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Ranks of submatrices and the off-diagonal indices of a square matrix

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Abstract

In the first part of the paper we determine bounds for the ranks of certain submatrices of square matrices taken from a prescribed similarity class. Then we discuss the concept of *off-diagonal indices* (defined in Section 1) which, very roughly speaking, measure, for each given integer s, how far we have to go off the main diagonal of a square matrix, to find an $s \times s$ nonzero minor. Some open problems are stated. © 2000 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

This paper is about matrices over an arbitrary field $\mathbf{F}$. The script letters $\mathscr{A}$ and $\mathscr{B}$ represent $n \times n$ similarity classes over $\mathbf{F}$. So, for any $M \in \mathscr{A}$, M is an n-square matrix over $\mathbf{F}$, and $\mathscr{A}$ is the set of all matrices over $\mathbf{F}$ similar to M. The *similarity invariant polynomials, eigenvalues, rank*, etc, of the class $\mathscr{A}$ are defined as the corresponding concepts referred to any $A \in \mathscr{A}$. It is well-known that there exists a matrix in $\mathscr{A}$ of the form $A_1 \oplus N$, where A_1 is nonsingular and N is nilpotent; moreover, the similarity classes of A_1 and N are well-defined and called the *nonsingular* and *nilpotent parts* of $\mathscr{A}$. The rank of $\mathscr{A}$ is denoted by $r_{\mathscr{A}}$ or rank $\mathscr{A}$.

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A subset $i \subset \{1, \dots, n\}$ will be called an *indexing set* of order $|i|$, where $|i|$ denotes the cardinality of i. We denote by i^{c} the complementary indexing set $\{1, \dots, n\} \backslash i$. Given two indexing sets, i and j, the symbol $A[i|j]$ represents the submatrix of A consisting of A's entries a_{vw}, with $v \in i$ and $w \in j$. Denote by $d(i, j)$, or just d_{ij}, the cardinality of $i \backslash j$. Clearly $j \backslash i = i^{\mathrm{c}} \backslash j^{\mathrm{c}}$ and, if i and j have the same order, $i \backslash j$ and $j \backslash i$ have the same cardinality. Therefore, in case $|i| = |j|$

$$d_{ij} = d_{ji} = d_{i^{\mathrm{c}} j^{\mathrm{c}}}.$$

In the sequel we study in some detail the following concepts:

Definition 1.1. For any nonnegative integer s, and n-square matrix A, the sth off-diagonal index of A is

$$d(A, s) := \inf\{d_{ij} : |i| = |j| = s,\ \det A[i|j] \neq 0\}.$$

The sth off-diagonal index of a similarity class $\mathscr{A}$ is defined by

$$d(\mathscr{A}, s) := \sup_{A \in \mathscr{A}} d(A, s). \tag{1}$$

We adopt the usual set-theoretical conventions according to which the 0×0 (empty) matrix has determinant 1, and $\inf \emptyset$ is $+\infty$. These conventions imply $d(A, 0) = 0$, as well as: $d(A, s) = +\infty$ iff $s > \operatorname{rank} A$.

Roughly speaking, $d(A, s)$ measures how far we have to go off the main diagonal of A to find an $s \times s$ nonzero minor. For instance, $d(A, 1) > 0$ iff all diagonal elements of A are zero; $d(A, n) = 0$ iff A is nonsingular. According to [2] (see also [3]) for any nonscalar class $\mathscr{A}$, the only constraint on the diagonal elements of $A \in \mathscr{A}$ is the trace condition; so $d(\mathscr{A}, 1) > 0$ iff the trace of $\mathscr{A}$ is zero.

The off-diagonal indices occur in problems connected with the pencil $xA + B$, where x is a variable, and A and B are supposed to run over two given similarity classes $\mathscr{A}$ and $\mathscr{B}$, respectively. For example, we may ask for the possible degrees of the polynomial $\det(xA + B)$, or the possible number of positive Kronecker indices of $xA + B$. These problems will be considered in forthcoming work.

In Section 4 we study the off-diagonal indices for their own sake, and, as we shall see, this will lead us to interesting properties and open problems.

Notation for similarity invariant polynomials. In the sequel $\mathscr{A}$ denotes a similarity class over $\mathbf{F}$, of order n, with similarity invariant polynomials $\alpha_1, \dots, \alpha_n$. The α's are monic polynomials, taken from the polynomial ring $\mathbf{F}[x]$, ordered so that $\alpha_1 | \cdots | \alpha_n$. Wherever needed, we use the conventions: $\alpha_v = 1$ for $v < 1$, and $\alpha_v = 0$ for $v > n$. The coefficients of the characteristic polynomial of $\mathscr{A}$, that we represent by $\chi_{\mathscr{A}}$, will be denoted by $\sigma_s(\mathscr{A})$, so that

$$\chi_{\mathscr{A}}(x) = x^n + \sigma_1(\mathscr{A}) x^{n-1} + \cdots + \sigma_n(\mathscr{A}).$$

The degree of a polynomial f is denoted $\deg f$.

2. ‘Minimal’ off-diagonal indices

In our definition (1) we considered the ‘sup’ over the class $\mathscr{A}$. We may now ask how interesting is the sequence of integers

$$\delta(\mathscr{A}, s) := \inf_{A\in\mathscr{A}} d(A, s),$$

that we may call the *minimal off-diagonal indices of the class* $\mathscr{A}$.

These numbers are not so interesting as those of Definition 1.1. In fact, the minimal off-diagonal indices have an extremely simple characterization, that is given, without proof, as a consequence of the following lemma.

