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Linear Algebra and its Applications 401 (2005) 211–220
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Three observations on the determinantal range

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Received 31 October 2003; accepted 7 June 2004
Available online 3 August 2004
Submitted by R. Merris

Dedicated to G. de Oliveira

Abstract

Let $A, C \in M_n$, the algebra of $n \times n$ complex matrices. The set of complex numbers

$$\Delta_C(A) = \{\det(A - UCU^*) : U^*U = I_n\}$$

is the C-*determinantal range* of A. In this note, it is proved that $\Delta_C(A)$ is an elliptical disc for $A, C \in M_2$. A necessary and sufficient condition for $\Delta_C(A)$ to be a line segment is given when A and C are normal matrices with pairwise distinct eigenvalues. The linear operators L that satisfy the linear preserver property $\Delta_C(A) = \Delta_C(L(A))$, for all $A, C \in M_n$, are characterized.

Keywords: Determinantal range; Cross ratio; Linear preserver

1. Introduction

Let M_n be the algebra of $n \times n$ complex matrices, and let U_n be the group of $n \times n$ unitary matrices. Let H_n denote the real space of $n \times n$ Hermitian matrices. For

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2004.06.006

$A, C \in M_n$ the *C-determinantal range of A* is the set of the complex plane denoted and defined by

$$\Delta_C(A) = \{\det(A - UCU^*) : U \in U_n\}. \tag{1}$$

This set is compact and connected, but it may not be simply connected [1].

The C-determinantal range of A can be viewed as a variation of the concept of *C-numerical range* of A, introduced in [10], and defined by

$$W_C(A) = \{\mathrm{Tr}\,(CU^*AU) : U \in U_n\}.$$

In fact, both sets are ranges of continuous functions defined on the *unitary similarity orbit* of $A \in M_n$, $\mathbb{O}(A) = \{U^*AU : U \in U_n\}$ and a certain parallelism exists between the properties of these notions [1,2]. Nevertheless, it seems much more complicated to deal with $\Delta_C(A)$ than with $W_C(A)$.

Let $\alpha_1, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$ be the eigenvalues of A and C, respectively. It can be easily seen that the $n!$ points (not necessarily distinct)

$$z_\sigma = \prod_{i=1}^{n}(\alpha_i - \gamma_{\sigma(i)}), \quad \sigma \in S_n,$$

S_n the symmetric group of degree n, belong to $\Delta_C(A)$. In the sequel, these points will be called *σ-points*.

It is easy to verify that $\Delta_C(A)$ is *unitarily invariant*, that is,

$$\Delta_C(A) = \Delta_{U^*CU}(V^*AV), \quad \text{for any } U, V \in U_n.$$

Let $A, C \in M_n$ be normal matrices. Since A and C are diagonalizable under unitary similarity transformations, and $\Delta_C(A)$ is unitarily invariant, we may consider $A = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)$ and $C = \mathrm{diag}\,(\gamma_1, \ldots, \gamma_n)$ in (1). Marcus [13] and de Oliveira [18] conjectured that

$$\Delta_C(A) \subseteq \mathrm{Co}\left\{\prod_{i=1}^{n}(\alpha_i - \gamma_{\sigma(i)}) : \sigma \in S_n\right\}, \tag{2}$$

