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Exponentiation in V-categories

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Abstract

For a Heyting algebra **V** which, as a category, is monoidal closed, we obtain characterizations of exponentiable objects and morphisms in the category of **V**-categories and apply them to some well-known examples. In the case $\mathbf{V} = \overline{\mathbb{R}}_+$ these characterizations of exponentiable morphisms and objects in the categories (**P**)**Met** of (pre)metric spaces and non-expansive maps show in particular that exponentiable metric spaces are exactly the almost convex metric spaces, while exponentiable complete metric spaces are the complete totally convex ones.

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0. Introduction

In 1973, Lawvere [10] observed that premetric spaces can be thought of as enriched categories over $[0,\infty]$: a premetric $d : X \times X \to [0,\infty]$ can be interpreted as the hom-functor of a category so that the inequalities

$$0 \geqslant d(x,x),$$

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$$d(x,y)+d(y,z)\geqslant d(x,z)$$

play the role of the unit and of the composition laws of a category.

Indeed, a **V**-category in the sense of Eilenberg and Kelly [8] is a set X endowed with a map $X\times X\to\mathbf{V}$, $(x,y)\mapsto X(x,y)$, and specifications of identity morphisms and composition law

$$I\to X(x,x),$$

$$X(x,y)\otimes X(y,z)\to X(x,z)$$

satisfying unity and associativity axioms, that can be expressed by commutative diagrams in the category **V**. (Here $\otimes$ is a tensor product in **V** with neutral element I.)

In our crucial example of premetric spaces, the category **V** is the complete real half-line $[0,\infty]$, with categorical structure $u\to v$ if $u\geqslant v$, with tensor product $+$ and unit 0. Since our category **V** is given by a lattice, commutativity of diagrams, and hence also the unity and associativity axioms, come for free.

Another interesting example arises for **V** the two-element chain $\mathbf{2}=\{\text{false}\vdash\text{true}\}$, with the tensor product $\wedge$, that coincides with the categorical one, and neutral element true. For a map $X\times X\to\mathbf{2}$, or equivalently for a binary relation $\leqslant$ in X, the unit and composition laws read as

$$\text{true}\vdash x\leqslant x,$$

$$(x\leqslant y)\wedge(y\leqslant z)\vdash(x\leqslant z),$$

so that $(X,\leqslant)$ is exactly a preordered set.

If $\mathbf{V}=\mathbf{Set}$, and if the tensor product is the categorical one, then a **V**-category is exactly a small category.

Moreover, in the former examples the notion of **V**-functor gives natural morphisms: an $\overline{\mathbb{R}}_+$-functor is a non-expansive map, a **2**-functor is a monotone map, while a **Set**-functor is just a functor.

The tensor product in **V** induces naturally a tensor product in the category **V**-**Cat** of **V**-categories and **V**-functors. Lawvere proved that with **V** also **V**-**Cat** is a monoidal closed category. Since in the examples $\mathbf{V}=\mathbf{2}$ and $\mathbf{V}=\mathbf{Set}$ the tensor and the categorical products coincide, this gives in particular that **2**-**Cat** and **Cat**(= **Set**-**Cat**) are Cartesian closed categories. Cartesian closedness of $\overline{\mathbb{R}}_+$-**Cat** does not follow from Lawvere's result, since the tensor product $+$ does not coincide with the categorical product max. In fact this category is not Cartesian closed, although $\overline{\mathbb{R}}_+$ is. Moreover, none of the categories listed is locally Cartesian closed. It is possible to avoid this problem working instead in the larger category **V**-**RGph** of reflexive **V**-graphs, which is a quasi-topos whenever **V** is locally Cartesian closed (see [5]).

The goal of this paper is to identify exponentiable morphisms and objects in **V**-**Cat** when **V** is a lattice, that is, a small category with at most one morphism between each pair of objects. Then the embedding $\mathbf{V}\text{-}\mathbf{Cat}\hookrightarrow\mathbf{V}\text{-}\mathbf{RGph}$ is full, a property that plays an essential role in this work. Our main result is Theorem 3.4 which characterizes exponentiable morphisms (hence in particular exponentiable objects) in **V**-**Cat** whenever the lattice **V** is complete and locally Cartesian closed, which means exactly that **V** is a complete Heyting algebra. This result gives in particular the characterization of exponentiable monotone maps in **POrd** obtained by Tholen [12] and new characterizations of exponentiable morphisms and objects in the category **Met** of metric spaces and non-expansive maps.

Finally we observe that some of the proofs presented in this paper follow the guidelines of the proof of the characterization of exponentiable continuous maps in **Top** obtained in [4], which raises the problem of whether these techniques can be applied to obtain such a characterization in the more general context of reflexive and transitive lax algebras described in [3] or [6].