Lemma 2.1. *There exists $M \in \mathscr{A}$ with all $s \times s$ leading principal minors nonsingular, for* $1 \leqslant s \leqslant \operatorname{rank} \mathscr{A}$.

Proof. We shall use the so-called *interlacing theorem* for the similarity invariant polynomials of principal submatrices [4,7]. The rank of $\mathscr{A}$ is the number of similarity invariant polynomials of $\mathscr{A}$, $\alpha_1|\cdots|\alpha_n$, that are not multiple of x. For an arbitrary but fixed $s \leqslant r_{\mathscr{A}}$, let φ_s be any monic polynomial of degree $s - \deg(\alpha_1\cdots\alpha_s)$, not multiple of x. Then the polynomials $\gamma_1|\cdots|\gamma_s$, given by $\gamma_i = \alpha_i$, for $i < s$, and $\gamma_s = \alpha_s\varphi_s$, are the invariant polynomials of a nonsingular $s \times s$ matrix, say A_s. The γ’s and α’s interlace, in the sense that $\alpha_i|\gamma_i|\alpha_{i+2n-2s}$ for all i. By the referred interlacing theorem, there exists $A \in \mathscr{A}$ having A_s as leading principal submatrix.

The lemma follows by an easy induction using suitably the procedure just described. The details are left to the reader. □

Theorem 2.2. $\delta(\mathscr{A}, s) = 0$ *for* $s \leqslant \operatorname{rank} \mathscr{A}$, *and* $\delta(\mathscr{A}, s) = +\infty$ *for* $s > \operatorname{rank} \mathscr{A}$. *There exists a matrix $M \in \mathscr{A}$ such that $d(M, s) = \delta(\mathscr{A}, s)$ for all s.*

3. Ranks of submatrices

Recall that $\mathscr{A}$ denotes a similarity class over **F**, of order n, with similarity invariant polynomials $\alpha_1|\cdots|\alpha_n$.

Theorem 3.1. *We are given two indexing sets, i and j, with the same cardinality r. The following conditions are equivalent*:
(a) $\operatorname{rank} \mathscr{A} \geqslant r$ *and* $\alpha_d = 1$, *where d denotes the cardinality of* $i\backslash j$.
(b) *There exists $A \in \mathscr{A}$ such that $A[i|j]$ is nonsingular.*
(c) *There exists $A \in \mathscr{A}$ such that $A[i|j]$ and $A[i\backslash j|j\backslash i]$ are nonsingular.*

Proof. For our purpose we may assume $i = \{1, \ldots, r\}$ and $j = \{d + 1, \ldots, d + r\}$. This means we are partitioning our matrices $A \in \mathscr{A}$ in the form

$$A=\begin{bmatrix} * & X & D & * \\ * & P & Y & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}, \tag{2}$$

where D and P are square blocks of orders d and $p := r - d$, respectively. We are looking for conditions on $\mathscr{A}$, equivalent to the existence of $A \in \mathscr{A}$ such that $A[i\,|\,j]$ – i.e., the corresponding '*XPDY*' block – is nonsingular (and, in item (c), with D nonsingular as well). Note that P is a principal submatrix of A.

(b) $\Rightarrow$ (a). Let $\psi_A(x)$ be the determinant of

$$(A - xI)[i\,|\,j] = \begin{bmatrix} X & D \\ P - xI & Y \end{bmatrix}.$$

(b) means that $\psi_A(0)$ is nonzero. Moreover deg $\psi_A \leqslant p$. This implies deg $\delta_r \leqslant p$, where δ_r is the rth determinantal divisor of $A - xI$. Recall $\delta_r = \alpha_1\alpha_2 \cdots \alpha_r$. Clearly $\alpha_d \neq 1$ implies $\deg(\alpha_d \cdots \alpha_r) > r - d = p$. So we must have $\alpha_d = 1$. The condition rank $\mathscr{A} \geqslant r$ is obvious.

(a) $\Rightarrow$ (c). By (a), and by the interlacing inequalities theorem [4,7], used as in the proof of Lemma 2.1, there exists $A \in \mathscr{A}$ having a leading principal submatrix A^*, of order $d + r$, satisfying rank $A^* \geqslant r$ and $\alpha_d^* = 1$. Clearly, if we prove (c) for the similarity class of A^*, then (c) will follow in general. This is the same thing as assuming, without loss of generality, that $n = d + r$. Thus, we shall assume that the last row and the last column of blocks in (2) are empty, that is

$$A=\begin{bmatrix} * & X & D \\ * & P & Y \\ * & * & * \end{bmatrix}.$$

As $\alpha_d = 1$, the main results of [6,8] (see also, [5, Theorem 5.2]) imply the existence of $A \in \mathscr{A}$ whose block $[X\ \ D]$ has rank d (*using the notation of* [6, p. 104], we are applying Theorems 1 and 2 to the case $p = r$; thus (I) and (II) of [6] do not hold). We may further assume that D is nonsingular. So we may zero out all blocks under D by means of similarity transformations, and get

$$\begin{bmatrix} * & * & D \\ U & P & 0 \\ V & W & 0 \end{bmatrix}. \tag{3}$$

For any $d \times p$ matrix Z, A is also similar to a matrix like

$$\begin{bmatrix} * & * & D \\ U & P + UZ & 0 \\ V & W + VZ & 0 \end{bmatrix}.$$

On the other hand, as

$$\begin{bmatrix} U & P \\ V & W \end{bmatrix}$$

has rank $\geqslant p$, there exists Z such that

$$\operatorname{rank}\begin{bmatrix} P+UZ \\ W+VZ \end{bmatrix} = p.$$

So, to simplify notations, we may assume that

$$\begin{bmatrix} P \\ W \end{bmatrix}$$

already has rank p. In (3), D may be used to zero out the (1, 2) block without changing the rank of the 'PW' block. So A is similar to a matrix like

