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Numerical ranges of unbounded operators arising in quantum physics

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Abstract

Creation and annihilation operators are used in quantum physics as the building blocks of linear operators acting on Hilbert spaces of many body systems. In quantum physics, pairing operators are defined in terms of those operators. In this paper, spectral properties of pairing operators are studied. The numerical ranges of pairing operators are investigated. In the context of matrix theory, the results give the numerical ranges of certain infinite tridiagonal matrices.

Keywords: Numerical range; Unbounded linear operator

1. Creation and annihilation operators

In quantum mechanics, states of a particle are described by vectors belonging to a Hilbert space, the so called *state space*. For physical systems composed of many identical particles, it is useful to define operators that *create* or *annihilate* a particle in a specified individual state. Operators of physical interest can be expressed in terms of these creation and annihilation operators [1,2]. Only totally symmetric and anti-symmetric states are observed in nature and particles occurring in these states

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are called *bosons* and *fermions*, respectively. If V is the state space of one boson and $m \in \mathbb{N}$, the mth completely symmetric space over V, denoted by $V_{(m)}$, is the appropriate state space to describe a system with m bosons. By convention, $V_{(0)} = \mathbb{C}$.

Let V be an n-dimensional vector space with inner product $(\cdot,\cdot)$, and let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of V. The *creation operator* associated with e_i, $i = 1,\ldots,n$, is the linear operator $f_i : V_{(m-1)} \to V_{(m)}$ defined by

$$f_i(x_1 * \cdots * x_{m-1}) = e_i * x_1 * \cdots * x_{m-1}, \tag{1}$$

for $x_1 * \cdots * x_{m-1}$ a decomposable tensor in $V_{(m-1)}$. The *annihilation operator* is the adjoint operator of the creation operator f_i, explicitly, it is the linear operator $g_i : V_{(m)} \to V_{(m-1)}$ defined by

$$g_i(x_1 * \cdots * x_m) = \sum_{k=1}^{m} (e_i, x_k) x_1 * \cdots * x_{k-1} * x_{k+1} * \cdots * x_m, \tag{2}$$

for $x_1 * \cdots * x_m$ in $V_{(m)}$. Denote by e_i^k the symmetric tensor product $e_i * \cdots * e_i$ with k factors. Clearly, $f_i(e_i^{m-1}) = e_i^m$ and $g_i(e_i^m) = m e_i^{m-1}$. These operators can also be defined on the symmetric algebra over V: $\Gamma^* = \bigoplus_{m=0}^{+\infty} V_{(m)}$. We consider Γ^* endowed with the norm induced by the standard inner product defined by $(x_1 * \cdots * x_m, y_1 * \cdots * y_m) = \mathrm{per}[(x_i, y_j)]$, for $x_1 * \cdots * x_m$ and $y_1 * \cdots * y_m$ decomposable tensors in $V_{(m)}$. Here, $\mathrm{per}X$ denotes the permanent of the matrix X.

The creation and annihilation operators satisfy the following *canonical commutation relations*: $[f_i, f_j] = [g_i, g_j] = 0, [g_i, f_j] = \delta_{ij}, i, j = 1,\ldots,n$, where $[f, g] = fg - gf$ denotes, as usual, the commutator of the operators f and g.

The *bosonic number operator in state i* is the linear operator $N_i : \Gamma^* \to \Gamma^*$ defined by $N_i = f_i g_i$, for $i = 1,\ldots,n$. It will be shown that the non-negative integers are the eigenvalues of this operator. This is related to the physical fact that an arbitrary number of bosons can occupy the same quantum state.

Let V be $\mathbb{C}^2$. For the symmetric algebra Γ^* over $\mathbb{C}^2$, the *pairing operator* $B : \Gamma^* \to \Gamma^*$ is the linear operator defined in terms of the creation and annihilation operators by

$$B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2, \qquad c, d, k, l \in \mathbb{C}. \tag{3}$$

These operators are unbounded. Moreover, B commutes with $f_1g_1 - f_2g_2$ and so, adding a multiple of this operator to B, we can take the coefficients of f_1g_1 and f_2g_2 equal. We can also substitute f_1 (f_2) by $\mathrm{e}^{\mathrm{i}\alpha} f_1$ ($\mathrm{e}^{\mathrm{i}\alpha} f_2$), $\alpha \in \mathbb{R}$, and choose α such that the arguments of k and l are equal.

The *numerical range* or *field of values* of a linear operator T on a complex Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$, is defined by

$$W(T) = \{(Tx, x) : x \in \mathcal{H}, (x, x) = 1\}.$$

One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toeplitz–Hausdorff Theorem (see e.g., [3,4]). In the finite dimensional case, $W(T)$ contains the spectrum of T, and it is a connected and compact subset of $\mathbb{C}$. In the infinite dimensional case, $W(T)$ does not have to be either bounded or closed.

We recall that a *tridiagonal matrix* is a matrix $A = (a_{ij})$ such that $a_{ij} = 0$ whenever $|i - j| > 1$. The numerical ranges of tridiagonal matrices deserved the attention of some authors (e.g., [5–8]). One of the main aims of this paper is the investigation of the numerical range of pairing operators B defined on the subspace $\Gamma^{(q)}$ of the symmetric algebra over $\mathbb{C}^2$. These operators admit well-structured infinite tridiagonal matrix representations. The numerical ranges of the pairing operators under consideration have an interesting relation with the numerical ranges of certain linear operators on an indefinite inner product space.

Let M_n be the algebra of $n \times n$ complex matrices, and let $S \in M_n$ be a selfadjoint matrix. The *positive S-numerical range* of $A \in M_n$ is denoted and defined by

$$V_S^+(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*Sx = 1\}.$$

This set is always a convex set [9]. If S is the $n \times n$ identity matrix I_n, then $V_S^+(A)$ reduces to the classical numerical range of $A \in M_n$. If S is a non-singular indefinite selfadjoint matrix, some authors use $W_S^+(A) = V_S^+(SA)$ as the definition of a numerical range of a matrix A associated with the indefinite inner product $\langle x, y\rangle_S = y^*Sx$. In this case, if A is not a S-scalar matrix, that is, $A \neq \lambda S$ where $\lambda \in \mathbb{C}$, $V_S^+(A)$ is unbounded and may not be closed [9,10].

This paper is organized as follows. In Section 2, some preliminary results concerning the Bogoliubov linear transformation are presented. In Section 3, spectral properties of certain pairing operators are investigated. In Section 4, the numerical ranges of the previously considered pairing operators are studied. In particular, the numerical ranges of the infinite tridiagonal matrix representations of the pairing operators are characterized.

2. The Bogoliubov transformation

For convenience, consider the annihilation and creation operators defined on the symmetric algebra over V arranged in a vector α with components

$$\alpha_i = g_i, \quad \alpha_{n+i} = f_i, \qquad i = 1, \ldots, n. \tag{4}$$

The invertible linear operator that maps the vector α into the vector β with components

$$\beta_i = \tilde{g}_i, \quad \beta_{n+i} = \tilde{f}_i, \qquad i = 1, \ldots, n, \tag{5}$$

is called a canonical transformation if it preserves the canonical commutation relations and it is usually called a *Bogoliubov transformation*.

We recall a useful characterization of a Bogoliubov transformation.

Proposition 2.1 [2]. *Let α and β be the column vectors with entries* (4) *and* (5), *respectively. The following conditions are equivalent*:

(i) *The linear operator that maps the vector* α *into the vector* β *is a Bogoliubov transformation*;

(ii) *The matrix* T *such that* $\beta = T\alpha$, *satisfies* $TLT^{\mathrm{T}} = L$ *and* $T^{\mathrm{T}}LT = L$, *where*

$$L = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

The linear operators $\tilde{g}_i$ are the adjoint operators of $\tilde{f}_i$ if the matrix T associated with the Bogoliubov transformation in Proposition 2.1 (ii) is a block matrix of the form

$$T = \begin{bmatrix} X & Y \\ \overline{Y} & \overline{X} \end{bmatrix}, \quad X, Y \in M_n. \tag{6}$$

Let the linear operator $\widetilde{N}_i : \Gamma^* \to \Gamma^*$ be defined by $\widetilde{N}_i = \tilde{f}_i \tilde{g}_i$, $i = 1, \ldots, n$. The following proposition is an easy consequence of the canonical commutation relations for the operators $\tilde{f}_i$ and $\tilde{g}_i$, $i = 1, \ldots, n$.

