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Continuity envelopes of spaces of generalised smoothness, entropy and approximation numbers ☆

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Abstract

We study continuity envelopes in spaces of generalised smoothness $B_{pq}^{(s,\Psi)}$ and $F_{pq}^{(s,\Psi)}$ and give some new characterisations for spaces $B_{pq}^{(s,\Psi)}$. The results are applied to obtain sharp asymptotic estimates for approximation numbers of compact embeddings of type $id : B_{pq}^{(s_1,\Psi)}(U) \to B_{\infty\,\infty}^{s_2}(U)$, where $\frac{n}{p} < s_1 - s_2 < \frac{n}{p} + 1$ and U stands for the unit ball in $\mathbb{R}^n$. In case of entropy numbers we can prove two-sided estimates.

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0. Introduction

Spaces of generalised smoothness have already been studied for long from different points of view, coming from the interpolation side (with a function parameter), see the results by Merucci [42] and Cobos and Fernandez [13], whereas the rather abstract approach (approximation by series of entire analytic functions and coverings) was independently developed by Gol'dman and Kalyabin, see [24–28,32,33]. Furthermore, the survey by Kalyabin and Lizorkin [36] and the appendix [41] cover the extensive (Russian) literature at that time. More recently, we mention the contributions of Gol'dman [27,28], Netrusov [45] and Burenkov [5]. The notion was revived and extended in the way we shall use it in connection with *limiting embeddings* and *spaces on fractals* by Edmunds and Triebel [21,22], Leopold [38,39] and Moura [43,44]. Closely linked, but slightly different is the approach to more general Lipschitz spaces as developed by Edmunds and Haroske [17,18,29]. The present state of the art is reviewed and covered in [23] by Farkas and Leopold linking function spaces of generalised smoothness with negative definite functions—and thus referring to applications for pseudo-differential operators (as generators of sub-Markovian semi-groups). Plainly, these latter applications and also the topic in its full generality are out of the scope of the present paper; it explains, however, the increased interest on function spaces of generalised smoothness quite recently. As a prototype one can think of spaces of Besov type $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$, where the function Ψ might behave like $\Psi(x) = (1 + |\log x|)^b$, $x \in (0,1]$, $b \in \mathbb{R}$; for example, we have for $1 < p \leqslant \infty$, $0 < q \leqslant \infty$, $0 < s < 1$, an easy characterisation by differences,

$$\|f|B^{(s,\Psi)}_{pq}\| \sim \|f|L_p\| + \left(\int_0^1 \left[t^{-s}\Psi(t) \sup_{|h|\leqslant t} \|f(\cdot+h) - f(\cdot)|L_p\|\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$

(with the usual modification if $q = \infty$).

In contrast to the above-described long history and variety of contributions devoted to spaces of generalised smoothness, continuity envelopes represent a very new tool for the characterisation of function spaces, developed only recently in [30,31,53]. Nevertheless, it promises by now already not only surprisingly sharp results based on classical concepts, but also a lot of applications, e.g. to the study of compact embeddings. We return to this point later. Roughly speaking, a continuity envelope $\mathfrak{E}_C(X)$ of a function space X consists of a so-called continuity envelope function

$$\mathscr{E}^X_C(t) \sim \sup_{\|f|X\|\leqslant 1} \frac{\omega(f,t)}{t}, \quad t > 0,$$

together with some fine index u_X; here $\omega(f,t)$ stands for the modulus of continuity, as usual. Forerunners of *continuity envelopes* in a wider sense are well-known for decades; among the big amount of work devoted to the study of *limiting or sharp embeddings* involving spaces that contain (at least) continuous functions we only want to mention a few: dealing with spaces of type B^s_{pq}, F^s_{pq} we refer to the result of Sickel and Triebel [49, Theorem 3.3.1] (also for further historical comments), the paper [34] by Kalyabin concerns the question of embeddings into C in the special context of spaces of generalised smoothness mentioned above, whereas the famous result [2] of Brézis and Wainger can be regarded as some origin of the idea of continuity envelopes at all. It states that some function $u \in H_p^{1+n/p}(\mathbb{R}^n)$, $1 < p < \infty$, is 'almost' Lipschitz-continuous in the sense that for all $x, y \in \mathbb{R}^n$, $0 < |x-y| < \frac{1}{2}$,

$$|u(x) - u(y)| \leqslant c|x-y||\log|x-y||^{1/p'} ||u|H_p^{1+n/p}(\mathbb{R}^n)||. \tag{1}$$

Here c is a constant independent of x, y and u, and $\frac{1}{p'} + \frac{1}{p} = 1$. We studied the sharpness of this assertion and parallel questions for more general spaces in [17,18]. These considerations led us finally to the introduction of continuity envelopes: obviously (1) results after some reformulation in

$$\mathscr{E}_C^{H_p^{1+n/p}}(t) \leqslant c|\log t|^{1/p'}, \quad 0 < t < \tfrac{1}{2}.$$

Turning to spaces defined on bounded domains, say, the unit ball $U \subset \mathbb{R}^n$ for simplicity, it is reasonable to consider compact embedding operators, $id : B_{pq}^{(s,\Psi)}(U) \to C(U)$, where C stands for the space of complex-valued bounded uniformly continuous functions. More precisely, we shall further inquire into the nature of this compactness and characterise the asymptotic behaviour of the corresponding approximation numbers; we prove

$$a_k(id : B_{pq}^{(s,\Psi)}(U) \to C(U)) \sim k^{-\frac{s}{n}+\frac{1}{p}} \Psi(k^{-\frac{1}{n}})^{-1}, \quad k \in \mathbb{N},$$

assuming that $2 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$ with $\frac{n}{p} < s < \frac{n}{p} + 1$, and Ψ as above. Studying entropy numbers instead of approximation numbers in the same context, we obtain two-sided estimates of the same type.

Let us finally mention that parallel studies, when questions of (Lipschitz-) continuity are replaced by inquiries about the unboundedness of functions, led to the concept of *growth envelopes* in [30,31,53] and were continued by Bricchi, Caetano, Haroske and Moura in different settings, cf. [4,7–9].

The paper is organised as follows. We collect the necessary background material in Section 1; in Section 2 we obtain different equivalent characterisations for spaces $B_{pq}^{(s,\Psi)}$. This is not only needed afterwards, but also of some interest of its own. Our main result on continuity envelopes in spaces of generalised smoothness can be found in Section 3. Section 4 contains entropy and approximation number estimates representing both an application of our envelope assertions, and the starting point for further possible applications in spectral theory; however, this is out of the scope of the present paper. We shall only give a brief account on the consequences we have in mind.

1. Preliminaries

1.1. General notation

As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 a_k \leqslant b_k \leqslant c_2 a_k \quad \text{or} \quad c_1 \varphi(x) \leqslant \psi(x) \leqslant c_2 \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and φ, ψ are non-negative functions. If $a \in \mathbb{R}$ then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of a.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with additional subscripts within the same formula. If not otherwise indicated, log is always taken with respect to base 2.

Apart from the last section we shall always deal with function spaces on $\mathbb{R}^n$; so for convenience we shall usually omit the "$\mathbb{R}^n$" from their notation.

1.2. Function spaces of generalised smoothness

Recall our introductory remarks on spaces of generalised smoothness, relating this topic with some historical background as well as the present state of the art. In our context, we shall be concerned with function spaces of generalised smoothness of Besov and Triebel–Lizorkin type, where the usual main smoothness parameter s is replaced by a couple (s, Ψ) and Ψ is a slowly varying function (in Karamata's sense).

