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Explicit inverses of some tridiagonal matrices

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Abstract

We give explicit inverses of tridiagonal 2-Toeplitz and 3-Toeplitz matrices which generalize some well-known results concerning the inverse of a tridiagonal Toeplitz matrix. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

In recent years the invertibility of nonsingular tridiagonal or block tridiagonal matrices has been quite investigated in different fields of applied linear algebra (for historical notes see [8]). Several numerical methods, more or less efficient, have risen in order to give expressions of the entries of the inverse of this kind of matrices. Though, explicit inverses are known only in a few cases, in particular when the tridiagonal matrix is symmetric with constant diagonals and subject to some restrictions (cf. [3,8,10]). Furthermore, Lewis [5] gave a different way to compute other explicit inverses of nonsymmetric tridiagonals matrices.

In [1], Gover defines a tridiagonal 2-Toeplitz matrix of order n, as a matrix $A = (a_{ij})$, where $a_{ij} = 0$ if $|i - j| > 1$ and $a_{ij} = a_{kl}$ if $(i, j) \equiv (k, l) \bmod 2$, i.e.,

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PII: S 0 0 2 4 - 3 7 9 5 (0 0) 0 0 2 8 9 - 5

$$A = \begin{pmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_1 & b_1 & & \\ & & c_1 & a_2 & b_2 & \\ & & & c_2 & a_1 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}_{n\times n}. \tag{1}$$

Similarly, a tridiagonal 3-Toeplitz matrix ($n \times n$) is of the form

$$B = \begin{pmatrix} a_1 & b_1 & & & & & & \\ c_1 & a_2 & b_2 & & & & & \\ & c_2 & a_3 & b_3 & & & & \\ & & c_3 & a_1 & b_1 & & & \\ & & & c_1 & a_2 & b_2 & & \\ & & & & c_2 & a_3 & b_3 & \\ & & & & & c_3 & a_1 & \ddots \\ & & & & & & \ddots & \ddots \end{pmatrix}_{n\times n}. \tag{2}$$

Making use of the theory of orthogonal polynomials, we will give the explicit inverse of tridiagonal 2-Toeplitz and 3-Toeplitz matrices, based on recent results from [1,6,7]. As a corollary, we will obtain the explicit inverse of a tridiagonal Toeplitz matrix, i.e., a tridiagonal matrix with constant diagonals (not necessarily symmetric). Different proofs of the results by Kamps [3,4] involving the sum of all the entries of the inverse can be simplified.

Throughout this paper, it is assumed that all the matrices are invertible.

2. Inverse of a tridiagonal matrix

Let T be an $n \times n$ real nonsingular tridiagonal matrix

$$T = \begin{pmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{pmatrix}. \tag{3}$$

Denote $T^{-1} = (\alpha_{ij})$. In [5] Lewis proved the following result.

Lemma 2.1. *Let T be the matrix* (3) *and assume that $b_\ell \neq 0$ for $\ell = 1, \ldots, n-1$. Then*

$$\alpha_{ji} = \gamma_{ij}\alpha_{ij} \quad \textit{when } i < j,$$

where

$$\gamma_{ij} = \prod_{\ell=i}^{j-1} \frac{c_\ell}{b_\ell}.$$

Of course this reduces to $\alpha_{ij} = \alpha_{ji}$, when T is symmetric.

Using this lemma, one can prove that there exist two finite sequences $\{u_i\}$ and $\{v_i\}$ $(i = 1, \ldots, n-1)$ such that

$$T^{-1} = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \cdots & u_1v_n \\ \gamma_{12}u_1v_2 & u_2v_2 & u_2v_3 & \cdots & u_2v_n \\ \gamma_{13}u_1v_3 & \gamma_{23}u_2v_3 & u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{1,n}u_1v_n & \gamma_{2,n}u_2v_n & \gamma_{3,n}u_3v_n & \cdots & u_nv_n \end{pmatrix}. \tag{4}$$

Since $\{u_i\}$ and $\{v_i\}$ are only defined up to a multiplicative constant, we make $u_1 = 1$. The following result shows how to compute $\{u_i\}$ and $\{v_i\}$. The determination of these sequences is very similar to the particular case studied in [8], where the reader may check for details.

