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The inertia of Hermitian block matrices with zero main diagonal

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Abstract

Let n_1, n_2, n_3 be nonnegative integers. We consider partitioned Hermitian matrices of the form

$$H = \begin{bmatrix} 0 & X_{12} & X_{13} \\ X_{12}^* & 0 & X_{23} \\ X_{13}^* & X_{23}^* & 0 \end{bmatrix},$$

where each X_{ij} is $n_i \times n_j$ and we characterize the set of the inertias

$$\left\{\mathrm{In}(H) \mid r_{ij} \leqslant \mathrm{rank}\, X_{ij} \leqslant R_{ij} \text{ for } 1 \leqslant i < j \leqslant 3\right\}$$

in terms of r_{ij}, R_{ij} and the block orders. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

Define the inertia of an $n \times n$ Hermitian matrix H as the triple $\mathrm{In}(H) = (\pi, \nu, \delta)$, where π is the number of positive eigenvalues of H, ν is the number of negative

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PII: S 0 0 2 4 - 3 7 9 5 (0 0) 0 0 0 8 9 - 6

eigenvalues and $\delta = n - \pi - \nu$, counting multiplicities. We will simply write $(\pi, \nu, *)$ for the inertia of H, when n is understood.

We denote by I_r the identity matrix of order r and by I the same matrix when we do not need to specify the order.

The characterization of the inertia of Hermitian matrices with prescribed 2×2 and 3×3 block decomposition has been quite thoroughly investigated. Cain and Marques de Sá [2] considered Hermitian matrices H of the form

$$H = \begin{bmatrix} H_1 & X_{12} & X_{13} \\ X_{12}^* & H_2 & 0 \\ X_{13}^* & 0 & 0 \end{bmatrix}$$

and characterized the set of the inertias

$$\left\{\mathrm{In}(H) \mid \mathrm{In}(H_j) = (\pi_j, \nu_j, *) \text{ and } r_{1,j+1} \leqslant \operatorname{rank} X_{1,j+1} \leqslant R_{1,j+1} \text{ for } j = 1, 2\right\}$$

in terms of $r_{1,2}$, $r_{1,3}$, $R_{1,2}$, $R_{1,3}$, the inertias of the nonzero diagonals blocks and the orders of the blocks. Their result can be stated as follows.

Theorem 1.1. *Let us assume that $\pi_1, \nu_1, \pi_2, \nu_2, n_1, n_2, n_3$ are nonnegative and*

$$\pi_i + \nu_i \leqslant n_i, \quad i = 1, 2,$$

$$0 \leqslant r_{1j} \leqslant R_{1j} \leqslant \min\left\{n_1, n_j\right\}, \quad j = 2, 3.$$

Then the following conditions are equivalent:

(i) *For $i = 1, 2$, and $j = 2, 3$, there exist $n_i \times n_i$ Hermitian matrices H_i and $n_1 \times n_j$ matrices X_{1j} such that*, $\mathrm{In}(H_i) = (\pi_i, \nu_i, *)$, $r_{1j} \leqslant \operatorname{rank} X_{1j} \leqslant R_{1j}$ *and*

$$H = \begin{bmatrix} H_1 & X_{12} & X_{13} \\ X_{12}^* & H_2 & 0 \\ X_{13}^* & 0 & 0 \end{bmatrix}$$

has inertia $(\pi, \nu, *)$.

(ii) *Let $k \in \{1, 2\}$. Let W_{kk} be any fixed $n_k \times n_k$ Hermitian matrix with inertia* $(\pi_k, \nu_k, *)$. (i) *holds with $H_k = W_{kk}$.*

(iii) *Let $k \in \{2, 3\}$. Let W_{1k} be any fixed $n_1 \times n_k$ matrix with* $r_{1k} \leqslant \operatorname{rank} W_{1k} \leqslant R_{1k}$. (i) *holds with $X_{1k} = W_{1k}$.*

(iv) *For $k = 1, 2$ let W_{kk} be any fixed $n_k \times n_k$ Hermitian matrix with inertia* $(\pi_k, \nu_k, *)$. (i) *holds with $H_1 = W_{11}$ and $H_2 = W_{22}$.*

(v) *Let W_{22} be any fixed $n_2 \times n_2$ Hermitian matrix with inertia* $(\pi_2, \nu_2, *)$, *and let W_{13} be any fixed $n_1 \times n_3$ matrix with* $r_{13} \leqslant \operatorname{rank} W_{13} \leqslant R_{13}$. (i) *holds with $H_2 = W_{22}$ and $X_{13} = W_{13}$.*