In Section 1 we present the basic results on **V**-categories we need throughout. Section 2 gives an account on exponentiability, including exponentiation in the category of reflexive **V**-graphs. Section 3 contains our main results, namely the characterizations of exponentiable morphisms and objects in **V**-**Cat**. The last section deals with examples, giving a special attention to exponentiability in the category **Met** of metric spaces and non-expansive maps.

1. The category of V-categories

1.1. Throughout **V** is a complete lattice equipped with a symmetric tensor product $\otimes$, with unit k, and with right adjoint hom; that is, for each $u, v, w \in \mathbf{V}$,

$$u \otimes v \leqslant w \iff v \leqslant \hom(u, w).$$

As a category, **V** is said to be a *closed symmetric monoidal category*.

A **V**-*enriched category* (or simply **V**-*category*) is a pair (X, a) with X a set and $a : X \times X \to \mathbf{V}$ a map such that:

(R) for each $x \in X$, $k \leqslant a(x, x)$;
(T) for each $x, x', x'' \in X$, $a(x, x') \otimes a(x', x'') \leqslant a(x, x'')$.

Given **V**-categories (X, a) and (Y, b), a **V**-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that, for each $x, x' \in X$, $a(x, x') \leqslant b(f(x), f(x'))$. They form the category **V**-**Cat** of **V**-categories and **V**-functors.

1.2. We list now two guiding examples of such categories, obtained when **V** is the two-element chain $\mathbf{2} = \{\text{false} \vdash \text{true}\}$, with the monoidal structure given by $\wedge$ and "true", and when **V** is the (complete) real half-line $[0, \infty]$, with the categorical structure induced by the relation $\geqslant$ (i.e., $a \to b$ means $a \geqslant b$) and with tensor product $+$, which we will denote by $\overline{\mathbb{R}}_+$.

For $\mathbf{V} = \mathbf{2}$, with the usual notation $x \leqslant x' :\iff a(x, x') = \text{true}$, axioms (R) and (T) read as:

$$\forall x \in X \quad x \leqslant x;$$

$$\forall x, x', x'' \in X \quad x \leqslant x' \ \&\ x' \leqslant x'' \implies x \leqslant x''$$

that is, $(X, \leqslant)$ is a preordered set. A **2**-functor is then a map $f : (X, \leqslant) \to (Y, \leqslant)$ between preordered sets such that

$$\forall x, x' \in X \quad x \leqslant x' \implies f(x) \leqslant f(x');$$

that is, f is a monotone map. Hence **2-Cat** is exactly the category **POrd** of preordered sets and monotone maps.

An $\overline{\mathbb{R}}_+$-category is a set X endowed with a map $a : X \times X \to [0, \infty]$ such that

$$\forall x \in X \quad 0 \geqslant a(x, x);$$

$$\forall x, x', x'' \in X \quad a(x, x') + a(x', x'') \geqslant a(x, x'');$$

that is, $a : X \times X \to [0, \infty]$ is a premetric in X. An $\overline{\mathbb{R}}_+$-functor is a map $f : (X, a) \to (Y, b)$ between premetric spaces satisfying the following inequality:

$$\forall x, x' \in X \quad a(x, x') \geqslant b\big(f(x), f(x')\big),$$

which means that f is a non-expansive map. Therefore the category $\overline{\mathbb{R}}_+$-**Cat** coincides with the category **PMet** of premetric spaces and non-expansive maps. (For more details, see [10,6].)

1.3. The **V**-functor $\hom : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ provides **V** with a structure of **V**-category. Indeed, for each $u, v, w \in \mathbf{V}$:

$$k \leqslant \hom(u, u),$$

$$\hom(u, v) \otimes \hom(v, w) \leqslant \hom(u, w).$$

The first inequality follows from $k \otimes u = u \leqslant u$, while the second is a consequence of

$$u \otimes \hom(u, v) \otimes \hom(v, w) \leqslant v \otimes \hom(v, w) \leqslant w.$$

1.4. The tensor product $\otimes$ of **V** induces a tensor product in **V**-**Cat**, which we will denote also by $\otimes$: for each pair $(X, a), (Y, b)$ of **V**-categories, $(X, a) \otimes (Y, b) := (X \times Y, a \otimes b)$, where $(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y')$; it is clear that, for each pair of **V**-functors $f : (X, a) \to (Y, b)$ and $g : (X', a') \to (Y', b')$, the map $f \times g : X \times X' \to Y \times Y'$ becomes a **V**-functor between the corresponding **V**-categories.