Lemma 2.2. (i) *The* $\{v_i\}$ $(i = 1, \ldots, n-1)$ *can be computed from*

$$v_1 = \frac{1}{d_1}, \qquad v_k = -\frac{b_{k-1}}{d_k} v_{k-1}, \quad k = 2, \ldots, n,$$

where

$$d_n = a_n, \qquad d_i = a_i - \frac{b_i c_i}{d_{i+1}}, \quad i = n-1, \ldots, 1.$$

(ii) *The* $\{u_i\}$ $(i = 1, \ldots, n-1)$ *can be computed from*

$$u_n = \frac{1}{\delta_n v_n}, \qquad u_k = -\frac{b_k}{\delta_k} u_{k+1}, \quad k = n-1, \ldots, 1,$$

where

$$\delta_1 = a_1, \qquad \delta_i = a_i - \frac{b_{i-1}c_{i-1}}{\delta_{i-1}}, \quad i = 2, \ldots, n-1.$$

Notice that for *centro-symmetric* matrices, i.e., $a_{ij} = a_{n+1-i,n+1-j}$, we have

$$v_i = u_{n+1-i}.$$

The following proposition is known in the literature. We give a proof based on the lecture of [8].

Theorem 2.1. *Let T be the* $n \times n$ *tridiagonal matrix defined in* (3) *and assume that* $b_\ell \neq 0$ *for* $\ell = 1, \ldots, n-1$. *Then*

$$(T^{-1})_{ij} = \begin{cases} (-1)^{i+j} b_i \cdots b_{j-1} \dfrac{d_{j+1} \cdots d_n}{\delta_i \cdots \delta_n} & \text{if } i \leqslant j, \\ (-1)^{i+j} c_j \cdots c_{i-1} \dfrac{d_{i+1} \cdots d_n}{\delta_j \cdots \delta_n} & \text{if } i > j \end{cases}$$

(*with the convention that empty product equals* 1).

Proof. Let us assume that $i \leqslant j$. Then

$$(T^{-1})_{ij} = u_i v_j = (-1)^{n-i} \frac{b_i \cdots b_{n-1}}{\delta_i \cdots \delta_n v_n} \times (-1)^{j-1} \frac{b_1 \cdots b_{j-1}}{d_1 \cdots d_j}.$$

But

$$v_n = (-1)^{n-1} \frac{b_1 \cdots b_{n-1}}{d_1 \cdots d_n}.$$

Therefore

$$\begin{aligned}(T^{-1})_{ij} &= (-1)^{n-i} \frac{b_i \cdots b_{n-1}}{\delta_i \cdots \delta_n} \times (-1)^{n-1} \frac{d_1 \cdots d_n}{b_1 \cdots b_{n-1}} \times (-1)^{j-1} \frac{b_1 \cdots b_{j-1}}{d_1 \cdots d_j} \\ &= (-1)^{i+j} b_i \cdots b_{j-1} \frac{d_{j+1} \cdots d_n}{\delta_i \cdots \delta_n}. \qquad \square\end{aligned}$$

If we set

$$\delta_i = \frac{\theta_i}{\theta_{i-1}}, \quad \theta_0 = 1, \quad \theta_1 = a_1, \tag{5}$$

we get the recurrence relation

$$\theta_i = a_i \theta_{i-1} - b_{i-1} c_{i-1} \theta_{i-2};$$

and if we now put

$$d_i = \frac{\phi_i}{\phi_{i+1}}, \quad \phi_{n+1} = 1, \quad \phi_n = a_n, \tag{6}$$

we get the recurrence relation

$$\phi_i = a_i \phi_{i+1} - b_i c_i \phi_{i+2}.$$

As a consequence, we will achieve

$$(T^{-1})_{ij} = \begin{cases} (-1)^{i+j} b_i \cdots b_{j-1} \theta_{i-1} \phi_{j+1} / \theta_n & \text{if } i \leqslant j, \\ (-1)^{i+j} c_j \cdots c_{i-1} \theta_{j-1} \phi_{i+1} / \theta_n & \text{if } i > j. \end{cases} \tag{7}$$

These are the general relations given by Usmani [9], which we will use throghout this paper. Notice that

$$\theta_n = \delta_1 \cdots \delta_n = \det T.$$

3. Chebyshev polynomials of the second kind

In the next it is useful to consider the set of polynomials $\{U_n\}_{n\geqslant 0}$, such that each U_n is of degree exactly n, satisfying the recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n \geqslant 1$$

with initial conditions

$$U_0 = 1 \quad \text{and} \quad U_1 = 2x.$$

These polynomials are called *Chebyshev polynomials of the second kind* and they have the form

