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Approximation of bone remodeling models

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Abstract

We present convergence results and error estimates concerning the numerical approximation of a class of bone remodeling models, that are elastic adaptive rod models. These are characterized by an elliptic variational equation, representing the equilibrium of the rod under the action of applied loads, coupled with an ordinary differential equation with respect to time, describing the physiological process of bone remodeling. We first consider the semi-discrete approximation, where only the space variables are discretized using the standard Galerkin method, and then, applying the forward Euler method for the time discretization, we focus on the fully discrete approximation.

Résumé

On présente des résultats de convergence et d'estimation d'erreur concernant l'approximation numérique d'une classe de modèles de remodelage des os, et qui correspondent à une classe de modèles de poutres en élasticité adaptative. Ces modèles sont caracterisés par une équation variationnelle elliptique, réprésentant l'équilibre d'une poutre sous l'action des forces appliquées, couplée avec une équation differentielle par raport au temps, décrivant le processus biologique de remodelage d'un os. On considère tout d'abord l'approximation semi-discrète, en discrétisant les variables spatiales et en utilisant une méthode de Galerkin standard, puis on applique la méthode d'Euler pour la discrétisation en temps, et finalement on analyse l'approximation discrète complète.

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1. Introduction

In this paper we analyze the numerical approximation of a class of bone remodeling models, which are elastic adaptive rod models. The theory of adaptive elasticity describes the physiological process of bone remodeling and was introduced by Cowin and Hegedus [4,12] (cf. also Cowin and Nachlinger [5], and Monnier and Trabucho [13], for uniqueness and existence results, respectively, for three-dimensional adaptive elasticity problems). We consider in this paper a class of models that corresponds to the class of simplified adaptive elastic rod models derived in Figueiredo and Trabucho [8] (see also Figueiredo, Leal and Pinto [9,10]) and that can be mathematically justified by the asymptotic expansion method (cf. also Trabucho and Viãno [14] for an explanation of the mathematical modeling of elastic rods with the asymptotic expansion method). More exactly, the class of models treated in this paper (cf. problem (2)) consists of a system of two coupled problems, the first one corresponds to generalized Bernoulli–Navier elastic equilibrium equations and describes the equilibrium of a rod, that is subjected to external applied loads, and the second problem is an ordinary differential equation with respect to time, which models the physiological process of bone remodeling. The unknown of this system is the pair (u, d), where u is a vector field and d a scalar field, such that $u(x, t)$ represents the equilibrium displacement at the point x of the rod and time t, and $d(x, t)$ is the change in volume fraction of the elastic material at the point x of the rod and time t. Moreover u depends on d and d depends on u. More precisely, u is the solution of a variational problem of elliptic type, whose associated bilinear and linear forms (the linearity understood with respect to u) depend nonlinearly on d. The other unknown d is the solution of the ordinary differential equation with respect to time, which depends on u. The major difficulty of this class of models is just this interdependence of the two unknowns u and d. Nevertheless we are able to overcome this difficulty, using convenient mathematical tools, in order to derive the theoretical results concerning the numerical approximation of this class of models.

We first consider the semi-discrete approximation of (2) (cf. problem (18)) where only the space variables are discretized. Denoting by h the space discretization parameter and by (u_h, d_h) the solution of the semi-discrete approximation we prove that (u_h, d_h) converges to (u, d), when $h \to 0^+$, in appropriate functional spaces of Sobolev type involving time (cf. Theorem 3.4). Then, we consider the fully discrete approximation (cf. problem (47)), using a numerical scheme to approximate the remodeling rate equation of the semi-discrete approximation. Because of the structure of this class of models we choose the forward Euler method, but we notice that other explicit one-step or multistep methods could also be used to approximate the remodeling rate equation. Denoting by (u_h^i, d_h^i) the solution of the fully discrete approximation problem at time t_i, for a finite number of time nodes t_i, with $i = 0, \ldots, N(\Delta t)$, where Δt is the time discretization parameter, we prove that the errors $u(., t_i) - u_h^i$ and $d(., t_i) - d_h^i$ converge to zero, as $h \to 0^+$ and $\Delta t \to 0^+$, for all $i = 0, \ldots, N(\Delta t)$, in appropriate Sobolev spaces (cf. Theorem 4.3).

To reach the two principal convergence results, established in Theorems 3.4 and 4.3, we essentially prove error estimates for $u - u_h$ and $u_h(.\,, t_i) - u_h^i$, which are generalizations of Céa's lemma (cf. Theorems 3.2 and 4.1), that together with the integral Gronwall's inequality and the error estimates of the forward Euler's method enable to deduce error estimates for $d - d_h$ and $d_h(.\,, t_i) - d_h^i$ (cf. Theorems 3.3 and 4.2).

Finally we briefly explain the contents of the paper. In Section 2 we introduce some notations and define the class of bone remodeling models. In Section 3 we prove the theoretical results for the semi-discrete approximation problem. In Section 4 we introduce the fully discrete approximation and prove more error estimates and convergence results. We also present some conclusions and future work.

2. Definition and properties of the class of models

2.1. Notations

Let ω be an open, bounded and connected subset of $\mathbb{R}^2$, with a boundary $\partial\omega$ regular enough. We denote by $\overline{\Omega}$ the set occupied by a cylindrical adaptive elastic rod, in its reference configuration, with length $L > 0$ and cross-section ω, that is $\overline{\Omega} = \overline{\omega} \times [0, L] \subset \mathbb{R}^3$. Moreover we define the three sets $\Gamma = \partial\omega \times]0, L[$ (where $\partial\omega$ is the boundary of ω), $\Gamma_0 = \overline{\omega} \times \{0\}$ and $\Gamma_L = \overline{\omega} \times \{L\}$, which represent, respectively, the lateral boundary and the two extremities of Ω. We assume that the rod is subjected to the action of external forces on Ω and $\Gamma \cup \Gamma_0 \cup \Gamma_L$. We also denote by $x = (x_1, x_2, x_3)$ a generic element of $\overline{\Omega}$ and we assume that the coordinate system (O, x_1, x_2, x_3) is a principal system of inertia associated with the rod Ω. Consequently, axis Ox_3 passes through the centroid of each section $\omega \times \{x_3\}$ and we have $\int_\omega x_1 \,\mathrm{d}\omega = \int_\omega x_2 \,\mathrm{d}\omega = \int_\omega x_1 x_2 \,\mathrm{d}\omega = 0$.

The set $C^m(\overline{\Omega})$ stands for the space of real functions m times continuously differentiable in $\overline{\Omega}$. The spaces $W^{m,q}(\Omega)$ and $W^{0,q}(\Omega) = L^q(\Omega)$ are the usual Sobolev spaces, where q is a real number satisfying $1 \leqslant q \leqslant \infty$ and m is a positive integer.

The set,

$$\mathcal{R} = \left\{ v \in \mathbb{R}^3 \colon\ v = a + b \wedge x,\ a, b \in \mathbb{R}^3 \right\}, \tag{1}$$

where $\wedge$ is the cross product in $\mathbb{R}^3$, is the set of infinitesimal rigid displacements. We denote by $[W^{m,q}(\Omega)]^3/\mathcal{R}$ the quotient space induced by the set $\mathcal{R}$ in the Sobolev space $[W^{m,q}(\Omega)]^3$.

Throughout the paper, the Latin indices $i, j, k, l, \ldots$ belong to the set $\{1, 2, 3\}$, the Greek indices $\alpha, \beta, \mu, \ldots$ vary in the set $\{1, 2\}$ and the summation convention with respect to repeated indices is employed, that is, for example, $a_i b_i = \sum_{i=1}^{3} a_i b_i$.