This tensor product has a unit element, $K = (\{*\}, k)$, a singleton set with $k(*, *) = k$. Moreover, it was shown in [10]:

1.5. Theorem. *The category* **V**-**Cat** *equipped with the tensor product* $\otimes$ *is a closed category.*

The description of its right adjoint hom, constructed in [10], becomes very simple in this context: for each pair (X, a) and (Y, b) of **V**-categories, $\hom((X, a), (Y, b))$ has as underlying set $X^Y = \{f : (X, a) \to (Y, b);\ f \text{ is a } \mathbf{V}\text{-functor}\}$, with structure d defined by:

$$d(f, g) = \bigwedge_{x \in X} b\big(f(x), g(x)\big).$$

1.6. In case $\otimes$ coincides with the categorical product, as it is the case when $\mathbf{V} = \mathbf{2}$, the category of **V**-categories is automatically Cartesian closed by the previous theorem. Whenever $\otimes$ is not the categorical product, this result is no longer valid. For instance, **PMet**, studied in detail in the last section (see 4.2), is not Cartesian closed.

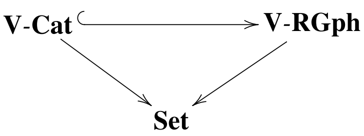
Even in the case ⊗ is the categorical product, Cartesian closedness is not inherited by the slice categories: for instance, **POrd** is not locally Cartesian closed (see [12] and Theorem 4.1).

2. The category of reflexive V-graphs

2.1. If we drop the axiom (T) in the definition of **V**-category, we end up with the *category* **V**-**RGph** *of reflexive* **V***-graphs and* **V***-functors*. That is, objects of **V**-**RGph** are pairs (X, a), where X is a set and $a : X \times X \to \mathbf{V}$ is a map such that $k \leqslant a(x, x)$ for every $x \in X$; morphisms $f : (X, a) \to (Y, b)$ in **V**-**RGph** are maps $f : X \to Y$ such that $a(x, x') \leqslant b(f(x), f(x'))$, for all $x, x' \in X$.

This category contains **V**-**Cat** as a full subcategory, and, moreover, as it was shown in [3]:

2.2. Theorem. *In the commutative diagram*



the (*full*) *embedding is reflective, with identity maps as reflections, and the forgetful functors are topological.*

In contrast with the situation in **V**-**Cat**, exponentiation in **V**-**RGph** is easily described, based on exponentiation in **Set** and **V**, as developed in [5].

2.3. For the forthcoming study of exponentiation in **V**-**Cat**, the description of exponentials in **V**-**RGph** is crucial. Our first observation in this direction is that the use of partial products as introduced in [7] turns out to simplify this study. Next we summarize the results needed for that.

2.4. Theorem. [11,7] *For a morphism $f : X \to Y$ in a category* **C**, *the following conditions are equivalent*:

(i) *f is exponentiable, i.e. $f \times - : \mathbf{C}/Y \to \mathbf{C}/Y$ has a right adjoint*;
(ii) *the 'change-of-base' functor $f \times_Y - : \mathbf{C}/Y \to \mathbf{C}/X$ has a right adjoint*;
(iii) *the 'pullback' functor $X \times_Y - : \mathbf{C}/Y \to \mathbf{C}$ has a right adjoint*;
(iv) **C** *has partial products over f.*

We recall that **C** *has partial products over* $f : X \to Y$ if, for each object Z in **C**, there is a diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\ \mathrm{ev}\ } & X \times_Y P & \xrightarrow{\ \pi_1\ } & X \\ & & {\scriptstyle\pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ & & P & \xrightarrow{\ p\ } & Y \end{array}$$

such that every diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\ \mathrm{ev}'\ } & X \times_Y Q & \xrightarrow{\ \pi_1'\ } & X \\ & & {\scriptstyle\pi_2'}\downarrow & & \downarrow{\scriptstyle f} \\ & & Q & \xrightarrow{\ q\ } & Y \end{array}$$

factors as $p \cdot t = q$ and $\mathrm{ev} \cdot (1_X \times t) = \mathrm{ev}'$ by a unique morphism $t : Q \to P$.

In [5] it is shown that, when **V** is a locally Cartesian category—which, in our situation just means **V** is a Heyting algebra—**V-RGph** has partial products over every **V**-functor $f : (X, a) \to (Y, b)$. In fact:

2.5. Theorem. [5] *If* **V** *is a Heyting algebra, then* **V**-**RGph** *is a quasitopos.*

Here we will sketch only the construction of partial products, since they will play a crucial role in the subsequent study.

