} q_j^n + \frac{3 - 2\rho}{6} q_{j+1}^n,$$

where ρ is a parameter and $q_j^n = H(x_j - 0.2)$. In Fig. 2 we present numerical solutions obtained at three distinct moments, $t = 0.2$, 0.4, 0.8, with two different couple of values of k, h ($h = \frac{1}{20}$, $k = 0.04$ and $h = \frac{1}{40}$, $k = 0.02$) and $\rho = 0.1$. We can prove that the scheme used is TV-*stable* in the sense of Theorem 3.3. As the method is consistent with the scalar nonhomogeneous conservation

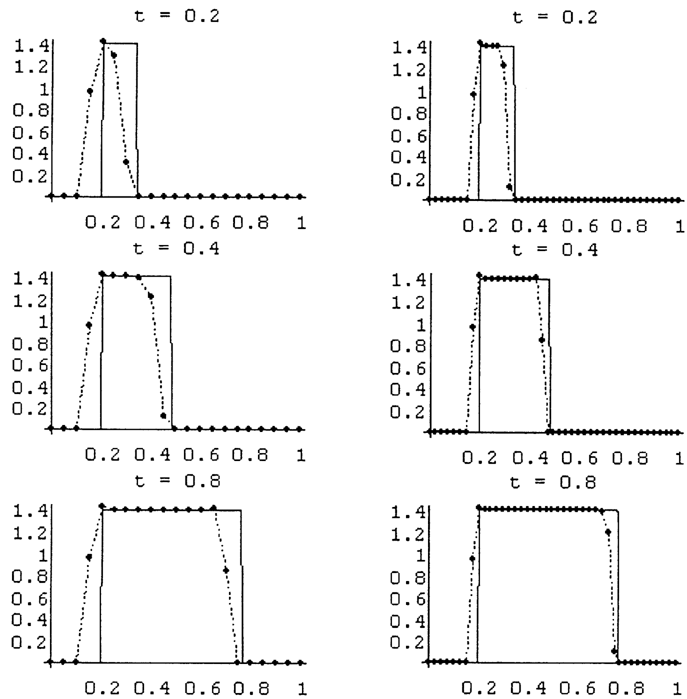

Fig. 2. Numerical solutions at $t=0.2$, 0.4 and 0.8 for $h=\frac{1}{20}$, $k=0.04$ (in the left side) and for $h=\frac{1}{40}$, $k=0.02$ (in the right side).

law, it converges. In this problem the "exact" solution has been obtained with a fine fixed mesh and it is represented with a continuous line. The numerical solution is represented with a dashed line.

5. Conclusions

In advection–reaction models the reaction term is often represented by time-dependent stiff sources. In this case, locally refined methods can be successfully used. If the advection–reaction equation models pollutant transport the reaction term is, in many cases, represented by time-dependent point sources. Nevertheless, in this case, locally refined methods lead to nonconvergent solutions as referred in Section 3. In order to approach this problem we construct a class of convergent methods by discretizing the problem in its integral form. As what concerns the point source this corresponds to considering a Dirac delta function as the derivative of a Heaviside function. By using Theorems 3.1 and 3.6, we can prove that the class of methods presented produces solutions that converge to a weak solution of (1).

As far as entropy is concerned, we have not succeeded in proving that the used numerical methods are consistent with the entropy condition. This fact is however in accordance with the results of several numerical tests performed. Further work must therefore be directed towards the building of a formal proof that the numerical methods considered satisfy the entropy condition.

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