$$\begin{bmatrix} * & 0 & D \\ * & P & 0 \\ * & W & 0 \end{bmatrix},$$

with the PW block of rank p. There exists a $p \times d$ matrix M such that $Q := P + MW$ is nonsingular. With one more block similarity we transform the last matrix into one of the following type

$$\begin{bmatrix} * & 0 & D \\ * & Q & * \\ * & W & * \end{bmatrix},$$

with Q and D nonsingular. This proves (c). □

Theorem 3.2. *We are given two indexing sets, i and j, and a nonnegative integer $r \leqslant \min\{|i|, |j|\}$. Define $d := r - |i \cap j|$. The following conditions are equivalent*:

(α) rank $\mathscr{A} \geqslant r$ *and* $\alpha_d = 1$.

(β) *There exists $A \in \mathscr{A}$ such that* rank $A[i|j] \geqslant r$.

Proof. First we consider the case $d \leqslant 0$. Then (α) reduces to rank $\mathscr{A} \geqslant r$; by Lemma 2.1 there exists $A \in \mathscr{A}$ having a nonsingular principal submatrix of order r. So (α) implies (β). The converse is obvious.

Now we prove the theorem when d is positive.

Assume (α) holds, and choose indexing subsets, $i^* \subset i$ and $j^* \subset j$, both of order r, such that $i^* \cap j^* = i \cap j$. By Theorem 3.1 there exists $A \in \mathscr{A}$ such that $A[i^*|j^*]$ is nonsingular. Then (β) holds.

Conversely, (β) implies the existence of indexing subsets, $i' \subset i$ and $j' \subset j$, both of order r, such that $A[i'|j']$ is nonsingular. Obviously

$$|i' \backslash j'| = |i' \backslash (i' \cap j')| \geqslant r - |i \cap j| = d.$$

Thus (α) follows easily from Theorem 3.1. □

At this stage it is only natural to consider the following problem.

Problem 1. Given indexing sets, i and j, of any orders, and an integer r, characterize the similarity classes $\mathscr{A}$ such that there exists $A \in \mathscr{A}$ satisfying rank $A[i|j] = r$.

A different problem, seemingly less difficult to handle, is obtained replacing "rank $A[i|j] = r$" by "rank $A[i|j] \leqslant r$". We could not solve these problems in the general case. One of the difficulties has to do with the peculiarities of the field. For example, with the help of [1, Theorem 2], we may obtain a very involved solution, only valid for infinite fields. The solution for the case when i and j are disjoint and $i \cup j = \{1, \dots, n\}$, given in [6, Theorem 2], also shows how hard it can be. We give three results solving Problem 1 for special kinds of indexing sets.

Theorem 3.3. *We are given two disjoint indexing sets, i and j, and a nonnegative integer r, satisfying $|i| + |j| < n$ and $r \leqslant \min\{|i|, |j|\}$. There exists $A \in \mathscr{A}$ such that* rank $A[i|j] = r$, *if and only if* $\alpha_r = 1$.

Proof. The *only if* part is obvious, because $xI - A$ has a nonzero, constant r-minor, and so rth determinantal divisor of $xI - A$ is 1.

We now prove the *if* part. Note that $i^{\mathrm{c}} \supset j$. By [6, Theorem 2], there exists $A \in \mathscr{A}$ such that rank $A[i|i^{\mathrm{c}}] = r$, except in either one of two cases. Of those cases we only need to know the following: in the first one, $|i|$ is odd; in the second exceptional case, $|i|$ is multiple of the degree of the minimal polynomial of $\mathscr{A}$, and that degree is $\geqslant 3$. If none of those exceptional cases occurs; the existing matrix A may be chosen in such a way that rank $A[i|j] = r$, because $A[i|j]$ is a submatrix of $A[i|i^{\mathrm{c}}]$. Now assume that one of those exceptional cases occurs; then we apply [6, Theorem 2] to submatrices in a slightly different position; namely, we choose an indexing set κ such that

$$\kappa \supset i, \quad \kappa^{\mathrm{c}} \supseteq j, \quad \text{and} \quad |\kappa| = |i| + 1.$$

By [6, Theorem 2], there exists $A \in \mathscr{A}$ such that rank $A[\kappa|\kappa^{\mathrm{c}}] = r$ (note that no exception occurs here). As $A[i|j]$ is a submatrix of $A[\kappa|\kappa^{\mathrm{c}}]$, A may be appropriately chosen so that rank $A[i|j] = r$. □

It is easy to see that if A is an $n \times m$ matrix over $\mathbf{F}$, and r is the rank of a $p \times t$ submatrix, then

$$r_A - (n - p) - (m - t) \leqslant r \leqslant \min\{r_A, p, t\}. \tag{4}$$

This is a best possible result, in the sense that, if these inequalities hold (for nonnegative integers, such that $p \leqslant n,\ t \leqslant m,\ r_{\mathscr{A}} \leqslant \min\{n, m\}$), then we may find a pair matrix–submatrix with the required sizes and ranks.

The proof of the next result is left to the reader.