Let $T > 0$ be a real parameter and we denote by t the time variable in the interval $[0, T]$. If V is a topological vectorial space, the set $C^m([0, T]; V)$ is the space of functions $g : t \in [0, T] \to g(t) \in V$, such that g is m times continuously differentiable with respect to t. If V is a Banach space we denote $\|.\|_{C^m([0,T];V)}$ the usual norm in $C^m([0, T]; V)$. Moreover, given a function $g(x, t)$ defined in $\overline{\Omega} \times [0, T]$ we denote by $\dot{g}$ its partial derivative with respect to time, by $\partial_\alpha g$ and $\partial_3 g$ its partial derivatives with respect to x_α and x_3, that is, $\dot{g} = \frac{\partial g}{\partial t}$, $\partial_\alpha g = \frac{\partial g}{\partial x_\alpha}$ and $\partial_3 g = \frac{\partial g}{\partial x_3}$.

2.2. A class of adaptive elastic rod models

In this paper we consider a class of bone remodeling models that correspond to the simplified adaptive elastic rod model derived in Figueiredo and Trabucho [8], for the case of a linear remodeling rate equation (cf. formulas (71)–(74) and (90) in [8]). For a rod represented by the set $\overline{\Omega} = \overline{\omega} \times [0, L]$ in its reference configuration this class of models is defined as follows:

$$\left[\begin{array}{l} \text{Find } (u,d) \text{ such that} \\ u = (u_1, u_2, u_3) : \overline{\Omega} \times [0,T] \to \mathbb{R}^3 \quad \text{and} \quad d : \overline{\Omega} \times [0,T] \to \mathbb{R}, \\ u(.,t) \in V, \\ a_d(u,v) = L_d(v), \quad \forall v \in V, \\ \dot{d} = c(d) e_{33}(u) + b(d), \quad \text{in } \Omega \times (0,T), \\ d(x,0) = \eta(x), \quad \text{in } \overline{\Omega}. \end{array}\right. \tag{2}$$

This problem (2) can be mathematically justified by the asymptotic expansion method as in Figueiredo and Trabucho [8]. It consists of a system of two nonlinear coupled problems, the variational equation $a_d(u,v) = L_d(v)$ representing the equilibrium equations of the rod (that are generalized Bernoulli–Navier elastic equilibrium equations), and the ordinary differential equation with respect to time $\dot{d} = c(d)e_{33}(u) + b(d)$, that is the remodeling rate equation and expresses the process of bone remodeling, due to external stimulus.

The unknowns of the model (2) are the vector field $u(x,t)$, corresponding to the displacement of the point x of the rod $\overline{\Omega}$ at time t, and the scalar field $d(x,t)$ that is the measure of change in volume fraction of the elastic material (from a reference volume fraction denoted in the sequel by ξ_0) at (x,t). The unknown displacement u is the solution of the variational inequality and depends on d; the unknown d depends on u and is the solution of the ordinary differential equation of parabolic type, that is the remodeling rate equation. In particular $e_{33}(u) = \partial_3 u_3$ is a component of the linear strain tensor $e(u)$ whose components are defined by $e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.

On the other hand, the data of the model (2) are the following: the space V of admissible displacements, the bilinear form $a_d(.,.) : V \times V \to \mathbb{R}$ and the linear form $L_d(.) : V \to \mathbb{R}$, that depend on the unknown d and represent the elastic equilibrium equations and the external forces acting on the rod, respectively, the initial value of the change in volume fraction $\eta(x) = d(x,0)$, and the coefficients $c(d)$ and $b(d)$ which are material coefficients depending upon the change in volume fraction d. We will describe next in detail all these data of the model (2). The space V of admissible displacements is the quotient space $V = V(\Omega)/\mathcal{R}$, where $V(\Omega)$ is the space of Bernoulli–Navier displacements defined by:

$$V(\Omega) = \left\{ v \in \left[W^{2,2}\left(]0,L[\right)\right]^2 \times W^{1,2}(\Omega) :\ e_{\alpha\beta}(v) = e_{3\beta}(v) = 0 \right\}, \tag{3}$$

or equivalently,

$$V(\Omega) = \big\{v = (v_1, v_2, v_3) \in \big[W^{2,2}\big(]0, L[\big)\big]^2 \times W^{1,2}(\Omega):\ v_\alpha(x) = v_\alpha(x_3),$$
$$v_3(x) = \underline{v}_3(x_3) - x_\alpha \partial_3 v_\alpha(x_3),\ \underline{v}_3 \in W^{1,2}\big(]0, L[\big)\big\}. \tag{4}$$

The bilinear form $a_d(.,.)$ is defined:

$$a_d(z, v) = \int_\Omega \frac{1}{b_{3333}(d)} e_{33}(z) e_{33}(v)\,\mathrm{d}\Omega, \quad \forall z, v \in V, \tag{5}$$

where $e_{33}(v) = \partial_3 v_3 = \partial_3 \underline{v}_3 - x_\alpha \partial_{33} v_\alpha$ and $b_{3333}(d)$ is a material coefficient that depends on d (in fact it is an element of the matrix $(b_{ijkl}(d))$ which is the inverse of the matrix composed of the three-dimensional elastic coefficients of the rod $\overline{\Omega}$, as explained in Figueiredo and Trabucho [8], formulas (47)–(48)). Moreover we assume that

$$0 < c^{\min} \leqslant \frac{1}{b_{3333}(d(x,t))} \leqslant c^{\max}, \quad \forall x \in \overline{\Omega},\ \forall t \in [0, T], \tag{6}$$

where $c^{\min}$ and $c^{\max}$ are strictly positive constants. The linear form $L_d(.)$ is defined for all $v \in V$ by:

$$L_d(v) = \int_\Omega \gamma\big(\xi_0 + \mathcal{P}_\varepsilon(d)\big) f_i v_i\,\mathrm{d}\Omega + \int_\Gamma g_i v_i\,\mathrm{d}\Gamma + \int_{\Gamma_0 \cup \Gamma_L} h_i v_i\,\mathrm{d}(\Gamma_0 \cup \Gamma_L). \tag{7}$$

The scalar γ is the density of the full elastic material, which is supposed to be a constant, ξ_0 is the reference volume fraction of the elastic material (already mentioned immediately after the definition of the problem (2)) that belongs to $C^1(\overline{\Omega})$, $f = (f_i)$, $g = (g_i)$ and $h = (h_i)$ are, respectively, the density of body loads and normal tractions on the lateral boundary Γ and on the two extremities $\Gamma_0 \cup \Gamma_L$ of the rod $\overline{\Omega}$, and finally $\mathcal{P}_\varepsilon(.)$ is a truncation operator. Moreover we assume that the resultant of the system of applied forces is null for rigid displacements, that is, for any $v = (v_i)$ in $\mathcal{R}$,

$$\int_\Omega \gamma\big(\xi_0 + \mathcal{P}_\eta(d)\big) f_i v_i\,\mathrm{d}\Omega + \int_\Gamma g_i v_i\,\mathrm{d}\Gamma + \int_{\Gamma_0 \cup \Gamma_L} h_i v_i\,\mathrm{d}(\Gamma_0 \cup \Gamma_L) = 0, \quad \text{in } [0, T]. \tag{8}$$