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{where } \cos\theta = x,$$

when $|x| < 1$. When $|x| > 1$, they have the form

$$U_n(x) = \frac{\sinh(n+1)\theta}{\sinh\theta}, \quad \text{where } \cosh\theta = x.$$

In particular

$$U_n(\pm 1) = (\pm 1)^n(n+1).$$

In any case, if $|x| \neq 1$, then

$$U_n(x) = \frac{r_+^{n+1} - r_-^{n+1}}{r_+ - r_-},$$

where

$$r_\pm = x \pm \sqrt{x^2 - 1}$$

are the two solutions of the quadratic equation $r^2 - 2xr + 1 = 0$.

The following proposition is a slight extension of Lemma 2.5 of Meurant [8].

Lemma 3.1. *Fix a positive integer number n and let a and b be nonzero real numbers such that $U_i(a/2b) \neq 0$ for all $i = 1, 2, \ldots, n$. Consider the recurrence relation*

$$\alpha_1 = a, \quad \alpha_i = a - \frac{b^2}{\alpha_{i-1}}, \quad i = 2, \ldots, n.$$

Under these conditions, if $a = \pm 2b$, then

$$\alpha_i = \frac{\pm b(i+1)}{i}.$$

Otherwise

$$\alpha_i = b\frac{r_+^{i+1} - r_-^{i+1}}{r_+^i - r_-^i},$$

where

$$r_{\pm} = \frac{a \pm \sqrt{a^2 - 4b^2}}{2b}$$

are the two solutions of the quadratic equation $r^2 - (a/b)r + 1 = 0$.

Proof. Let us assume that $a \neq \pm 2b$ (otherwise it is trivial) and set $\alpha_i = \beta_i/\beta_{i-1}$. Then

$$\beta_i - a\beta_{i-1} + b^2\beta_{i-2} = 0, \quad \beta_0 = 1, \ \ \beta_1 = a,$$

hence $\beta_i = b^i U_i(a/2b)$ and the conclusion follows. $\square$

4. Inverse of a tridiagonal 2-Toeplitz matrix

In this section we will determine the explicit inverse of a matrix of type (1). By Theorem 2.1, this is equivalent to determine δ_i and d_i, i.e., by (7) and transformations (5) and (6), θ_i and ϕ_i. So, to evaluate θ_i, we have

$$\begin{aligned} &\theta_0 = 1, \quad \theta_1 = a_1, \\ &\theta_{2i} = a_2\theta_{2i-1} - b_1c_1\theta_{2i-2}, \\ &\theta_{2i+1} = a_1\theta_{2i} - b_2c_2\theta_{2i-1}. \end{aligned}$$

Making the change

$$z_i = (-1)^i\theta_i,$$

we get the relations

$$\begin{aligned} &z_0 = 1, \quad z_1 = -a_1, \\ &z_{2i} = -a_2z_{2i-1} - b_1c_1z_{2i-2}, \\ &z_{2i+1} = -a_1z_{2i} - b_2c_2z_{2i-1}. \end{aligned}$$

According to the results in [6],

$$z_i = Q_i(0),$$

where Q_i is a polynomial of degree exactly i, defined by the recurrence relation

$$Q_{i+1}(x) = (x - \tilde{\beta}_i)Q_i(x) - \tilde{\gamma}_i Q_{i-1}(x), \tag{8}$$

where

$$\begin{cases} \tilde{\beta}_{2j} = a_1, \\ \tilde{\beta}_{2j+1} = a_2, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\gamma}_{2j} = b_2c_2, \\ \tilde{\gamma}_{2j+1} = b_1c_1, \end{cases} \tag{9}$$

and initial conditions $Q_0(x) = 1$ and $Q_1(x) = x - \tilde{\beta}_0$.