Theorem 3.4. *Let i be an indexing set with p elements, and r a nonnegative integer. There exists $A \in \mathscr{A}$ such that $A[i|\{1, 2, \dots, n\}]$ has* rank r, *iff*

$$r_{\mathscr{A}} - (n - p) \leqslant r \leqslant \min\{r_{\mathscr{A}}, p\}. \tag{5}$$

Theorem 3.5. *Let r and p be nonnegative integers,* $p \leqslant n$. *There exists* $A \in \mathscr{A}$ *having a principal* $p \times p$ *submatrix of* rank r, *if and only if the following inequalities hold*:

$$r_{\mathscr{A}} - 2(n-p) \leqslant r \leqslant \min\{r_{\mathscr{A}}, p\}, \tag{6}$$

$$\min\{r_{\mathscr{A}}, p\} - p + \deg(\alpha_1 \cdots \alpha_p) \leqslant r. \tag{7}$$

Proof. Let M be a principal $p \times p$ submatrix of a matrix $A \in \mathscr{A}$, of rank r, and let $\mu_1 | \cdots | \mu_p$ be the invariant factors of M. The inequalities (6) were already considered. By [4,7] we have $\alpha_i | \mu_i$, for $i = 1, \ldots, p$. Let a_0 $[m_0]$ be the number of polynomials among $\alpha_1, \ldots, \alpha_p$ $[\mu_1, \ldots, \mu_p]$ that are multiple of x. Note that $a_0 = p - \min\{r_{\mathscr{A}}, p\}$ and $m_0 = p - r$. Clearly $a_0 \leqslant m_0$ and there are at least $m_0 - a_0$ indices s $(s \leqslant p)$ such that α_s strictly divides μ_s; therefore

$$m_0 - a_0 \leqslant \deg(\mu_1 \cdots \mu_p) - \deg(\alpha_1 \cdots \alpha_p).$$

This is equivalent to the inequality (7).

Conversely, assume (6)–(7) hold. Define the polynomials $\mu_1, \ldots, \mu_p$ by

$$\mu_i = \begin{cases} x\alpha_i & \text{if } r < i \leqslant \min\{r_{\mathscr{A}}, p\}, \\ \alpha_i & \text{otherwise.} \end{cases}$$

Note that α_i is multiple [not multiple] of x for $i > \min\{r_{\mathscr{A}}, p\}$ $[i \leqslant \min\{r_{\mathscr{A}}, p\}]$; therefore $\mu_1 | \cdots | \mu_p$. Trivially $\alpha_i | \mu_i$, for all i, and the relations $\mu_i | \alpha_{i+2n-2p}$ follow easily from the definition of the μ's and the left inequality in (6).

Now redefine μ_p multiplying it by any monic polynomial of degree

$$p + r - \min\{r_{\mathscr{A}}, p\} - \deg(\alpha_1 \cdots \alpha_p), \tag{8}$$

not multiple of x. Note that (8) is nonnegative, because of (7). The new value of μ_p yeilds $\deg(\mu_1 \cdots \mu_p) = p$, and these μ's obviously satisfy $\alpha_i | \mu_i | \alpha_{i+2n-2p}$ for all i. Let M be any $p \times p$ matrix with invariant factors (μ_i). Then M has rank r, and, by [4,7], there exists $A \in \mathscr{A}$ having M as a principal submatrix. □

If i and j are arbitrary indexing sets, and $A[i|j]$ has rank r, then $A[i \cap j | i \cap j]$ is a principal submatrix of A of rank $\leqslant r$. We therefore have the following immediate consequence of (4) and the previous theorem.

Corollary 3.6. *If* $A \in \mathscr{A}$ *and* $A[i|j]$ *has* rank r *we have*

$$r_{\mathscr{A}} - (n - |i|) - (n - |j|) \leqslant r \leqslant \min\{r_{\mathscr{A}}, |i|, |j|\},$$
$$\min\{r_{\mathscr{A}}, p\} - p + \deg(\alpha_1 \cdots \alpha_p) \leqslant r,$$

where p denotes $|i \cap j|$.

4. The off-diagonal indices

We now consider a second similarity class $\mathscr{B}$, with similarity invariant polynomials $\beta_1|\cdots|\beta_n$, and the pencils of the form

$$xA+B,$$

where $A\in\mathscr{A}$ and $B\in\mathscr{B}$. Define $\varDelta(x):=\det(xA+B)$. Let us expand $\varDelta(x)$ in powers of x, say $\varDelta(x)=C_nx^n+\cdots+C_0$. Clearly, the coefficients C_s are given by

$$C_s=\sum_{|v|=|w|=s}\epsilon_{vw}\det(A[v|w])\det(B[v^{\mathrm{c}}|w^{\mathrm{c}}]),\tag{9}$$

where v and w run over the set of indexing sets of order s and ϵ_{vw} is the sign of the $[v|w]$-minor.

Theorem 4.1. *If $\mathscr{A}$ and $\mathscr{B}$ are nilpotent classes and $r_{\mathscr{A}}+r_{\mathscr{B}}\geqslant n$, then there exist $A\in\mathscr{A}$ and $B\in\mathscr{B}$ such that $xA+B$ is nonsingular.*

Proof. We let A be a Jordan normal form of $\mathscr{A}$. Let i $[j]$ be the set of indices of the zero rows [resp. columns] of A. Clearly $n-r_{\mathscr{A}}$ is the cardinality of i, that we denote by r.