We suppose that $0 < \xi_0^{\min} \leqslant \xi_0(x) \leqslant \xi_0^{\max} < 1$, for all $x \in \overline{\Omega}$, and the truncation operator $\mathcal{P}_\varepsilon$ is of class C^1 and satisfies $0 < \varepsilon/2 \leqslant (\xi_0 + \mathcal{P}_\varepsilon(d))(x) \leqslant 1$ for all $x \in \overline{\Omega}$, where $\varepsilon > 0$ is a small parameter. We also assume that $f_i \in C^1([0, T])$, $g_i \in C^1([0, T]; W^{1-1/p,p}(\Gamma))$ and $h_i \in C^1([0, T]; W^{1-1/p,p}(\Gamma_0 \cup \Gamma_L))$, with $p > 3$. These regularity hypotheses on the forces are necessary to obtain existence results (cf. Theorem 2.1 in Figueiredo and Trabucho [8] and also Theorem 1 in Monnier and Trabucho [13]). Finally, we suppose that the initial value $d(x, 0) = \eta(x)$ of the change in volume fraction verifies $\eta \in C^0(\overline{\Omega})$ and the material coefficients $c(d)$ and $b(d)$ appearing in the second term of the remodeling rate equation are continuously differentiable with respect to d and that there exist positive constants c_1, c_2, c_3, c_4, such that, for all $x \in \overline{\Omega}$ and for all $t \in [0, T]$:

$$\left|c\big(d(x,t)\big)\right| \leqslant c_1, \quad \left|b\big(d(x,t)\big)\right| \leqslant c_2, \quad \left|c'\big(d(x,t)\big)\right| \leqslant c_3, \quad \left|b'\big(d(x,t)\big)\right| \leqslant c_4, \quad (9)$$

where $c'(.)$ and $b'(.)$ are the derivatives of the scalar functions $c(.)$ and $b(.)$.

We also notice that we have the following Korn's type inequality in the space $V = V(\Omega)/\mathcal{R}$ (cf. Ciarlet [1] or Valent [15])

$$\exists c > 0: \quad \|v\|^2_{[W^{1,2}(\Omega)]^3} \leqslant c\left\|e_{33}(v)\right\|^2_{L^2(\Omega)}, \quad \forall v \in V, \tag{10}$$

where

$$\left\|e_{33}(v)\right\|^2_{L^2(\Omega)} = \text{meas}(\omega)\|\partial_3 \underline{v}_3\|^2_{L^2(0,L)} + \left(\int_\omega x_\alpha^2 \,\mathrm{d}\omega\right)\|\partial_{33} v_\alpha\|^2_{L^2(0,L)}, \tag{11}$$

and meas(ω) is the measure of the set ω. Then, we conclude that $\|e_{33}(.)\|_{L^2(\Omega)}$ is a norm in the space V, equivalent to the usual norm induced in V by $\|.\|_{[W^{1,2}(\Omega)]^3}$. So in the sequel and for all $v \in V$, we denote by $\|v\|_V$ the norm $\|e_{33}(v)\|_{L^2(\Omega)}$ or equivalently the norm $\|v\|_{[W^{1,2}(\Omega)]^3}$. Moreover, V is a Hilbert space with the norm $\|e_{33}(.)\|_{L^2(\Omega)}$. In addition, for each d, the bilinear form $a_d(.,.)$ is continuous and elliptic in V (these two properties of $a(.,.)$ are also a consequence of the condition (6) imposed on the coefficient $b_{3333}(d)$), that is, for all z and v in V:

$$\begin{aligned} &\left|a_d(z,v)\right| \leqslant c^{\max}\left\|e_{33}(z)\right\|_{L^2(\Omega)}\left\|e_{33}(v)\right\|_{L^2(\Omega)} = c^{\max}\|z\|_V\|v\|_V \quad \text{(continuity)},\\ &a_d(v,v) \geqslant c^{\min}\left\|e_{33}(v)\right\|^2_{L^2(\Omega)} = c^{\min}\|v\|^2_V \quad \text{(ellipticity)}. \end{aligned} \tag{12}$$

The existence and uniqueness of solution of the class of bone remodeling models defined by (2) is established in Theorem 3.5 of Figueiredo and Trabucho [8]. The proof of existence relies on Schauder's fixed point theorem together with the Cauchy–Lipschitz–Picard theorem (used to solve the remodeling rate equation, for a fixed displacement), the Lax–Milgram lemma (that is necessary to guarantee the existence of solution of the variational equation, for a fixed change of volume fraction) and regularity results. The proof of uniqueness is based on arguments similar to those of Cowin and Nachlinger [5]. The next theorem summarizes this statement of existence and uniqueness.

Theorem 2.1 (Solution of (2)). *We assume that, for each fixed $\hat{d}$, the unique solution $\hat{u}$ of the equilibrium problem,*

$$\textit{Find } \hat{u}(.,t) \in V, \textit{ such that } a_{\hat{d}}(\hat{u}, v) = L_{\hat{d}}(v), \ \forall v \in V, \tag{13}$$

has components with the regularity $\hat{u}_\alpha(.,t) \in W^{3,2}(]0,L[)$ and $\underline{\hat{u}}_3(.,t) \in W^{2,2}(]0,L[)$, for any $t \in [0,T]$ (which implies that $\hat{u}(.,t) \in W^{2,2}(\Omega)$). Then, there exists a unique pair (u,d) solution of problem (2)*, verifying*:

$$u \in C^1\big([0,T];V\big) \quad \textit{and} \quad d \in C^1\big([0,T];C^0\big(\overline{\Omega}\big)\big). \tag{14}$$

3. Semi-discrete approximation

In this section we prove error estimates and convergence results for the semi-discrete approximation of (2), where only the space variables are discretized using the standard Galerkin method.

We consider first the space $Y = W^{2,2}(\Omega) \cap V$, which is a real separable Hilbert space, endowed with the inner product $(.,.)_Y$ defined by:

$$(u, v)_Y = \big(e_{33}(u), e_{33}(v)\big)_{L^2(\Omega)} + \big(\partial_3 e_{33}(u), \partial_3 e_{33}(v)\big)_{L^2(\Omega)}, \tag{15}$$

for any u and v in Y, and where $(.,.)_{L^2(\Omega)}$ is the usual inner product in $L^2(\Omega)$.

We introduce a family $\{V_h\}$ of finite dimensional subspaces of V, where $h > 0$ is a space discretization parameter and $\dim V_h = n(h) \to \infty$, as $h \to 0^+$. We assume that V_h is smooth enough, and such that, for any element (v_{h1}, v_{h2}, v_{h3}) in V_h, with $v_{h3} = \underline{v}_{h3} - x_\alpha \partial_3 v_{h\alpha}$, we have:

$$v_{h\alpha} \in W^{3,2}\big(]0, L[\big) \quad \text{and} \quad \underline{v}_{h3} \in W^{2,2}\big(]0, L[\big), \tag{16}$$

and consequently $V_h \subset Y = W^{2,2}(\Omega) \cap V$. Moreover we also assume that this family $\{V_h\}$ has the following approximation property (meaning that $\bigcup_{h>0} V_h$ is dense in Y for the norm $\|.\|_Y$, induced by the inner product $(.,.)_Y$):

$$\forall v \in Y, \quad \exists \{v_h\}_{h>0},\ v_h \in V_h\colon \quad \|v - v_h\|_Y \to 0, \quad \text{when } h \to 0^+. \tag{17}$$

The semi-discrete approximation of problem (2) is defined by:

$$\left[\begin{array}{l} \text{Find } (u_h, d_h) \text{ such that} \\ u_h = (u_{h1}, u_{h2}, u_{h3})\colon \overline{\Omega} \times [0, T] \to \mathbb{R}^3 \quad \text{and} \quad d_h\colon \overline{\Omega} \times [0, T] \to \mathbb{R}, \\ u_h(.,t) \in V_h, \\ a_{d_h}(u_h, v_h) = L_{d_h}(v_h), \quad \forall v_h \in V_h, \\ \dot{d}_h = c(d_h) e_{33}(u_h) + b(d_h), \quad \text{in } \Omega \times (0, T), \\ d_h(x, 0) = \eta_h(x), \quad \text{in } \overline{\Omega}. \end{array}\right. \tag{18}$$

By Theorem 2.1 and assuming the regularity (16) on $V_h \subset Y = W^{2,2}(\Omega) \cap V$, there exists a unique solution of problem (18).