(vi) *The following inequalities hold*:

$$\pi \geqslant \max\{\pi_1, \pi_2 + r_{13}, \pi_1 + \pi_2 - R_{12}, r_{12} - \nu_1, r_{12} - \nu_2\},$$

$$\nu \geqslant \max\{\nu_1, \nu_2 + r_{13}, \nu_1 + \nu_2 - R_{12}, r_{12} - \pi_1, r_{12} - \pi_2\},$$
$$\pi \leqslant \min\{n_1 + \pi_2, \pi_1 + n_2 + R_{13}, \pi_1 + \pi_2 + R_{12} + R_{13}\},$$
$$\nu \leqslant \min\{n_1 + \nu_2, \nu_1 + n_2 + R_{13}, \nu_1 + \nu_2 + R_{12} + R_{13}\},$$
$$\pi - \nu \leqslant \min\{\pi_1 + \pi_2, \pi_1 + \pi_2 + R_{12} - \nu_2\},$$
$$\nu - \pi \leqslant \min\{\nu_1 + \nu_2, \nu_1 + \nu_2 + R_{12} - \nu_2\},$$
$$\pi + \nu \geqslant \pi_1 + \pi_2 + \nu_1 + \nu_2 - R_{12},$$
$$\pi + \nu \leqslant \min\left\{\begin{matrix} n_1 + n_2 + R_{13}, n_1 + \pi_2 + \nu_2 + R_{12} + R_{13}, \\ \pi_1 + \nu_1 + n_2 + R_{12} + 2R_{13} \end{matrix}\right\}.$$

This paper gives a similar characterization when H is decomposed into 3×3 blocks whose diagonal blocks are zeros, namely:

Theorem 1.2. *For $i = 1, 2, 3$, let n_i be a nonnegative integer, and for $i < j \leqslant 3$, let*

$$0 \leqslant r_{ij} \leqslant R_{ij} \leqslant \min\{n_i, n_j\}.$$

Then the following conditions are equivalent:

(i) *For $i = 1, 2$ and for $i < j \leqslant 3$, there exist $n_i \times n_j$ matrices X_{ij} such that $r_{ij} \leqslant \operatorname{rank} X_{ij} \leqslant R_{ij}$ and*

$$H = \begin{bmatrix} 0 & X_{12} & X_{13} \\ X_{12}^* & 0 & X_{23} \\ X_{13}^* & X_{23}^* & 0 \end{bmatrix}$$

has inertia $(\pi, \nu, *)$.

(ii) *Let $(k, t) \in \{(1, 2), (1, 3), (2, 3)\}$ and let W_{kt} be any fixed $n_k \times n_t$ matrix with $r_{kt} \leqslant \operatorname{rank} W_{kt} \leqslant R_{kt}$.* (i) *holds with $X_{kt} = W_{kt}$.*

(iii) *The following inequalities hold*:

$$\max\{r_{12}, r_{13}, r_{23}\} \leqslant \pi, \nu \leqslant \min\{R_{12} + n_3, R_{13} + n_2, R_{23} + n_1\},$$

$$\pi - \nu, \nu - \pi \leqslant \min\{R_{12}, R_{13}, R_{23}\},$$

$$2\pi - \nu, 2\nu - \pi \leqslant R_{12} + R_{13} + R_{23},$$

$$\pi + \nu \leqslant n_1 + n_2 + n_3.$$

2. Preliminary results

Lemma 2.1. *Let n and p denote nonnegative integers such that $p \leqslant n$. There exists an $n \times n$ matrix P of rank p such that $M = P + P^*$ has inertia $(\pi, \nu, *)$ if and only if*

$$0 \leqslant \pi \leqslant p, \quad 0 \leqslant \nu \leqslant p, \quad \pi + \nu \leqslant n. \tag{2.1}$$

Proof. Let us prove the necessity of (2.1). Let N and Q be, respectively, the leading $\pi \times \pi$ principal submatrices of U^*MU and U^*PU, where U is a unitary matrix such that U^*MU is diagonal and N is positive definite. Then if $\pi > p$, there will exist a $x \neq 0$ such that $Qx = 0$; therefore

$$x^*Nx = x^*\left(Q + Q^*\right)x = 0,$$

which is a contradiction. Thus $\pi \leqslant p$. In a similar way one proves $\nu \leqslant p$.