The solution of the recurrence relation (8) with coefficients (9) is

$$Q_{2i}(x) = P_i\,[\pi_2(x)]\,, \quad Q_{2i+1}(x) = (x-a_1)\,P_i^*\,[\pi_2(x)]\,, \tag{10}$$

where

$$\pi_2(x) = (x-a_1)(x-a_2), \tag{11}$$

$$P_i^*(x) = \left(\sqrt{b_1b_2c_1c_2}\right)^i U_i\left(\frac{x-b_1c_1-b_2c_2}{2\sqrt{b_1b_2c_1c_2}}\right), \tag{12}$$

and

$$\begin{aligned} P_i(x) = \left(\sqrt{b_1b_2c_1c_2}\right)^i \Bigg[& U_i\left(\frac{x-b_1c_1-b_2c_2}{2\sqrt{b_1b_2c_1c_2}}\right) \\ & + \beta U_{i-1}\left(\frac{x-b_1c_1-b_2c_2}{2\sqrt{b_1b_2c_1c_2}}\right)\Bigg], \end{aligned} \tag{13}$$

with $\beta^2 = b_2c_2/b_1c_1$. We may conclude that

$$\theta_i = (-1)^i Q_i(0).$$

Now, to evaluate ϕ_i we put

$$\psi_i = \phi_{n+1-i},$$

for $i = 1, \dots, n$, and we have to distinguish the cases when the order n of the matrix (1) is odd and when it is even.

First let us suppose that n is odd. Then we have

$$\begin{aligned} &\psi_0 = 1, \quad \psi_1 = a_1, \\ &\psi_{2i} = a_2\psi_{2i-1} - b_2c_2\psi_{2i-2}, \\ &\psi_{2i+1} = a_1\psi_{2i} - b_1c_1\psi_{2i-1}. \end{aligned}$$

Following the same steps as in the previous case, we obtain

$$\psi_i = (-1)^i Q_i(0),$$

i.e.,

$$\phi_i = (-1)^i Q_{n+1-i}(0),$$

where $Q_i(x)$ is the same as (10) but with $\beta^2 = b_1c_1/b_2c_2$ in (13). Notice that if $b_1c_1 = b_2c_2$, then $\theta_i = \phi_{n+1-i}$.

If n is even, we get

$$\begin{aligned} &\psi_0 = 1, \quad \psi_1 = a_2, \\ &\psi_{2i} = a_1\psi_{2i-1} - b_1c_1\psi_{2i-2}, \\ &\psi_{2i+1} = a_2\psi_{2i} - b_2c_2\psi_{2i-1}. \end{aligned}$$

Therefore

$$\psi_i = (-1)^i Q_i(0),$$

i.e.,

$$\phi_i = (-1)^{i+1} Q_{n+1-i}(0),$$

where

$$Q_{2i}(x) = P_i\left[\pi_2(x)\right], \quad Q_{2i+1}(x) = (x-a_2)P_i^*\left[\pi_2(x)\right],$$

with $\pi_2(x)$, $P_i(x)$ and $P_i^*(x)$ the same as in (11), (13) and (12), respectively. Observe that if $a_1 = a_2$, then $\theta_i = \phi_{n+1-i}$.

We have determined completely the θ_i's and ϕ_i's of (7) – thus the inverse of the matrix – in the case of a tridiagonal 2-Toeplitz matrix (1).

Theorem 4.1. *Let A be the tridiagonal matrix* (1), *with* $a_1a_2 \neq 0$ *and* $b_1b_2c_1c_2 > 0$. *Put*

$$\pi_2(x) := (x-a_1)(x-a_2), \quad \beta := \sqrt{b_2c_2/b_1c_1}$$

and let $\{Q_i(\cdot;\alpha,\gamma)\}$ *be the sequence of polynomials defined by*

$$Q_{2i}(x;\alpha,\gamma) = \left(\sqrt{b_1b_2c_1c_2}\right)^i \left[U_i\left(\frac{\pi_2(x)-b_1c_1-b_2c_2}{2\sqrt{b_1b_2c_1c_2}}\right) + \gamma U_{i-1}\left(\frac{\pi_2(x)-b_1c_1-b_2c_2}{2\sqrt{b_1b_2c_1c_2}}\right)\right]$$

$$Q_{2i+1}(x;\alpha,\gamma) = (x-\alpha)\left(\sqrt{b_1b_2c_1c_2}\right)^i U_i\left(\frac{\pi_2(x)-b_1c_1-b_2c_2}{2\sqrt{b_1b_2c_1c_2}}\right),$$

where α *and* γ *are some parameters. Under these conditions*,

$$(A^{-1})_{ij} = \begin{cases} (-1)^{i+j}\, b_{p_i}^{\lfloor (j-i)/2\rfloor}\, b_{q_i}^{\lfloor (j-i+1)/2\rfloor}\, \theta_{i-1}\,\phi_{j+1}/\theta_n & \text{if } i \leqslant j, \\ (-1)^{i+j}\, c_{p_j}^{\lfloor (j-i)/2\rfloor}\, c_{q_j}^{\lfloor (j-i+1)/2\rfloor}\, \theta_{j-1}\,\phi_{i+1}/\theta_n & \text{if } i > j, \end{cases} \tag{14}$$