Proposition 2.2. *If the operators* $\tilde{f}_i$ *and* $\tilde{g}_i$ *satisfy the canonical commutation relations, then*

$$[\widetilde{N}_i, \tilde{f}_j^r] = r\delta_{ij}\tilde{f}_i^r \quad \textit{and} \quad [\widetilde{N}_i, \tilde{g}_j^r] = -r\delta_{ij}\tilde{g}_i^r, \quad i, j = 1, \ldots, n, \quad r \in \mathbb{N}_0.$$

Proof. Let $r \in \mathbb{N}_0$. By induction on k, we prove that

$$\widetilde{N}_i \tilde{f}_j^r = k\delta_{ij}\tilde{f}_i^r + \tilde{f}_j^k \widetilde{N}_i \tilde{f}_j^{r-k}, \quad i, j = 1, \ldots, n, \quad k = 0, \ldots, r. \tag{7}$$

In fact, if $k = 0$, (7) is trivial. Suppose that (7) is true for $k - 1$. Then we successively have:

$$\begin{aligned} \widetilde{N}_i \tilde{f}_j^r &= (k-1)\delta_{ij}\, \tilde{f}_i^r + \tilde{f}_j^{k-1}\widetilde{N}_i \tilde{f}_j^{r-k+1} \\ &= (k-1)\delta_{ij}\tilde{f}_i^r + \tilde{f}_j^{k-1}\tilde{f}_i(\delta_{ij} + \tilde{f}_j\tilde{g}_i)\tilde{f}_j^{r-k} \qquad (8) \\ &= (k-1)\delta_{ij}\tilde{f}_i^r + \tilde{f}_j^k(\delta_{ij} + \tilde{f}_i\tilde{g}_i)\tilde{f}_j^{r-k} \qquad (9) \\ &= k\delta_{ij}\tilde{f}_i^r + \tilde{f}_j^k\, \widetilde{N}_i \tilde{f}_j^{r-k}, \end{aligned}$$

where (8) is a consequence of $[\tilde{g}_i, \tilde{f}_j] = \delta_{ij}$, and (9) follows from $[\tilde{f}_i, \tilde{f}_j] = 0$ and $\tilde{f}_i\delta_{ij} = \tilde{f}_j\delta_{ij}$. Hence, (7) holds for $k = 0, \ldots, r$. The case $k = r$ gives the asserted set of relations on the left-hand side. By transconjugation of these relations, the result follows. □

3. Spectral properties of pairing operators

The symmetric space $\mathbb{C}^2_{(m)}$ is spanned by the vectors $e_1^k * e_2^{m-k}$, $k = 0, \ldots, m$. For $q \geqslant 0$, denote by $\Gamma^{(q)}$ the subspace of the symmetric algebra over $\mathbb{C}^2$ spanned by the vectors $e_1^n * e_2^{n+q}$, $n \in \mathbb{N}_0$, and, for $q < 0$, the subspace spanned by the vectors $e_1^{n-q} * e_2^n$, $n \in \mathbb{N}_0$. It is clear that any two subspaces $\Gamma^{(q)}$ are disjoint. It can be easily seen that the symmetric algebra Γ^* over $\mathbb{C}^2$ is given by $\Gamma^* = \bigoplus_{q=-\infty}^{+\infty} \Gamma^{(q)}$. The subspaces $\Gamma^{(q)}$, $q \in \mathbb{Z}$, satisfy the following property.

Proposition 3.1. *For $q \in \mathbb{Z}$, the subspace $\Gamma^{(q)}$ is invariant under the pairing operator B.*

Proof. For $q \geqslant 0$ and $n \in \mathbb{N}_0$, we have

$$\begin{aligned} B(e_1^n * e_2^{n+q}) &= (cn + d(n+q))e_1^n * e_2^{n+q} \\ &\quad + ke_1^{n+1} * e_2^{n+1+q} + ln(n+q)e_1^{n-1} * e_2^{n-1+q} \in \Gamma^{(q)}. \end{aligned}$$

Analogously, for $q < 0$ and $n \in \mathbb{N}_0$, we find

$$\begin{aligned} B(e_1^{n-q} * e_2^{n}) &= (c(n-q) + dn)e_1^{n-q} * e_2^{n} \\ &\quad + ke_1^{n+1-q} * e_2^{n+1} + ln(n-q)e_1^{n-1-q} * e_2^{n-1} \in \Gamma^{(q)}. \end{aligned}$$

Since B is a linear operator, it satisfies $B(\Gamma^{(q)}) \subseteq \Gamma^{(q)}$, for any integer q. □

Remark 3.1. The matrix representation, in the standard basis, of the pairing operator $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$ restricted to $\Gamma^{(q)}$, $q \geqslant 0$, is the infinite tridiagonal matrix $T_{c,d}^q$ given by

$$\begin{bmatrix} dq & l\sqrt{1+q} & 0 & 0 & \cdots \\ k\sqrt{1+q} & c+d+dq & l\sqrt{2(2+q)} & 0 & \cdots \\ 0 & k\sqrt{2(2+q)} & 2(c+d)+dq & l\sqrt{3(3+q)} & \cdots \\ 0 & 0 & k\sqrt{3(3+q)} & 3(c+d)+dq & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad c,d,k,l \in \mathbb{C}.$$

For $q < 0$, the matrix representation, in the standard basis, of the pairing operator $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$ restricted to $\Gamma^{(q)}$ is the tridiagonal matrix $T_{d,c}^{-q}$.

In the sequel, we adopt the following notation: $D = \{z \in \mathbb{C} : |z| < 1\}$.
For $z \in D$, let $\tilde{f}_1$ and $\tilde{f}_2$ be the linear operators on Γ^* defined by

$$\tilde{f}_1 = \frac{1}{\sqrt{1-|z|^2}}(f_1 - \bar{z}g_2), \quad \tilde{f}_2 = \frac{1}{\sqrt{1-|z|^2}}(f_2 - \bar{z}g_1). \tag{10}$$

Their adjoint operators are

$$\tilde{g}_1 = \frac{1}{\sqrt{1-|z|^2}}(g_1 - zf_2), \quad \tilde{g}_2 = \frac{1}{\sqrt{1-|z|^2}}(g_2 - zf_1), \tag{11}$$

respectively. The linear operator that maps the vector $\alpha^{\mathrm{T}} = (g_1, g_2, f_1, f_2)$ into the vector $\beta^{\mathrm{T}} = (\tilde{g}_1, \tilde{g}_2, \tilde{f}_1, \tilde{f}_2)$ is a Bogoliubov transformation.

Proposition 3.2. *The Bogoliubov transformation defined by* (10) *and* (11) *maps the pairing operator* $B : \Gamma^* \to \Gamma^*$ *defined by* $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$, $c, d, k, l \in \mathbb{C}$, *into* $B = \lambda_0\iota + \tilde{c}\tilde{f}_1\tilde{g}_1 + \tilde{d}\tilde{f}_2\tilde{g}_2 + \tilde{k}\tilde{f}_1\tilde{f}_2 + \tilde{l}\tilde{g}_1\tilde{g}_2$, *where* ι *denotes the identity map*, $z \in D$, *and*

$$\lambda_0 = \frac{1}{1-|z|^2}((c+d)|z|^2 + k\bar{z} + lz), \tag{12}$$

$$\tilde{c} = \frac{1}{1-|z|^2}(c + d|z|^2 + k\bar{z} + lz), \tag{13}$$

$$\tilde{d} = \frac{1}{1-|z|^2}(c|z|^2 + d + k\bar{z} + lz), \tag{14}$$

$$\tilde{k} = \frac{1}{1-|z|^2}((c+d)z + k + lz^2), \tag{15}$$

$$\tilde{l} = \frac{1}{1-|z|^2}((c+d)\bar{z} + k\bar{z}^2 + l). \tag{16}$$

Moreover,

$$\tilde{c} = c + \lambda_0 \quad \textit{and} \quad \tilde{d} = d + \lambda_0. \tag{17}$$

Proof. The Bogoliubov transformation defined by (10) and (11) is associated with a matrix T of the form (6), where the submatrices X and Y are

$$X = \frac{1}{\sqrt{1-|z|^2}} I_2, \quad Y = \frac{1}{\sqrt{1-|z|^2}} \begin{bmatrix} 0 & -z \\ -z & 0 \end{bmatrix}.$$

Since $\alpha = T^{-1}\beta$ and

$$T^{-1} = \frac{1}{\sqrt{1-|z|^2}} \begin{bmatrix} 1 & 0 & 0 & z \\ 0 & 1 & z & 0 \\ 0 & \bar{z} & 1 & 0 \\ \bar{z} & 0 & 0 & 1 \end{bmatrix},$$

the following inverse relations hold:

$$f_1 = \frac{1}{\sqrt{1-|z|^2}}(\tilde{f}_1 + \bar{z}\tilde{g}_2), \quad f_2 = \frac{1}{\sqrt{1-|z|^2}}(\tilde{f}_2 + \bar{z}\tilde{g}_1) \tag{18}$$

and

$$g_1 = \frac{1}{\sqrt{1-|z|^2}}(\tilde{g}_1 + z\tilde{f}_2), \quad g_2 = \frac{1}{\sqrt{1-|z|^2}}(\tilde{g}_2 + z\tilde{f}_1). \tag{19}$$

Taking into account (18) and (19) in $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$, the result easily follows. □

The pairing operator B in (3) is a selfadjoint operator if and only if $c, d \in \mathbb{R}$ and $l = \bar{k}$.

Proposition 3.3. *The pairing operator* $B = \lambda_0 \iota + \tilde{c}\, \tilde{f}_1 \tilde{g}_1 + \tilde{d}\, \tilde{f}_2 \tilde{g}_2 + \tilde{k}\, \tilde{f}_1 \tilde{f}_2 + \tilde{l}\, \tilde{g}_1 \tilde{g}_2$ *is a selfadjoint operator if and only if* λ_0, $\tilde{c}$ *and* $\tilde{d}$ *are real numbers and* $\tilde{l} = \bar{\tilde{k}}$.