Theorem 3.1. *We suppose that the initial conditions of problems* (2) *and* (18) *verify* $\|\eta_h - \eta\|_{L^2(\Omega)} \to 0$ *as* $h \to 0^+$. *Then the solution* (u_h, d_h) *of problem* (18) *satisfies*:

$$\exists \hat{c} > 0\colon \quad \|u_h\|_{C^0([0,T];V)} \leqslant \hat{c}, \quad \forall h > 0, \tag{19}$$

$$\exists \bar{c} > 0\colon \quad \|d_h\|_{C^0([0,T];L^2(\Omega))} \leqslant \bar{c}, \quad \forall h > 0, \tag{20}$$

where $\hat{c}$ *and* $\bar{c}$ *are constants independent of* h *and* t.

Proof. Taking $v_h = u_h$ in (18) we obtain $a_{d_h}(u_h, u_h) = L_{d_h}(u_h)$. Then, due to the condition $0 < \varepsilon/2 \leqslant (\xi_0 + \mathcal{P}_\varepsilon(d_h))(x) \leqslant 1$, the continuity of $L_{d_h}(.)$ and the uniform ellipticity of $a_{d_h}(.,.)$ independently of h, cf. (12), we obtain, for each t, the following norm estimate from below and from above,

$$c^{\min}\big\|u_h(.,t)\big\|_V^2 \leqslant a_{d_h}(u_h, u_h) \leqslant c\big\|u_h(.,t)\big\|_V, \tag{21}$$

where c is a constant independent of h and t. This implies that u_h is bounded in $C^0([0,T];V)$, because from (21) we deduce that

$$\|u_h\|_{C^0([0,T];V)} \leqslant \hat{c} = \frac{c}{c^{\min}}. \tag{22}$$

Taking now the integral with respect to time in the remodeling rate equation of problem (18) we get:

$$d_h(x,t) = \int_0^t \big[c(d_h)e_{33}(u_h) + b(d_h)\big]\,\mathrm{d}s + \eta_h(x). \tag{23}$$

But by (21)–(22) $\|e_{33}(u_h)\|_{C^0([0,T];L^2(\Omega))}$ is bounded from above by a constant and by (9) the sequences of material coefficients $c(d_h)$ and $b(d_h)$ are also bounded in $C^0([0,T];L^2(\Omega))$, thus we have that

$$\big\|d_h(.,t)\big\|_{L^2(\Omega)} \leqslant c + \|\eta_h\|_{L^2(\Omega)}, \tag{24}$$

with c a constant independent of h and t. But as η_h is bounded in $L^2(\Omega)$, the latter estimate clearly implies the estimate (20), and the proof is complete. □

The next theorem provides an estimate from above for the error $(u_h - u)(.,t)$ in the norm of the space V, which is fundamental for further proofs in this paper.

Theorem 3.2. *We suppose that the initial conditions of problems* (2) *and* (18) *verify* $\|\eta_h - \eta\|_{L^2(\Omega)} \to 0$ *as* $h \to 0^+$. *Then, there exists a constant* c, *independent of* h *and* t, *such that, for each* $t \in [0,T]$, *the first components of the solutions of problems* (2) *and* (18) *satisfy*:

$$\big\|(u_h - u)(.,t)\big\|_V \leqslant c\Big[\big\|u(.,t) - v_h\big\|_V + \big\|(d_h - d)(.,t)\big\|_{L^2(\Omega)}\big(1 + \big\|e_{33}(v_h)\big\|_{C^0(\overline{\Omega})}\big)\Big],$$
$$\forall v_h \in V_h. \tag{25}$$

Proof. The theorem is essentially a consequence of the first Strang lemma (which is a generalization of Céa's lemma, cf. Ciarlet [2,3]) which yields:

$$\begin{aligned}\big\|(u_h-u)(.\,,t)\big\|_V \leqslant c\bigg[\big\|u(.\,,t)-v_h\big\|_V + \frac{|L_{d_h}(u_h(.\,,t)-v_h)-L_d(u_h(.\,,t)-v_h)|}{\|u_h(.\,,t)-v_h\|_V}\\ + \frac{|a_{d_h}(v_h,u_h(.\,,t)-v_h)-a_d(v_h,u_h(.\,,t)-v_h)|}{\|u_h(.\,,t)-v_h\|_V}\bigg], \quad \forall v_h\in V_h,\end{aligned} \tag{26}$$

with $c>0$ a constant independent of h and t. We analyse now the consistency errors,

$$\frac{|L_{d_h}(u_h(.\,,t)-v_h)-L_d(u_h(.\,,t)-v_h)|}{\|u_h(.\,,t)-v_h\|_V}, \tag{27}$$

and

$$\frac{|a_{d_h}(v_h,u_h(.\,,t)-v_h)-a_d(v_h,u_h(.\,,t)-v_h)|}{\|u_h(.\,,t)-v_h\|_V}. \tag{28}$$

Using the mean value theorem for the operator $\mathcal{P}_\varepsilon$ we deduce that

$$\begin{aligned}\big|L_{d_h}(w_h)-L_d(w_h)\big| &= \bigg|\int_\Omega \gamma\big(\mathcal{P}_\varepsilon(d_h)-\mathcal{P}_\varepsilon(d)\big) f w_h\bigg|\\ &\leqslant c\big\|(d_h-d)(.\,,t)\big\|_{L^2(\Omega)}\|w_h\|_V, \quad \forall w_h\in V_h,\end{aligned} \tag{29}$$

where c is another strictly positive constant, independent of h and t and depending on f, γ and $\mathcal{P}_\varepsilon$. Therefore, considering $w_h=u_h(.\,,t)-v_h$, we have for each t, the consistency error estimate for the loads,

$$\frac{|L_{d_h}(u_h(.\,,t)-v_h)-L_d(u_h(.\,,t)-v_h)|}{\|u_h(.\,,t)-v_h\|_V} \leqslant c\big\|(d_h-d)(.\,,t)\big\|_{L^2(\Omega)}. \tag{30}$$