where $p_\ell = (3-(-1)^\ell)/2$, $q_\ell = (3+(-1)^\ell)/2$, $\lfloor z\rfloor$ *denotes the greater integer less or equal to the real number* z,

$$\theta_i = (-1)^i Q_i(0; a_1, \beta), \tag{15}$$

and

$$\phi_i = \begin{cases} (-1)^i Q_{n+1-i}(0; a_1, 1/\beta) & \text{if } n \text{ is odd}, \\ (-1)^{i+1} Q_{n+1-i}(0; a_2, \beta) & \text{if } n \text{ is even}. \end{cases}$$

Remark. As we already noticed, under the conditions of Theorem 4.1, if n is odd and $b_1c_1 = b_2c_2$, then $\theta_i = \phi_{n+1-i}$; and if n is even and $a_1 = a_2$, then also $\theta_i = \phi_{n+1-i}$.

As a first application, let us consider a classical problem on the inverse of a tridiagonal Toeplitz matrix. Suppose that $a_1 = a_2 = a$, $b_1 = b_2 = b$ and $c_1 = c_2 = c$ in (1), with $a \neq 0$ and $bc > 0$. Therefore, we have the $n \times n$ tridiagonal Toeplitz matrix

$$T = \begin{pmatrix} a & b & & & 0 \\ c & a & b & & \\ & c & a & \ddots & \\ & & \ddots & \ddots & b \\ 0 & & & c & a \end{pmatrix}. \tag{16}$$

According to (14),

$$(T^{-1})_{ij} = \begin{cases} (-1)^{i+j}\, b^{j-i}\, \theta_{i-1}\, \phi_{j+1}/\theta_n & \text{if } i \leqslant j, \\ (-1)^{i+j}\, c^{i-j}\, \theta_{j-1}\, \phi_{i+1}/\theta_n & \text{if } i > j. \end{cases}$$

It is well-known and we can easily check using elementary trigonometry that

$$U_{2i+1}(x) = 2xU_i\left(2x^2 - 1\right).$$

Since $\beta = 1$ and

$$U_i\left(\frac{a^2 - 2bc}{2bc}\right) = U_i\left(2\left(\frac{a}{2\sqrt{bc}}\right)^2 - 1\right) = \frac{\sqrt{bc}}{a} U_{2i+1}\left(\frac{a}{2\sqrt{bc}}\right),$$

from (15) we get

$$\theta_i = \left(\sqrt{bc}\right)^i U_i\left(\frac{a}{2\sqrt{bc}}\right),$$

and taking into account the above remark the ϕ_i's can be computed using the θ_i's, giving

$$\phi_i = \theta_{n+1-i} = \left(\sqrt{bc}\right)^{n+1-i} U_{n+1-i}\left(\frac{a}{2\sqrt{bc}}\right).$$

Therefore, the next result comes immediately.

Corollary 4.1. *Let T be the matrix in* (16) *and define $d := a/2\sqrt{bc}$. The inverse is given by*

$$(T^{-1})_{ij} = \begin{cases} (-1)^{i+j} \dfrac{b^{j-i}}{\left(\sqrt{bc}\right)^{j-i+1}} \dfrac{U_{i-1}(d)U_{n-j}(d)}{U_n(d)} & \textit{if } i \leqslant j, \\[2ex] (-1)^{i+j} \dfrac{c^{i-j}}{\left(\sqrt{bc}\right)^{i-j+1}} \dfrac{U_{j-1}(d)U_{n-i}(d)}{U_n(d)} & \textit{if } i > j. \end{cases}$$

This result is well-known. It can be deduced, e.g., using results in the book of Heinig and Rost [2, p. 28]. The next corollary is Theorem 3.1 of Kamps [3].