where Co $\{\cdot\}$ is the convex hull of $\{\cdot\}$. This conjecture was proved in certain special cases (see [3–5,8,11,15,16]), but even the case $n = 4$ remains open. While it seems difficult to prove (or disprove) (2), to give the complete characterization of $\Delta_C(A)$ it is much more difficult. Indeed, even the statement of a necessary and sufficient condition for $\Delta_C(A)$ to be a line segment is not trivial. Sufficient conditions for $\Delta_C(A)$ to be a line segment are known. Fiedler [9] proved that if $A = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)$, $C = \mathrm{diag}\,(\gamma_1, \ldots, \gamma_n)$ and all the α_i and γ_j belong to the same straight line through the origin, that is, $\arg(\alpha_1) = \cdots = \arg(\alpha_n) = \arg(\gamma_1) = \cdots = \arg(\gamma_n) (\mathrm{mod}\,\pi)$, then $\Delta_C(A)$ is a line segment through the origin and the equality in (2) holds. In [3] it was proved that if $A = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)$, $C = \mathrm{diag}\,(\gamma_1, \ldots, \gamma_n)$ and all the α_i and γ_j belong to the same circle with center at the origin, that is, $|\alpha_1| = \cdots = |\alpha_n| = |\gamma_1| = \cdots = |\gamma_n|$, then $\Delta_C(A)$ is a line segment through the origin and the equality in (2) holds.

One of the motivations of this note is the exploitation of the parallelism between the properties of $\Delta_C(A)$ and $W_C(A)$ [14], as well as to emphasize the analogies of the proof techniques in both situations. The note is organized as follows. In Section 2, we prove that $\Delta_C(A)$ is an elliptical disc when $A, C \in M_2$. In Section 3, a necessary and sufficient condition for $\Delta_C(A)$ to be a line segment is given, when $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, $C = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$ and $\alpha_1, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$ are pairwise distinct. We conjecture that this restriction on the eigenvalues may be relaxed and that if $\Delta_C(A)$, $n \geqslant 2$, is a line segment, then the line containing it passes through the origin. In Section 4, linear operators L on M_n (and on H_n) that satisfy the linear preserver property $\Delta_C(A) = \Delta_C(L(A))$, for all $A, C \in M_n$ (for all $A \in H_n$ and $C \in M_n$) are characterized.

2. The elliptical range theorem for $\Delta_C(A)$

For $C = \operatorname{diag}(\gamma_1, \gamma_2)$ and $A \in M_2$, it was proved [2] that $\Delta_c(A)$ is an elliptical disc with foci $(\alpha_1 - \gamma_1)(\alpha_2 - \gamma_2)$ and $(\alpha_1 - \gamma_2)(\alpha_2 - \gamma_1)$, and with the length of the minor axis equal to $|\gamma_1 - \gamma_2|\sqrt{\operatorname{Tr} AA^* - |\alpha_1|^2 - |\alpha_2|^2}$. This result is generalized for $A, C \in M_2$ in Theorem 2.1 which treats the general case. In fact, following analogous steps to those used in Section 2 of [12] we can prove that for $A, C \in M_2$, $\Delta_C(A) = \Delta_{C_1}(A_1)$, A_1 and C_1 being matrices of the form (3).

Theorem 2.1. *Let*

$$A = \xi_1 \begin{bmatrix} \alpha & a \\ b & \alpha \end{bmatrix} \quad \text{and} \quad C = \xi_2 \begin{bmatrix} \gamma & c \\ d & \gamma \end{bmatrix}, \tag{3}$$

where $\alpha, \gamma, \xi_1, \xi_2 \in \mathbb{C}$ with $|\xi_1| = |\xi_2| = 1$, $a \geqslant b \geqslant 0$ and $c \geqslant d \geqslant 0$. Let α_1, α_2 and γ_1, γ_2 be the eigenvalues of A and C, respectively. Then

$$\begin{aligned} \Delta_C(A) = {} & (\xi_1\alpha - \xi_2\gamma)^2 - (\xi_1^2 ab + \xi_2^2 dc) \\ & + \xi_1\xi_2\{r[(ac+bd)\cos t + \mathrm{i}(ac-bd)\sin t] : r \in [0,1], \quad t \in [0, 2\pi)\}, \end{aligned}$$

that is, $\Delta_C(A)$ is an elliptical disc with $(\alpha_1 - \gamma_1)(\alpha_2 - \gamma_2)$ and $(\alpha_1 - \gamma_2)(\alpha_2 - \gamma_1)$ as foci and $\{(\xi_1\alpha - \xi_2\gamma)^2 - (\xi_1^2 ab + \xi_2^2 dc) + \xi_1\xi_2 s(ac + bd) : s \in [-1, 1]\}$ and $\{(\xi_1\alpha - \xi_2\gamma)^2 - (\xi_1^2 ab + \xi_2^2 dc) + \mathrm{i}\xi_1\xi_2 s(ac - bd) : s \in [-1, 1]\}$ as major and minor axis, respectively.