We remark now that $e_{33}(v_h)\in C^0(\overline{\Omega})$, for any $v_h\in V_h$. In fact $\partial_3\underline{v}_{h3}$ and $\partial_{33}v_{h\alpha}$ belong to the space $W^{1,2}(]0,L[)$, because of the regularity hypotheses described in (16), and the inclusion $W^{1,2}(]0,L[)\subset C^0([0,L])$ is continuous. Thus, using this property, and the mean value theorem for the scalar function $b_{3333}(.)$, we immediately deduce the following estimate:

$$\begin{aligned}\big|a_{d_h}(v_h,w_h)-a_d(v_h,w_h)\big| &= \bigg|\int_\Omega\bigg[\frac{1}{b_{3333}(d_h)}-\frac{1}{b_{3333}(d)}\bigg]e_{33}(v_h)e_{33}(w_h)\bigg|\\ &= \bigg|\int_\Omega\bigg[\frac{b_{3333}(d)-b_{3333}(d_h)}{b_{3333}(d_h)b_{3333}(d)}\bigg]e_{33}(v_h)e_{33}(w_h)\bigg|\\ &\leqslant c\big\|(d_h-d)(.\,,t)\big\|_{L^2(\Omega)}\big\|e_{33}(v_h)\big\|_{C^0(\overline{\Omega})}\|w_h\|_V, \quad \forall w_h\in V_h,\end{aligned} \tag{31}$$

where c is another strictly positive constant, independent of h and t. So, considering $w_h = u_h(.\,,t) - v_h$

$$\frac{|a_{d_h}(v_h, u_h(.\,,t) - v_h) - a_d(v_h, u_h(.\,,t) - v_h)|}{\|u_h(.\,,t) - v_h\|_V} \leqslant c\big\|(d_h - d)(.\,,t)\big\|_{L^2(\Omega)}\big\|e_{33}(v_h)\big\|_{C^0(\overline{\Omega})}. \tag{32}$$

Gathering estimates (26), (30) and (32) we obtain (25). □

As a consequence of Theorem 3.2 we prove now an estimate, in the norm of the space $L^2(\Omega)$, for the error $(d_h - d)(.\,,t)$.

Theorem 3.3. *We suppose that the initial conditions of problems* (2) *and* (18) *verify* $\|\eta_h - \eta\|_{L^2(\Omega)} \to 0$ *as* $h \to 0^+$. *Then, there exist constants* c_1 *and* c_2, *independent of* h *and* t, *such that, for each* $t \in [0, T]$, *the second components of the solutions of problems* (2) *and* (18) *satisfy*:

$$\begin{aligned}\big\|(d_h - d)(.\,,t)\big\|_{L^2(\Omega)} &\leqslant \Big(c_1 T \max_{t\in[0,T]}\big\|u(.\,,t) - v_h\big\|_V + \|\eta_h - \eta\|_{L^2(\Omega)}\Big)\\ &\quad\times \big(1 + c_2\big(2 + \big\|e_{33}(v_h)\big\|_{C^0(\overline{\Omega})}\big)\big)T\mathrm{e}^{Tc_2(2+\|e_{33}(v_h)\|_{C^0(\overline{\Omega})})}, \quad \forall v_h \in V_h. \end{aligned}\tag{33}$$

Proof. Subtracting the two remodeling rate equations and integrating in time, we have:

$$\begin{aligned}(d_h - d)(x,t) &= \int_0^t \big[\big(c(d_h) - c(d)\big)e_{33}(u_h) + c(d)e_{33}(u_h - u) + b(d_h) - b(d)\big]\,\mathrm{d}s\\ &\quad + (\eta_h - \eta).\end{aligned}\tag{34}$$

Using the mean value theorem for the scalar functions $c(.)$ and $b(.)$, and Theorem 3.1, we have that

$$\begin{aligned}\big\|(d_h - d)(.\,,t)\big\|_{L^2(\Omega)} &\leqslant \|\eta_h - \eta\|_{L^2(\Omega)}\\ &\quad + \hat{c}\int_0^t \big[\big\|(d_h - d)(.\,,s)\big\|_{L^2(\Omega)} + \big\|(u_h - u)(.\,,s)\big\|_V\big]\,\mathrm{d}s,\end{aligned}\tag{35}$$

where $\hat{c} > 0$ is a constant independent of h and t. Using the estimate (25) we get:

$$\begin{aligned}&\big\|(d_h - d)(.\,,t)\big\|_{L^2(\Omega)}\\ &\quad\leqslant \Big[\hat{c}cT \max_{t\in[0,T]}\big\|u(.\,,t) - v_h\big\|_V + \|\eta_h - \eta\|_{L^2(\Omega)}\Big]\\ &\qquad + \int_0^t \big\|(d_h - d)(.\,,s)\big\|_{L^2(\Omega)}\big(\hat{c} + \hat{c}c\big(1 + \big\|e_{33}(v_h)\big\|_{C^0(\overline{\Omega})}\big)\big)\,\mathrm{d}s, \quad \forall v_h \in V_h.\end{aligned}\tag{36}$$

Applying now the integral Gronwall's inequality (cf. Evans [7, p. 625]) we have the estimate (33), with the constant $c_1 = \hat{c}c$ and $c_2 = \max\{\hat{c}, \hat{c}c\}$. □

Using again the estimates (25) and (33) the following convergence holds for the semi-discrete approximation of (2).

Theorem 3.4. *We suppose that the initial conditions of problems* (2) *and* (18) *verify* $\|\eta_h - \eta\|_{L^2(\Omega)} \to 0$, *as* $h \to 0^+$. *Then the solutions* (u, d) *and* (u_h, d_h) *of problems* (2) *and* (18) *verify*:

$$\|u - u_h\|_{C^0([0,T];V)} \to 0, \quad \text{as } h \to 0^+, \tag{37}$$

$$\|d - d_h\|_{C^1([0,T];L^2(\Omega))} \to 0, \quad \text{as } h \to 0^+. \tag{38}$$

Proof. We first note that the space of polynomials $p_h : [0, T] \to V_h$,

$$p_h(t) = a_h^0 + a_h^1 t + \cdots + a_h^n t^n, \tag{39}$$

with $a_h^i \in V_h$, for all $i = 0, \ldots, n$, and n a positive integer, is dense in the space $W(Y)$ defined by:

$$W(Y) = \left\{v \in L^2\big([0, T]; Y\big): \ \dot{v} \in L^2\big([0, T]; Y'\big)\right\}, \tag{40}$$

where Y' is the dual of the space Y and $L^2([0, T]; X)$, with $X = Y$ or $X = Y'$, denotes the space of functions $v : t \to v(t) \in X$, equipped with the norm $\|v\|_{L^2([0,T];X)} = (\int_0^T \|v(t)\|_X^2)^{1/2}$. This statement is a consequence of the assumption (17) which establishes that $\bigcup_{h>0} V_h$ is dense in Y and a property of $W(Y)$ (cf. Haslinger, Miettinen and Panagiotopoulos [11, p. 17, Remark 1.3]).

So, as u belongs to $W(Y)$, there exists a sequence of polynomials $\{p_h\}$ converging strongly to u in $W(Y)$. Moreover since the embedding $W(Y) \subset C^0([0, T]; Y)$ is continuous (cf. Haslinger, Miettinen and Panagiotopoulos [11, p. 17, Proposition 1.4]) we have:

$$\|u - p_h\|_{C^0([0,T];Y)} \to 0, \quad \text{as } h \to 0^+, \tag{41}$$

and hence

$$\|u - p_h\|_{C^0([0,T];V)} \to 0, \quad \text{as } h \to 0^+. \tag{42}$$

In addition, for each t, $e_{33}(p_h(t)) = \partial_3 \underline{p}_{h3}(t) - x_\alpha \partial_{33} p_{h\alpha}(t)$, where $\partial_3 \underline{p}_{h3}(t)$ and $\partial_{33} p_{h\alpha}(t)$ are in the space $W^{1,2}(]0, L[)$, which is continuously embedded in $C^0([0, L])$. Consequently, the sequence $e_{33}(p_h)$ is bounded in $C^0([0, T]; C^0(\overline{\Omega}))$, because from (41),

$$\exists c > 0: \quad \|p_h\|_{C^0([0,T];Y)} \leqslant c, \tag{43}$$

and

$$\left\| e_{33}(p_h) \right\|_{C^0([0,T];C^0(\overline{\Omega}))} \leqslant c_1 \left\| e_{33}(p_h) \right\|_{C^0([0,T];W^{1,2}(\Omega))} \leqslant c_2 \| p_h \|_{C^0([0,T];Y)}, \quad (44)$$

with c_1 and c_2 two constants independent of h and t. Therefore, taking $v_h = p_h(.\,,t)$, firstly in (33) and then in (25), we obtain the aimed convergence results in $C^0([0,T];L^2(\Omega))$ for $\{d_h\}$ and in $C^0([0,T];V)$ for the sequence $\{u_h\}$. Using this latter convergences and the equality,

$$(\dot{d}_h - \dot{d})(x,t) = \big(c(d_h) - c(d)\big)e_{33}(u_h) + c(d)e_{33}(u_h - u) + b(d_h) - b(d), \quad (45)$$

we obtain as well the convergence of $\{d_h\}$ to d in $C^1([0,T];L^2(\Omega))$, as $h \to 0^+$. □

4. Fully discrete approximation

In this section we describe the fully discrete approximation of problem (2) and we prove also some error estimates and convergence results.