We apply Theorem 3.1 to the class $\mathscr{B}$. Our assumption $r_{\mathscr{A}}+r_{\mathscr{B}}\geqslant n$ reads $r_{\mathscr{B}}\geqslant r$. As $\mathscr{B}$ is nilpotent, $r_{\mathscr{B}}$ is the number of $\beta_1,\ldots,\beta_n$ equal to 1; therefore $\beta_d=1$. So there exists $B\in\mathscr{B}$ such that $B[i|j]$ is nonsingular.

Now observe that $A[i^{\mathrm{c}}|j^{\mathrm{c}}]$ is an identity matrix, and it is the only nonsingular submatrix of A of order $r_{\mathscr{A}}$. Therefore, the coefficient $C_{r_{\mathscr{A}}}$ equals ϵ_{ij} det $B[i|j]$, and so $xA+B$ is nonsingular. □

Theorem 4.2. *If the coefficient C_s in* (9) *is nonzero, then*

$$n-r_{\mathscr{B}}\leqslant s\leqslant r_{\mathscr{A}},\tag{10}$$

$$\beta_{d(\mathscr{A},s)}=\alpha_{d(\mathscr{B},n-s)}=1.\tag{11}$$

Proof. Without loss of generality we may assume A satisfies $d(A,s)=d(\mathscr{A},s)$. There exist i and j such that $|i|=|j|=s$, and $A[i|j]$ and $B[i^{\mathrm{c}}|j^{\mathrm{c}}]$ are nonsingular. We have

$$d_{ij}=d_{i^{\mathrm{c}}j^{\mathrm{c}}}\geqslant d(\mathscr{A},s).$$

Theorem 3.1 applied to B and to its $[i^{\mathrm{c}}|j^{\mathrm{c}}]$-minor gives $\beta_{d(\mathscr{A},s)}=1$. The other identity in (11) is obtained by exchanging the roles of A and B. The rank conditions (10) are obvious. □

Next we prove some basic properties of the off-diagonal indices. Recall that $\sigma_s(\mathscr{A})$ denotes the sth coefficient of the characteristic polynomial of $\mathscr{A}$.

Theorem 4.3. *Let* $\mathbf{A}$ *denote an* $n \times n$ *matrix or similarity class. For all relevant values of* s:

(a) *If* $d(\mathbf{A}, s) > d(\mathbf{A}, s+1)$, *then* $d(\mathbf{A}, s) = 1$.
(b) *Let* K *be a positive integer. If* $d(\mathbf{A}, s) \leqslant K$, *then* $d(\mathbf{A}, t) \leqslant K$ *for all* $t \leqslant s$.
(c) $d(\mathbf{A}, s+1) \leqslant d(\mathbf{A}, s) + 1$, *whenever* $s <$ rank $\mathbf{A}$.
(d) $\sigma_s(\mathbf{A}) \neq 0 \;\Rightarrow\; d(\mathbf{A}, s) = 0$.
(e) *For two square matrices* V *and* W:

$$d(V \oplus W, s) = \min_{0 \leqslant k \leqslant s} \{d(V, k) + d(W, s-k)\}.$$

Proof. (a) Let A be an $n \times n$ matrix, and let i and j be distinct indexing sets of order $s+1$, such that $A[i|j]$ is nonsingular. Let i' be any subset of i, of cardinality s, containing $i \cap j$. There exists $j' \subset j$ such that $A[i'|j']$ is square nonsingular; as $d_{i'j'} \leqslant d_{ij}$, we have $d(A, s) \leqslant d_{ij}$. This argument shows that $d(A, s+1) > 0$ implies $d(A, s) \leqslant d(A, s+1)$.

Now assume $d(A, s) > d(A, s+1)$. Then $d(A, S+1) = 0$. This means that A has a nonsingular principal submatrix of order $s+1$; this principal submatrix has a nonsingular submatrix, say $A[u|v]$, of order s. Clearly $d_{uv} \leqslant 1$ and, therefore, $d(A, s) \leqslant 1$. This forces $d(A, s) = 1$.

We use this to prove (a) for a similarity class $\mathscr{A}$. Assume $d(\mathscr{A}, s) > d(\mathscr{A}, s+1)$. There exists $M \in \mathscr{A}$ satisfying $d(M, s) = d(\mathscr{A}, s)$; we have $d(M, s) > d(\mathscr{A}, s+1 \geqslant d(M, s+1)$; using (a) for individual matrices, we get $d(M, s) = 1$, etc.

(b) We get a contradiction from the assumption: $d(\mathbf{A}, s) \leqslant K$ and $d(\mathbf{A}, t) > K$, for some $t \leqslant s$. As a matter of fact, if τ denotes the largest t' such that $t \leqslant t' \leqslant s$ and $d(\mathbf{A}, t) \leqslant d(\mathbf{A}, t')$, then we have $d(\mathbf{A}, \tau) > d(\mathbf{A}, \tau+1)$ and $d(\mathbf{A}, \tau) > K$; this contradicts (a). So (b) holds.

(c) The assumption $s <$ rank A implies that any nonsingular submatrix $A[i|j]$ of order s is contained in a nonsingular submatrix of A, of order $s+1$. That is, there exist indexing sets of order $s+1$, i^* and j^*, such that $i^* \supseteq i$, $j^* \supseteq j$ and $A[i^*|j^*]$ is nonsingular. The property follows from $d_{i^*j^*} \leqslant d_{ij} + 1$.

The extension to a similarity class $\mathscr{A}$ is straightforward.

(d) is obvious, because, for a square matrix A, $\sigma_s(A)$ is (up to the sign) the sum of all principal minors of order s of A.