Definition 1.1. A positive and measurable function Ψ defined on the interval $(0, 1]$ is said to be *slowly varying* if

$$\lim_{t \to 0} \frac{\Psi(st)}{\Psi(t)} = 1, \quad s \in (0, 1]. \tag{2}$$

Example 1.2. Any function of the form

$$\Psi(t) = \exp\left\{ -\int_t^1 \varepsilon(s) \frac{\mathrm{d}s}{s} \right\}, \quad t \in (0, 1],$$

where ε is a measurable function with $\lim_{s \to 0} \varepsilon(s) = 0$, is slowly varying (actually this is a characterisation: any slowly varying function is equivalent to a function Ψ of the above type for an appropriate function ε); in particular,

$$\Psi_b(x) = (1 + |\log x|)^b, \quad x \in (0, 1], \quad b \in \mathbb{R}, \tag{3}$$

is slowly varying; we return to this particular choice in the sequel for illustration. We remark that Ψ_b is also an example of an admissible function in the sense of [21,22]. We recall that an admissible function Ψ is a positive monotone function defined on $(0,1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$, $j \in \mathbb{N}$. An admissible function is, up to equivalence, a slowly varying function (cf. Proposition 1.9.7 of [3]).

The proposition below gives some properties of slowly varying functions that will be useful in what follows. We refer to the monograph [1] for details and further properties; see also [19,54, Chapter V], and, quite recently, [46,47].

Proposition 1.3. *Let Ψ be a slowly varying function.*

(i) *For any $\delta > 0$ there exists $c = c(\delta) > 1$ such that*

$$\frac{1}{c} s^{\delta} \leqslant \frac{\Psi(st)}{\Psi(t)} \leqslant c s^{-\delta}, \quad t, s \in (0,1].$$

(ii) *For each $\alpha > 0$ there is a decreasing function ϕ and an increasing function φ with*

$$t^{-\alpha} \Psi(t) \sim \phi(t) \quad \text{and} \quad t^{\alpha} \Psi(t) \sim \varphi(t).$$

(iii) *Let $\delta \in \mathbb{R}$ and $g(t) = t^{\delta} \Psi(t)$, $t \in (0,1]$. There exists a positive C^{∞} function h such that $h \sim g$ and*

$$\lim_{t \to 0} t^k \frac{h^{(k)}(t)}{h(t)} = \delta(\delta - 1) \cdots (\delta - k + 1), \quad k \in \mathbb{N}.$$

Before introducing the function spaces under consideration we need to recall some notation. By $\mathscr{S}$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathscr{S}'$ the dual space of all tempered distributions on $\mathbb{R}^n$. Furthermore, L_p with $0 < p \leqslant \infty$, stands for the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f|L_p\| := \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x \right)^{1/p},$$

with the obvious modification if $p = \infty$. If $\varphi \in \mathscr{S}$ then

$$\widehat{\varphi}(\xi) \equiv (\mathscr{F}\varphi)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x)\, \mathrm{d}x, \quad x \in \mathbb{R}^n, \tag{4}$$

denotes the Fourier transform of φ. As usual, $\mathscr{F}^{-1}\varphi$ or $\varphi^{\vee}$, stands for the inverse Fourier transform, given by the right-hand side of (4) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are extended to $\mathscr{S}'$ in the standard way. Let $\varphi_0 \in \mathscr{S}$ be such that

$$\varphi_0(x) = 1 \quad \text{if } |x| \leqslant 1 \quad \text{and} \quad \operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \leqslant 2\}, \tag{5}$$

and for each $j \in \mathbb{N}$ let

$$\varphi_j(x) := \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \quad x \in \mathbb{R}^n. \tag{6}$$

Then the sequence $(\varphi_j)_{j=0}^{\infty}$ forms a dyadic resolution of unity.

Definition 1.4. Let $0 < p, q \leqslant \infty$, $s \in \mathbb{R}$ and Ψ be a slowly varying function.

(i) Then $B_{pq}^{(s,\Psi)}$ is the collection of all $f \in \mathscr{S}'$ such that

$$\|f|B_{pq}^{(s,\Psi)}\| := \left(\sum_{j=0}^{\infty} 2^{jsq}\Psi(2^{-j})^q \|(\varphi_j\widehat{f})^\vee|L_p\|^q\right)^{1/q}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty$. Then $F_{pq}^{(s,\Psi)}$ is the collection of all $f \in \mathscr{S}'$ such that

$$\|f|F_{pq}^{(s,\Psi)}\| := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq}\, \Psi(2^{-j})^q\, |(\varphi_j\widehat{f})^\vee(\cdot)|^q\right)^{1/q} \Bigg| L_p \right\|$$

(with the usual modification if $q = \infty$) is finite.

Remark 1.5. The above spaces were introduced by Edmunds and Triebel in [21,22] and also considered by Moura [43,44] when Ψ is an admissible function. For further basic properties, like the independence of the spaces from the chosen dyadic resolution of unity (in the sense of equivalent norms) we refer to [23] in a more general setting. As already mentioned in the introduction, the extensive Russian literature can be found in the survey by Kalyabin and Lizorkin [36] and the appendix [41]. If $\Psi \equiv 1$ then the spaces $B_{pq}^{(s,\Psi)}$ and $F_{pq}^{(s,\Psi)}$ coincide with the usual Besov and Triebel–Lizorkin spaces, B_{pq}^s and F_{pq}^s, respectively, and the following elementary embeddings hold:

$$A_{pq}^{s+\varepsilon} \hookrightarrow A_{pq}^{(s,\Psi)} \hookrightarrow A_{pq}^{s-\varepsilon}, \tag{7}$$

for all $\varepsilon > 0$ and $A \in \{B, F\}$, in view of Proposition 1.3(i); see also [4, Proposition 4.6]. For convenience, we shall continue writing A_{pq}^s or $A_{pq}^{(s,\Psi)}$, respectively, when both B- and F-spaces are concerned and no distinction is needed.

Example 1.6. With the particular choice of Ψ_b given by (3) we obtain spaces $B_{pq}^{s,b}$ consisting of those $f \in \mathscr{S}'$ for which

$$\|f|B_{pq}^{s,b}\| = \left(\sum_{j=0}^{\infty} 2^{jsq}(1+j)^{bq}\|(\varphi_j\widehat{f})^\vee|L_p\|^q\right)^{1/q}$$

is finite (usual modification for $q = \infty$); similarly for $F_{pq}^{s,b}$. These spaces were studied by Leopold [38,39].

For later use we also recall a special lift property for spaces $B_{pq}^{(s,\Psi)}$, obtained in [8] in case of Ψ being an admissible function and in [23] for a more general situation. Let Ψ be a slowly varying function and $(\varphi_j)_{j\in\mathbb{N}_0}$ a smooth dyadic partition of unity according to (5), (6). Denote by

$$\widetilde{\Psi}(\xi) = \sum_{j=0}^{\infty} \Psi(2^{-j})\varphi_j(\xi), \quad \xi\in\mathbb{R}^n,$$

and

$$J^{\widetilde{\Psi}}f := (\widetilde{\Psi}(\cdot)\widehat{f})^{\vee}, \quad f\in\mathcal{S}'.$$

Proposition 1.7. *Let* $0<p,q\leqslant\infty$ (*with* $p<\infty$ *in* F-*case*), $s\in\mathbb{R}$, *and* Ψ *be a slowly varying function. Then* $J^{\widetilde{\Psi}}$ *is a topological isomorphism from* $A_{pq}^{(s,\Psi)}$ *onto* A_{pq}^{s}.

A proof is given in [8, Proposition 3.2] for Ψ an admissible function and in [23, Theorem 3.1.8] for a more general situation. The essential advantage of this result is that it enables us to gain from the wider knowledge concerning embeddings and spaces of type $A_{p,q}^{s}$.

1.3. Continuity envelopes

The concept of continuity envelopes has been introduced by Haroske [30] and Triebel [53]. Here we quote the basic definitions and results concerning continuity envelopes. However, we shall be rather concise and we mainly refer to [30,31,53] for heuristics, motivations and details on this subject.

Let C be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. Recall that the classical Lipschitz space Lip^1 is defined as the space of all functions $f\in C$ such that

$$\|f|\mathrm{Lip}^1\| = \|f|C\| + \sup_{t\in(0,1)} \frac{\omega(f,t)}{t} \tag{8}$$

is finite, where $\omega(f,t)$ stands for the modulus of continuity,

$$\omega(f,t) = \sup_{|h|\leqslant t}\ \sup_{x\in\mathbb{R}^n} |f(x+h)-f(x)|, \quad t>0.$$

Definition 1.8. Let $X\hookrightarrow C$ be some function space on $\mathbb{R}^n$.