Corollary 4.2. *Let Σ be the n-square matrix*

$$\Sigma = \begin{pmatrix} a & b & & & 0 \\ b & a & b & & \\ & b & a & \ddots & \\ & & \ddots & \ddots & b \\ 0 & & & b & a \end{pmatrix}. \tag{17}$$

The inverse is given by

$$(\Sigma^{-1})_{ij} = \begin{cases} (-1)^{i+j}\dfrac{1}{b}\dfrac{U_{i-1}(a/2b)U_{n-j}(a/2b)}{U_n(a/2b)} & \text{if } i \leqslant j, \\[2ex] (-1)^{i+j}\dfrac{1}{b}\dfrac{U_{j-1}(a/2b)U_{n-i}(a/2b)}{U_n(a/2b)} & \text{if } i > j. \end{cases}$$

We consider now a second application. In [3,4], the matrices of type (17), with $a > 0$, $b \neq 0$ and $a > 2|b|$, arose as the covariance matrix of one-dependent random variables $Y_1, \ldots, Y_n$, with same expectation. Let us consider the least squares estimator

$$\hat{\mu}_{\text{opt}} = \frac{\mathbf{1}^{\mathrm{t}}\Sigma^{-1}Y}{\mathbf{1}^{\mathrm{t}}\Sigma^{-1}\mathbf{1}},$$

where $\mathbf{1} = (1, \ldots, 1)$ and $Y = (Y_1, \ldots, Y_n)^{\mathrm{t}}$, which estimates the parameter μ equal to the common expectation of the Y_i's, with variance

$$V\left(\hat{\mu}_{\text{opt}}\right) = \frac{1}{\mathbf{1}^{\mathrm{t}}\Sigma^{-1}\mathbf{1}}.$$

According to Kamps, the estimator $\hat{\mu}_{\text{opt}}$ is the best unbiased estimator based on Y. So, the sum of all the entries of the inverse of Σ, $\mathbf{1}^{\mathrm{t}}\Sigma^{-1}\mathbf{1}$, has an important role in the determination of this estimator and therefore in the computation of the variance $V(\hat{\mu}_{\text{opt}})$. Kamps used in [3] some sum and product formulas involving different kinds of Chebyshev polynomials. We will prove the same results in a more concise way.

Corollary 4.3. *The sum s_i of the* ith *row (or column) of Σ^{-1}, for $i = 1, \ldots, n$ is given by*

$$s_i = \frac{1 + b(\sigma_{1i} + \sigma_{1,n-i+1})}{a + 2b},$$

where $\sigma_{ij} = (\Sigma^{-1})_{ij}$, when $i \leqslant j$.

Proof. By Corollary 4.2, for $i \leqslant j$,

$$\sigma_{ij} = (-1)^{i+j} \frac{1}{b} \frac{\left(r_+^i - r_-^i\right)\left(r_+^{n-j+1} - r_-^{n-j+1}\right)}{(r_+ - r_-)\left(r_+^{n+1} - r_-^{n+1}\right)},$$

where

$$r_\pm = \frac{a \pm \sqrt{a^2 - 4b^2}}{2b}.$$

Thus

$$s_i = \sum_{j=1}^{i-1} \sigma_{ji} + \sum_{j=i}^{n} \sigma_{ij}.$$

By the sum of a geometric progression and the fact that

$$r_+ r_- = 1,$$

we have

$$\sum_{j=1}^{i-1} \sigma_{ji} = \frac{b\,(\sigma_{1i} + \sigma_{ii})}{a + 2b} + \frac{1}{a + 2b} \frac{\left(r_+^{i+1} - r_-^{i+1}\right)\left(r_+^{n-i+1} - r_-^{n-i+1}\right)}{(r_+ - r_-)\left(r_+^{n+1} - r_-^{n+1}\right)}$$

and

$$\sum_{j=i}^{n} \sigma_{ij} = \frac{b\left(\sigma_{1,n-i+1} - \sigma_{ii}\right)}{a + 2b} - \frac{1}{a + 2b} \frac{\left(r_+^i - r_-^i\right)\left(r_+^{n-i} - r_-^{n-i}\right)}{(r_+ - r_-)\left(r_+^{n+1} - r_-^{n+1}\right)}.$$

Finally

$$\begin{aligned}&\left(r_+^{i+1} - r_-^{i+1}\right)\left(r_+^{n-i+1} - r_-^{n-i+1}\right) - \left(r_+^i - r_-^i\right)\left(r_+^{n-i} - r_-^{n-i}\right) \\ &\quad = (r_+ - r_-)\left(r_+^{n+1} - r_-^{n+1}\right). \qquad \Box\end{aligned}$$

The previous theorem says that s_i depends only on the first row of Σ^{-1}. In fact (as the following result says) the sum of all entries of Σ^{-1} depends only on σ_{11} and σ_{1n}.