(e) The *nonzero* minors of $A = V \oplus W$ are precisely the products

$$\det(A[i_V|j_V]) \det(A[i_W|j_W]),$$

with $A[i_V|j_V]$ a nonsingular submatrix of V, and $A[i_W|j_W]$ a nonsingular submatrix of W. The property is an easy consequence of $d(i_V \cup i_W, j_V \cup j_W) = d(i_V, j_V) + d(i_W, j_W)$. □

Theorem 4.4. *Assume $\mathscr{A}$ has nonsingular and nilpotent parts $\mathscr{A}_1$ and $\mathscr{N}$, of dimensions n_1 and n_0, respectively, and let $k_1 \geqslant \cdots \geqslant k_u > 0$ be the orders of the nilpotent Jordan blocks of $\mathscr{A}$. Then*

$$d(\mathscr{A}, s) \leqslant d(\mathscr{A}_1, s) \quad \text{if } s < n_1, \tag{12}$$

$$d(\mathscr{A}, s) = d(\mathscr{N}, s - n_1) \quad \text{if } s \geqslant n_1. \tag{13}$$

$d(\mathscr{A}, 1), d(\mathscr{A}, 2), \ldots, d(\mathscr{A}, n_1)$ *is a sequence of zeroes and ones, ending up with* 0. *Moreover, if $N := J_{k_1} \oplus \cdots \oplus J_{k_u}$ (J_t is a Jordan block of order t) is the Jordan normal form of $\mathscr{N}$,*

$$d(\mathscr{N}, t) = d(N, t) \tag{14}$$

$$= \min\{w : t \leqslant k_1 + \cdots + k_w - w\}. \tag{15}$$

Proof. It is not difficult to prove, by simple calculation, that $d(N, t)$ is given by formula (15). We omit the details.

Clearly, $\sigma_{n_1}(\mathscr{A})$ is (up to the sign) the determinant of $\mathscr{A}_1$ which is nonzero. So $d(\mathscr{A}, n_1) = d(\mathscr{A}_1, n_1) = 0$ and, therefore, the identity (13) holds for $s = n_1$.

By Theorem 4.3(b) (applied to the case $s := n_1$, $K := 1$), $d(\mathscr{A}, 1), \ldots, d(\mathscr{A}, n_1)$ and $d(\mathscr{A}_1, 1), \ldots, d(\mathscr{A}_1, n_1)$, are sequences of zeroes and ones.

For $s < n_1$, let $M_s \in \mathscr{A}_1$ satisfy $d(M_s, s) = d(\mathscr{A}_1, s)$. As $d(M_s, s) \in \{0, 1\}$ and $d(N, t)$ is positive except for $t = 0$, Theorem 4.3(e) yields $d(M_s \oplus N, s) = d(M_s, s)$. So (12) holds.

Now we prove that

$$d(A, s) \leqslant d(N, s - n_1), \tag{16}$$

holds, for any $A \in \mathscr{A}$, and any $s > n_1$. Let $w \in \{0, \ldots, u\}$, and define $K := k_1 + \cdots + k_w$ ($K := 0$ if $w = 0$). Denote by $\delta_q(\mathscr{A})$ the qth determinantal divisor of $xI - A$. The polynomials $x^{k_1}, \ldots, x^{k_u}$ are the x-powered elementary divisors of $xI - A$. As $\delta_{n-w}(\mathscr{A})$ is the product $\alpha_1 \cdots \alpha_{n-w}$, we have

$$\delta_{n-w}(\mathscr{A}) = \psi_w(x) x^{n_0 - K},$$

where $\psi_w(x)$ is not a multiple of x. Therefore, there exists an $(n - w)$-minor of $xI - A$ with a nonzero coefficient for the term of degree $n_0 - K$; assume it is the minor corresponding to the row set μ and column set ν, $|\mu| = |\nu| = n - w$. This implies the existence of a subset σ of $\mu \cap \nu$, with $n_0 - K$ elements, such that $A[\mu\backslash\sigma|\nu\backslash\sigma]$ is nonsingular; this matrix is of order $n - w - n_0 + K = n_1 + K - w$. As $|\mu| = |\nu| = n - w$, we have $d(\mu, \nu) \leqslant w$. On the other hand $d(\mu, \nu) = d(\mu\backslash\sigma, \nu\backslash\sigma)$, and thus $d(\mu\backslash\sigma, \nu\backslash\sigma) \leqslant w$. This proves the formula

$$d(A, n_1 + k_1 + \cdots + k_w - w) \leqslant w, \quad \text{for any } w \in \{0, \ldots, u\}. \tag{17}$$

Next we take any s such that $n_1 < s \leqslant \operatorname{rank} \mathscr{A}$, and use inequality (17) with $w := d(N, s - n_1)$. As $d(N, t)$ is given by (15), we have $s \leqslant n_1 + k_1 + \cdots + k_w - w$. If

$d(A, s) > d(A, n_1 + k_1 + \cdots + k_w - w)$, then Theorem 4.3 implies $d(A, s) \leqslant 1 \leqslant w$; otherwise, (17) implies $d(A, s) \leqslant w$. So (16) holds.

The inequality (16) applied to the nilpotent case ($\mathscr{A} = \mathscr{N}$ and $n_1 = 0$) gives the identity (14).