We divide the interval $[0,T]$ into $N = N(\Delta t)$ intervals of length Δt, where Δt tends to zero as $N(\Delta t) \to +\infty$, and

$$0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_N = T, \quad (46)$$

where $\{t_i\}$ is the sequence of discretization time nodes and $t_i = t_{i-1} + \Delta t$, for $i = 0, \ldots, N$. We seek an approximation (u_h^i, d_h^i) at node t_i to the solution (u_h, d_h) of problem (18), and therefore an approximation to the exact solution (u,d) of problem (2). In order to obtain (u_h^i, d_h^i) we consider a numerical scheme to approximate the remodeling rate equation, such that, for each time node t_i the change in volume fraction $d_h(.\,,t_i)$ is replaced by its approximation d_h^i and consequently introducing d_h^i in the variational inequality of (18) $u_h(.\,,t_i)$ is substituted by the approximation u_h^i. Since the remodeling rate equation is an ordinary differential equation and due to the structure of the model (2) we choose the forward Euler method to approximate this equation. Then, the fully discrete approximation of problem (2) is defined by:

$$\begin{aligned}
&\text{Given } d_h^0 = d_{h,\Delta t}^0,\ \text{determine } u_h^0 \in V_h \text{ by } a_{d_h^0}(u_h^0, v_h) = L_{d_h^0}(v_h),\ \forall v_h \in V_h,\\
&\text{and find } (u_h^i, d_h^i),\ \text{for } i = 1, \ldots, N \text{ such that}\\
&\left[\begin{array}{l}
u_h^i \in V_h,\\
a_{d_h^i}(u_h^i, v_h) = L_{d_h^i}(v_h), \quad \forall v_h \in V_h,\\
d_h^i = d_h^{i-1} + [c(d_h^{i-1})e_{33}(u_h^{i-1}) + b(d_h^{i-1})]\Delta t.
\end{array}\right.
\end{aligned} \quad (47)$$

We remark that $d_h^0 = d_{h,\Delta t}^0$ is an approximation of the initial value $\eta_h = d_h(.\,,0)$ (cf. (18)) and may depend on the time discretization parameter Δt (we can also choose $d_{h,\Delta t}^0 = \eta_h$, which is independent of Δt).

We notice that other explicit one-step or multistep methods, like Runge–Kutta methods or Adams–Bashford methods, for example, could also be used to approximate the remodeling rate equation. We believe that, the theoretical convergence analysis, that we carry out in this section for the discrete problem (47), can also apply if the forward Euler method is replaced by one of the above mentioned explicit methods.

We shall show in this section that the solution of the fully discrete problem (47) approximates the exact solution of problem (2).

In order to present a preliminary result, we introduce first some notations concerning the semi-discrete variational equation of (18), that is, $a_{d_h}(u_h, v_h) = L_{d_h}(v_h)$, for all $v_h \in V_h$. Denoting by $n(h)$ the dimension of the space V_h and by $\{\varphi_h^k\}_{k=1}^{n(h)}$ a basis of V_h, then V_h is isometrically isomorphic to $\mathbb{R}^{n(h)}$. This isomorphism associates to each $v_h \in V_h$ a vector $\hat{v}_h = (v_k)_{k=1}^{n(h)}$, that is

$$v_h = \sum_{k=1}^{n(h)} v_k \varphi_h^k \quad \Longrightarrow \quad \hat{v}_h = (v_k)_{k=1}^{n(h)} \in \mathbb{R}^{n(h)}. \tag{48}$$

Hence, inserting $u_h = \sum_{k=1}^{n(h)} u_k \varphi_h^k$ and $v_h = \sum_{k=1}^{n(h)} v_k \varphi_h^k$ in the variational equation $a_{d_h}(u_h, v_h) = L_{d_h}(v_h)$, we obtain that this variational equation is equivalent to the following matrix equation:

$$\hat{u}_h = (u_k)_{k=1}^{n(h)} \in \mathbb{R}^{n(h)}\colon\ A_{d_h}\hat{u}_h = F_{d_h}. \tag{49}$$

The solution $\hat{u}_h$ depends on t (that is, for each k, u_k depends on t), A_{d_h} is a symmetric and positive definite matrix of order $n(h)$ and F_{d_h} is a vector with $n(h)$ components, both depending on d_h. More exactly, we have:

$$A_{d_h} = \big(a_{d_h}\big(\varphi_h^k, \varphi_h^l\big)\big)_{k,l=1}^{n(h)} \quad \text{with } a_{d_h}\big(\varphi_h^k, \varphi_h^l\big) = \int_\Omega \frac{1}{b_{3333}(d_h)} e_{33}\big(\varphi_h^k\big) e_{33}\big(\varphi_h^l\big)\,\mathrm{d}\Omega, \tag{50}$$

and

$$\begin{aligned} &F_{d_h} = \big(L_{d_h}\big(\varphi_h^k\big)\big)_{k=1}^{n(h)} \quad \text{with} \\ &L_{d_h}\big(\varphi_h^k\big) = \int_\Omega \gamma\big(\xi_0 + \mathcal{P}_\varepsilon(d_h)\big) f_k \varphi_h^k\,\mathrm{d}\Omega + \int_\Gamma g_k \varphi_h^k\,\mathrm{d}\Gamma + \int_{\Gamma_0\cup\Gamma_L} h_k \varphi_h^k\,\mathrm{d}(\Gamma_0 \cup \Gamma_L). \end{aligned} \tag{51}$$

Proposition 4.1. *We suppose that* $\frac{1}{b_{3333}(d_h)} = r + \mathcal{O}(h)$, *where* r *is a scalar function, such that* r *is independent of* h, $0 < |r| \leqslant c$ *with* $c > 0$ *a constant, and* $\mathcal{O}(h)$ *is a term of order* h (*cf. Monnier and Trabucho* [13], *formulas* (6) *and* (2), *for a justification of this condition on the material coefficient* $b_{3333}(d_h)$). *In addition we also assume that, for each* k, *the sequence of basis functions* $\{\varphi_h^k\}$ *verifies*:

$$\exists c > 0\colon \quad \big\|\varphi_h^k\big\|_Y \leqslant c, \quad \forall h > 0, \tag{52}$$

where c is independent of k and h. Then, the sequence of solutions $\{u_h\}_{h>0}$ of the semi-discrete variational equations $a_{d_h}(u_h, v_h) = L_{d_h}(v_h)$ (cf. (18)), *verifies*:

$$\exists c_0 > 0\colon \quad \|u_h\|_{C^0([0,T];Y)} \leqslant c_0, \tag{53}$$

and consequently

$$\exists c_1 > 0\colon \quad \big\|e_{33}(u_h)\big\|_{C^0([0,T];C^0(\overline{\Omega}))} \leqslant c_1, \tag{54}$$

with c_0 and c_1 two different constants independent of h and t.