Corollary 4.4.

$$\mathbf{1}^{\mathrm{t}} \Sigma^{-1} \mathbf{1} = \frac{n + 2bs_1}{a + 2b}.$$

5. Inverse of a tridiagonal 3-Toeplitz matrix

In an analogous way of the 2-Toeplitz matrices, we may ask about the inverse of a tridiagonal 3-Toeplitz matrix. To solve this case, we have to compute θ_i and ϕ_i in (7).

For θ_i, we have

$$\begin{aligned}
&\theta_0 = 1, \quad \theta_1 = a_1, \\
&\theta_{3i} = a_3\theta_{3i-1} - b_2c_2\theta_{3i-2}, \\
&\theta_{3i+1} = a_1\theta_{3i} - b_3c_3\theta_{3i-1}, \\
&\theta_{3i+2} = a_2\theta_{3i+1} - b_1c_1\theta_{3i}.
\end{aligned}$$

Making the change

$$z_i = (-1)^i\theta_i,$$

we get the relations

$$\begin{aligned}
&z_0 = 1, \quad z_1 = -a_1, \\
&z_{3i} = -a_3z_{3i-1} - b_2c_2z_{3i-2}, \\
&z_{3i+1} = -a_1z_{3i} - b_3c_3z_{3i-1}, \\
&z_{3i+2} = -a_2z_{3i+1} - b_1c_1z_{3i}.
\end{aligned}$$

According to the results in [7],

$$z_i = Q_i(0),$$

where Q_i is a polynomial of degree exactly i, defined according to

$$Q_{i+1}(x) = (x - \tilde{\beta}_i)Q_i(x) - \tilde{\gamma}_i Q_{i-1}(x),$$

where

$$\begin{cases} \tilde{\beta}_{3j} = a_1, \\ \tilde{\beta}_{3j+1} = a_2, \\ \tilde{\beta}_{3j+2} = a_3, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\gamma}_{3j} = b_3c_3, \\ \tilde{\gamma}_{3j+1} = b_1c_1, \\ \tilde{\gamma}_{3j+2} = b_2c_2. \end{cases} \tag{18}$$

The solution of the recurrence relation (8) with coefficients (18) is

$$\begin{aligned}
&Q_{3i}(x) = P_i\left[\pi_3\left(x\right)\right] + b_3c_3\left(x - a_2\right)P_{i-1}\left[\pi_3\left(x\right)\right], \\
&Q_{3i+1}(x) = (x - a_1)P_i\left[\pi_3\left(x\right)\right] + b_1c_1b_3c_3P_{i-1}\left[\pi_3\left(x\right)\right], \\
&Q_{3i+2}(x) = \left[(x - a_1)(x - a_2) - b_1c_1\right]P_i\left[\pi_3\left(x\right)\right],
\end{aligned}$$

where

$$\pi_3(x) = \begin{vmatrix} x - a_1 & 1 & 1 \\ b_1c_1 & x - a_2 & 1 \\ b_3c_3 & b_2c_2 & x - a_3 \end{vmatrix}$$

and

$$P_i(x) = \left(2\sqrt{b_1b_2b_3c_1c_2c_3}\right)^i \left[U_i\left(\frac{x - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)\right].$$

We may conclude that

$$\theta_i = (-1)^i Q_i(0).$$

In order to compute the ϕ_i's, we define

$$\psi_i = \phi_{n+1-i}$$

for $i = 1, \ldots, n$. In this situation, we have to distinguish three cases. If $n \equiv 0 \pmod 3$, then

$$\begin{aligned}
&\psi_0 = 1, \quad \psi_1 = a_3,\\
&\psi_{3i} = a_1\psi_{3i-1} - b_1c_1\psi_{3i-2},\\
&\psi_{3i+1} = a_3\psi_{3i} - b_3c_3\psi_{3i-1},\\
&\psi_{3i+2} = a_2\psi_{3i+1} - b_2c_2\psi_{3i}.
\end{aligned}$$