Lemma 2.1 [12,17]. *Let A, C be matrices of the form (3) with $\xi = \xi_2 = 1$. Then*

$$\begin{aligned} W_C(A) = \{ & 2\alpha\gamma + r[(ac + bd)\cos t \\ & + \mathrm{i}(ac - bd)\sin t] : r \in [0, 1],\ t \in [0, 2\pi)\}, \end{aligned}$$

which is the elliptical disc with $\{2\alpha\gamma + s(ac + bd) : s \in [-1, 1]\}$ as the major axis and with $\{2\alpha\gamma + \mathrm{i}s(ac - bd) : s \in [-1, 1]\}$ as the minor axis.

We now prove Theorem 2.1.

Proof. The following expansion can be easily obtained

$$\det(A - U^*CU) = \det A + \det C - \operatorname{Tr}(Z^{\mathrm{T}}AZU^*CU), \tag{4}$$

where

$$Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We can assume $A, C \in M_2$ of the form

$$A = \xi_1 \begin{bmatrix} \alpha & -b \\ -a & \alpha \end{bmatrix} \quad \text{and} \quad C = \xi_2 \begin{bmatrix} \gamma & c \\ d & \gamma \end{bmatrix}, \tag{5}$$

where $\alpha, \gamma, \xi_1, \xi_2 \in \mathbb{C}$ are such that $|\xi_1| = |\xi_2| = 1$, $-a \leqslant -b \leqslant 0$ and $c \geqslant d \geqslant 0$. Having in mind (4), and applying Lemma 2.1 to the matrices A and C defined in (5), we can conclude that $\Delta_C(A)$ is the elliptical disc centered at $(\xi_1\alpha - \xi_2\gamma)^2 - (\xi_1^2 ab + dc\xi_2^2)$, with $2(ac + bd)$ and $2(ac - bd)$ as the lengths of its major and minor axis, respectively. Hence, the semi-focal distance is given by $2\sqrt{ab}\sqrt{cd}$. The direction of the major axis is $u = 2\xi_1\xi_2(ac + bd)$. Since $|\xi_1| = |\xi_2| = 1$, we have $|u| = 2(ac + bd)$. The foci of the elliptical disc are

$$f_i = (\xi_1\alpha - \xi_2\gamma)^2 - (\xi_1^2 ab + dc\xi_2^2) \pm 2\sqrt{ab}\sqrt{cd}\frac{u}{|u|}, \quad i = 1, 2.$$

We observe that the eigenvalues of A and C are $\alpha_i = \xi_1(\alpha \pm \sqrt{ab})$ and $\gamma_i = \xi_2(\gamma \pm \sqrt{cd})$, $i = 1, 2$, respectively. By straightforward computations, we have that $f_1 = (\alpha_1 - \gamma_1)(\alpha_2 - \gamma_2)$ and $f_2 = (\alpha_1 - \gamma_2)(\alpha_2 - \gamma_1)$, and so the theorem follows. □

Remark. Li [12] proved that $W_C(A)$, $A, C \in M_2$, is an elliptical disc. Following analogous steps, an alternative proof for Theorem 2.1 can be obtained.