Proof. Trivial. □

Throughout this section, let $\varDelta = (c+d)^2 - 4|k|^2$, for $c, d \in \mathbb{R}$ and $k \in \mathbb{C}$.

Proposition 3.4. *If* $B = c f_1 g_1 + d f_2 g_2 + k f_1 f_2 + \bar{k} g_1 g_2$, *with* $c, d \in \mathbb{R}$ *and* $k \in \mathbb{C}$, *is a selfadjoint pairing operator and* $\varDelta > 0$, *then* B *can be reduced by a Bogoliubov transformation to the form* $B = \lambda_0 \iota + \tilde{c}\, \tilde{f}_1 \tilde{g}_1 + \tilde{d}\, \tilde{f}_2 \tilde{g}_2$, *where* ι *denotes the identity map and* λ_0, $\tilde{c}$, $\tilde{d}$ *are given by* (12)–(14), *respectively. Moreover*,

(i) *If* $c + d > 0$, *then* $\tilde{c} + \tilde{d} = \sqrt{\varDelta}$ *and* $\lambda_0 = -\frac{1}{2}(c+d) + \frac{1}{2}\sqrt{\varDelta}$;
(ii) *If* $c + d < 0$, *then* $\tilde{c} + \tilde{d} = -\sqrt{\varDelta}$ *and* $\lambda_0 = -\frac{1}{2}(c+d) - \frac{1}{2}\sqrt{\varDelta}$.

Proof. By Proposition 3.2, under a Bogoliubov transformation, we can take the selfadjoint pairing operator $B = c\, f_1 g_1 + d f_2 g_2 + k f_1 f_2 + \bar{k} g_1 g_2$, where $c, d \in \mathbb{R}$ and $k \in \mathbb{C}$, into the form $B = \lambda_0\, \iota + \tilde{c}\, \tilde{f}_1 \tilde{g}_1 + \tilde{d}\, \tilde{f}_2 \tilde{g}_2 + \tilde{k}\, \tilde{f}_1 \tilde{g}_2 + \bar{\tilde{k}}\, \tilde{f}_2 \tilde{g}_1$, where λ_0, $\tilde{c}$, $\tilde{d}$ and $\tilde{k}$ are given by (12), (13), (14) and (15), respectively. If $\varDelta > 0$, it is possible to find $z \in D$ such that $\tilde{k} = 0$. In fact, we can choose a solution z of the quadratic equation

$$\bar{k} z^2 + (c+d) z + k = 0, \tag{20}$$

for which $\tilde{k}$ vanishes. The choice can be made as follows. For $k = 0$ and $c + d \neq 0$, we take $z = 0$. For $k \neq 0$, we have

$$z = \frac{-(c+d) \pm \sqrt{(c+d)^2 - 4|k|^2}}{2\bar{k}}. \tag{21}$$

The product of the roots of the quadratic equation in (20) is $k/\bar{k}$, a complex number of modulus 1. Therefore, one of these roots has modulus less than 1 and for this root $\tilde{k} = 0$. Thus, we may concentrate on $B = \lambda_0 \iota + \tilde{c}\, \tilde{f}_1 \tilde{g}_1 + \tilde{d}\, \tilde{f}_2 \tilde{g}_2$. From (13) and (14), we find

$$\tilde{c} + \tilde{d} = \frac{(c+d)(1+|z|^2) + 2k\bar{z} + 2\bar{k} z}{1 - |z|^2}. \tag{22}$$

From (21) and (22), we get $\tilde{c} + \tilde{d} = \mp\sqrt{\varDelta}$. From (17), we have $\tilde{c} + \tilde{d} = c + d + 2\lambda_0$. Hence, $\lambda_0 = -\frac{1}{2}(c+d) \pm \frac{1}{2}\sqrt{\varDelta}$. If $c + d > 0$, we consider the plus sign for the $\pm$ sign in (21), so that z belongs to D. Thus, (i) holds. If $c + d < 0$, we take the minus sign for the $\pm$ sign in (21), otherwise z does not belong to D. Hence, (ii) follows. □

Remark 3.2. If $\Delta = k = 0$, then $\tilde{k} = 0$ for any $z \in D$. If $\Delta \leqslant 0$ and $k \neq 0$, it can be easily seen that both roots of the quadratic equation in (20) have modulus 1 and so we cannot choose $z \in D$ such that $\tilde{k} = 0$. As observed in the proof of Proposition 3.4, if $\Delta > 0$ one of the roots of (20) has modulus less than 1, while the other one has modulus greater than 1.

Proposition 3.5. *Let $B = cf_1g_1 + df_2g_2 + kf_1f_2 + \bar{k}g_1g_2$, with $c, d \in \mathbb{R}$ and $k \in \mathbb{C}$, be a selfadjoint pairing operator defined on the symmetric algebra Γ^* over $\mathbb{C}^2$. A complex number z satisfies $[B, g_1 - zf_2] = \frac{1}{2}(d - c \pm \sqrt{\Delta})(g_1 - zf_2)$ and $[B, g_2 - zf_1] = \frac{1}{2}(c - d \pm \sqrt{\Delta})(g_2 - zf_1)$ if and only if z is a root of* (20).

Proof. ($\Rightarrow$) We have

$$[B, g_1 - zf_2] = -(c + \bar{k}z)g_1 - (k + dz)f_2. \tag{23}$$

It is not difficult to see that there exists $w \in \mathbb{C}$ such that

$$[B, g_1 - zf_2] = w(g_1 - zf_2). \tag{24}$$

In fact, from (23) and (24), we obtain

$$\begin{bmatrix} c & \bar{k} \\ k & d \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} = w \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix}. \tag{25}$$

The solutions w of (25) are such that

$$\det \begin{bmatrix} -c - w & -\bar{k} \\ k & d - w \end{bmatrix} = 0,$$

that is, $w = \frac{1}{2}(d - c) \pm \frac{1}{2}\sqrt{\Delta}$. From (25), we get $z = -(c + w)/\bar{k}$.

($\Leftarrow$) It is a straightforward computation. □

Proposition 3.6. *For $z \in \mathbb{C}$, there exists a non-zero vector u in the Hilbert space Γ^* such that $(g_1 - zf_2)u = 0$ and $(g_2 - zf_1)u = 0$ if and only if $|z| < 1$, and the respective vector u is given by the formula*

$$u = \sum_{n=0}^{+\infty} c_0 \frac{z^n}{n!} f_1^n f_2^n(1), \quad c_0 \in \mathbb{C} \setminus \{0\}.$$

Proof. ($\Rightarrow$) Consider an arbitrary element $u = \sum_{n,m=0}^{+\infty} c_{nm} f_1^n f_2^m(1) \in \Gamma^*$, $c_{nm} \in \mathbb{C} \setminus \{0\}$. Since we are assuming $(g_1 - zf_2)u = 0$, it follows that

$$\sum_{n,m=0}^{+\infty} (c_{n+1m+1}(n+1) - c_{nm}z) f_1^n f_2^{m+1}(1) = 0.$$

Hence,

$$c_{n+1m+1}(n+1) - c_{nm}z = 0. \tag{26}$$

By the hypothesis $(f_2 - zg_1)u = 0$, and so we also have

$$c_{n+1m+1}(m+1) - c_{nm}z = 0. \tag{27}$$

From (26) and (27) we get $(n-m)c_{n+1m+1} = 0$, that is, $c_{nm} = c_n\delta_{nm}$. Thus, $u = \sum_{n=0}^{+\infty} c_n f_1^n f_2^n(1) \in \Gamma^{(0)}$. From (27) it follows that $c_{n+1}(n+1) - c_n z = 0$, $n \in \mathbb{N}_0$. By induction on n, it can easily be proved that $c_n = c_0 z^n/n!$, $n \in \mathbb{N}_0$, $c_0 \in \mathbb{C} \setminus \{0\}$. The vector u belongs to the Hilbert space Γ^* if and only if $|z| < 1$.