Next we show the sufficiency of (2.1). If $\pi + \nu \leqslant p$ then we can make

$$P = \left[\begin{array}{cc|cc} I_\pi & 0 & 0 & 0 \\ 0 & -I_\nu & 0 & 0 \\ \hline 0 & 0 & \mathrm{i}I_{p-\pi-\nu} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

If $\pi + \nu \geqslant p$, we may assume, without loss of generality, that $\pi \geqslant \nu$, and set

$$P = \left[\begin{array}{cc|c|cc} 0 & 0 & \multirow{2}{*}{I_ν} & 0 & 0 \\ 0 & \mathrm{i}I_{p-\pi} & & 0 & 0 \\ \cline{1-3} \multicolumn{2}{c|}{0_\nu} & \mathrm{i}I_\nu & & \\ \hline \multicolumn{2}{c}{0} & 0 & I_{\pi-\nu} & 0 \\ \multicolumn{2}{c}{0} & 0 & 0 & 0 \end{array}\right]$$

since the inertia of

$$\begin{bmatrix} 0 & I_\nu \\ I_\nu & 0 \end{bmatrix}$$

is $(\nu, \nu, 0)$. $\square$

If we consider A and B, $n \times m$ and $m \times n$ matrices, respectively, with ranks a and b, then the rank of the product AB may be r if and only if

$$\max\{0, a + b - m\} \leqslant r \leqslant \min\{a, b\}.$$

Lemma 2.2. *If a Hermitian block matrix contains the* 2×2 *block principal submatrix*

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

then it is congruent to a matrix with this submatrix undisturbed and these I's being the only nonzero blocks in the rows and columns containing them.

Since the matrix

$$H = \begin{bmatrix} 0 & X & I_m \\ X^* & 0_n & Z \\ I_m & Z^* & 0 \end{bmatrix} \tag{2.2}$$

is congruent to

$$\begin{bmatrix} 0 & 0 & I_m \\ 0 & H' & 0 \\ I_m & 0 & 0 \end{bmatrix}$$

by the Lemma 2.2, where $H' = -ZX - (ZX)^*$, it follows from (2.1) and the remark following Lemma 2.1 that:

Corollary 2.3. *Let n, m, x and z denote nonnegative integers such that* $x, z \leqslant \min\{m, n\}$. *There exist an* $m \times n$ *matrix X and an* $n \times m$ *matrix Z such that* rank $X = x$, rank $Z = z$ *and H in* (2.2) *has inertia* $(\pi, \nu, *)$ *if and only if*

$$m \leqslant \pi, \nu \leqslant \min\{x + m, z + m\}$$

and

$$\pi + \nu \leqslant n + 2m.$$

3. Proof

We are ready to prove Theorem 1.2. The equivalence between (i) and (ii) is straightforward (see e.g. the beginning of the proof of the Theorem 2.1 in [2]).

Let us denote $X_{12} = X$, $X_{13} = Y$, $X_{23} = Z$ and fix their ranks: rank $X = x$, rank $Y = y$, rank $Z = z$. We will prove that (ii) is equivalent to (iii). Without loss of generality we may choose

$$Y = \begin{bmatrix} 0 & I_y \\ 0 & 0 \end{bmatrix}.$$

Then

$$H = \begin{bmatrix} 0 & 0 & X_1 & 0 & I_y \\ 0 & 0 & X_2 & 0 & 0 \\ X_1^* & X_2^* & 0 & Z_1 & Z_2 \\ 0 & 0 & Z_1^* & 0 & 0 \\ I_y & 0 & Z_2^* & 0 & 0 \end{bmatrix},$$

where

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix}.$$

Suppose that the rank of Z_1 is a. Then there exist nonsingular matrices U and V such that

$$UZ_1V = \begin{bmatrix} 0 & I_a \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & V^* & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} H \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & U^* & 0 & 0 \\ 0 & 0 & 0 & V & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

$$= \left[\begin{array}{cc|c|cc} 0 & 0 & X_1U^* & 0 & I_y \\ 0 & 0 & X_2U^* & 0 & 0 \\ \hline (X_1U^*)^* & (X_2U^*)^* & 0 & UZ_1V & UZ_2 \\ \hline 0 & 0 & (UZ_1V)^* & 0 & 0 \\ I_y & 0 & (UZ_2)^* & 0 & 0 \end{array}\right].$$

This is congruent to

$$\left[\begin{array}{cc|cc|ccc} 0 & 0 & 0 & \bar{X}_1 & 0 & 0 & I_y \\ 0 & 0 & 0 & \bar{X}_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_a & 0 \\ \bar{X}_1^* & \bar{X}_2^* & 0 & 0 & 0 & 0 & \bar{Z}_2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 & 0 & 0 & 0 \\ I_y & 0 & 0 & \bar{Z}_2^* & 0 & 0 & 0 \end{array}\right].$$