3. A necessary and sufficient condition for $\Delta_c(A)$ to be a line segment

In order to prove the main result of this section, we recall the definition of *cross ratio* of $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$. Define $S : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ by

$$S(z) = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)} \quad \text{if } z_2, z_3, z_4 \in \mathbb{C};$$

$$S(z) = \frac{z - z_3}{z - z_4} \quad \text{if } z_2 = \infty;$$

$$S(z) = \frac{z_2 - z_4}{z - z_4} \quad \text{if } z_3 = \infty;$$

$$S(z) = \frac{z - z_3}{z_2 - z_3} \quad \text{if } z_4 = \infty.$$

We have $S(z_2) = 1$, $S(z_3) = 0$ and $S(z_4) = \infty$, and S is the unique Möbius transformation which satisfies the previous conditions. The *cross ratio* of $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$

denoted by (z_1, z_2, z_3, z_4), is the image of z_1 under the unique Möbius transformation which takes z_2 to 1, z_3 to 0, and z_4 to ∞.

The following lemma will be used in the proof of Lemma 3.2. For a proof, see e.g. [7].

Lemma 3.1. *Let z_1, z_2, z_3 and z_4 be four distinct points in $\mathbb{C}_\infty$. Then (z_1, z_2, z_3, z_4) is a real number if and only if z_1, z_2, z_3 and z_4 belong to the same straight line or to the same circle.*

We give a solution to Problem 10484 in [6].

Lemma 3.2. *Let $n \geqslant 2$, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ be complex row vectors such that $\alpha_1, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$ are pairwise distinct. Consider the $n!$ complex numbers (counting multiplicities)*

$$z_\sigma = \prod_{i=1}^{n} (\alpha_i - \gamma_{\sigma(i)}), \quad \sigma \in S_n.$$

For $P(\alpha, \gamma) = \mathrm{Co}\{z_\sigma : \sigma \in S_n\}$, $P(\alpha, \gamma)$ is a line segment of a line passing through the origin if and only if all the α_i and γ_j lie on a common circle or straight line.

Proof. ($\Rightarrow$) For $\alpha_i, \alpha_j, \gamma_i, \gamma_j \in \mathbb{C}_\infty$, $i, j = 1, \ldots, n$, by the definition of cross ratio, we have

$$(\alpha_i, \alpha_j, \gamma_i, \gamma_j) = S(\alpha_i),$$

where S is the unique Möbius transformation such that $S(\alpha_j) = 1$, $S(\gamma_i) = 0$ and $S(\gamma_j) = \infty$. By definition of S,

$$S(\alpha_i) = \frac{(\alpha_i - \gamma_i)(\alpha_j - \gamma_j)}{(\alpha_i - \gamma_j)(\alpha_j - \gamma_i)}.$$

For $\sigma = (id)$ and $\tau = (ij)$, we clearly have

$$\frac{(\alpha_i - \gamma_i)(\alpha_j - \gamma_j)}{(\alpha_i - \gamma_j)(\alpha_j - \gamma_i)} = \frac{z_\sigma}{z_\tau},$$

and so

$$S(\alpha_i) = \frac{z_\sigma}{z_\tau} = \frac{|z_\sigma| \mathrm{e}^{\mathrm{i}\arg(z_\sigma)}}{|z_\tau| \mathrm{e}^{\mathrm{i}\arg(z_\tau)}}.$$

Since z_σ and z_τ belong to $P(\alpha, \gamma)$ and $P(\alpha, \gamma)$ is a line segment of a line passing through the origin, then $\arg(z_\sigma) = \arg(z_\tau) (\operatorname{mod} \pi)$. Thus,

$$S(\alpha_i) = \frac{|z_\sigma|}{|z_\tau|} \quad \text{or} \quad S(\alpha_i) = -\frac{|z_\sigma|}{|z_\tau|}.$$

Since $S(\alpha_i)$ is a real number, we conclude that $(\alpha_i, \alpha_j, \gamma_i, \gamma_j)$ is a real number for $i, j = 1, \ldots, n$. By Lemma 3.1, $\alpha_i, \alpha_j, \gamma_i$ and γ_j belong to the same circle or to the same straight line.