($\Leftarrow$) Clear. □

Corollary 3.1. *Let $\tilde{g}_1, \tilde{g}_2 : \Gamma^* \to \Gamma^*$ be defined by* (11), *with $z \in D$ satisfying* (20). *If $\Delta > 0$ and $c_0 \in \mathbb{C}$, the vector $u = \sum_{n=0}^{+\infty} c_0 \frac{z^n}{n!} f_1^n f_2^n(1) \in \Gamma^{(0)}$ satisfies $\tilde{g}_1 u = \tilde{g}_2 u = 0$.*

Proof. The corollary is an obvious consequence of Proposition 3.6. □

Proposition 3.7. *Let $B = cf_1g_1 + df_2g_2 + kf_1f_2 + \bar{k}g_1g_2$, with $c, d \in \mathbb{R}$ and $k \in \mathbb{C}$, be a selfadjoint pairing operator defined on Γ^*. If $\Delta < 0$, then B does not have eigenvectors in the Hilbert space Γ^*.*

Proof. (By contradiction) Suppose that there exists in Γ^* an eigenvector u of B associated with the eigenvalue $\lambda \in \mathbb{R}$, that is, $Bu = \lambda u$. By Proposition 3.5, there exists $z \in \mathbb{C}$ such that $[B, g_1 - zf_2] = \frac{1}{2}(d - c + \mathrm{i}\sqrt{-\Delta})(g_1 - zf_2)$ and $[B, g_2 - zf_1] = \frac{1}{2}(c - d + \mathrm{i}\sqrt{-\Delta})(g_2 - zf_1)$ if and only if z is a root of (20). Easy computations yield

$$\begin{aligned} B(g_1 - zf_2)u &= [B, g_1 - zf_2]u + (g_1 - zf_2)Bu \\ &= \left(\lambda + \frac{1}{2}(c - d + \mathrm{i}\sqrt{-\Delta})\right)(g_1 - zf_2)u \end{aligned}$$

and

$$B(g_2 - zf_1)u = \left(\lambda + \frac{1}{2}(c - d + \mathrm{i}\sqrt{-\Delta})\right)(g_2 - zf_1)u.$$

Then, either $(g_1 - zf_2)u$ vanishes or it is an eigenvector of B corresponding to the eigenvalue $\lambda + \frac{1}{2}(d - c + \mathrm{i}\sqrt{-\Delta})$. Since a selfadjoint operator does not have complex eigenvalues, this hypothesis does not hold and so $(g_1 - zf_2)u = 0$. In an analogous way, we conclude that $(g_2 - zf_1)u = 0$. By Proposition 3.6, the conditions $(g_1 - zf_2)u = 0$ and $(g_2 - zf_1)u = 0$ hold if and only if $|z| < 1$. The assumption $\Delta < 0$ implies that $|z| = 1$, a contradiction. □

Proposition 3.8. *The eigenvalues of the operators $\tilde{N}_1 = \tilde{f}_1\tilde{g}_1$ and $\tilde{N}_2 = \tilde{f}_2\tilde{g}_2$ defined on Γ^* are the non-negative integers and the common eigenvectors corresponding to the eigenvalues n_1 and n_2 are of the form $c_0\tilde{f}_1^{n_1}\tilde{f}_2^{n_2}\mathrm{e}^{zf_1f_2}(1)$, where $c_0 \in \mathbb{C} \setminus \{0\}$ and z is the root of* (20) *in D.*

Proof. Since the operators $\widetilde{N}_1$ and $\widetilde{N}_2$ commute, they have common eigenvectors. Let u be a non-zero vector in Γ^* such that $\widetilde{N}_1 u = \lambda_1 u$ and $\widetilde{N}_2 u = \lambda_2 u$. Replacing u by $\tilde{g}_1 u$ in $\widetilde{N}_1 u$ and u by $\tilde{g}_2 u$ in $\widetilde{N}_2 u$, we obtain

$$\widetilde{N}_1 \tilde{g}_1 u = (\lambda_1 - 1)\tilde{g}_1 u \quad \text{and} \quad \widetilde{N}_2 \tilde{g}_2 u = (\lambda_2 - 1)\tilde{g}_2 u. \tag{28}$$

From the left-hand side equation in (28), we conclude that either $\tilde{g}_1 u = 0$ or $\tilde{g}_1 u$ is an eigenvector of $\widetilde{N}_1$ associated with $(\lambda_1 - 1)$. From the right-hand side equation in (28), we conclude that either $\tilde{g}_2 u = 0$ or $\tilde{g}_2 u$ is an eigenvector of $\widetilde{N}_2$ associated with $(\lambda_2 - 1)$. If $\tilde{g}_1 u = 0$ and $\tilde{g}_2 u = 0$, by Proposition 3.6, u is of the asserted form and $\lambda_1 = \lambda_2 = 0$. In this case, the result follows. If $\tilde{g}_1 u \neq 0$ or $\tilde{g}_2 u \neq 0$, we repeat the previous procedure. Indeed, there exist integers k_1, k_2 such that $v = \tilde{g}_1^{k_1} \tilde{g}_2^{k_2} u \neq 0$ and $\tilde{g}_1^{k_1+1} \tilde{g}_2^{k_2} u = \tilde{g}_1^{k_1} \tilde{g}_2^{k_2+1} u = 0$. Since $\widetilde{N}_1$ and $\widetilde{N}_2$ are positive semidefinite operators, the eigenvalues $\lambda_1 - k_1$ and $\lambda_2 - k_2$ associated with the eigenvector v are non-negative. The process stops when $\lambda_1 - k_1 = \lambda_2 - k_2 = 0$, and so λ_1 and λ_2 are non-negative integers. Since $\tilde{g}_1 v = \tilde{g}_2 v = 0$, we find that $(g_1 - z f_2)v = (g_2 - z f_1)v = 0$. By Proposition 3.6, $v = c_0 \sum_{n=0}^{+\infty} \frac{z^n}{n!} f_1^n f_2^n (1) \in \Gamma^{(0)}$, $c_0 \in \mathbb{C} \setminus \{0\}$. It can be easily verified that $v = \tilde{g}_1^{k_1} \tilde{g}_2^{k_2} u$ implies $k_1! k_2! u = \tilde{f}_1^{k_1} \tilde{f}_2^{k_2} v$ and the result follows. □

In the following theorem, the eigenvalues and the eigenvectors of the selfadjoint pairing operator B restricted to the subspace $\Gamma^{(0)}$ are obtained.

Theorem 3.1. *Let the selfadjoint pairing operator $B = c f_1 g_1 + d f_2 g_2 + k f_1 f_2 + \bar{k} g_1 g_2$, with $c, d \in \mathbb{R}$ and $k \in \mathbb{C}$, be restricted to the subspace $\Gamma^{(0)}$, and let $\Delta > 0$. The eigenvalues of B are*

$$\lambda_n = \begin{cases} -\frac{1}{2}(c+d) + \frac{2n+1}{2}\sqrt{\Delta}, & \text{if } c+d > 0 \\ -\frac{1}{2}(c+d) - \frac{2n+1}{2}\sqrt{\Delta}, & \text{if } c+d < 0 \end{cases}, \quad n \in \mathbb{N}_0.$$

The eigenvectors of B associated with the eigenvalue λ_n are the vectors $v_n = c_0 \tilde{f}_1^n \tilde{f}_2^n \mathrm{e}^{z f_1 f_2}(1)$, where c_0 is a non-zero complex number and z is the root of (20) *in D.*

Proof. Consider the Bogoliubov transformation that maps the annihilation operators g_i and the creation operators f_i into their adjoint operators $\tilde{g}_i$ and $\tilde{f}_i$, $i = 1, 2$, respectively. By Proposition 3.4, under this Bogoliubov transformation, B can be taken in the form $B = \lambda_0 \iota + \tilde{c} \tilde{f}_1 \tilde{g}_1 + \tilde{d} \tilde{f}_1 \tilde{g}_1$, where λ_0, $\tilde{c}$ and $\tilde{d}$ are given by (12), (13) and (14), respectively. It can be easily seen that the operators $\widetilde{N}_1 - \widetilde{N}_2$ and $N_1 - N_2$ coincide in Γ^*, and so the operators $\widetilde{N}_1$ and $\widetilde{N}_2$ are equal in $\Gamma^{(0)}$. Therefore, their eigenvalues are the non-negative integers. Since $B - \lambda_0 \iota$ is a linear combination of the commuting operators $\widetilde{N}_1$ and $\widetilde{N}_2$, by Proposition 3.8, the eigenvalues of the selfadjoint pairing operator B are $\lambda_n = \lambda_0 + (\tilde{c} + \tilde{d})n$, $n \in \mathbb{N}_0$. If $c + d > 0$, then $\tilde{c} + \tilde{d}$ and λ_0 are given by Proposition 3.4 (i). Thus, $\lambda_n = -\frac{c+d}{2} + \frac{2n+1}{2}\sqrt{\Delta}$, $n \in \mathbb{N}_0$. If $c + d < 0$, then $\tilde{c} + \tilde{d}$ and λ_0 are given by Proposition 3.4 (ii). Thus,

$\lambda_n = -\frac{c+d}{2} - \frac{2n+1}{2}\sqrt{\Delta}$, $n \in \mathbb{N}_0$. The common eigenvectors of $\widetilde{N}_1$ and $\widetilde{N}_2$ are the eigenvectors of B and, by Proposition 3.8, the theorem follows. □

Theorem 3.1 can be easily generalized as follows.

Theorem 3.2. *Let the selfadjoint pairing operator* $B = cf_1g_1 + df_2g_2 + kf_1f_2 + \bar{k}g_1g_2$, *with* $c, d \in \mathbb{R}$ *and* $k \in \mathbb{C}$, *be defined on* Γ^*, *and let* $\Delta > 0$. *The eigenvalues of* B *are*

$$\lambda_{n_1n_2} = \begin{cases} \frac{1}{2}(c-d)(n_1-n_2) - \frac{1}{2}(c+d) + \frac{n_1+n_2+1}{2}\sqrt{\Delta}, & \text{if } c+d > 0 \\ \frac{1}{2}(c-d)(n_1-n_2) - \frac{1}{2}(c+d) - \frac{n_1+n_2+1}{2}\sqrt{\Delta}, & \text{if } c+d < 0 \end{cases},$$

$n_1, n_2 \in \mathbb{N}_0$. *The eigenvectors of* B *associated with the eigenvalue* $\lambda_{n_1n_2}$ *are* $v_{n_1n_2} = c_0\tilde{f}_1^{n_1}\tilde{f}_2^{n_2}\mathrm{e}^{zf_1f_2}(1)$, *where* c_0 *is a non-zero complex number and* z *is the root of* (20) *in* D.