2.6. For each **V**-functor $f : (X, a) \to (Y, b)$ between reflexive **V**-graphs and for each reflexive **V**-graph (Z, c), the partial product (P, d) of (Z, c) over f is defined as follows:

$$P = \big\{(s, y);\ y \in Y \text{ and } s : \big(f^{-1}(y), a\big) \to (Z, c) \text{ is a } \mathbf{V}\text{-functor}\big\}$$

(here $a : f^{-1}(y) \times f^{-1}(y) \to \mathbf{V}$ is just the restriction of $a : X \times X \to \mathbf{V}$); to obtain the structure d on P, for each $(s, y), (s', y') \in P$, $x \in f^{-1}(y)$ and $x' \in f^{-1}(y')$, we first form in **V** the partial product $v((s, x), (s', x'))$ of $c(s(x), s'(x'))$ over $a(x, x') \leqslant b(y, y')$; then $d((s, y), (s', y'))$ is the multiple pullback of the $(v((s, x), (s', x')))_{x \in f^{-1}(y), x' \in f^{-1}(y')}$ in the lattice $\downarrow b(y, y')$; that is

$$d\big((s, y), (s', y')\big) = \begin{cases} \displaystyle\bigwedge_{x \in f^{-1}(y),\ x' \in f^{-1}(y')} v((s, x), (s', x')), \\ \quad \text{if } f^{-1}(y) \neq \emptyset \neq f^{-1}(y'), \\ b(y, y'), \quad \text{otherwise.} \end{cases}$$

Since

$$v\big((s, x), (s', x')\big) = \bigwedge_{x \in f^{-1}(y), x' \in f^{-1}(y')} \bigvee \big\{v \in \mathbf{V};\ v \leqslant b(y, y') \text{ and } \\ a(x, x') \wedge v \leqslant c\big(s(x), s'(x')\big)\big\},$$

using the distributivity of binary meets over arbitrary joins, we obtain the following description of $d((s,y),(s',y'))$, which is easier to handle is this context:

$$d\big((s,y),(s',y')\big) = \max\big\{v \in \mathbf{V};\ v \leqslant b(y,y')\ \&\ \forall x \in f^{-1}(y) \forall x' \in f^{-1}(y')\ a(x,x') \wedge v \leqslant c\big(s(x), s'(x')\big)\big\}.$$

3. Exponentiation in V-Cat

3.1. In order to characterize exponentiable morphisms, and consequently exponentiable objects, in **V**-**Cat**, we first state some auxiliary results.

3.2. Lemma. *The following conditions are equivalent*:

(i) *k is the top element of* $\mathbf{V}$;
(ii) *for every $u, v, w \in \mathbf{V}$,*

$$(u \wedge v) \otimes w \leqslant (u \otimes w) \wedge v.$$

3.3. Proposition. *If* $\mathbf{V}$ *is a Heyting algebra, k is the top element of* $\mathbf{V}$, *$f:(X,a) \to (Y,b)$ is an exponentiable morphism in* **V**-**Cat** *and (Z,c) is a* **V**-*category, then the partial products of (Z,c) over f in* **V**-**Cat** *and in* **V**-**RGph** *coincide.*

Proof. Let

$$\begin{array}{ccccc} (Z,c) & \xleftarrow{\ \mathrm{ev}\ } & (X \times_Y P, \tilde{d}) & \xrightarrow{\ \pi_1\ } & (X,a) \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ & & (P,d) & \xrightarrow{\ p\ } & (Y,b) \end{array}$$

and

$$\begin{array}{ccccc} (Z,c) & \xleftarrow{\ \mathrm{ev}'\ } & (X \times_Y P', \tilde{d}') & \xrightarrow{\ \pi_1'\ } & (X,a) \\ & & \downarrow{\scriptstyle \pi_2'} & & \downarrow{\scriptstyle f} \\ & & (P',d') & \xrightarrow{\ p'\ } & (Y,b) \end{array}$$

be the partial products of (Z,c) over f in **V**-**RGph** and **V**-**Cat**, respectively. Since **V**-**Cat** is closed under pullbacks and the latter diagram lies in particular in **V**-**RGph**, there exists a unique **V**-functor $t:(P',d') \to (P,d)$ such that $p \cdot t = p'$ and $\mathrm{ev} \cdot (1 \times t) = \mathrm{ev}'$. It is easy to check that $t: P' \to P$ is a bijection, using the universal properties of both diagrams. We therefore assume, for simplicity, that t is an identity map. To show it is an isomorphism, that is, $d = d'$, let $(s_0, y_0), (s_1, y_1) \in P$. The pair $(\{0,1\}, e)$ with $e(0,1) = d((s_0,y_0),(s_1,y_1))$ and $e(0,0) = e(1,1) = k$ is a **V**-category. Hence, the diagram in **V**-**Cat**