(i) The *continuity envelope function* $\mathscr{E}_{\mathsf{C}}^{X}:(0,\infty)\to[0,\infty)$ is defined by

$$\mathscr{E}_{\mathsf{C}}^{X}(t) := \sup_{\|f|X\|\leqslant 1} \frac{\omega(f,t)}{t}, \quad t>0.$$

(ii) Assume $X \not\hookrightarrow \mathrm{Lip}^1$. Let $\varepsilon \in (0,1)$, $H(t) := -\log \mathscr{E}_{\mathsf{C}}^X(t)$, $t \in (0,\varepsilon]$, and let μ_H be the associated Borel measure. The number u_X, $0 < u_X \leqslant \infty$, is defined as the infimum of all numbers v, $0 < v \leqslant \infty$, such that

$$\left(\int_0^\varepsilon \left(\frac{\omega(f,t)}{t\mathscr{E}_{\mathsf{C}}^X(t)} \right)^v \mu_H(\mathrm{d}t) \right)^{1/v} \leqslant c \| f|X \| \tag{9}$$

(with the usual modification if $v = \infty$) holds for some $c > 0$ and all $f \in X$. The couple

$$\mathfrak{E}_{\mathsf{C}}(X) = (\mathscr{E}_{\mathsf{C}}^X(\cdot), u_X)$$

is called *continuity envelope* for the function space X.

As it will be useful in the sequel, we recall some properties of the continuity envelopes. In view of (i) we obtain—strictly speaking—equivalence classes of continuity envelope functions when working with equivalent (quasi-) norms in X as we shall do in the sequel. However, for convenience we do not want to distinguish between representative and equivalence class in what follows and thus stick at the notation introduced in (i). Note that $\mathscr{E}_{\mathsf{C}}^X$ is equivalent to some monotonically decreasing function; for a proof and further properties we refer to [30,31]. Concerning (ii) it is obvious that (9) holds for $v = \infty$ and any X. Moreover, one verifies that

$$\begin{aligned} \sup_{0<t\leqslant\varepsilon} \frac{g(t)}{\mathscr{E}_{\mathsf{C}}^X(t)} &\leqslant c_1 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathscr{E}_{\mathsf{C}}^X(t)} \right)^{v_1} \mu_H(\mathrm{d}t) \right)^{1/v_1} \\ &\leqslant c_2 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathscr{E}_{\mathsf{C}}^X(t)} \right)^{v_0} \mu_H(\mathrm{d}t) \right)^{1/v_0} \end{aligned} \tag{10}$$

for $0 < v_0 < v_1 < \infty$ and all non-negative monotonically decreasing functions g on $(0,\varepsilon]$; cf. [53, Proposition 12.2, pp. 183–184]. So—passing to a monotonically decreasing function equivalent to $\frac{\omega(f,t)}{t}$, see [15, Chapter 2, Lemma 6.1, p. 43]—we observe that the left-hand sides in (9) are monotonically ordered in v and it is natural to look for the smallest possible one.

Proposition 1.9. (i) *Let $X_i \hookrightarrow C$, $i = 1,2$, be some function spaces on $\mathbb{R}^n$. Then $X_1 \hookrightarrow X_2$ implies that there is some positive constant c such that for all $t > 0$,*

$$\mathscr{E}_{\mathsf{C}}^{X_1}(t) \leqslant c \mathscr{E}_{\mathsf{C}}^{X_2}(t).$$

(ii) *Let $X_i \hookrightarrow C$, $i = 1,2$, be some function spaces on $\mathbb{R}^n$ with $X_1 \hookrightarrow X_2$. Assume for their continuity envelope functions*

$$\mathscr{E}_{\mathsf{C}}^{X_1}(t) \sim \mathscr{E}_{\mathsf{C}}^{X_2}(t), \quad t \in (0,\varepsilon),$$

for some $\varepsilon > 0$. Then we get for the corresponding indices u_{X_i}, $i = 1,2$, that

$$u_{X_1} \leqslant u_{X_2}.$$

Remark 1.10. Plainly, by (8) and Definition 1.8(i) the above assertion (i) implies that $\mathscr{E}_{\mathsf{C}}^{X}$ is bounded when $X \hookrightarrow \mathrm{Lip}^1$; those spaces will be of no further interest for us.

2. Equivalent characterisations of $B_{pq}^{(s,\boldsymbol{\Psi})}$

We present three different approaches to characterise $B_{pq}^{(s,\Psi)}$, where the first method—atomic decomposition—is already known [3,23,43,44]; the latter two—by approximation and differences, respectively—are new.

2.1. Characterisation by atomic decompositions

An important tool in our later considerations is the characterisation of the spaces of generalised smoothness by means of atomic decompositions. We state this here for the B-spaces only. We refer to [43,44] for a complete description in case of Ψ being an admissible function and to [3,23] for a more general situation. Recall that all spaces are defined on $\mathbb{R}^n$ unless otherwise stated.

We need some preparation. As for $\mathbb{Z}^n$, it stands for the lattice of all points in $\mathbb{R}^n$ with integer-valued components, Q_{vm} denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-v}m = (2^{-v}m_1, \ldots, 2^{-v}m_n)$, and with side length 2^{-v}, where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $v \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q.

Definition 2.1. (i) Let $K \in \mathbb{N}_0$ and $c > 1$. A K times differentiable complex-valued function $a(x)$ in $\mathbb{R}^n$ (continuous if $K = 0$) is called *an* 1_K*-atom* if

$$\operatorname{supp} a \subset cQ_{0m}, \quad \text{for some } m \in \mathbb{Z}^n,$$

and

$$|D^\alpha a(x)| \leqslant 1, \quad \text{for } |\alpha| \leqslant K.$$

(ii) Let $K \in \mathbb{N}_0$, $L + 1 \in \mathbb{N}_0$ and $c > 1$. A K times differentiable complex-valued function $a(x)$ in $\mathbb{R}^n$ (continuous if $K = 0$) is called *an* $(s, p, \Psi)_{K,L}$*-atom* if for some $v \in \mathbb{N}_0$,

$$\operatorname{supp} a \subset cQ_{vm}, \quad \text{for some } m \in \mathbb{Z}^n,$$

$$|D^\alpha a(x)| \leqslant 2^{-v\left(s-\frac{n}{p}\right)+|\alpha| v} \Psi(2^{-v})^{-1}, \quad \text{for } |\alpha| \leqslant K,$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\, \mathrm{d}x = 0, \quad \text{if } |\beta| \leqslant L.$$

If the atom $a(x)$ is located at $Q_{\nu m}$, that means

$$\operatorname{supp} a \subset cQ_{\nu m}, \quad \text{with } \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,$$

then we write it as $a_{\nu m}(x)$. The sequence spaces b_{pq} are defined as follows:

Definition 2.2. Let $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, and $0 < p, q \leqslant \infty$. Then

$$b_{pq} = \left\{ \lambda : \|\lambda \mid b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}$$

(with the usual modification if $p = \infty$ or/and $q = \infty$).

For $0 < p \leqslant \infty$ we put $\sigma_p := n(1/p - 1)_+$.

Theorem 2.3. *Let $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with*

$$K \geqslant (1 + [s])_+ \quad \textit{and} \quad L \geqslant \max(-1, [\sigma_p - s])$$

be fixed. Then $f \in \mathcal{S}'$ belongs to $B_{pq}^{(s,\Psi)}$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m}(x), \quad \textit{convergence being in } \mathcal{S}', \tag{11}$$

where $\lambda \in b_{pq}$ and $a_{\nu m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s, p, \Psi)_{K,L}$-atoms ($\nu \in \mathbb{N}$) according to Definition 2.1. *Furthermore*

$$\inf \|\lambda | b_{pq}\|,$$

where the infimum is taken over all admissible representations (11), *is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}$.*

This theorem coincides with [43, Theorem 1.18 (ii)] when Ψ is an admissible function. The general case is covered by Farkas and Leopold [23, Theorem 4.4.3], and Bricchi [3, Theorem 2.3.7(i)].