Proof. We remark that the solution of Eq. (49) is equal to:

$$\hat{u}_h = A_{d_h}^{-1} F_{d_h} \in \mathbb{R}^{n(h)}. \tag{55}$$

But, because of (51)–(52), the components of the vector F_{d_h} are bounded from above by a constant independent of h, and, also because of the hypothesis on $\frac{1}{b_{3333}(d_h)}$, any matrix norm of $A_{d_h}^{-1}$ is bounded from above by a constant independent of h. Therefore, any component of the vector $\hat{u}_h$ is bounded from above by a constant independent of h. Consequently the sequence $\{u_h\}_{h>0}$ verifies (53), since $u_h = \sum_{k=1}^{n(h)} u_k \varphi_h^k$, the condition (52) is verified and $u_h \in C^1([0,T];Y)$ (this regularity of u_h is a consequence of the existence theorem, as explained in the sentence immediately after the semi-discrete formulation (18)). Moreover, for each $h > 0$, $e_{33}(u_h) \in C^0([0,T]; C^0(\overline{\Omega}))$, because of the regularity hypothesis (16) required to the space V_h and also because the space $W^{1,2}(\,]0,L[\,)$ is continuously embedded in the space $C^0(\,]0,L[\,)$. Consequently we have:

$$\big\|e_{33}(u_h)\big\|_{C^0([0,T];C^0(\overline{\Omega}))} \leqslant c_1 \big\|e_{33}(u_h)\big\|_{C^0([0,T];W^{1,2}(\Omega))} \leqslant c_2 \|u_h\|_{C^0([0,T];Y)}, \tag{56}$$

with c_1 and c_2 strictly positive constants independent of h and t, which proves (54). □

We prove now two preliminary norm estimates.

Theorem 4.1. *There exist constants $c_1 > 0$ and $c_2 > 0$, independent of h and t_i, such that, the first components u_h and u_h^i of the solutions of problems* (18) *and* (47) *verify*:

$$\big\|u_h^i\big\|_V \leqslant c_1, \tag{57}$$

$$\big\|u_h(.\,,t_i) - u_h^i\big\|_V \leqslant c_2\big[\big\|u_h(.\,,t_i) - v_h\big\|_V + \big\|d_h(.\,,t_i) - d_h^i\big\|_{L^2(\Omega)}\big(1 + \big\|e_{33}(v_h)\big\|_{C^0(\overline{\Omega})}\big)\big], \quad \forall v_h \in V_h, \tag{58}$$

for all $i = 0, 1, \ldots, N$ and for all $h > 0$. Moreover, assuming the hypothesis of Proposition 4.1, *then, there exists a constant $c_3 > 0$, such that, the estimate* (58) *becomes*:

$$\big\|u_h(.\,,t_i) - u_h^i\big\|_V \leqslant c_3 \big\|d_h(.\,,t_i) - d_h^i\big\|_{L^2(\Omega)}. \tag{59}$$

Proof. The result (57) can be proven in the same way as in Theorem 3.1. We can also repeat the arguments used in Theorem 3.2 for the two following problems:

$$u_h \in V_h\colon \quad a_{d_h}(u_h, v_h) = L_{d_h}(v_h), \quad \forall v_h \in V_h, \tag{60}$$

$$u_h^i \in V_h\colon \quad a_{d_h^i}\big(u_h^i, v_h\big) = L_{d_h^i}(v_h), \quad \forall v_h \in V_h, \tag{61}$$

considering the time $t = t_i$ in problem (60). Consequently we obtain the estimate (25), where t is replaced by t_i, u by u_h, u_h by u_h^i, d by d_h, and d_h by d_h^i, that is:

$$\big\|u_h(.\,,t_i) - u_h^i\big\|_V \leqslant c\Big[\big\|u_h(.\,,t_i) - v_h\big\|_V + \big\|d_h(.\,,t_i) - d_h^i\big\|_{L^2(\Omega)}\big(1 + \big\|e_{33}(v_h)\big\|_{C^0(\overline{\Omega})}\big)\Big], \quad \forall v_h \in V_h. \tag{62}$$

Assuming the hypothesis of Proposition 4.1, and taking $v_h = u_h(.\,,t_i)$ in estimate (62), then the property (54) is verified, and we have (59). □

Theorem 4.2. *We assume the hypotheses of Proposition* 4.1. *Then, there exists a constant* $c > 0$, *independent of* h *and* t_i, *such that, the second components* d_h *and* d_h^i *of the solutions of problems* (18) *and* (47) *verify, for all* $i = 0, 1, \ldots, N$ *and for all* $h > 0$,

$$\big\|d_h(.\,,t_i) - d_h^i\big\|_{L^2(\Omega)} \leqslant \mathrm{e}^{c(t_i - t_0)}\big\|\eta_h - d_h^0\big\|_{L^2(\Omega)} + g(\Delta t)\frac{\mathrm{e}^{c(t_i - t_0)} - 1}{c}, \tag{63}$$

where $g(.)$ *is a scalar function independent of* h, *that tends to zero as* $\Delta t \to 0$.

Proof. We first introduce, for the forward Euler method, the error e_h^i between d_h and d_h^i:

$$e_h^i(x) = d_h(x, t_i) - d_h^i(x), \tag{64}$$

and the corresponding consistency error ε_h^i,

$$\varepsilon_h^i(x) = d_h(x, t_{i+1}) - d_h(x, t_i) - \Delta t\, F\big(t_i, d_h(x, t_i)\big), \tag{65}$$

where

$$F\big(t_i, d_h(.\,,t_i)\big) = c\big(d_h(.\,,t_i)\big)e_{33}\big(u_h(.\,,t_i)\big) + b\big(d_h(.\,,t_i)\big). \tag{66}$$

From (64)–(66) and (47) we have that

$$e_h^{i+1} = e_h^i + \Delta t\Big[F\big(t_i, d_h(.\,,t_i)\big) - F\big(t_i, d_h^i\big)\Big] + \varepsilon_h^i. \tag{67}$$

Using the definition (66) we obtain the following estimate:

$$\begin{aligned}\left\|F\left(t_i, d_h(.\,,t_i)\right) - F\left(t_i, d_h^i\right)\right\|_{L^2(\Omega)} \leqslant{}& \left\|b\left(d_h(.\,,t_i)\right) - b\left(d_h^i\right)\right\|_{L^2(\Omega)} \\ &+ \left\|\left[c\left(d_h(.\,,t_i)\right) - c\left(d_h^i\right)\right]e_{33}\left(u_h^i\right)\right\|_{L^2(\Omega)} \\ &+ \left\|c\left(d_h(.\,,t_i)\right)e_{33}\left(u_h(.\,,t_i) - u_h^i\right)\right\|_{L^2(\Omega)}, \quad (68)\end{aligned}$$

therefore applying the mean value theorem to the scalar functions $b(.)$ and $c(.)$ and Theorem 4.1, we get:

$$\begin{aligned}\left\|F\left(t_i, d_h(.\,,t_i)\right) - F\left(t_i, d_h^i\right)\right\|_{L^2(\Omega)} &\leqslant c_1\left\|d_h(.\,,t_i) - d_h^i\right\|_{L^2(\Omega)} + c_2\left\|u_h(.\,,t_i) - u_h^i\right\|_V \\ &\leqslant c\left\|d_h(.\,,t_i) - d_h^i\right\|_{L^2(\Omega)}, \quad (69)\end{aligned}$$

where the last inequality is a consequence of (59) and c_1, c_2, c are different strictly positive constants independent of h and t_i, for all $i = 0, 1, \ldots, N$. We also conclude, from (69), that $F(t,.)$ is Lipschitz continuous with respect to the second argument with c the Lipschitz constant. From (67) and (69) we have:

$$\left\|e_h^{i+1}\right\|_{L^2(\Omega)} \leqslant (1 + \Delta t c)\left\|e_h^i\right\|_{L^2(\Omega)} + \left\|\varepsilon_h^i\right\|_{L^2(\Omega)}. \quad (70)$$