Proof. The selfadjoint pairing operator B can be taken in the form $B = \lambda_0\iota + \tilde{c}\tilde{f}_1\tilde{g}_1 + \tilde{d}\tilde{f}_1\tilde{g}_1$, where $\tilde{c} = c + \lambda_0$ and $\tilde{d} = d + \lambda_0$, according to (17) in Proposition 3.2. By Proposition 3.8, the eigenvalues of the operator B are $\lambda_{n_1n_2} = \lambda_0 + \tilde{c}n_1 + \tilde{d}n_2$, $n_1, n_2 \in \mathbb{N}_0$. For $n_1, n_2 \in \mathbb{N}_0$ and $c + d > 0$, λ_0 is given by Proposition 3.4 (i), and so

$$\lambda_{n_1n_2} = \frac{1}{2}(c-d)(n_1-n_2) - \frac{1}{2}(c+d) + \frac{n_1+n_2+1}{2}\sqrt{\Delta}.$$

For $n_1, n_2 \in \mathbb{N}_0$ and $c + d < 0$, λ_0 is given by Proposition 3.4 (ii). Thus,

$$\lambda_{n_1n_2} = \frac{1}{2}(c-d)(n_1-n_2) - \frac{1}{2}(c+d) - \frac{n_1+n_2+1}{2}\sqrt{\Delta}.$$

The common eigenvectors of $\widetilde{N}_1$ and $\widetilde{N}_2$ corresponding to the eigenvalues n_1 and n_2 are eigenvectors of B and, by Proposition 3.8, the theorem follows. □

4. The numerical range of pairing operators

The aim of this section is the characterization of the numerical range of the pairing operator B restricted to $\Gamma^{(q)}$, $q \in \mathbb{Z}$. An inclusion relation for $W(B|_{\Gamma^{(q)}})$ is presented in Lemma 4.1. This lemma will be used in the proofs of Theorems 4.2, 4.3 and 4.6.

Lemma 4.1. *Let the pairing operator* $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$, $c, d, k, l \in \mathbb{C}$, *be restricted to* $\Gamma^{(q)}$, $q \in \mathbb{Z}$, *and let*

$$W = \left\{ \frac{(c+d)|z|^2 + k\bar{z} + lz}{1-|z|^2} : z \in D \right\}. \tag{29}$$

Then $(1+|q|)W + \tau_q \subseteq W(B|_{\Gamma^{(q)}})$, *where* $\tau_q = qd$, *if* $q \geqslant 0$, *and* $\tau_q = -qc$, *if* $q < 0$.

Proof. Let $q \geqslant 0$. For an arbitrary element $\psi \in \Gamma^{(q)}$,

$$\psi = \sum_{n=0}^{+\infty} c_n e_1^n * e_2^{n+q}, \; c_n \in \mathbb{C},$$

the following holds:

$$(\psi, \psi) = \sum_{n=0}^{+\infty} |c_n|^2 n!(n+q)!,$$

$$(f_1 f_2 \psi, \psi) = \sum_{n=0}^{+\infty} c_n \bar{c}_{n+1}(n+1)!(n+q+1)!,$$

$$(g_1 g_2 \psi, \psi) = \sum_{n=0}^{+\infty} c_{n+1} \bar{c}_n (n+1)!(n+q+1)!,$$

$$(f_1 g_1 \psi, \psi) = \sum_{n=0}^{+\infty} n|c_n|^2 n!(n+q)!,$$

$$(f_2 g_2 \psi, \psi) = \sum_{n=0}^{+\infty} (n+q)|c_n|^2 n!(n+q)!.$$

If $c_n = z^n/n!$, $z \in D$, the above series converge. We have

$$(\psi, \psi) = \sum_{n=0}^{+\infty} \prod_{j=1}^{q} (n+j)|z|^{2n} = q! \frac{1}{(1-|z|^2)^{1+q}},$$

$$(f_1 f_2 \psi, \psi) = \bar{z} \sum_{n=0}^{+\infty} \prod_{j=1}^{1+q} (n+j)|z|^{2n} = (1+q)! \frac{\bar{z}}{(1-|z|^2)^{2+q}},$$

$$(g_1 g_2 \psi, \psi) = z \sum_{n=0}^{+\infty} \prod_{j=1}^{1+q} (n+j)|z|^{2n} = (1+q)! \frac{z}{(1-|z|^2)^{2+q}},$$

$$(f_1 g_1 \psi, \psi) = \sum_{n=0}^{+\infty} \prod_{j=0}^{q} (n+j)|z|^{2n} = (1+q)! \frac{|z|^2}{(1-|z|^2)^{2+q}},$$

$$(f_2 g_2 \psi, \psi) = \sum_{n=0}^{+\infty} \prod_{j=0}^{q} (n+j)|z|^{2n} + q \sum_{n=0}^{+\infty} \prod_{j=1}^{q} (n+j)|z|^{2n}$$

$$= (1+q)! \frac{|z|^2}{(1-|z|^2)^{2+q}} + q q! \frac{1}{(1-|z|^2)^{1+q}}.$$

Thus, for $q \geqslant 0$, the complex numbers

$$\frac{(B\psi,\psi)}{(\psi,\psi)} = (1+q)\frac{(c+d)|z|^2 + k\bar{z} + lz}{1-|z|^2} + qd, \quad z \in D,$$

belong to $W(B|_{\Gamma^{(q)}})$.

If $q < 0$, the proof is analogous. □

Given a convex subset K of $\mathbb{C}$, a point $\mu \in K$ is called a *corner* of K if K is contained in an angle with vertex at μ, and magnitude less than π.

The following result on the corners of the numerical range of unbounded linear operators will be used in the proof of Theorem 4.2. The proof for bounded operators in [3, Theorem 1.5-5] can be easily adapted to this case.

Theorem 4.1 [3]. *If $\mu \in W(T)$ is a corner of $\overline{W(T)}$, then μ is an eigenvalue of the operator T.*

We now characterize the numerical range of the selfadjoint pairing operator B restricted to $\Gamma^{(0)}$.

Theorem 4.2. *Let the selfadjoint pairing operator $B = cf_1g_1 + df_2g_2 + kf_1f_2 + \bar{k}g_1g_2$, with $c, d \in \mathbb{R}$ and $k \in \mathbb{C}$, be restricted to the subspace $\Gamma^{(0)}$ and $\Delta = (c+d)^2 - 4|k|^2$. Then $W(B|_{\Gamma^{(0)}})$ is:*

(i) $\left[-\frac{1}{2}(c+d) + \frac{1}{2}\sqrt{\Delta}, +\infty\right)$, *if $\Delta > 0$ and $c+d > 0$;*
(ii) $\left(-\infty, -\frac{1}{2}(c+d) - \frac{1}{2}\sqrt{\Delta}\right]$, *if $\Delta > 0$ and $c+d < 0$;*
(iii) $\left(-\frac{1}{2}(c+d), +\infty\right)$, *if $\Delta = 0$ and $c+d > 0$;*
(iv) $\left(-\infty, -\frac{1}{2}(c+d)\right)$, *if $\Delta = 0$ and $c+d < 0$;*
(v) $\{0\}$, *if $\Delta = c+d = 0$;*
(vi) *the whole $\mathbb{R}$, if $\Delta < 0$.*

Proof. Since the pairing operator B is selfadjoint, $c+d \in \mathbb{R}$ and $l = \bar{k}$. Obviously, $W(B|_{\Gamma^{(0)}})$ is a subset of the real line. Since it is a connected set, $W(B|_{\Gamma^{(0)}})$ is an interval. Now, we characterize the extreme points of this interval. If an extremum point of the interval is a corner of $\overline{W(B|_{\Gamma^{(0)}})}$, by Theorem 4.1 it is an eigenvalue of the operator.

(i) If $\Delta > 0$, then $c+d \neq 0$. Let $c+d > 0$. By Theorem 3.1, the minimum eigenvalue of the selfadjoint pairing operator $B|_{\Gamma^{(0)}}$ is $\lambda_0 = -\frac{1}{2}(c+d) + \frac{1}{2}\sqrt{\Delta}$ and there does not exist a maximum eigenvalue. By Theorem 4.1, (i) follows.

(ii) If $\Delta > 0$ and $c+d < 0$, the proof proceeds analogously to (i).