2.2. Characterisation by approximation

For each $p \in (0, \infty]$ we consider the class

$$\mathcal{U}_p : \{a = (a_j)_{j=0}^{\infty} : a_j \in \mathcal{S}' \cap L_p, \operatorname{supp} \widehat{a_j} \subset \{y : |y| \leqslant 2^{j+1}\},\ j \in \mathbb{N}_0\},$$

cf. [50, 2.5.3/(4), p. 80].

Taking advantage of Proposition 1.3(i), the proof of Theorem 2.5.3(i) in [50, p. 81] can be appropriately modified in order to obtain the following:

Theorem 2.4. *Let $0<p,q\leqslant\infty$, Ψ be a slowly varying function and $s>\sigma_p$. Then*

$$B_{pq}^{(s,\Psi)} = \Bigg\{ f\in\mathscr{S}' : \exists\, a = (a_j)_{j=0}^{\infty}\in\mathscr{U}_p \text{ such that } f = \lim_{k\to\infty} a_k \text{ in } \mathscr{S}' \text{ and}$$

$$\|f|B_{pq}^{(s,\Psi)}\|^a := \|a_0|L_p\| + \left(\sum_{k=1}^{\infty} 2^{skq}\Psi(2^{-k})^q\|f-a_k|L_p\|^q\right)^{1/q} < \infty \Bigg\}$$

(*with the usual modification if $q=\infty$*). *Furthermore,*

$$\|f|B_{pq}^{(s,\Psi)}\|^X := \inf\|f|B_{pq}^{(s,\Psi)}\|^a,$$

where the infimum is taken over all admissible systems $a\in\mathscr{U}_p$, is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}$.

2.3. Characterisation by differences

Next, we recall the definition of differences of functions. If f is an arbitrary function on $\mathbb{R}^n$, $h\in\mathbb{R}^n$ and $k\in\mathbb{N}$, then

$$(\Delta_h^k f)(x) := \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f(x+jh), \quad x\in\mathbb{R}^n.$$

Note that Δ_h^k can also be defined iteratively via

$$(\Delta_h^1 f)(x) = f(x+h)-f(x) \quad \text{and} \quad (\Delta_h^{k+1}f)(x) = \Delta_h^1(\Delta_h^k f)(x), \quad k\in\mathbb{N}.$$

For convenience we may write Δ_h instead of Δ_h^1. Furthermore, the *kth modulus of smoothness* of a function $f\in L_p$, $1\leqslant p\leqslant\infty$, $k\in\mathbb{N}$, is defined by

$$\omega_k(f,t)_p = \sup_{|h|\leqslant t}\|\Delta_h^k f|L_p\|, \quad t>0. \tag{12}$$

We shall simply write $\omega(f,t)_p$ instead of $\omega_1(f,t)_p$ and $\omega(f,t)$ instead of $\omega(f,t)_\infty$.

Theorem 2.5. *Let $0<p,q\leqslant\infty$, $s>\sigma_p$, and Ψ be a slowly varying function. If k is an integer such that $k>s$, then*

$$\|f|B_{pq}^{(s,\Psi)}\|^{(k)} := \|f|L_p\| + \left(\int_0^1\left(\frac{\omega_k(f,t)_p}{t^s\Psi(t)^{-1}}\right)^q\frac{\mathrm{d}t}{t}\right)^{1/q}$$

(*with the usual modification if $q=\infty$*) *is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}$.*

The proof follows closely the proof of the analogous assertion when $\Psi\equiv 1$ given in [50, 2.5.12, pp. 110–112], with the appropriate modifications in view of Proposition 1.3(i). Again, many results of the above type can be found in the already mentioned papers by Gol'dman and Kalyabin, for instance.

In connection with this type of characterisation we refer also to [36, Theorem 8.2].

Example 2.6. We return to our example Ψ_b given by (3). Assume $b \in \mathbb{R}$, $0 < p, q \leqslant \infty$, $s > \sigma_p$, and $k \in \mathbb{N}$ with $k > s$. Then

$$\|f|B^{s,b}_{pq}\| \sim \|f|L_p\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega_k(f,t)_p}{t^s |\log t|^{-b}} \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q}$$

(with the usual modification if $q = \infty$), see Example 1.6.

3. Continuity envelopes: the main result

Recall that we shall write $A^{(s,\Psi)}_{p,q}$ for both $B^{(s,\Psi)}_{p,q}$ and $F^{(s,\Psi)}_{p,q}$ as long as no distinction is needed. Note that $A^{(s,\Psi)}_{pq} \hookrightarrow C$ for $0 < p, q \leqslant \infty$ (with $p < \infty$ in the F-case), $s > \frac{n}{p}$, and Ψ a slowly varying function; this follows immediately from (7) together with the corresponding well-known results for spaces A^s_{pq}. Thus, by Definition 1.8, it is reasonable to study continuity envelopes in that situation. Moreover, by an analogous argument it turns out that $A^{(s,\Psi)}_{pq} \hookrightarrow \mathrm{Lip}^1$ for $s > \frac{n}{p} + 1$; according to Remark 1.10 these spaces are not investigated further. Hence—postponing the tricky limiting situations $s = \frac{n}{p}$ and $s = \frac{n}{p} + 1$ (where the remaining indices p and q have to interplay with Ψ appropriately) to separate studies in the future—we are left to consider spaces $A^{(s,\Psi)}_{pq}$ with $\frac{n}{p} < s < \frac{n}{p} + 1$. Our result is the following.

Theorem 3.1. *Let $0 < p, q \leqslant \infty$ (with $p < \infty$ in the F-case), $0 < \sigma < 1$, $s = \frac{n}{p} + \sigma$, and Ψ be a slowly varying function. Then*:

(i) $\mathfrak{E}_{\mathsf{C}}(B^{(s,\Psi)}_{pq}) = (t^{-(1-\sigma)}\Psi(t)^{-1}, q)$;
(ii) $\mathfrak{E}_{\mathsf{C}}(F^{(s,\Psi)}_{pq}) = (t^{-(1-\sigma)}\Psi(t)^{-1}, p)$.

Proof. The proof follows the one of Haroske [31, Theorem 6.2.1] for A^s_{pq} ($\Psi \equiv 1$), with the appropriate modifications.

Step 1: We show

$$\mathscr{E}_{\mathsf{C}}^{B^{(\sigma,\Psi)}_{\infty q}}(t) \leqslant c t^{-(1-\sigma)} \Psi(t)^{-1}, \quad t \in (0,1); \tag{13}$$

as the elementary embedding

$$B^{(s,\Psi)}_{pq} \hookrightarrow B^{(\sigma,\Psi)}_{\infty q}, \quad s = \frac{n}{p} + \sigma, \ 0 < p < \infty, \ 0 < q \leqslant \infty, \tag{14}$$

(consequence of Proposition 1.7 and the corresponding well-known assertion for $\Psi \equiv 1$), then implies

$$\mathscr{E}_{\mathsf{C}}^{B^{(s,\Psi)}_{pq}}(t) \leqslant c t^{-(1-\sigma)} \Psi(t)^{-1}, \quad t \in (0,1),$$

in view of Proposition 1.9(i). Recall that by Theorem 2.5 (where we can choose $k = 1 > \sigma$ now)

$$\|f|B^{(\sigma,\Psi)}_{\infty q}\| \sim \|f|L_\infty\| + \left(\int_0^1 \left(\frac{\omega(f,t)}{t^\sigma \Psi(t)^{-1}}\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}. \tag{15}$$