$$\begin{array}{ccccc} (Z,c) & \xleftarrow{\ \mathrm{ev}\cdot\iota\ } & (X \times_Y \{0,1\}, \tilde{e}) & \longrightarrow & (X,a) \\ & & \downarrow & & \downarrow{\scriptstyle f} \\ & & (\{0,1\},e) & \xrightarrow{\ p\cdot\iota\ } & (Y,b) \end{array}$$

where $\iota : (\{0,1\}, e) \to (P, d)$ is the inclusion map, with $\iota(0) = (s_0, y_0)$ and $\iota(1) = (s_1, y_1)$, induces a **V**-functor $(\{0,1\}, e) \to (P', d')$ by the universal property of (P', d'). Hence $d((s_0, y_0), (s_1, y_1)) = e(0,1) \leqslant d'((s_0, y_0), (s_1, y_1))$ and the conclusion follows. □

3.4. Theorem. *If* **V** *is a Heyting algebra and k is the top element of* **V**, *then a* **V***-functor $f : (X, a) \to (Y, b)$ is exponentiable in* **V-Cat** *if and only if, for each $x_0, x_2 \in X$, $y_1 \in Y$ and for each $v_0, v_1 \in \mathbf{V}$ such that $v_0 \leqslant b(f(x_0), y_1)$ and $v_1 \leqslant b(y_1, f(x_2))$,*

$$\bigvee_{x \in f^{-1}(y_1)} \big(a(x_0, x) \wedge v_0\big) \otimes \big(a(x, x_2) \wedge v_1\big) \geqslant a(x_0, x_2) \wedge (v_0 \otimes v_1). \qquad (*)$$

Proof. We first show that condition $(*)$ above guarantees the exponentiability of f. More precisely, we show that, for any **V**-category (Z, c), its partial product (P, d) over f, formed in **V-RGph**, is a **V**-category; that is, it satisfies axiom (T):

$$\forall (s_0, y_0), (s_1, y_1), (s_2, y_2) \in P$$
$$d\big((s_0, y_0), (s_1, y_1)\big) \otimes d\big((s_1, y_1), (s_2, y_2)\big) \leqslant d\big((s_0, y_0), (s_2, y_2)\big).$$

Let $u_0 := d((s_0, y_0), (s_1, y_1))$, $u_1 := d((s_1, y_1), (s_2, y_2))$ and $u := d((s_0, y_0), (s_2, y_2))$. For every $x_i \in f^{-1}(y_i)$, $i = 0, 1, 2$, we have

$$u_0 \leqslant b(y_0, y_1) \quad \text{and} \quad a(x_0, x_1) \wedge u_0 \leqslant c\big(s_0(y_0), s_1(y_1)\big),$$
$$u_1 \leqslant b(y_1, y_2) \quad \text{and} \quad a(x_1, x_2) \wedge u_1 \leqslant c\big(s_1(y_1), s_2(y_2)\big).$$

Hence, for every $x_0 \in f^{-1}(y_0)$ and $x_2 \in f^{-1}(y_2)$,

$$u_0 \otimes u_1 \leqslant b(y_0, y_1) \otimes b(y_1, y_2) \leqslant b(y_0, y_2), \quad \text{and}$$

$$\begin{aligned} a(x_0, x_2) \wedge (u_0 \otimes u_1) &\leqslant \bigvee_{x \in f^{-1}(y_1)} \big(a(x_0, x) \wedge u_0\big) \otimes \big(a(x, x_2) \wedge u_1\big) \\ &\leqslant \bigvee_{x \in f^{-1}(y_1)} c\big(s_0(x_0), s_1(x)\big) \otimes c\big(s_1(x), s_2(x_2)\big) \\ &\leqslant c\big(s_0(x_0), s_2(x_2)\big). \end{aligned}$$

Therefore, $u_0 \otimes u_1 \leqslant u$ as claimed.

To show the necessity of the condition, we consider $x_0, x_2 \in X$, $y_0 = f(x_0)$, $y_2 = f(x_2)$, $y_1 \in Y$, $v_0, v_1 \in \mathbf{V}$ as in $(*)$, and we define a triple of maps

$$\begin{aligned} s_0 : f^{-1}(y_0) &\to \mathbf{V} \\ x &\mapsto a(x_0, x), \\ s_1 : f^{-1}(y_1) &\to \mathbf{V} \\ x &\mapsto a(x_0, x) \wedge v_0, \\ s_2 : f^{-1}(y_2) &\to \mathbf{V} \\ x &\mapsto \bigvee_{x_1 \in f^{-1}(y_1)} \big(a(x_0, x_1) \wedge v_0\big) \otimes \big(a(x_1, x) \wedge v_1\big). \end{aligned}$$