($\Leftarrow$) Suppose that all the α_i and γ_j, $i, j = 1, \ldots, n$, belong to the same circle or to the same straight line. By Lemma 3.1, there is a unique Möbius transformation S such that $S(\alpha_j) = 1$, $S(\gamma_{\sigma(i)}) = 0$ and $S(\gamma_{\sigma(j)}) = \infty$ and

$$S(\alpha_i) = \frac{(\alpha_i - \gamma_{\sigma(i)})(\alpha_j - \gamma_{\sigma(j)})}{(\alpha_i - \gamma_{\sigma(j)})(\alpha_j - \gamma_{\sigma(i)})}$$

is a real number, for $i = 1, \ldots, n$ and $\sigma \in S_n$. Consider the transposition $\tau \in S_n$ such that $\tau(k) = \sigma(k)$ for $k \neq i, j$, $\tau(i) = \sigma(j)$ and $\tau(j) = \sigma(i)$. Then

$$S(\alpha_i) = \frac{z_\sigma}{z_\tau} = \frac{|z_\sigma| \mathrm{e}^{\mathrm{i} \arg z_\sigma}}{|z_\tau| \mathrm{e}^{\mathrm{i} \arg z_\tau}}.$$

By the hypothesis, $S(\alpha_i)$ is a real number for $i = 1, \ldots, n$, and so $\arg(z_\sigma) = \arg(z_\tau) (\operatorname{mod} \pi)$ for all $\sigma, \tau \in S_n$. As $P(\alpha, \gamma) = \operatorname{Co}\{z_\sigma : \sigma \in S_n\}$ is a compact and connected subset of $\mathbb{R}$ (or of $\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}$, θ real), it follows that $P(\alpha, \gamma)$ is a line segment. $\square$

Theorem 3.1. *Let A and C be $n \times n$ normal matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$, respectively, such that $\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n$, are pairwise distinct. The set $\Delta_C(A)$ is a line segment of a line passing through the origin if and only if all the α_i and γ_j lie on a common circle or straight line.*

Proof. ($\Rightarrow$) Let $\Delta_C(A)$ be a line segment of a line through the origin. The endpoints of this segment are corners (z belonging to the boundary of $\Delta_C(A)$ is a corner, if in the neighborhood of z, $\Delta_C(A)$ is contained in an angle with vertex at z and measuring less than π.) It is known [2,8] that if z is a corner, then $z = z_\sigma$ for some $\sigma \in S_n$. Thus, $\Delta_C(A) = \operatorname{Co}\{z_\sigma : \sigma \in S_n\} = P(\alpha, \gamma)$ is a line segment of a line through the origin, and by Lemma 3.2 all the α_i and γ_j lie on a common circle or straight line.

($\Leftarrow$) If all the α_i and γ_j lie on a common circle or straight line, by Lemma 3.2 $P(\alpha, \gamma)$ is a line segment of a line passing through the origin. In these cases, (2) holds with equality [3,9]. $\square$

Observation. For the sufficient condition the relaxation on the eigenvalues is trivial. We conjecture that this relaxation is still possible for the necessary condition. Moreover, we conjecture that if $\Delta_C(A)$, $n \geqslant 2$, is a line segment, then the line containing it passes through the origin.

4. A linear preserver property

We investigate the structure of those linear operators $L : M_n \longrightarrow M_n$ that satisfy the relation $\Delta_C(A) = \Delta_C(L(A))$, for all $A, C \in M_n$. We start with some useful lemmas.

Lemma 4.1. *Let $A \in M_n$. The following conditions are equivalent*:

(i) *For any $C \in M_n$, $\Delta_C(A)$ is a singleton.*
(ii) *A is a scalar matrix.*

Proof. The proof (ii) $\Rightarrow$ (i) is trivial. We prove the direct implication.