Let $f \in B^{(\sigma,\Psi)}_{\infty q}$ with $\|f|B^{(\sigma,\Psi)}_{\infty q}\| \leqslant 1$. Then by (15) and the fact that $\frac{\omega(f,t)}{t}$ is equivalent to a monotonically decreasing function, see [15, Chapter 2, Lemma 6.1, p. 43] or [31, Proposition 4.3.3(i)], we obtain for any $\tau \in (0,1)$,

$$\begin{aligned}\frac{\omega(f,\tau)}{\tau}\left(\int_0^\tau t^{(1-\sigma)q-1}\,\Psi(t)^q\,\mathrm{d}t\right)^{1/q} &\leqslant c_1\left(\int_0^\tau \left(\frac{\omega(f,t)}{t^\sigma \Psi(t)^{-1}}\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\\ &\leqslant c_2\|f|B^{(\sigma,\Psi)}_{\infty q}\| \leqslant c_2.\end{aligned} \tag{16}$$

Taking into account Proposition 1.3(ii), for $\varepsilon > 0$ there is a decreasing function h such that $t^{-\varepsilon}\,\Psi(t) \sim h(t)$, $t \in (0,1]$. Then the left-hand side of (16) can be further estimated from below by

$$\begin{aligned}c_1\,\frac{\omega(f,\tau)}{\tau}\,h(\tau)\left(\int_0^\tau t^{(1+\varepsilon-\sigma)q-1}\,\mathrm{d}t\right)^{1/q} &\geqslant c_2\,\frac{\omega(f,\tau)}{\tau}\,h(\tau)\,\tau^{1+\varepsilon-\sigma}\\ &\geqslant c_3\,\frac{\omega(f,\tau)}{\tau}\,\tau^{1-\sigma}\,\Psi(\tau),\end{aligned}$$

leading to

$$\frac{\omega(f,\tau)}{\tau} \leqslant c\tau^{-(1-\sigma)}\,\Psi(\tau)^{-1},$$

for all $f \in B^{(\sigma,\Psi)}_{\infty q}$ with $\|f|B^{(\sigma,\Psi)}_{\infty q}\| \leqslant 1$, and hence (13).

Step 2: We verify

$$\mathscr{E}_C^{B^{(s,\Psi)}_{pq}}(t) \geqslant ct^{-(1-\sigma)}\,\Psi(t)^{-1}, \quad t \in (0, 2^{-1}], \tag{17}$$

and adapt the corresponding proof in [31, Theorem 6.2.1] appropriately. For that reason recall the atomic decomposition of spaces $B^{(s,\Psi)}_{pq}$ given in Theorem 2.3. According to this we know that functions

$$f_j(x) = 2^{-j\sigma}\,\Psi(2^{-j})^{-1}\,\varphi(2^j x), \quad j \in \mathbb{N},$$

are atoms in $B^{(s,\Psi)}_{pq}$ (no moment conditions needed), where φ is a compactly supported C^∞ function, thought as a mollified version of

$$\widetilde{\varphi}(x) = \begin{cases} 0, & |x| \geqslant 1,\\ 1-|x|, & |x| \leqslant 1,\end{cases} \qquad x \in \mathbb{R}^n.$$

Then clearly

$$\frac{\omega(\widetilde{\varphi}(2^j\cdot),t)}{t} = 2^j, \quad t \sim 2^{-j}, \quad j \in \mathbb{N},$$

and hence

$$\frac{\omega(f_j, t)}{t} \sim 2^{j(1-\sigma)} \Psi(2^{-j})^{-1}, \quad t \sim 2^{-j}, \ j \in \mathbb{N}.$$

Moreover, $\|f_j | B_{pq}^{(s,\Psi)}\| \sim 1$. Consequently, we arrive at

$$\mathscr{E}_C^{B_{pq}^{(s,\Psi)}}(2^{-j}) \geqslant c \frac{\omega(f_j, 2^{-j})}{2^{-j}} \geqslant c' 2^{j(1-\sigma)} \Psi(2^{-j})^{-1}, \quad j \in \mathbb{N},$$

and this leads to (17) finally, since $\Psi(t) \sim \Psi(2^{-j})$, $t \in [2^{-(j+1)}, 2^{-j}]$, $j \in \mathbb{N}$, due to Proposition 1.3(i).

Step 3: For simplicity, we shall write u_B instead of u_X when $X = B_{pq}^{(s,\Psi)}$ in the sequel. We first prove $u_B \leqslant q$. In view of our results in the preceding steps together with (14) and another application of Proposition 1.9(ii) it is sufficient to prove $u_X \leqslant q$ for $X = B_{\infty q}^{(\sigma,\Psi)}$. But this follows immediately from (15) and the already established equivalence

$$\mathscr{E}_C^{B_{\infty q}^{(\sigma,\Psi)}}(t) \sim t^{-(1-\sigma)} \Psi(t)^{-1}, \quad t \in (0,1),$$

granted that μ_H from Definition 1.8(ii) behaves like

$$\mu_H(\mathrm{d}t) \sim \frac{\mathrm{d}t}{t}. \tag{18}$$

Note that by Proposition 1.3(iii), there exists a positive C^∞ function h such that $h(t) \sim t^{1-\sigma} \Psi(t)$ in $(0,1]$ and

$$\lim_{t \to 0} t \frac{h'(t)}{h(t)} = 1 - \sigma > 0.$$

Therefore, h is increasing in some neighbourhood $(0, \varepsilon]$ of the origin with

$$\frac{h'(t)}{h(t)} \sim \frac{1}{t}, \quad t \in (0, \varepsilon], \tag{19}$$

and, moreover, by Steps 1 and 2,

$$h(t)^{-1} \sim t^{-(1-\sigma)} \Psi(t)^{-1} \sim \mathscr{E}_C^{B_{pq}^{(s,\Psi)}}(t), \quad t \in (0, \varepsilon].$$

In view of Definition 1.8(ii) we then obtain

$$H(t) = -\log \mathscr{E}_C^{B_{pq}^{(s,\Psi)}}(t) \sim \log h(t), \quad t \in (0, \varepsilon],$$

and (19) yields (18).

Step 4: In case of the B-spaces it remains to derive from

$$\left(\int_0^\varepsilon \left[\frac{\omega(f,t)}{t^\sigma \Psi(t)^{-1}} \right]^v \frac{\mathrm{d}t}{t} \right)^{1/v} \leqslant c \|f | B_{pq}^{(s,\Psi)}\| \tag{20}$$

for some $c > 0$ and all $f \in B_{pq}^{(s,\Psi)}$ that $v \geqslant q$; recall (9) and (18). We make use of a suitable combination of our extremal functions f_j from Step 2, see the corresponding proof in [31, Theorem 6.2.1]. Put

$$f(x) = \sum_{j=1}^{\infty} b_j 2^{-j\sigma} \Psi(2^{-j})^{-1} \varphi(2^j x - y^j), \quad x \in \mathbb{R}^n,$$

where φ is as above, $b_j \geqslant 0$, $j \in \mathbb{N}$, $(b_j)_{j\in\mathbb{N}} \in \ell_q$, and the y^j are chosen such that the supports of $\varphi(2^j \cdot - y^j)$ and $\varphi(2^k \cdot - y^k)$ are disjoint for $j \neq k$, $j,k \in \mathbb{N}$. Then by Theorem 2.3 f belongs to $B^{(s,\Psi)}_{pq}$ with

$$\|f|B^{(s,\Psi)}_{pq}\| \leqslant c\|b|\ell_q\|. \tag{21}$$

Let $J \in \mathbb{N}$ be such that $2^{-J} \sim \varepsilon$ for ε given by (20). For simplicity, we may assume $b_j \equiv 0$, $j = 1, \ldots, J-1$. By our assumptions,

$$\omega(f, 2^{-j}) \geqslant b_j 2^{-j\sigma} \Psi(2^{-j})^{-1} \omega(\varphi(2^j \cdot - y^j), 2^{-j}) \geqslant c b_j 2^{-j\sigma} \Psi(2^{-j})^{-1}$$

and consequently, involving (20), (21) and $b_j \equiv 0$, $j < J$, additionally,

$$\left(\sum_{j=J}^{\infty} b_j^v \right)^{1/v} \leqslant \left(\sum_{j=J}^{\infty} \left[\frac{\omega(f, 2^{-j})}{2^{-j\sigma} \Psi(2^{-j})^{-1}} \right]^v \right)^{1/v} \sim \left(\int_0^{\varepsilon} \left[\frac{\omega(f,t)}{t^{\sigma} \Psi(t)^{-1}} \right]^v \frac{\mathrm{d}t}{t} \right)^{1/v}$$

$$\leqslant c_1 \|f|B^{(s,\Psi)}_{pq}\| \leqslant c_2 \|b|\ell_q\| \sim \left(\sum_{j=J}^{\infty} b_j^q \right)^{1/q}$$

from whence we conclude $v \geqslant q$. Thus part (i) of the theorem is proved.