Suppose now that the rank of the block $\bar{X}_2$ is b. Then

$$\left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & \hat{X} & 0 & 0 & I_y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_b & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_a & 0 \\ 0 & 0 & I_b & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{X}^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{Z} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_a & 0 & 0 & 0 & 0 & 0 \\ I_y & 0 & 0 & 0 & 0 & \hat{Z}^* & 0 & 0 & 0 \end{array}\right].$$

is congruent to H by the Lemma 2.2, and is permutation similar to

$$\widehat{H} = \begin{bmatrix} 0 & \widehat{X} & I_y \\ \widehat{X}^* & 0 & \widehat{Z} \\ I_y & \widehat{Z}^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & I_a \\ I_a & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & I_b \\ I_b & 0 \end{bmatrix} \oplus 0, \tag{3.1}$$

where

$$\widehat{X} \text{ is } y \times (n_2 - a - b), \quad \operatorname{rank} \widehat{X} \stackrel{\text{def}}{=} \beta - b$$

$$\widehat{Z} \text{ is } (n_2 - a - b) \times y, \quad \operatorname{rank} \widehat{Z} \stackrel{\text{def}}{=} \alpha$$

with

$$\max\{0, z - y\} \leqslant a \leqslant \min\{z, n_3 - y\}, \tag{3.2}$$

$$\max\{0, x - a\} \leqslant \beta \leqslant \min\{x, n_2 - a\}, \tag{3.3}$$

$$\max\{0, \beta - y\} \leqslant b \leqslant \min\{\beta, n_1 - y\} \tag{3.4}$$

and

$$\max\{0, z - a - b\} \leqslant \alpha \leqslant \min\{z - a, n_2 - a - b\}. \tag{3.5}$$

Denote the first matrix of the decomposition (3.1) by H'. By Corollary 2.3, the set of inertias $(\pi', \nu', *)$ of H' is described by

$$y \leqslant \pi', \nu' \leqslant \min\{\beta - b + y, \alpha + y\}, \quad \pi' + \nu' \leqslant n_2 - a - b + 2y. \tag{3.6}$$

From (3.1), we have

$$\pi = \pi' + a + b, \quad \nu = \nu' + a + b. \tag{3.7}$$

Combining (3.7) and (3.6), we get

$$y + a + b \leqslant \pi, \nu \leqslant \min\{\beta + y + a, \alpha + y + a + b\} \tag{3.8}$$

and

$$\pi + \nu \leqslant n_2 + 2y + a + b. \tag{3.9}$$

Eliminating α, from (3.5) and (3.8), and β, from (3.3), (3.4), and (3.8), (see [1,2]), and after eliminating redundancies, we get

$$\pi, \nu \leqslant \min\{y + n_2, x + y + a, 2y + a + b, y + z + b\}, \tag{3.10}$$

$$\pi, \nu \geqslant y + a + b, \tag{3.11}$$

$$\pi + \nu \leqslant n_2 + 2y + a + b \tag{3.12}$$

and

$$a + b \geqslant x - y. \tag{3.13}$$

Finally, we combine (3.2), the inequality $0 \leqslant b \leqslant n_1 - y$ (from (3.4)), and (3.10)–(3.13) in order to eliminate a and b. For example, the inequality $\pi, \nu \leqslant x + y + z$ is redundant since $2\pi \leqslant \nu + x + y + z$ and $2\nu \leqslant \pi + x + y + z$ is equivalent to $\pi, \nu \leqslant x + y + z - |\pi - \nu|$.

Then, we get

$$\max\{x, y, z\} \leqslant \pi, \nu \leqslant \min\{x + n_3, y + n_2, z + n_1\},$$
$$\pi - \nu, \nu - \pi \leqslant \min\{x, y, z\},$$
$$2\pi - \nu, 2\nu - \pi \leqslant x + y + z,$$
$$\pi + \nu \leqslant n_1 + n_2 + n_3.$$

Combining these inequalities with $r_{12} \leqslant x \leqslant R_{12}, r_{13} \leqslant y \leqslant R_{13}$ and $r_{23} \leqslant z \leqslant R_{23}$ we obtain inequalities of (iii).

To see that we have actually proven the equivalence note that H and $\widehat{H}$ are congruent, that the Corollary 2.3 gives us a necessary and sufficient condition and that the elimination of the several constants does not change the equivalence. Details are left to the reader.

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References

[1] B.E. Cain, E. Marques de Sá, The inertia of Hermitian matrices with a prescribed 2 × 2 block decomposition, Linear and Multilinear Algebra 31 (1992) 119–130.
[2] B.E. Cain, E. Marques de Sá, The inertia of certain skew-triangular block matrices, Linear Algebra Appl. 160 (1992) 75–85.
[3] E.V. Haynesworth, A.M. Ostrowski, On the inertia of some classes of partitioned matrices, Linear Algebra Appl. 1 (1968) 299–316.
[4] S. Perlis, Theory of Matrices, Addison-Wesley, Reading, MA, 1952.