Let $A \in M_n$. Since $\Delta_C(A)$ reduces to a singleton for any $C \in M_n$, there exists a matrix $C \in M_n$ such that the eigenvalues of A and C are pairwise distinct and the corresponding determinantal range is a singleton. Thus, all the σ-points, $\sigma \in S_n$, coincide with the singleton $\Delta_C(A)$, and so

$$\frac{\alpha_i - \gamma_i}{\alpha_j - \gamma_i} = \frac{\alpha_i - \gamma_j}{\alpha_j - \gamma_j} = \frac{\gamma_i - \gamma_j}{\gamma_i - \gamma_j} = 1, \quad i, j = 1, \ldots, n.$$

We conclude that $\alpha_i = \alpha_j$, for all $i, j = 1, \ldots, n$.

Because the set $\Delta_C(A)$ is unitarily invariant, we may use Schur's Lemma and consider A in upper triangular form. Suppose that $A = (a_{jh})$ is not a normal matrix, and so there exists $a_{jh} \neq 0$, with $j < h$. By the hypothesis, $\Delta_C(A)$ is a singleton for any C. Hence, there exists a matrix C such that $\gamma_{\sigma(j)} \neq \gamma_{\sigma(h)}$, $\sigma \in S_n$, and for which $\Delta_C(A)$ is a singleton. Let $V = P_\sigma P_{(1j)\circ(2h)} \in M_n$, where $P_\sigma = (\delta_{j\sigma(h)})$ and $P_{(1j)\circ(2h)}$ is the permutation matrix associated with $\tau = (1j) \circ (2h) \in S_n$. It is easy to see that

$$\{\det (A - V(W_2 \oplus I_{n-2})V^{\mathrm{T}}CV(W_2 \oplus I_{n-2})^*V^{\mathrm{T}} : \; W_2 \in U_2\} \subset \Delta_C(A).$$

This region is an elliptical disc with foci z_σ and $z_{\sigma\circ(jh)}$, and with the length of the minor axis $|a_{jh}||\gamma_{\sigma(j)} - \gamma_{\sigma(h)}|$. Since $\Delta_C(A)$ is a singleton, this is a contradiction. It follows that A is a normal matrix, namely, a scalar matrix. $\square$

Lemma 4.2. *Let $\xi \in \mathbb{C}$. The equality* $\det (I_n - C) = \det (\xi I_n - C)$ *is valid for all $C \in M_n$ if and only if $\xi = 1$.*

Proof. The part ($\Leftarrow$) is clear.

Suppose that $\xi \neq 1$. Taking $C = I_n$ we obtain $\det (I_n - C) = 0$. We also have $\det (\xi I_n - C) = \det ((\xi - 1)I_n) = (\xi - 1)^n$. As $\xi \neq 1$, then $\det (\xi I_n - C) \neq 0$, a contradiction. $\square$

Lemma 4.3 [19]. *A linear operator on H_n mapping the cone of positive semi-definite matrices onto itself must be of the form $A \mapsto U^*AU$ or $A \mapsto U^*A^{\mathrm{T}}U$, for some invertible $U \in M_n$.*

Theorem 4.1. *A linear operator $L : H_n \longrightarrow H_n$ satisfies*

$$\Delta_C(A) = \Delta_C(L(A)), \quad \textit{for all } A \in H_n \quad \textit{and} \quad \textit{for all } C \in M_n,$$

if and only if there exists a unitary matrix U such that L is of the form

$$A \mapsto UAU^* \quad \textit{or} \quad A \mapsto UA^{\mathrm{T}}U^*.$$

Proof. The implication ($\Leftarrow$) is clear.

We prove the converse. First of all, if $L(A) = 0$, then A is a scalar matrix. In fact, this is a consequence of Lemma 4.1, since $\Delta_C(A) = \Delta_C(L(A))$ is a singleton for any $C \in M_n$. Suppose $A \neq 0$. Then there exists a non-singular matrix C such that $\Delta_C(A) = 0$. But, if C is non-singular, then $\Delta_C(0) = \Delta_C(L(A)) \neq 0$, which contradicts $\Delta_C(A) = \Delta_C(L(A)) = 0$. Hence $A = 0$ and so L is non-singular.