Step 5: Let $s_0 > s > \sigma$ and $s_0 - \frac{n}{p_0} = s - \frac{n}{p} = \sigma$. As a consequence of Proposition 1.7 and the embedding assertion in e.g. [52, 11.4, p. 55] it holds

$$B^{(s_0,\Psi)}_{p_0 p} \hookrightarrow F^{(s,\Psi)}_{pq} \hookrightarrow B^{(\sigma,\Psi)}_{\infty p}.$$

Then, due to Proposition 1.9, (ii) is a consequence of (i). □

Remark 3.2. When $\Psi \equiv 1$, Theorem 3.1 coincides with [31, Theorem 6.2.1].

Example 3.3. Using our particular choice Ψ_b given by (3), Theorem 3.1 reads as

$$\mathfrak{E}_C(B^{s,b}_{pq}) = (t^{-(1-\sigma)} |\log t|^{-b}, q),$$

and

$$\mathfrak{E}_C(F^{s,b}_{pq}) = (t^{-(1-\sigma)} |\log t|^{-b}, p),$$

where $b \in \mathbb{R}$, $0 < p, q \leqslant \infty$ (with $p < \infty$ in the F-case), $0 < \sigma < 1$ and $s = \frac{n}{p} + \sigma$.

As a first application we can conclude some Hardy-type inequalities. This follows immediately from our above assertions together with the monotonicity (10), see [53, Proposition 12.2, pp. 183–184], and the fact that

$$\sup_{t>0} \frac{\varkappa(t)}{\mathscr{E}^X_C(t)} \frac{\omega(f,t)}{t} \leqslant c$$

holds for some $c>0$ and all $f \in X$, $\|f|X\| \leqslant 1$, if, and only if, $\varkappa$ is bounded, see [31, Proposition 4.3.3(iv)].

Corollary 3.4. *Let p, q, s, σ and Ψ as in Theorem* 3.1.

(i) *Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0,\varepsilon]$ and let $0<u\leqslant\infty$. Then*

$$\left(\int_0^\varepsilon (\varkappa(t)t^{-\sigma}\Psi(t)\omega(f,t))^u \frac{dt}{t}\right)^{1/u} \leqslant c\|f|B_{pq}^{(s,\Psi)}\|$$

for some $c>0$ and all $f \in B_{pq}^{(s,\Psi)}$ if, and only if, $\varkappa$ is bounded and $q\leqslant u\leqslant\infty$, with the modification

$$\sup_{t\in(0,\varepsilon)} \varkappa(t)t^{-\sigma}\Psi(t)\omega(f,t) \leqslant c\|f|B_{pq}^{(s,\Psi)}\| \tag{22}$$

if $u=\infty$. In particular, if $\varkappa$ is an arbitrary non-negative function on $(0,\varepsilon]$, then (22) *holds if, and only if, $\varkappa$ is bounded.*

(ii) *Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0,\varepsilon]$ and let $0<u\leqslant\infty$. Then*

$$\left(\int_0^\varepsilon (\varkappa(t)t^{-\sigma}\Psi(t)\omega(f,t))^u \frac{dt}{t}\right)^{1/u} \leqslant c\|f|F_{pq}^{(s,\Psi)}\|$$

for some $c>0$ and all $f \in F_{pq}^{(s,\Psi)}$ if, and only if, $\varkappa$ is bounded and $p\leqslant u\leqslant\infty$, with the modification

$$\sup_{t\in(0,\varepsilon)} \varkappa(t)t^{-\sigma}\Psi(t)\omega(f,t) \leqslant c\|f|F_{pq}^{(s,\Psi)}\| \tag{23}$$

if $u=\infty$. In particular, if $\varkappa$ is an arbitrary non-negative function on $(0,\varepsilon]$, then (23) *holds if, and only if, $\varkappa$ is bounded.*

4. Entropy and approximation numbers

We study compact embeddings of function spaces of generalised smoothness and qualify their compactness further by means of entropy numbers and approximation numbers, respectively; here we shall essentially gain from our above envelope results. First, we briefly recall these concepts.

Let A_1 and A_2 be two complex (quasi-) Banach spaces and let T be a linear and continuous operator from A_1 into A_2. If T is compact then for any given $\varepsilon>0$ there are finitely many balls in A_2 of radius ε which cover the image $T(U_{A_1})$ of the unit ball $U_{A_1}=\{a\in A_1 : \|a|A_1\|\leqslant 1\}$.

Definition 4.1. Let $k\in\mathbb{N}$ and let $T : A_1 \to A_2$ be the above continuous operator.

(i) The *kth entropy number* e_k of T is the infimum of all numbers $\varepsilon>0$ such that there exist 2^{k-1} balls in A_2 of radius ε which cover $T(U_{A_1})$.

(ii) The *kth approximation number* a_k of T is the infimum of all numbers $\|T - S\|$ where S runs through the collection of all continuous linear maps from A_1 to A_2 with rank $S < k$,

$$a_k(T) = \inf\{\|T - S\| : S \in \mathscr{L}(A_1, A_2), \operatorname{rank} S < k\}.$$

For details and properties of entropy and approximation numbers we refer to [11,16,37,48] (restricted to the case of Banach spaces), and [20] for some extensions to quasi-Banach spaces. Obviously, entropy numbers 'measure' the compactness of operators in geometrical terms whereas approximation numbers characterise it by approximation with finite-rank operators.

Remark 4.2. A strong motivation to study entropy numbers as well as approximation numbers comes from spectral theory, in particular, the investigation of eigenvalues of compact operators. Though these consequences are out of the scope of the present paper, we briefly recall some ideas. Let A be a complex (quasi-) Banach space and $T \in \mathscr{L}(A)$ compact. Then the spectrum of T (apart from the point 0) consists only of eigenvalues of finite algebraic multiplicity, $\{\mu_k(T)\}_{k \in \mathbb{N}}$, ordered as usual $|\mu_1(T)| \geqslant |\mu_2(T)| \geqslant \cdots \geqslant 0$. Carl's inequality gives an excellent link between entropy numbers and eigenvalues of T:

$$\left(\prod_{m=1}^{k} |\mu_m(T)|\right)^{1/k} \leqslant \inf_{n \in \mathbb{N}} 2^{\frac{n}{2k}} e_n(T), \quad k \in \mathbb{N}, \tag{24}$$

in particular,

$$|\mu_k(T)| \leqslant \sqrt{2} e_k(T). \tag{25}$$

This result was originally proved by Carl [10] and Carl and Triebel [12] when A is a Banach space. An extension to quasi-Banach spaces can be found in [20, Theorem 1.3.4, p. 18]. Conversely, we may also gain from the study of approximation numbers when dealing with eigenvalue estimates, where it is reasonable to concentrate on the Hilbert space setting first. Let $\mathscr{H}$ be a complex Hilbert space and $T \in \mathscr{L}(\mathscr{H})$ compact, the non-zero eigenvalues of which are denoted by $\{\mu_k(T)\}_{k \in \mathbb{N}}$ again; then T^*T has a non-negative, self-adjoint, compact square root $|T|$, and for all $k \in \mathbb{N}$,

$$a_k(T) = \mu_k(|T|), \tag{26}$$

see [16, Theorem II.5.10, p. 91]. Hence, if in addition T is non-negative and self-adjoint, then the approximation numbers of T coincide with its eigenvalues. Outside Hilbert spaces the results are less good but still very interesting, cf. [11,16,37,48] for further details.

The interplay between continuity envelopes and approximation numbers relies on the following outcome.