(iii) If $\Delta = 0$ and $c + d > 0$, then $c + d = 2|k|$ and easy computations show that B can be reduced to the form

$$B = \frac{c-d}{2}(f_1 g_1 - f_2 g_2) + \frac{c+d}{2}(f_2 + g_1)^*(f_2 + g_1) - \frac{c+d}{2}\iota.$$

When B is restricted to $\Gamma^{(0)}$, the first summand vanishes. Then $B|_{\Gamma^{(0)}}$ is a positive semidefinite selfadjoint operator translated by $-\frac{1}{2}(c+d)$. We show that the numerical range of $B + \frac{1}{2}(c+d)\iota$ restricted to $\Gamma^{(0)}$ is $(0, +\infty)$, or equivalently, $W(C|_{\Gamma^{(0)}}) = (0, +\infty)$, where $C = (f_2 + g_1)^*(f_2 + g_1)$. Indeed, let $w_N = \sum_{n=1}^{N} \frac{u_n}{n!} f_1^n f_2^n(1) \in \Gamma^{(0)}$. Let $u_0 = u_{N+1} = 0$. We have

$$\frac{(Cw_N, w_N)}{(w_N, w_N)} = \frac{\sum_{n=0}^{N}(n+1)|u_n + u_{n+1}|^2}{\sum_{n=1}^{N}|u_n|^2} \geqslant 0$$

and 0 may be approached as closely as desired. In fact, if $u_n = (-1)^n(N-n)$, $n = 1, \dots, N$,

$$\lim_{N\to\infty} \frac{(Cw_N, w_N)}{(w_N w_N)} = \lim_{N\to\infty} \frac{1 + 2 + \cdots + (N+1)}{1 + 4 + \cdots + (N-1)^2} = 0.$$

Suppose that $0 \in W(C|_{\Gamma^{(0)}})$. Thus, 0 is a corner of $\overline{W(C|_{\Gamma^{(0)}})}$ and, by Theorem 4.1, it is an eigenvalue of C. Then there exists a non-zero vector $u \in \Gamma^{(0)}$ such that $Cu = 0$, and so $(Cu, u) = ((f_2 + g_1)u, (f_2 + g_1)u) = 0$. Therefore, $(f_2 + g_1)u = 0$, which is impossible by Proposition 3.6. Hence, $0 \notin W(C|_{\Gamma^{(0)}})$. Thus, $W(B|_{\Gamma^{(0)}}) = (-\frac{1}{2}(c+d), +\infty)$.

(iv) If $\Delta = 0$ and $c + d < 0$, the proof proceeds analogously to (iii).

(v) If $\Delta = c + d = 0$, then $k = 0$ and $B|_{\Gamma^{(0)}} = 0$. Thus, its numerical range is the singleton $\{0\}$.

(vi) Let $\Delta < 0$. Since B is selfadjoint, by Lemma 4.1 we have

$$W = \left\{ \frac{(c+d)|z|^2 + k\bar{z} + \bar{k}z}{1 - |z|^2} : z \in D \right\} \subseteq W(B|_{\Gamma^{(0)}}) \subseteq \mathbb{R}.$$

Considering $r = (1 + |z|^2)/(1 - |z|^2)$ and $\phi = \arg z - \arg k$, we easily verify that

$$W = \left\{ \frac{c+d}{2}(r-1) + |k|\sqrt{r^2 - 1}\cos\phi : \phi \in \mathbb{R},\ r \geqslant 1 \right\} = \mathbb{R}.$$

Therefore, $W(B|_{\Gamma^{(0)}}) = \mathbb{R}$. □

Remark 4.1. Theorem 4.2 describes the numerical range of the following infinite tridiagonal selfadjoint matrix, which is the matrix representation, in the standard basis, of the selfadjoint pairing operator $B = cf_1 g_1 + df_2 g_2 + kf_1 f_2 + \bar{k} g_1 g_2$ restricted to the subspace $\Gamma^{(0)}$,

$$\begin{bmatrix} 0 & \bar{k} & 0 & 0 & \cdots \\ k & c+d & 2\bar{k} & 0 & \cdots \\ 0 & 2k & 2(c+d) & 3\bar{k} & \cdots \\ 0 & 0 & 3k & 3(c+d) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad c+d \in \mathbb{R},\ k \in \mathbb{C}. \tag{30}$$

For $q \in \mathbb{Z}$, we have the following result.

Theorem 4.3. *Let the selfadjoint pairing operator* $B = cf_1g_1 + df_2g_2 + kf_1f_2 + \bar{k}g_1g_2$, $c, d \in \mathbb{R}$ *and* $k \in \mathbb{C}$, *be restricted to the subspace* $\Gamma^{(q)}$, $q \in \mathbb{Z}$. *Let* $\Delta = (c+d)^2 - 4|k|^2$ *and*

$$\alpha^{\kappa} = \begin{cases} \frac{1+q}{2}(d - c + \kappa\sqrt{\Delta}) - d, & \text{if } q \geqslant 0 \\ \frac{1-q}{2}(c - d + \kappa\sqrt{\Delta}) - c, & \text{if } q < 0 \end{cases}, \quad \kappa \in \{-1, 0, 1\}.$$

Then $W(B|_{\Gamma^{(q)}})$ *is*:

(i) $[\alpha^1, +\infty)$, *if* $\Delta > 0$ *and* $c + d > 0$;
(ii) $(-\infty, \alpha^{-1}]$, *if* $\Delta > 0$ *and* $c + d < 0$;
(iii) $(\alpha^0, +\infty)$, *if* $\Delta = 0$ *and* $c + d > 0$;
(iv) $(-\infty, \alpha^0)$, *if* $\Delta = 0$ *and* $c + d < 0$;
(v) $\{\alpha^0\}$, *if* $\Delta = c + d = 0$;
(vi) *the whole* $\mathbb{R}$, *if* $\Delta < 0$.

Proof. The proof follows similar steps to the proof of Theorem 4.2, using Theorem 3.2 instead of Theorem 3.1. □

Remark 4.2. If $q \geqslant 0$, Theorem 4.3 describes the numerical range of the tridiagonal selfadjoint matrix $S^q_{c,d}$ given by

$$\begin{bmatrix} dq & \bar{k}\sqrt{1+q} & 0 & 0 & \cdots \\ k\sqrt{1+q} & c+d+dq & \bar{k}\sqrt{2(2+q)} & 0 & \cdots \\ 0 & k\sqrt{2(2+q)} & 2(c+d)+dq & \bar{k}\sqrt{3(3+q)} & \cdots \\ 0 & 0 & k\sqrt{3(3+q)} & 3(c+d)+dq & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad c, d \in \mathbb{R},\ k \in \mathbb{C}.$$

If $q < 0$, Theorem 4.3 characterizes $W(S^{-q}_{d,c})$.

The Hyperbolical Range Theorem will be used in the proof of Theorem 4.5 and has the following statement:

Theorem 4.4 (Hyperbolical Range Theorem)[11]. *Let* $A = (a_{ij}) \in M_2$ *and* $J = \mathrm{diag}(1, -1)$. *Let* α_1, α_2 *be the eigenvalues of* JA, *and let*

$$M = |\lambda_1|^2 + |\lambda_2|^2 - \mathrm{Tr}(A^*JAJ), \quad N = \mathrm{Tr}(A^*JAJ) - 2\mathrm{Re}(\bar{\alpha}_1\alpha_2).$$

Denote by l_1 the line perpendicular to the line defined by α_1 and α_2 and passing through $\alpha = \frac{1}{2}\mathrm{Tr}(JA)$. Denote by l_2 the line defined by a_{11} and $-a_{22}$.

(a) *If $M > 0$ and $N > 0$, then $V_J^+(A)$ is bounded by a branch of the hyperbola with α_1 and α_2 as foci, transverse and non-transverse axis of length $\sqrt{N}$ and $\sqrt{M}$, respectively.*
(b) *If $M > 0$ and $N = 0$, then $V_J^+(A)$ is*
 (i) *the line l_1, if $|a_{12}| = |a_{21}|$;*
 (ii) *an open half-plane defined by the line l_1, if $|a_{12}| \neq |a_{21}|$.*
(c) *If $M > 0$ and $N < 0$, then $V_J^+(A)$ is the whole complex plane.*
(d) *If $M = 0$ and $N > 0$, then $V_J^+(A)$ is a closed half-line in l_2 with endpoint α_1 or α_2.*
(e) *If $M = N = 0$, then $V_J^+(A)$ is*
 (i) *the singleton $\{\alpha\}$, if $\mathrm{Tr}(A) = 0$;*
 (ii) *an open half-line in l_2 with endpoint α, if $\mathrm{Tr}(A) \neq 0$.*

Next, we generalize Theorem 4.2 for non-selfadjoint pairing operators. We will denote by $\mathrm{Re}(A)$ the selfadjoint operator $\frac{1}{2}(A + A^*)$.

Theorem 4.5. *Let the pairing operator $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$, $c, d, k, l \in \mathbb{C}$, be restricted to $\Gamma^{(0)}$. Let $\Delta = (c+d)^2 - 4kl$, and let*

$$M = \frac{1}{2}|\Delta| + |k|^2 + |l|^2 - \frac{1}{2}|c+d|^2, \quad N = \frac{1}{2}|\Delta| - |k|^2 - |l|^2 + \frac{1}{2}|c+d|^2.$$

Denote by l_1 the line perpendicular to the line defined by $\alpha_1 = -\frac{1}{2}(c+d) + \frac{1}{2}\sqrt{\Delta}$ and $\alpha_2 = -\frac{1}{2}(c+d) - \frac{1}{2}\sqrt{\Delta}$, and passing through $-\frac{1}{2}(c+d)$. Denote by l_2 the line defined by 0 and $c+d$.