Next, note that $\Delta_C(L(I_n)) = \Delta_C(I_n) = \det(I_n - C)$, for all $C \in M_n$ and, by Lemma 4.1, $L(I_n) = \xi I_n$, for some $\xi \in \mathbb{C}$. For all $C \in M_n$, we have $\Delta_C(\xi I_n) = \Delta_C(I_n)$ and, by Lemma 4.2, it follows that $\xi = 1$. Therefore, the operator L preserves I_n and $\Delta_C(A)$. Now, we show that this operator L maps the set of positive definite matrices onto itself. To this end, let A be positive definite. Suppose $L(A)$ is not positive definite. Then there exists $r \geqslant 0$ such that $L(A) + rI_n$ is singular. We know that

$$\Delta_C(L(A + rI_n)) = \Delta_C(L(A) + rI_n) = \Delta_C(A + rI_n),$$

for all Hermitian C. In particular for $C = 0$, we have

$$\Delta_0(A + rI_n) = \Delta_0(L(A) + rI_n).$$

Since $A + rI_n$ is positive definite, $\Delta_0(A + rI_n) = \det(A + rI_n) > 0$. On the other hand,

$$\Delta_0(L(A) + rI_n) = \det(L(A) + rI_n) = 0,$$

because $L(A) + rI_n$ is a singular matrix. Hence, there does not exist $r \geqslant 0$ such that $L(A) + rI_n$ is singular, and so $L(A)$ is positive definite.

Since L is invertible, one can apply the previous arguments to L^{-1} to conclude that L^{-1} maps the set of positive definite matrices into itself. Thus, it preserves I_n and $\Delta_C(A)$. Hence, L maps the set of positive definite matrices onto itself. By Lemma 4.3, there exists an invertible matrix U such that the operator L is of the form

$$A \mapsto UAU^* \quad \text{or} \quad A \mapsto UA^{\mathrm{T}}U^*.$$

Since $L(I_n) = I_n$, we have $UU^* = I_n$. $\square$

The following result will be used in the proof of Theorem 4.2.

Lemma 4.4 [2]. *Let C be an $n \times n$ normal matrix with simple eigenvalues. If there exists at least one corner on the boundary of $\Delta_C(A)$, then $A \in M_n$ is a normal matrix.*

Theorem 4.2. *A linear operator $L : M_n \longrightarrow M_n$ satisfies*

$$\Delta_C(A) = \Delta_C(L(A)), \quad \textit{for all } A, C \in M_n$$

if and only if there exists a unitary matrix U such that L is of the form

$$A \mapsto UAU^* \quad \textit{or} \quad A \mapsto UA^{\mathrm{T}}U^*.$$

Proof. ($\Rightarrow$) Let L be a linear operator in M_n such that $\Delta_C(A) = \Delta_C(L(A))$ for all $A, C \in M_n$. Suppose that A is a Hermitian matrix. We prove that $L(A)$ is Hermitian.

By the hypothesis, $\Delta_C(A) = \Delta_C(L(A))$ for all $C \in M_n$. In particular, there exists $C \in H_n$ such that the eigenvalues of C and A are pairwise distinct and also the eigenvalues of C and $L(A)$ are pairwise distinct. Since A and C are Hermitian, it follows that

$$\Delta_C(A) = [\min_{\sigma} z_\sigma, \max_{\sigma} z_\sigma], \quad \sigma \in S_n.$$

As $\Delta_C(A) = \Delta_C(L(A))$, using Lemma 3.2 we can conclude that the eigenvalues of C and $L(A)$ belong to the same straight line or to the same circle. Since C is Hermitian it has real eigenvalues, so $L(A)$ has real eigenvalues.