Proposition 4.3. *Let X be some Banach space defined on the unit ball U in $\mathbb{R}^n$ with $X(U) \hookrightarrow C(U)$. Then there is some $c>0$ such that for all $k \in \mathbb{N}$,*

$$a_k(id : X(U) \to C(U)) \leqslant ck^{-\frac{1}{n}} \mathscr{E}_{\mathsf{C}}^X(k^{-\frac{1}{n}}). \tag{27}$$

This result can be found in [7]; it is essentially based on an estimate obtained by Carl and Stephani [11, Theorem 5.6.1, p. 178].

We return to the function spaces studied above. Note that there cannot be a compact embedding between spaces on $\mathbb{R}^n$; the counterpart for spaces $A_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ follows immediately from the well-known fact for spaces $A_{pq}^s(\mathbb{R}^n)$ and (7). Let U be the unit ball in $\mathbb{R}^n$; we deal with spaces $A_{pq}^{(s,\Psi)}(U)$ now defined by restriction from their $\mathbb{R}^n$-counterparts. Checking the argument in our proof one immediately verifies that Theorem 3.1 can be transferred to spaces on domains without any difficulty, i.e. we have for the local continuity envelopes $\mathfrak{E}_{\mathsf{C}}(A_{pq}^{(s,\Psi)}(U)) = \mathfrak{E}_{\mathsf{C}}(A_{pq}^{(s,\Psi)}(\mathbb{R}^n))$. Combining (the counterpart of) Theorem 3.1 with (27) immediately leads to the upper estimate in the following proposition.

Proposition 4.4. *Let $2 \leqslant p \leqslant \infty$ (with $p<\infty$ in the F-case), $0<q\leqslant\infty$, $s \in \mathbb{R}$ with $\frac{n}{p}<s<\frac{n}{p}+1$, and Ψ be a slowly varying function. Then*

$$a_k(id : B_{pq}^{(s,\Psi)}(U) \to C(U)) \sim k^{-\frac{s}{n}+\frac{1}{p}} \Psi(k^{-\frac{1}{n}})^{-1}, \quad k \in \mathbb{N}. \tag{28}$$

The same is true with $B_{pq}^{(s,\Psi)}(U)$ replaced by $F_{pq}^{(s,\Psi)}(U)$.

Proof. Note that the restriction $p \geqslant 2$ is due to the lower estimate; it is, however, to expect, in view of related situations, say, when $\Psi \equiv 1$; see Remark 4.6.

Step 1: The upper estimate in (28) is a direct consequence of Theorem 3.1 and (27). Note that the difficulty with $0<p,q<1$, when the spaces $B_{pq}^{(s,\Psi)}(U)$ are not Banach spaces and hence Proposition 4.3 cannot be applied directly, can easily be surmounted by a continuous embedding argument $B_{pq}^{(s,\Psi)}(U) \hookrightarrow B_{r\hat{q}}^{(\sigma,\Psi)}(U)$, where $p<1<r$, $s-\frac{n}{p}=\sigma-\frac{n}{r}$, $\hat{q}=\max(q,1)$, in view of the multiplicativity of approximation numbers.

Step 2: For the estimate from below we make use of the special lift property Proposition 1.7 together with related results for $\Psi \equiv 1$. Let $\mu \in \mathbb{R}$ be such that $0<\mu<1$. Then by the multiplicativity of approximation numbers,

$$\begin{aligned} &a_{2k}(id : B_{pq}^{(s,\Psi)}(U) \to B_{\infty\infty}^{(-\mu,\Psi)}(U)) \\ &\quad \leqslant a_k(id : B_{pq}^{(s,\Psi)}(U) \to C(U)) a_k(id : C(U) \to B_{\infty\infty}^{(-\mu,\Psi)}(U)). \end{aligned}$$

It is thus sufficient to show that

$$a_{2k}(id : B_{pq}^{(s,\Psi)}(U) \to B_{\infty\infty}^{(-\mu,\Psi)}(U)) \geqslant c_1 k^{-\frac{s+\mu}{n}+\frac{1}{p}}, \tag{29}$$

for $2 \leqslant p \leqslant \infty$, and

$$a_k(id : C(U) \to B^{(-\mu,\Psi)}_{\infty\infty}(U)) \leqslant c_2 k^{-\frac{\mu}{n}} \Psi\left(k^{-\frac{1}{n}}\right) \tag{30}$$

in order to verify the estimate from below. In the spirit of Proposition 1.7 we can simplify (29) by

$$a_{2k}(id : B^{(s,\Psi)}_{pq}(U) \to B^{(-\mu,\Psi)}_{\infty\infty}(U)) \sim a_{2k}(id : B^{s}_{pq}(U) \to B^{-\mu}_{\infty\infty}(U)),$$

the rest being a consequence of the well-known result [20, Theorem 3.3.4, p. 119] for $\Psi \equiv 1$, see also (32) below. For the required extension operators we refer to [35,41]; see also the survey article [36]. These papers cover more general settings, too; we proceed by extension and restriction in the usual way. Concerning (30), $C(U) \hookrightarrow B^{0}_{\infty\infty}(U)$ leads to

$$a_k(id : C(U) \to B^{(-\mu,\Psi)}_{\infty\infty}(U)) \leqslant c a_k(id : B^{0}_{\infty\infty}(U) \to B^{(-\mu,\Psi)}_{\infty\infty}(U)).$$

The usual lift operator

$$I_\sigma f = ((1+|\xi|^2)^{\sigma/2} \widehat{f})^\vee, \quad f \in \mathscr{S}', \tag{31}$$

maps $A^{(s,\Psi)}_{pq}$ isomorphically onto $A^{(s-\sigma,\Psi)}_{pq}$, cf. [3, Proposition 2.2.19]; here we use again a result on $\mathbb{R}^n$, but it can be adapted to our setting in the above-described way. Thus this lifting argument together with another application of Proposition 1.7 and $C(U) \hookrightarrow B^{0}_{\infty\infty}(U)$ provide

$$\begin{aligned}
&a_k(id : B^{0}_{\infty\infty}(U) \to B^{(-\mu,\Psi)}_{\infty\infty}(U)) \\
&\quad \leqslant c_1 a_k(id : B^{\mu}_{\infty\infty}(U) \to B^{(0,\Psi)}_{\infty\infty}(U)) \\
&\quad \leqslant c_2 a_k(id : B^{(\mu,\Psi^{-1})}_{\infty\infty}(U) \to B^{0}_{\infty\infty}(U)) \\
&\quad \leqslant c_3 a_k(id : B^{(\mu,\Psi^{-1})}_{\infty\infty}(U) \to C(U)) \\
&\quad \leqslant c_4 k^{-\frac{\mu}{n}} \left[\Psi\left(k^{-\frac{1}{n}}\right)^{-1}\right]^{-1} = c_4 k^{-\frac{\mu}{n}} \Psi\left(k^{-\frac{1}{n}}\right),
\end{aligned}$$

where we finally applied Step 1. This yields (30) and finishes the proof. $\square$

Remark 4.5. Following the above proof it is clear that the target space $C(U)$ can be replaced by $B^{0}_{\infty,\infty}(U)$ without any difficulty. When Ψ is an admissible function, taking the special features of admissible functions into account—cf. [8, Lemma 2.3(ii)]—, we could simplify (28) by

$$a_k(id : B^{(s,\Psi)}_{pq}(U) \to C(U)) \sim k^{-\frac{s}{n}+\frac{1}{p}} \Psi(k^{-1})^{-1}, \quad k \in \mathbb{N},$$

thus hiding the influence of the underlying measure space $\mathbb{R}^n$ equipped with the Lebesgue measure ℓ_n.