The map s_0 is a **V**-functor, since, for each $x, x' \in f^{-1}(y_0)$,

$$\begin{aligned}&a(x_0,x)\otimes a(x,x')\leqslant a(x_0,x')\\&\quad\Longrightarrow a(x,x')\leqslant \hom\big(a(x_0,x),a(x_0,x')\big)=\hom\big(s_0(x),s_0(x')\big).\end{aligned}$$

In order to check that s_1 is a **V**-functor, let $x, x' \in f^{-1}(y_1)$. Then

$$\begin{aligned}s_1(x)\otimes a(x,x')&=\big(a(x_0,x)\wedge v_0\big)\otimes a(x,x')\\&\leqslant \big(a(x_0,x)\otimes a(x,x')\big)\wedge v_0\leqslant a(x_0,x')\wedge v_0=s_1(x')\\&\Longrightarrow a(x,x')\leqslant \hom\big(s_1(x),s_1(x')\big).\end{aligned}$$

Finally we have to check that s_2 is also a **V**-functor: for $x, x' \in f^{-1}(y_2)$,

$$\begin{aligned}s_2(x)\otimes a(x,x')&=\bigvee_{x_1\in f^{-1}(y_1)}\big(a(x_0,x_1)\wedge v_0\big)\otimes\big(a(x_1,x)\wedge v_1\big)\otimes a(x,x')\\&\leqslant\bigvee_{x_1\in f^{-1}(y_1)}\big(a(x_0,x_1)\wedge v_0\big)\otimes\big(\big(a(x_1,x)\otimes a(x,x')\big)\wedge v_1\big)\\&\leqslant\bigvee_{x_1\in f^{-1}(y_1)}\big(a(x_0,x_1)\wedge v_0\big)\otimes\big(a(x_1,x')\wedge v_1\big)\\&=s_2(x');\end{aligned}$$

hence, $a(x,x') \leqslant \hom(s_2(x), s_2(x'))$.

Finally, it is easy to check that

$$v_0\leqslant d\big((s_0,y_0),(s_1,y_1)\big)\quad\text{and}\quad v_1\leqslant d\big((s_1,y_1),(s_2,y_2)\big),$$

since: $v_0 \leqslant b(y_0, y_1)$ by hypothesis, and, for every $x'_0 \in f^{-1}(y_0)$, $x'_1 \in f^{-1}(y_1)$,

$$a(x'_0,x'_1)\wedge v_0\leqslant \hom\big(s_0(x'_0),s_1(x'_1)\big)$$

follows from

$$\begin{aligned}s_0(x'_0)\otimes\big(a(x'_0,x'_1)\wedge v_0\big)&=a(x_0,x'_0)\otimes\big(a(x'_0,x'_1)\wedge v_0\big)\\&\leqslant\big(a(x_0,x'_0)\otimes a(x'_0,x'_1)\big)\wedge v_0\\&\leqslant a(x_0,x'_1)\wedge v_0=s_1(x'_1);\end{aligned}$$

analogously, $v_1 \leqslant b(y_1, y_2)$ by hypothesis, and, for each $x'_1 \in f^{-1}(y_1)$, $x'_2 \in f^{-1}(y_2)$,

$$a(x'_1,x'_2)\wedge v_1\leqslant \hom\big(s_1(x'_1),s_2(x'_2)\big)$$

follows easily from the inequalities

$$\begin{aligned}s_1(x'_1)\otimes\big(a(x'_1,x'_2)\wedge v_1\big)&=\big(a(x_0,x'_1)\wedge v_0\big)\otimes\big(a(x'_1,x'_2)\wedge v_1\big)\\&\leqslant\bigvee_{x_1\in f^{-1}(y_1)}\big(a(x_0,x_1)\wedge v_0\big)\otimes\big(a(x_1,x'_2)\wedge v_1\big)=s_2(x'_2).\end{aligned}$$

Therefore, since we are assuming that (P, d) is a **V**-category, we may conclude that

$$d\big((s_0,y_0),(s_2,y_2)\big)\geqslant v_0\otimes v_1,$$

which means in particular that

$$a(x_0,x_2)\wedge(v_0\otimes v_1)\leqslant \hom\big(s_0(x_0),s_2(x_2)\big)=\hom\big(k,s_2(x_2)\big)=s_2(x_2);$$

that is,

$$a(x_0,x_2)\wedge(v_0\otimes v_1)\leqslant \bigvee_{x_1\in f^{-1}(y_1)}\big(a(x_0,x_1)\wedge v_0\big)\otimes\big(a(x_1,x_2)\wedge v_1\big),$$

which completes the proof. □

3.5. Corollary. *If* **V** *is a Heyting algebra with k its top element, then a* **V***-category (X,a) is exponentiable in* **V**-**Cat** *if and only if*

$$\forall x_0,x_2\in X\ \forall v_0,v_1\in \mathbf{V}$$
$$\bigvee_{x\in X}\big(a(x_0,x)\wedge v_0\big)\otimes\big(a(x,x_2)\wedge v_1\big)\geqslant a(x_0,x_2)\wedge(v_0\otimes v_1).$$