Since $\Delta_C(L(A)) = [\min_\sigma z_\sigma, \max_\sigma z_\sigma]$, the endpoints of this line segment are corners and, by Lemma 4.4, $L(A)$ is normal. Thus, $L(A)$ is Hermitian and $L(H_n) \subseteq H_n$.

By Theorem 4.1, we have

(i) $L(A) = UAU^*$ or (ii) $L(A) = UA^{\mathrm{T}}U^*$.

Consider $A \in M_n$ in the cartesian decomposition, that is, $A = \operatorname{Re} A + \mathrm{i} \operatorname{Im} A$, where $\operatorname{Re} A = (A + A^*)/2$ and $\operatorname{Im} A = (A - A^*)/2\mathrm{i}$ are Hermitian matrices. If (i) holds, then

$$\begin{aligned} L(A) &= L(\operatorname{Re} A) + \mathrm{i}L(\operatorname{Im} A) \\ &= U(\operatorname{Re} A)U^* + \mathrm{i}U(\operatorname{Im} A)U^* \\ &= UAU^*. \end{aligned}$$

If (ii) holds, a similar argument can be used. The converse implication follows directly. □

References

[1] N. Bebiano, Some variations on the concept of the c-numerical range, Port. Math. 43 (1985–1986) 189–200.

[2] N. Bebiano, Some analogies between the c-numerical range and a certain variation of this concept, Linear Algebra Appl. 81 (1986) 47–54.

[3] N. Bebiano, J. da Providência, Some remarks on a conjecture of Oliveira, Linear Algebra Appl. 102 (1988) 241–246.

[4] N. Bebiano, New developments on the Marcus–Oliveira conjecture, Linear Algebra Appl. 197–198 (1994) 791–844.

[5] N. Bebiano, A. Kovačec, J. da Providência, The validity of the Marcus–Oliveira conjecture for essentially Hermitian matrices, Linear Algebra Appl. 197–198 (1994) 411–427.

[6] N. Bebiano, J. da Providência, Problem P 10484, Amer. Math. Monthly 9 (1995) 841.

[7] J.B. Conway, Functions of One Complex Variable I, second ed., Springer Verlag, 1991.

[8] S.W. Drury, B. Cload, On the determinantal conjecture of Marcus and de Oliveira, Linear Algebra Appl. 177 (1992) 105–109.

[9] M. Fiedler, Bounds for the determinant of the sum of Hermitian matrices, Proc. Amer. Math. Soc. 30 (1971) 27–31.

[10] M. Goldberg, E.G. Straus, Elementary inclusion relations of generalized numerical ranges, Linear Algebra Appl. 18 (1977) 1–24.

[11] A. Kovačec, On a conjecture of Marcus and de Oliveira, Linear Algebra Appl. 201 (1994) 91–97.

[12] C.-K. Li, Some convexity theorems for the generalized numerical ranges, Linear and Multilinear Algebra 40 (1996) 235–240.

[13] M. Marcus, Derivations, Plücker relations and the numerical range, Indiana Univ. Math. J. 22 (1973) 1137–1149.

[14] M. Marcus, M. Sandy, Conditions for the generalized numerical range to be real, Linear Algebra Appl. 71 (1985) 219–239.

[15] J.K. Merikoski, A. Virtanen, Some notes on de Oliveira's determinantal conjecture, Linear Algebra Appl. 121 (1989) 345–352.

[16] J.K. Merikoski, A. Virtanen, Some further notes on the Marcus–de Oliveira determinantal conjecture, Linear Algebra Appl. 187 (1993) 259–262.

[17] H. Nakazato, The C-numerical range of a 2×2 matrix, Sci. Rep. Hirosaki Univ. 41 (1994) 197–206.

[18] G.N. de Oliveira, Normal matrices (research problem), Linear and Multilinear Algebra 12 (1982) 153–154.

[19] H. Schneider, Positive operators and an inertia theorem, Numer. Math. 7 (1965) 11–17.