As (16) holds for any $A \in \mathscr{A}$, we have $d(\mathscr{A}, s) \leqslant d(\mathscr{N}, s - n_1)$. Finally, let us choose $A_1 \in \mathscr{A}_1$, and $s > n_1$; taking into account Theorem 4.3(e) and the fact that $d(N, k)$ is nondecreasing with k, we get

$$d(A_1 \oplus N, s) \geqslant \min_{k \geqslant s - n_1} d(N, k) = d(N, s - n_1) = d(\mathscr{N}, s - n_1).$$

Therefore $d(\mathscr{A}, s) \geqslant d(\mathscr{N}, s - n_1)$, and (13) holds. □

Theorem 4.5. *Let $\mathscr{A}$ be a nonderogatory class. Then*

$$d(\mathscr{A}, s) = \begin{cases} 0 & \textit{if } s \leqslant \operatorname{rank} \mathscr{A} \quad \textit{and} \quad \sigma_s(\mathscr{A}) \neq 0, \\ 1 & \textit{if } s \leqslant \operatorname{rank} \mathscr{A} \quad \textit{and} \quad \sigma_s(\mathscr{A}) = 0, \\ +\infty & \textit{if } s > \operatorname{rank} \mathscr{A}, \end{cases}$$

and $d(C, s) = d(\mathscr{A}, s)$, for all s, where C denotes a companion matrix of $\mathscr{A}$.

Proof. It is easy to check that C has off-diagonal indices as displayed in the statement. The rest is a simple matter, based on Theorems 4.4 and 4.3(d) and the fact that $\mathscr{A}$ has at most one nilpotent Jordan block. □

Combining the two previous theorems, we obtain the following characterization of all possible sequences of off-diagonal indices of individual matrices and similarity classes. The straightforward proof is left to the reader.

Theorem 4.6. *For any $n \times n$ matrix A, $d(A, 1), \ldots, d(A, n)$ may be split into three consecutive sections: the first one is a 0–1 sequence ending with 0; the second section (that may be empty) is a nondecreasing sequence of positive integers, which starts with 1 and has jumps not greater than 1; the third section (that may be empty) is a tail of $+\infty$'s. Any sequence of this kind is the sequence of off-diagonal indices of an $n \times n$ matrix. The result also holds when we replace the individual matrix by a similarity class.*

5. Examples and further results

The aim of this section is to show that the converse to property (d) of Theorem 4.3 is not true in general and that we may have strict inequality in (12).

In the next theorem we single out the following

Exceptional case: $\mathscr{A}$ is a $n \times n$ scalar, nonzero class, and n is multiple of the characteristic of **F**.

Clearly, in this case, the converse to Theorem 4.3(d) is not true for $s \in \{1, n-1\}$.

Theorem 5.1. *Suppose $\mathscr{A}$ is not in the above exceptional case. If $\sigma_1(\mathscr{A}) = 0$ $[\sigma_{n-1}(\mathscr{A}) = 0]$, there exists $A \in \mathscr{A}$ with all diagonal elements [resp., all principal minors of order $n-1$] equal to zero.*

Proof. For a scalar $\mathscr{A}$ the theorem is obvious. So assume $\mathscr{A}$ is not scalar.

According to [2,3], if $\mathscr{A}$ has zero trace, and $d_1, \ldots, d_n$ are any elements of $\mathbf{F}$ with sum 0, there exists $A \in \mathscr{A}$ with diagonal elements $d_1, \ldots, d_n$. This settles the case $\sigma_1(\mathscr{A}) = 0$.

Now assume $\sigma_{n-1}(\mathscr{A}) = 0$. If $\mathscr{A}$ is nonsingular, then $\mathscr{A}^{-1}$ has zero trace, and therefore there exists $M \in \mathscr{A}^{-1}$ with zero diagonal; then M^{-1} lies in $\mathscr{A}$ and has all principal minors of order $n-1$ equal to zero. If $\mathscr{A}$ is singular, let $\mathscr{A}$ be any matrix in $\mathscr{A}$ of the form $A = A_1 \oplus N$, with A_1 nonsingular and N a direct sum of nilpotent Jordan blocks. All principal minors of N are zero. Therefore, $\sigma_{n-1}(\mathscr{A}) = 0$ implies N has order $\geqslant 2$; so all principal minors of A of order $n-1$ are zero. □

For elements α and β of $\mathbf{F}$, we let $\mathscr{A}_{\alpha\beta}$ be the $n \times n$ similarity class with invariant polynomials $1, x-\alpha, \ldots, x-\alpha, (x-\alpha)(x-\beta)$. We shall determine the off-diagonal indices of $\mathscr{A}_{\alpha\beta}$. As this is a simple task in case $\alpha = 0$, we shall henceforth assume that α is nonzero. We denote by p the characteristic of the field $\mathbf{F}$. An integer m is said to be *nonzero* (or *invertible*) *in* $\mathbf{F}$ if it is not multiple of p.

Theorem 5.2. *Let s be an integer such that $1 \leqslant s < n$.*

(i) *There exists $G \in \mathscr{A}_{\alpha\beta}$ with all s-by-s principal minors equal to zero iff*

$$s \text{ is nonzero in } \mathbf{F}, \text{ and } n\alpha = s(\alpha - \beta). \tag{18}$$

(ii) *Assume (18) holds. An $n \times n$ matrix G satisfies the previous conditions iff G is diagonally similar to $\alpha I - (\alpha/s)\Omega_n$, where Ω_n denotes the $n \times n$ matrix with all entries equal to 1.*