3.6. Corollary. *If the tensor product $\otimes$ and the categorical product coincide in* **V**, *then a* **V***-functor $f:(X,a)\to(Y,b)$ is exponentiable in* **V**-**Cat** *if and only if*

$$\forall x_0,x_2\in X\ \forall y_1\in Y$$
$$\bigvee_{x\in f^{-1}(y_1)}\big(a(x_0,x)\wedge a(x,x_2)\big)\geqslant a(x_0,x_2)\wedge b\big(f(x_0),y_1\big)\wedge b\big(y_1,f(x_2)\big).$$

Proof. If $\otimes=\wedge$, one has

$$\bigvee_{x\in f^{-1}(y_1)}\big(a(x_0,x)\wedge v_0\big)\otimes\big(a(x,x_2)\wedge v_1\big)$$
$$=v_0\wedge v_1\wedge\bigvee_{x\in f^{-1}(y_1)}\big(a(x_0,x)\wedge a(x,x_2)\big),$$

and

$$v_0\wedge v_1\wedge\bigvee_{x\in f^{-1}(y_1)}\big(a(x_0,x)\wedge a(x,x_2)\big)\geqslant a(x_0,x_2)\wedge v_0\wedge v_1$$
$$\iff\bigvee_{x\in f^{-1}(y_1)}\big(a(x_0,x)\wedge a(x,x_2)\big)\geqslant a(x_0,x_2)\wedge v_0\wedge v_1.$$

Finally it is clear that it is enough to check the latter inequality for $v_0=b(f(x_0),y_1)$ and $v_1=b(y_1,f(x_2))$. □

3.7. Remark. The description of quotients obtained in [9] gives an alternative way of showing that $(*)$ is sufficient for exponentiability. First, one can prove that a pullback of a quotient map $g:Z\to Y$ along f is again a quotient map. Roughly speaking, $(*)$ guarantees that, for x_0,x_2 in X, each zigzag in Z mapped to $f(x_0)$, $f(x_2)$ can be lifted in an appropriate way to a zigzag in $X\times_Y Z$ mapped to x_0,x_2. Finally, it is easy to check that property $(*)$ is pullback-stable, hence the change of base functor $f\times_Y -$ preserves quotients, and therefore, by Freyd's Adjoint Functor Theorem, it has a right adjoint.

4. Examples

4.1. Exponentiation in **POrd**

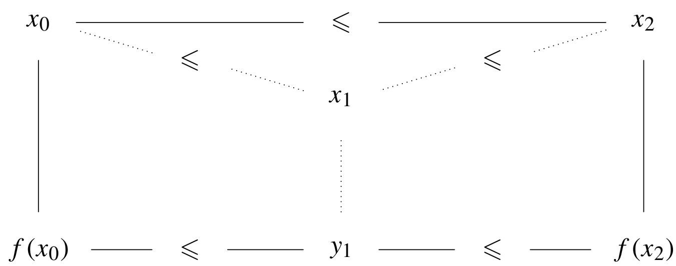
We consider first exponentiation in **V-Cat** for $\mathbf{V} = \mathbf{2}$, that is, in the category **POrd** of preordered sets and monotone maps. In this category the condition of Corollary 3.6 trivializes except for $b(f(x_0), y_1) = b(y_1, f(x_2)) = a(x_0, x_2) = \text{true}$. Then it means that:

$$\bigvee_{x \in f^{-1}(y_1)} a(x_0, x) \wedge a(x, x_2) = \text{true};$$

that is, there exists $x \in X$ such that $f(x) = y_1$ and $x_0 \leqslant x \leqslant x_2$. Therefore we have:

Theorem. [12] *A monotone map* $f : (X, \leqslant) \to (Y, \leqslant)$ *between preordered sets is exponentiable in* **POrd** *if and only if*

$$\forall x_0, x_2 \in X \ \forall y_1 \in Y \ x_0 \leqslant x_2 \ \& \ f(x_0) \leqslant y_1 \leqslant f(x_2)$$
$$\implies \exists x_1 \in X\colon \ f(x_1) = y_1 \ \& \ x_0 \leqslant x_1 \leqslant x_2\colon$$



We observe that, if $f : (X, \leqslant) \to (Y, \leqslant)$ is an embedding, this condition can be restated as:

Corollary. *An embedding* $f : (X, \leqslant) \to (Y, \leqslant)$ *is exponentiable in* **POrd** *if and only if*

$$\downarrow f(X) \cap \uparrow f(X) = f(X).$$

Using the closure defined by $\uparrow$, this just means that $f(X)$ is the intersection of an open and a closed subset of Y, which resembles the characterization of exponentiable embeddings in **Top** as the locally closed embeddings (see [11, Corollary 2.7]).