Moreover due to the definition of ε_h^i we clearly have:

$$\varepsilon_h^i(x) = \int_{t_i}^{t_{i+1}} \left[\dot{d}_h(x,s) - \dot{d}_h(x,t_i)\right] \mathrm{d}s, \quad (71)$$

so we get the following bound from above for $\|\varepsilon_h^i\|_{L^2(\Omega)}$,

$$\left\|\varepsilon_h^i\right\|_{L^2(\Omega)} \leqslant \max_{s\in[t_i,t_{i+1}]}\left\|\dot{d}_h(.\,,s) - \dot{d}_h(.\,,t_i)\right\|_{L^2(\Omega)} \int_{t_i}^{t_{i+1}} 1\,\mathrm{d}s \leqslant g(\Delta t)\Delta t, \quad (72)$$

where $g(\Delta t)$ is independent of h and converges to 0 as $\Delta t \to 0^+$, because $\dot{d}_h \in C^0([0,T]; L^2(\Omega))$ and $\dot{d}_h$ converges to $\dot{d}$ in $C^0([0,T]; L^2(\Omega))$.

Using (70) and (72) we can argue, for instance, as in Crouzeix and Mignot [6, p. 76], and we directly obtain (63). We repeat here this argument for convenience of the reader. It relies on the following lemma (whose proof is immediate by induction in i, because of the inequality $1 + x \leqslant \mathrm{e}^x$, for all $x \in \mathbb{R}$):

Lemma. *Let $\theta_i \geqslant 0$ and $\alpha_i \geqslant 0$ be two sequences of real numbers, such that*

$$\theta_{i+1} \leqslant (1 + c\Delta t_i)\theta_i + \alpha_i, \quad \forall i \geqslant 0, \quad (73)$$

where $c > 0$ *is a positive constant and* $\Delta t_i = t_{i+1} - t_i$, *then*

$$\theta_i \leqslant \mathrm{e}^{c(t_i - t_0)}\theta_0 + \sum_{j=0}^{i-1} \mathrm{e}^{c(t_i - t_{j+1})}\alpha_j, \quad \forall i \geqslant 0. \tag{74}$$

Considering $\theta_i = \|e_h^i\|_{L^2(\Omega)}$ and $\alpha_i = \|\varepsilon_h^i\|_{L^2(\Omega)}$ we may apply (74) to (70) and also using (72) we obtain:

$$\begin{aligned} \|e_h^i\|_{L^2(\Omega)} &\leqslant \mathrm{e}^{c(t_i - t_0)}\|e_h^0\|_{L^2(\Omega)} + g(\Delta t)\sum_{j=0}^{i-1} \mathrm{e}^{c(t_i - t_{j+1})} \int_{t_j}^{t_{j+1}} 1\,\mathrm{d}s \\ &\leqslant \mathrm{e}^{c(t_i - t_0)}\|e_h^0\|_{L^2(\Omega)} + \sum_{j=0}^{i-1} g(\Delta t) \int_{t_j}^{t_{j+1}} e^{c(t_i - s)}\,\mathrm{d}s \\ &\leqslant \mathrm{e}^{c(t_i - t_0)}\|e_h^0\|_{L^2(\Omega)} + g(\Delta t)\frac{\mathrm{e}^{c(t_i - t_0)} - 1}{c}, \end{aligned} \tag{75}$$

and the proof is complete. □

We are able now to conclude the following convergence result:

Theorem 4.3. *We suppose that the initial conditions* η, η_h *and* $d_h^0 = d_{h,\Delta t}^0$ *of problems* (2), (18) *and* (47) *verify* $\|\eta - \eta_h\|_{L^2(\Omega)} \to 0$ *and* $\|\eta_h - d_h^0\|_{L^2(\Omega)} \to 0$ *as* $h \to 0^+$ *and* $\Delta t \to 0^+$. *Moreover, we assume the hypotheses of Proposition* 4.1. *Then, the solutions* (u, d) *and* (u_h^i, d_h^i) *of problems* (2) *and* (47) *satisfy*:

$$\lim_{h\to 0^+} \lim_{\Delta t \to 0^+} \Big(\max_{0\leqslant i \leqslant N} \big\|u(.\,, t_i) - u_h^i\big\|_V\Big) = 0, \tag{76}$$

$$\lim_{h\to 0^+} \lim_{\Delta t \to 0^+} \Big(\max_{0\leqslant i \leqslant N} \big\|d(.\,, t_i) - d_h^i\big\|_{L^2(\Omega)}\Big) = 0. \tag{77}$$

Proof. It is a trivial consequence of the following triangular norm inequalities:

$$\begin{aligned} &\big\|u(.\,, t_i) - u_h^i\big\|_V \leqslant \big\|u(.\,, t_i) - u_h(.\,, t_i)\big\|_V + \big\|u_h(.\,, t_i) - u_h^i\big\|_V, \\ &\big\|d(.\,, t_i) - d_h^i\big\|_{L^2(\Omega)} \leqslant \big\|d(.\,, t_i) - d_h(.\,, t_i)\big\|_{L^2(\Omega)} + \big\|d_h(.\,, t_i) - d_h^i\big\|_{L^2(\Omega)} \end{aligned} \tag{78}$$

and Theorems 3.4 and 4.1–4.2. □

5. Conclusion and future work

We have proved theoretical error estimates and convergence results for the approximation of a class of bone remodeling models, that consists of a quasi-static system coupling a variational equation, of elliptic type, with an ordinary differential equation with respect to time. We have used a Galerkin method for the space discretization and a forward Euler method for the time discretization. The structure of this class of models enables the use of the integral Gronwall's inequality and a generalization of Céa's lemma, which are the fundamental mathematical tools for the proofs presented in this paper. The choice of an explicit method, as the forward Euler method, to approximate the remodeling rate equation, is also suggested by the structure of this class of models.

We observe that we could have considered in (2) a remodeling rate equation depending nonlinearly on $e_{33}(u)$, that is (cf. Figueiredo and Trabucho [8], formula (74)):

$$\dot{d} = \frac{1}{b_{3333}(d)} e_{33}(u) e_{33}(u) + c(d) e_{33}(u) + b(d), \tag{79}$$

which is an equation that seems to be more suitable to represent the remodeling rate process, from the mechanical view-point, even in the case of small strains (cf. Hegedus and Cowin [12]). In fact, all the convergence and error estimates results presented in this paper can also be derived for this type of nonlinear remodeling rate equation; the nonlinear term $\frac{1}{b_{3333}(d)} e_{33}(u) e_{33}(u)$ in (79) only originates more complicated calculus.

Moreover we intend to do some numerical experiments in order to confirm the theoretical convergence results presented in this paper.

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