(a) *If $M > 0$ and $N > 0$, then $W(B|_{\Gamma^{(0)}})$ is bounded by a branch of the hyperbola with α_1 and α_2 as foci, transverse and non-transverse axis of length $\sqrt{N}$ and $\sqrt{M}$, respectively.*
(b) *If $M > 0$ and $N = 0$, then $W(B|_{\Gamma^{(0)}})$ is*
 (i) *the line l_1, if $|k| = |l|$;*
 (ii) *an open half-plane defined by the line l_1, if $|k| \neq |l|$.*
(c) *If $M > 0$ and $N < 0$, then $W(B|_{\Gamma^{(0)}})$ is the whole complex plane.*
(d) *If $M = 0$ and $N > 0$, then $W(B|_{\Gamma^{(0)}})$ is a closed half-line in l_2 with endpoint α_1 or α_2.*
(e) *If $M = N = 0$, then $W(B|_{\Gamma^{(0)}})$ is*
 (i) *the singleton $\{0\}$, if $c + d = 0$;*
 (ii) *an open half-line in l_2 with endpoint $-\frac{1}{2}(c+d)$, if $c + d \neq 0$.*

Proof. By Lemma 4.1, W is a subset of $W(B|_{\Gamma(0)})$. Let $J = \mathrm{diag}(1, -1)$ and

$$A = \begin{bmatrix} 0 & l \\ k & c+d \end{bmatrix}.$$

It can be easily verified that

$$W = \left\{ \frac{1}{1-|z|^2}(1\ \bar{z})A(1\ z)^{\mathrm{T}} :\ z \in D \right\} = V_J^+(A),$$

and so the subset W is described by the Hyperbolical Range Theorem. Let $\Delta = (c+d)^2 - 4kl$ and $P = 2|k|^2 + 2|l|^2 - |c+d|^2$. The eigenvalues α_1 and α_2 of the matrix JA are $-\frac{1}{2}(c+d) \pm \frac{1}{2}\sqrt{\Delta}$, and we have

$$M = |\alpha_1|^2 + |\alpha_2|^2 - \mathrm{Tr}(A^* JAJ) = \frac{1}{2}(|\Delta| + P),$$

$$N = \mathrm{Tr}(A^* JAJ) - 2\mathrm{Re}(\bar{\alpha}_1\alpha_2) = \frac{1}{2}(|\Delta| - P).$$

It can be easily seen that $M \geqslant 0$ and

$$|\Delta|^2 = |c+d|^4 + 16|k|^2|l|^2 - 8|k||l||c+d|^2\cos(2\alpha - 2\beta), \tag{31}$$

where $2\alpha = \arg(kl)$ and $\beta = \arg(c+d)$. By the Hyperbolical Range Theorem, the subset W of $W(B|_{\Gamma(0)})$ is bounded by a branch of a possibly degenerate hyperbola. The following cases may occur:

Case 1. $M > 0$ and $N > 0$. We prove the *claim* that $W(B|_{\Gamma(0)}) = W$. The unit eigenvectors associated with an extremum eigenvalue of $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}B)$, $\theta \in [0, 2\pi)$, give rise to boundary points of the numerical range of B. The real part of $\mathrm{e}^{\mathrm{i}\theta}B$ is $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}B) = c_\theta f_1g_1 + d_\theta f_2g_2 + k_\theta f_1f_2 + \bar{k}_\theta g_1g_2$, where $c_\theta = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}c)$, $d_\theta = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}d)$ and $2k_\theta = (k+\bar{l})\cos\theta + \mathrm{i}(k-\bar{l})\sin\theta$. Moreover, $c_\theta + d_\theta = |c+d|\cos(\beta+\theta)$. Let $\Delta_\theta = (c_\theta + d_\theta)^2 - 4|k_\theta|^2$. After some computations, we get $\Delta_\theta = \frac{1}{2}|\Delta|\cos(2\theta+\psi) - \frac{1}{2}P$, where $\tan\psi = \mathrm{Im}\Delta/\mathrm{Re}\Delta$. It follows that $-M \leqslant \Delta_\theta \leqslant N$, for all $\theta \in [0, 2\pi)$. Let θ be such that $\Delta_\theta > 0$. If $c_\theta + d_\theta > 0$, by Theorem 3.1, the minimum eigenvalue of the selfadjoint pairing operator $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}B)$ is $\lambda_0^\theta = -\frac{1}{2}(c_\theta + d_\theta) + \frac{1}{2}\sqrt{\Delta_\theta}$. The eigenvectors associated with λ_0^θ are $v_0^\theta = c_0\mathrm{e}^{z_\theta f_1f_2}(1)$, where c_0 is a non-zero complex number, $z_\theta = 0$, if $k_\theta = 0$, and $z_\theta = \lambda_0^\theta/\bar{k}_\theta$, if $k_\theta \neq 0$. Then $z_\theta \in D$ and, as in the proof of Lemma 4.1 (i), for $q = 0$, we have

$$\frac{(Bv_0^\theta, v_0^\theta)}{(v_0^\theta, v_0^\theta)} = \frac{(c+d)|z_\theta|^2 + k\bar{z}_\theta + lz_\theta}{1-|z_\theta|^2}.$$

This point belongs to the boundary of $W(B|_{\Gamma(0)})$ and also belongs to W. As θ varies in $[0, 2\pi)$, all the boundary points of $W(B|_{\Gamma(0)})$ belong to W. If $c_\theta + d_\theta < 0$, the discussion follows along similar lines. Thus, $W(B|_{\Gamma(0)}) = W$ is bounded by a branch of the hyperbola with foci α_1 and α_2, transverse axis of length $\sqrt{N}$ and non-transverse axis of length $\sqrt{M}$.

Case 2. $M > 0$ and $N = 0$. Since $N = 0$, we have $M = |\Delta| = P$. Therefore, $\Delta_\theta = \frac{1}{2}M(\cos(2\theta + \psi) - 1)$ and it can be easily seen that there exists $\theta' = -\psi/2 \in [0, 2\pi)$ such that the real sinusoidal function $f(\theta) := \Delta_\theta$ satisfies $f(\theta) < 0$, for $\theta \neq \theta'$ and $f(\theta') = 0$. In this case, there is a unique supporting line of W, specifically the line l_1 passing through $-(c + d)/2$ and perpendicular to the line defined by α_1 and α_2. If $|k| \neq |l|$, then W is an open half-plane defined by the line l_1. By Theorem 4.2 iii) or iv), the boundary of the half-plane does not belong to $W(B|_{\Gamma^{(0)}})$ and so $W(B|_{\Gamma^{(0)}})$ coincides with W. If $|k| = |l| \neq 0$, then W is the line l_1. In this case, Δ_θ and $c_\theta + d_\theta$ vanish only in the direction $\theta = (\pi/2 - \beta) \bmod \pi$. By Theorem 4.2 v), it follows that $W(B|_{\Gamma^{(0)}})$ coincides with W. If $k = l = 0$, then $M = 0$, contradicting the hypothesis.

Case 3. $M > 0$ and $N < 0$. Since $N < 0$, there does not exist any supporting line for the set W, which is the whole complex plane. Hence, $W(B|_{\Gamma^{(0)}}) = \mathbb{C}$.

Case 4. $M = 0$ and $N > 0$. Since $M = 0$, we have $N = |\Delta| = -P > 0$. In this case, there are infinite supporting lines of the set W and the branch of the hyperbola given by the Hyperbolical Range Theorem degenerates into a closed half-line in the line defined by 0 and $c + d$, with endpoint either α_1 or α_2. For $\theta \in [0, 2\pi)$, $\Delta_\theta = \frac{1}{2}N(\cos(2\theta + \psi) + 1) \geqslant 0$. Using analogous arguments to those in the proof of the Case 2, we conclude that $W(B|_{\Gamma^{(0)}}) = W$.

Case 5. $M = 0$ and $N = 0$. It can be easily seen that $N = \Delta = 0$ and straightforward computations yield $|k| = |l| = \frac{1}{2}|c + d|$. If $k = 0$, having in mind Theorem 4.2 (v), we conclude that $W(B) = \{0\}$. If $k \neq 0$, W is an open half-line in the line defined by 0 and $c + d$ and with endpoint $-\frac{1}{2}(c + d)$. In this case, $\Delta_\theta = 0$ for $\theta \in [0, 2\pi)$, and $c_\theta + d_\theta$ vanishes only in the direction $\theta = (\frac{\pi}{2} - \alpha) \bmod \pi$. By similar arguments to those used above, it can be shown that $W(B|_{\Gamma^{(0)}}) = W$.