If $n \equiv 1 \pmod 3$, then

$$\begin{aligned}
&\psi_0 = 1, \quad \psi_1 = a_1,\\
&\psi_{3i} = a_2\psi_{3i-1} - b_2c_2\psi_{3i-2},\\
&\psi_{3i+1} = a_1\psi_{3i} - b_1c_1\psi_{3i-1},\\
&\psi_{3i+2} = a_3\psi_{3i+1} - b_3c_3\psi_{3i},
\end{aligned}$$

and if $n \equiv 2 \pmod 3$, then

$$\begin{aligned}
&\psi_0 = 1, \quad \psi_1 = a_2,\\
&\psi_{3i} = a_3\psi_{3i-1} - b_3c_3\psi_{3i-2},\\
&\psi_{3i+1} = a_2\psi_{3i} - b_2c_2\psi_{3i-1},\\
&\psi_{3i+2} = a_1\psi_{3i+1} - b_1c_1\psi_{3i}.
\end{aligned}$$

It is clear that these three recurrence relations can be solved by a similar process as for the computation of the θ_i's (*mutatis mutandis*). Following in this way, we can state the next proposition.

Theorem 5.1. *Let B be the tridiagonal matrix* (2), *with $a_1a_2a_3 \neq 0$ and $b_1b_2b_3c_1c_2c_3 > 0$. Let*

$$\pi_3\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}; x \Bigg) := \begin{vmatrix} x-a & 1 & 1 \\ \alpha & x-b & 1 \\ \gamma & \beta & x-c \end{vmatrix},$$

$$\begin{aligned}
P_i\left(\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \end{matrix}\;; x\right) :=& \, 2\sqrt{\alpha\beta\gamma}\\
&\times U_i\left(\frac{1}{2\sqrt{\alpha\beta\gamma}}\left[\pi_3\left(\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \end{matrix}\;; x\right) - (a+b+c)\right]\right),
\end{aligned}$$

$$
\begin{aligned}
Q_{3i}\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}; x\Big) &:= P_i\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}; x\Big) \\
&\quad +\gamma(x-b)\, P_{i-1}\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}; x\Big),
\end{aligned}
$$

$$
\begin{aligned}
Q_{3i+1}\left(\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \end{matrix}; x\right) &:= (x-a)\, P_i\left(\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \end{matrix}; x\right) \\
&\quad +\alpha\gamma\, P_{i-1}\left(\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \end{matrix}; x\right),
\end{aligned}
$$

$$
Q_{3i+2}\left(\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \end{matrix}; x\right) := [(x-a)(x-b)-\alpha]\, P_i\left(\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \end{matrix}; x\right),
$$

where a, b, c, α, β *and* γ *are some parameters, subject to the restriction* $\alpha\beta\gamma > 0$. *Under these conditions,*

$$
(B^{-1})_{ij} = \begin{cases} (-1)^{i+j}\, \beta_i\beta_{i+1}\cdots\beta_j\, \theta_{i-1}\,\phi_{j+1}/\theta_n & \text{if } i \leqslant j, \\ (-1)^{i+j}\, \gamma_j\gamma_{j+1}\cdots\gamma_i\, \theta_{j-1}\,\phi_{i+1}/\theta_n & \text{if } i > j, \end{cases} \tag{19}
$$

where

$$
\beta_{3\ell+s+1} = b_{s+1}, \quad \gamma_{3\ell+s+1} = c_{s+1} \quad (s = 0, 1, 2;\ \ell = 0, 1, 2, \ldots),
$$

and the θ_i*'s and* ϕ_i*'s are explicitly given by*

$$
\theta_i = (-1)^i\, Q_i\left(\begin{matrix} a_1 & a_2 & a_3 \\ b_1c_1 & b_2c_2 & b_3c_3 \end{matrix}; 0\right) \tag{20}
$$

and

$$
\phi_i = \begin{cases} (-1)^{n+1-i}\, Q_{n+1-i}\left(\begin{matrix} a_3 & a_2 & a_1 \\ b_2c_2 & b_1c_1 & b_3c_3 \end{matrix}; 0\right) & \text{if } n \equiv 0, \pmod 3, \\ (-1)^{n+1-i}\, Q_{n+1-i}\left(\begin{matrix} a_1 & a_3 & a_2 \\ b_3c_3 & b_2c_2 & b_1c_1 \end{matrix}; 0\right) & \text{if } n \equiv 1, \pmod 3, \\ (-1)^{n+1-i}\, Q_{n+1-i}\left(\begin{matrix} a_2 & a_1 & a_3 \\ b_1c_1 & b_3c_3 & b_2c_2 \end{matrix}; 0\right) & \text{if } n \equiv 2, \pmod 3. \end{cases}
$$

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