Remark 4.6. When $\Psi \equiv 1$, the situation is well-known apart from some limiting cases: let $0<p,q\leqslant\infty$ (with $p<\infty$ in the F-case), $s>\frac{n}{p}$, then for all $k\in\mathbb{N}$,

$$a_k(id : B^s_{pq}(U)\to C(U))\sim\begin{cases} k^{-(\frac{s}{n}-\frac{1}{p})}, & 2\leqslant p\leqslant\infty, \\ k^{-(\frac{s}{n}-\frac{1}{p})\frac{p'}{2}}, & 1<p<2,\ s<n, \\ k^{-(\frac{s}{n}-\frac{1}{2})}, & 1<p<2,\ s>n, \end{cases} \tag{32}$$

where p' is given by $\frac{1}{p}+\frac{1}{p'}=1$, $1<p<\infty$, as usual; cf. [6,20, Theorem 3.3.4, p. 119]. In particular, (28) coincides with the first line of (32) for $\Psi\equiv 1$. Moreover, (32) also suggests that for $p<2$ the upper estimate obtained in Step 1 of the proof of Proposition 4.4 for all $0<p\leqslant\infty$ (i.e. also for $0<p<2$) is certainly not asymptotically sharp; it differs for $p<2$ and $p\geqslant 2$. This phenomenon is well-known for approximation numbers.

Dealing with entropy numbers instead, sharp asymptotic estimates in case of $\Psi\equiv 1$ are completely covered by [20, Theorem 3.3.3/2, p. 118],

$$e_k(id : B^s_{pq}(U)\to C(U))\sim k^{-\frac{s}{n}}, \tag{33}$$

where $0<p,q\leqslant\infty$ and $s\in\mathbb{R}$ with $s>\frac{n}{p}$.

Corollary 4.7. *Let $2\leqslant p\leqslant\infty$ (with $p<\infty$ in the F-case), $0<q\leqslant\infty$, $s_1,s_2\in\mathbb{R}$ with $\frac{n}{p}<s_1-s_2<\frac{n}{p}+1$, $a_1,a_2\in\mathbb{R}$ and Ψ be a slowly varying function. Then for all $k\in\mathbb{N}$,*

$$a_k(id : B^{(s_1,\Psi^{a_1})}_{pq}(U)\to B^{(s_2,\Psi^{a_2})}_{\infty\infty}(U))\sim k^{-\frac{s_1-s_2}{n}+\frac{1}{p}}\Psi(k^{-\frac{1}{n}})^{a_2-a_1}. \tag{34}$$

The same is true with $B^{(s_1,\Psi^{a_1})}_{pq}(U)$ replaced by $F^{(s_1,\Psi^{a_1})}_{pq}(U)$.

Proof. Note that for a slowly varying function Ψ any function Ψ^a, $a\in\mathbb{R}$, is slowly varying, too. Then the assertion is an immediate consequence of Propositions 4.4, 1.7, Remark 4.5 and an application of the lift operator I_σ from (31) mapping $A^{(s,\Psi)}_{pq}$ isomorphically onto $A^{(s-\sigma,\Psi)}_{pq}$, cf. [3, Proposition 2.2.19; 50, Theorem 2.3.8, p. 58] for $\Psi\equiv 1$,

$$\begin{aligned} &a_k(id : B^{(s_1,\Psi^{a_1})}_{pq}(U)\to B^{(s_2,\Psi^{a_2})}_{\infty\infty}(U)) \\ &\quad\sim a_k(id : B^{(s_1-s_2,\Psi^{a_1-a_2})}_{pq}(U)\to B^0_{\infty\infty}(U))\sim k^{-\frac{s_1-s_2}{n}+\frac{1}{p}}\Psi(k^{-\frac{1}{n}})^{a_2-a_1} \end{aligned}$$

for $k\in\mathbb{N}$. $\square$

Example 4.8. We return to our particular choice Ψ_b given by (3), in particular, with $a_1 := b_1$, $a_2 := b_2$. Then (34) reads as

$$a_k(id : B^{s_1,b_1}_{pq}(U)\to B^{s_2,b_2}_{\infty\infty}(U))\sim k^{-\frac{s_1-s_2}{n}+\frac{1}{p}}(1+\log k)^{b_2-b_1}, \quad k\in\mathbb{N},$$

where $2\leqslant p\leqslant\infty$, $0<q\leqslant\infty$, $\frac{n}{p}<s_1-s_2<\frac{n}{p}+1$, and $b_1,b_2\in\mathbb{R}$.

Using a rather rough general estimate between entropy and approximation numbers we can reformulate Proposition 4.4 in terms of entropy numbers.

Corollary 4.9. *Let* $0<p,q\leqslant\infty$ (*with* $p<\infty$ *in the* F-*case*), $s_1,s_2\in\mathbb{R}$ *with* $\frac{n}{p}<s_1-s_2<\frac{n}{p}+1$, $a_1,a_2\in\mathbb{R}$, *and* Ψ *be a slowly varying function. Then there are numbers* $c_2>c_1>0$ *such that for all* $k\in\mathbb{N}$,

$$c_1 k^{-\frac{s_1-s_2}{n}}\Psi(k^{-\frac{1}{n}})^{a_2-a_1}\leqslant e_k(id: B_{pq}^{(s_1,\Psi^{a_1})}(U)\to B_{\infty\infty}^{(s_2,\Psi^{a_2})}(U))$$
$$\leqslant c_2 k^{-\frac{s_1-s_2}{n}+\frac{1}{p}}\Psi(k^{-\frac{1}{n}})^{a_2-a_1}. \tag{35}$$

The same is true with $B_{pq}^{(s_1,\Psi^{a_1})}(U)$ *replaced by* $F_{pq}^{(s_1,\Psi^{a_1})}(U)$.

Proof. For the upper estimate we combine (34) and the following: Let $T\in\mathscr{L}(A_1,A_2)$ be a compact operator between quasi-Banach spaces A_1, A_2. Assume that there is some $c'>0$ with $a_{2j-1}(T)\leqslant c' a_{2j}(T)$ for all $j\in\mathbb{N}$. Then

$$e_k(T)\leqslant c a_k(T)$$

for some $c>0$ and all $k\in\mathbb{N}$. Details can be found in [51]; there is a forerunner in [11, p. 96] restricted to Banach spaces. The extension to $p<2$—comparing Corollaries 4.9 and 4.7—is covered by embedding arguments again or directly by Step 1 of the proof of Proposition 4.4.

The lower estimate is obtained in exactly the same way as in Step 2 of the proof of Proposition 4.4 with the only modification that (29) has to be replaced by

$$e_{2k}(id: B_{pq}^{(s,\Psi)}(U)\to B_{\infty\infty}^{(-\mu,\Psi)}(U))\geqslant ck^{-\frac{s+\mu}{n}}, \tag{36}$$

valid for all $0<p\leqslant\infty$, see [20, Theorem 3.3.3/2, p. 118] and (33). The rest is lifting. □

Remark 4.10. Comparison with the situation $\Psi\equiv 1$ recalled in (33) suggests that the upper estimate in (35) is not sharp apart from the case $p=\infty$. Moreover, this assumption is supported by Leopold's results on entropy numbers and approximation numbers in spaces of generalised smoothness; concerning approximation numbers he obtained a sequence space result similar to Proposition 4.4 in [40, Corollary 1] with a (function space) forerunner in [38, Remark 4]. As far as we know there are no further results in this direction. Similarly, in the context of entropy numbers, Leopold proved in [40, Theorems 3,4] (with forerunners in [38,39]) sharp asymptotic estimates, using (sub-) atomic decomposition techniques and sequence space assertions. For instance, with Ψ_b given by (3), [39, Theorem 3] states that

$$e_k(id: B_{pq}^{s_1,b}(U)\to B_{\infty\infty}^{s_2}(U))\sim k^{-\frac{s_1-s_2}{n}}(1+\log k)^{-b},\quad k\in\mathbb{N},$$

where $0<p,q\leqslant\infty$, $s_1-s_2>\frac{n}{p}$, and $b\in\mathbb{R}$.

Further entropy and approximation number results (related to the case $s_1-s_2=\frac{n}{p}+1$, which is out of the scope of the present paper) can also be found in [14,17,18,38,39].

Finally, entropy numbers for embeddings of spaces of generalised smoothness were already studied in [43, Theorem 3.13], but in the context of spaces defined on (fractal) (d, Ψ)-sets.

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