Case 6. $M = 0$ and $N < 0$. Under these hypothesis, it can easily be seen that $0 = -M \leqslant \Delta_\theta \leqslant N < 0$, which is impossible. □

Using Theorem 3.2, Lemma 4.1 and the ideas in the proof of Theorem 4.5, we may characterize the numerical range of the pairing operator B, restricted to the subspace $\Gamma^{(q)}$, $q \in \mathbb{Z}$. We shall prove that these sets are homothetic, that is, they are bounded by (possibly degenerate) homothetic hyperbolas.

Theorem 4.6. *Let the pairing operator* $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$, $c, d, k, l \in \mathbb{C}$, *be restricted to* $\Gamma^{(q)}$, $q \in \mathbb{Z}$. *Let* $\Delta = (c + d)^2 - 4kl$ *and let*

$$M = \frac{1}{2}|\Delta| + |k|^2 + |l|^2 - \frac{1}{2}|c + d|^2, \quad N = \frac{1}{2}|\Delta| - |k|^2 - |l|^2 + \frac{1}{2}|c + d|^2.$$

Let $\kappa \in \{-1, 0, 1\}$ and $\varepsilon\varepsilon' \in \{11, 02, 20\}$. Denote by l_1 the line passing through α_{11}^0 and perpendicular to the line defined by α_{11}^1 and α_{11}^{-1}, and denote by l_2 the line defined by α_{20}^0 and α_{02}^0, where

$$\alpha_{\varepsilon\varepsilon'}^{\kappa} = \begin{cases} \frac{1+q}{2}(\varepsilon d - \varepsilon' c + \kappa\sqrt{\Delta}) - d, & \text{if } q \geqslant 0, \\ \frac{1-q}{2}(\varepsilon c - \varepsilon' d + \kappa\sqrt{\Delta}) - c, & \text{if } q < 0. \end{cases}$$

(a) *If $M > 0$ and $N > 0$, then $W(B|_{\Gamma^{(q)}})$ is bounded by a branch of the hyperbola with α_{11}^1 and α_{11}^{-1} as foci, transverse and non-transverse axis of length $(1 + |q|)\sqrt{N}$ and $(1 + |q|)\sqrt{M}$, respectively.*

(b) *If $M > 0$ and $N = 0$, then $W(B|_{\Gamma^{(q)}})$ is*
  (i) *the line l_1, if $|k| = |l|$;*
  (ii) *an open half-plane defined by the line l_1, if $|k| \neq |l|$.*

(c) *If $M > 0$ and $N < 0$, then $W(B|_{\Gamma^{(q)}})$ is the whole complex plane.*

(d) *If $M = 0$ and $N > 0$, then $W(B|_{\Gamma^{(q)}})$ is a closed half-line in l_2 with endpoint α_{11}^1 or α_{11}^{-1}.*

(e) *If $M = N = 0$, then $W(B|_{\Gamma^{(q)}})$ is*
  (i) *the singleton $\{\alpha_{11}^0\}$, if $c + d = 0$;*
  (ii) *an open half-line in l_2 with endpoint α_{11}^0, if $c + d \neq 0$.*

Proof. We prove that

$$W(B|_{\Gamma^{(q)}}) = (1 + |q|)W(B|_{\Gamma^{(0)}}) + \tau_q, \quad q \in \mathbb{Z}, \tag{32}$$

where $\tau_q = qd$, if $q \geqslant 0$, and $\tau_q = -qc$, if $q < 0$. By Lemma 4.1, $W(B|_{\Gamma^{(q)}})$ contains $(1 + |q|)W + \tau_q$, and by Theorem 4.5, we have that $W = W(B|_{\Gamma^{(0)}})$. Thus, $(1 + |q|)W(B|_{\Gamma^{(0)}}) + \tau_q \subseteq W(B|_{\Gamma^{(q)}})$, $q \in \mathbb{Z}$. Let $q \geqslant 0$. As in the proof of Theorem 4.4, we consider $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}B) = c_\theta f_1 g_1 + d_\theta f_2 g_2 + k_\theta f_1 f_2 + \bar{k}_\theta g_1 g_2$, with $c_\theta = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}c)$, $d_\theta = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}d)$ and $2k_\theta = (k + \bar{l})\cos\theta + \mathrm{i}(k - \bar{l})\sin\theta$.

(a) Let $\theta \in [0, 2\pi)$ be such that $\Delta_\theta = (c_\theta + d_\theta)^2 - 4|k_\theta|^2 > 0$. If $c_\theta + d_\theta > 0$, by Theorem 3.2, the minimum eigenvalue of the selfadjoint pairing operator $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}B)$ restricted to $\Gamma^{(q)}$, $q \geqslant 0$, is

$$\lambda_{0q}^\theta = \frac{q}{2}(d_\theta - c_\theta) - \frac{1}{2}(c_\theta + d_\theta) + \frac{1+q}{2}\sqrt{\Delta_\theta} = (1+q)\lambda_{00}^\theta + qd_\theta,$$

and the eigenvectors of $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}B)$ associated with the eigenvalue λ_{0q}^θ are the vectors $v_{0q}^\theta = c_0\tilde{f}_2^q \mathrm{e}^{z_\theta f_1 f_2}(1)$, where c_0 is a non-zero complex number, $z_\theta = 0$, if $k_\theta = 0$, $z_\theta = \lambda_{00}^\theta/\bar{k}_\theta$, if $k_\theta \neq 0$, and $\tilde{f}_2 = \frac{1}{\sqrt{1-|z_\theta|^2}}(f_2 - \bar{z}_\theta g_1)$. Using analogous arguments to those in the proof of Lemma 4.1, we find

$$w_q^\theta = \frac{(Bv_{0q}^\theta, v_{0q}^\theta)}{(v_{0q}^\theta, v_{0q}^\theta)} = (1+q)\frac{(c+d)|z_\theta|^2 + k\bar{z}_\theta + lz_\theta}{1 - |z_\theta|^2} + qd, \tag{33}$$

which is a boundary point of $W(B|_{\Gamma^{(q)}})$, $q \geqslant 0$. If $c_\theta + d_\theta < 0$, the reasoning is similar. From (33), we get the following relation between the boundary points w_q^θ of $W(B|_{\Gamma^{(q)}})$, $q > 0$, and the boundary points w_0^θ of $W(B|_{\Gamma^{(0)}})$: $w_q^\theta = (1+q)w_0^\theta + qd$. This means that the boundary generating curve of $W(B|_{\Gamma^{(q)}})$, $q > 0$, is obtained from the boundary generating curve of $W(B|_{\Gamma^{(0)}})$ by a dilation of ratio $1+q$ and a translation associated with qd. Hence, the equality in (32) holds for $q \geqslant 0$. That is, $W(B|_{\Gamma^{(q)}})$, $q \geqslant 0$, is bounded by a branch of the hyperbola with α_{11}^1 and α_{11}^{-1} as foci, and transverse and non-transverse axis of length $(1+q)\sqrt{N}$ and $(1+q)\sqrt{M}$, respectively.

(b) If $|k| \neq |l|$, then $(1+q)W + qd$ is an open half-plane defined by the line l_1. By similar arguments to those in the proof of Theorem 4.3 iii), it can be shown that the boundary of this half-plane does not belong to $W(B|_{\Gamma^{(q)}})$ and so $W(B|_{\Gamma^{(q)}})$ coincides with $(1+q)W + qd$, for $q \geqslant 0$. If $|k| = |l| \neq 0$, then $(1+q)W + dq$ is the line l_1. In this case, $\Delta_\theta = (c_\theta + d_\theta)^2 - 4|k_\theta|^2$ and $c_\theta + d_\theta$ vanish only in one direction, and so the equality in (32), $q \geqslant 0$, follows.

(c) Since $W = \mathbb{C}$, it is clear that $W(B|_{\Gamma^{(q)}}) = \mathbb{C}$.

(d) In this case, the set $(1+q)W + qd$ degenerates into a closed half-line in l_2 with endpoint α_{11}^1 or α_{11}^{-1}. Since $\Delta_\theta \geqslant 0$ for $\theta \in [0, 2\pi)$, by analogous arguments to those used above, the equality in (32), $q \geqslant 0$, is proved to hold.

(e) As in the proof of Theorem 4.5, we have $|k| = |l| = \frac{1}{2}|c+d|$. If $k = 0$, we conclude that $W(B|_{\Gamma^{(q)}}) = \{qd\}$. If $k \neq 0$, $(1+q)W + qd$ is an open half-line in l_2 with endpoint α_{11}^0 and we may conclude that $W(B|_{\Gamma^{(q)}}) = (1+q)W + qd$.

If $q < 0$, the proof is similar. □

Remark 4.3. The pairing operator $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$ restricted to $\Gamma^{(q)}$ is represented by the tridiagonal matrix $T_{c,d}^q$ in Remark 3.1. Thus, $W(T_{c,d}^q)$, $q \geqslant 0$, is characterized by Theorem 4.6. For $q < 0$, the pairing operator $B = cf_1g_1 + df_2g_2 + kf_1f_2 + lg_1g_2$ restricted to $\Gamma^{(q)}$ is represented by the tridiagonal matrix $T_{d,c}^{-q}$, and so $W(T_{c,d}^q)$ is given by the same theorem, replacing q, c and d by $-q$, d and c, respectively.

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