Proof. First we prove the 'only if' claims of (i) and (ii). Assume $G \in \mathscr{A}_{\alpha\beta}$ has all s-by-s principal minors equal to zero. Let M be any $s \times s$ principal submatrix of G. By the interlacing theorem [4,7] the s similarity invariant polynomials of M are

$$1, x-\alpha, \ldots, x-\alpha, (x-\alpha)(x-\mu)$$

for $s > 1$, or just $x - \mu$ in case $s = 1$, where μ is an element of $\mathbf{F}$. As M is singular, $\mu = 0$ and, therefore, the trace of M is $(s-1)\alpha$. So the diagonal entries of G are pairwise equal; let Δ be the common value of these entries. We have

$$\begin{cases} s\Delta = (s-1)\alpha, \\ n\Delta = (n-1)\alpha + \beta. \end{cases}$$

As $\alpha \neq 0$, these equations imply (18). The similarity invariant polynomials of $\alpha I - G$ are $1, x, \ldots, x, x(x - \alpha + \beta)$. Therefore, $\alpha I - G$ is a rank-one matrix which, as we have just seen, has all its diagonal entries equal to α/s. This means that $\alpha I - G$ is diagonally similar to $(\alpha/s)\Omega_n$. This proves that G satisfies the 'only if' condition of (ii).

We now prove the 'if' claims. Assume (18) holds. As $(\alpha/s)\Omega_n$ has rank 1 and trace $n\alpha/s$, its similarity invariant polynomials are $1, x, \ldots, x, x(x - n\alpha/s)$. Therefore, $G_0 := \alpha I - (\alpha/s)\Omega_n$ lies in $\mathscr{A}_{\alpha\beta}$. On the other hand, any $s \times s$ submatrix of $(\alpha/s)\Omega_n$ has rank 1 and trace α; therefore, such a submatrix has α as an eigenvalue. So any $s \times s$ principal submatrix of G_0 is singular. This completes the proof of (i). Now (ii) follows in a simple way. □

The first part of this theorem obviously determines the off-diagonal indices of $\mathscr{A}_{\alpha\beta}$: we have, for $1 < s < n$,

$$d(\mathscr{A}_{\alpha\beta}, s) = \begin{cases} 1 & \text{if (18)}, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

It is easily to compute the coefficients of the characteristic polynomial of $\mathscr{A}_{\alpha\beta}$ and check the following

$$\sigma_s(\mathscr{A}_{\alpha\beta}) = 0 \iff \binom{n-1}{s}\alpha + \binom{n-1}{s-1}\beta = 0. \tag{20}$$

Comparing (19) with (20), the reader may find classes of examples where the converse to Theorem 4.3(d) fails. An interesting case occurs when $\beta = 0$ *and n is not a multiple of p*. We then have, for $1 < s < n$

$$d(\mathscr{A}_{\alpha 0}, s) = 1 \iff p \text{ divides } n - s, \tag{21}$$

$$\sigma_s(\mathscr{A}_{\alpha 0}) = 0 \iff p \text{ divides } \binom{n-1}{s}. \tag{22}$$

The examples we are looking for are based on the fact that the right-hand sides of (21) and (22) are not equivalent; this is the case when $p = 5$, $n = 7$ and $s = 4$; so the converse to Theorem 4.3(d) is not true in general.

Note that, if n is invertible in $\mathbf{F}$ we have

$$\sigma_s(\mathscr{A}_{\alpha\beta}) = 0 \iff \binom{n}{s}[n\alpha - s(\alpha - \beta)] = 0. \tag{23}$$

So, if $\mathbf{F}$ has zero characteristic, $\sigma_s(\mathscr{A}_{\alpha\beta}) = 0$ is equivalent to $d(\mathscr{A}_{\alpha\beta}, s) = 1$. Therefore, the counterexamples we have to the converse of Theorem 4.3(d) all live in nonzero characteristic. So we ask:

Problem 2. For fields of characteristic zero, if the sth coefficient of the characteristic polynomial of $\mathscr{A}$ is zero, may we always find a matrix $A \in \mathscr{A}$ with all $s \times s$ principal minors zero?

Our examples also show that we may have strict inequality in (12). Consider again the case when $\beta = 0$ and n is not a multiple of p. Then $\mathscr{A}_1$, the nonsingular part of $\mathscr{A}_{\alpha 0}$, is the Singleton class of the scalar matrix αI_{n-1}, and so we have

$d(\mathscr{A}_1, s) = 0$ for $s \leqslant n - 1$. However, according to (21), we have $d(\mathscr{A}_{\alpha 0}, s) = 1$ for all s congruent with n modulo p.

Again, all counterexamples we found live in fields of nonzero characteristic. We then ask:

Problem 3. Is it true $d(\mathscr{A}, s) = d(\mathscr{A}_1, s)$, for fields of zero characteristic?

An interesting fact occurs with our examples $\mathscr{A}_{\alpha\beta}$. When n is invertible in $\mathbf{F}$, the matrix $G_0 := \alpha I - ((\alpha - \beta)/n) \cdot \Omega_n$ satisfies

$$d(G_0, s) = d(\mathscr{A}_{\alpha\beta}, s) \quad \text{for all } s \geqslant 1.$$

This suggests:

Problem 4. Which similarity classes satisfy the property: there exists $A \in \mathscr{A}$ such that $d(A, s) = d(\mathscr{A}, s)$ for all s?

Nilpotent and nonderogatory matrices also satisfy this property. However, if n is multiple of p, $\alpha = \beta$ and $p > 2$, no matrix $A \in \mathscr{A}_{\alpha\alpha}$ satisfies $d(A, s) = d(\mathscr{A}_{\alpha\alpha}, s)$ for all s. We still have no clue for a general answer.

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