4.2. Exponentiation in **PMet**

Now we consider $\mathbf{V} = \overline{\mathbb{R}}_+ = ([0, \infty], \geqslant)$ the (complete) real half-line. As we have already observed, the category **V-Cat** is the category **PMet** of premetric spaces and non-expansive maps. Here condition $(*)$ of Theorem 3.4 may be also simplified, as we explain next. In the sequel we use the natural order in the real numbers, which is the opposite of the categorical one, so that $(*)$ reads as: for each $x_0, x_2 \in X$, $y_1 \in Y$, and for each $v_0, v_1 \in \overline{\mathbb{R}}_+$ such that $v_0 \geqslant b(f(x_0), y_1)$ and $v_1 \geqslant b(y_1, f(x_2))$,

$$\inf_{x \in f^{-1}(y_1)} \big(a(x_0, x) \vee v_0\big) + \big(a(x, x_2) \vee v_1\big) \leqslant a(x_0, x_2) \vee (v_0 + v_1).$$

We recall from Section 2 that, given a non-expansive map $f : (X, a) \to (Y, b)$ and a premetric space (Z, c), the structure d in the partial product P is given by

$$d\big((s, y), (s', y')\big) = \min\big\{v \in \overline{\mathbb{R}}_+;\ v \geqslant b(y, y')\ \&\ \forall x \in f^{-1}(y)\ \forall x' \in f^{-1}(y')\ a(x, x') \vee v \geqslant c\big(s(x), s'(x')\big)\big\},$$

for every $(s, y), (s', y') \in P$.

Theorem. *A non-expansive map $f : (X, a) \to (Y, b)$ between premetric spaces is exponentiable in* **PMet** *if and only if, for each $x_0, x_2 \in X$, $y_1 \in Y$ and $u_0, u_1 \in \overline{\mathbb{R}}_+$ such that $u_0 \geqslant b(f(x_0), y_1)$, $u_1 \geqslant b(y_1, f(x_2))$ and $u_0 + u_1 = \max\{a(x_0, x_2), b(f(x_0), y_1) + b(y_1, f(x_2))\} < \infty$,*

$$\forall \varepsilon > 0\ \exists x_1 \in f^{-1}(y_1)\colon\ a(x_0, x_1) < u_0 + \varepsilon \text{ and } a(x_1, x_2) < u_1 + \varepsilon. \qquad (**)$$

Proof. First we show that condition $(**)$ is necessary for exponentiability. For u_0, u_1 as above, Theorem 3.4 together with $u_0 + u_1 \geqslant a(x_0, x_2)$ assure that

$$\inf_{x \in f^{-1}(y_1)} \big(a(x_0, x) \vee u_0\big) + \big(a(x, x_2) \vee u_1\big) \leqslant u_0 + u_1,$$

and the conclusion follows.

Conversely, assume that condition $(**)$ is satisfied.

We first observe that, when $\max\{a(x_0, x_2), b(f(x_0), y_1) + b(y_1, f(x_2))\} = \infty$, $(*)$ is immediately verified.

Let $v_0, v_1 \in \overline{\mathbb{R}}_+$ be such that $v_0 \geqslant b(f(x_0), y_1)$ and $v_1 \geqslant b(y_1, f(x_2))$.

If $a(x_0, x_2) \leqslant b(f(x_0), y_1) + b(y_1, f(x_2))$, then necessarily $u_0 = b(f(x_0), y_1)$ and $u_1 = b(y_1, f(x_2))$ in $(**)$. We therefore have

$$\forall \varepsilon > 0\ \exists x_1 \in f^{-1}(y_1)\colon\ a(x_0, x_1) < u_0 + \varepsilon \leqslant v_0 + \varepsilon \text{ and } a(x_1, x_2) < u_1 + \varepsilon \leqslant v_1 + \varepsilon,$$

which implies that

$$\inf_{x \in f^{-1}(y_1)} \big(a(x_0, x) \vee v_0\big) + \big(a(x, x_2) \vee v_1\big) = v_0 + v_1 = a(x_0, x_2) \vee (v_0 + v_1).$$

Consider now the case $a(x_0, x_2) \geqslant b(f(x_0), y_1) + b(y_1, f(x_2))$. If $v_0 + v_1 \geqslant a(x_0, x_2)$, using $(**)$ for $u_0 \leqslant v_0$ and $u_1 \leqslant v_1$ such that $u_0 + u_1 = a(x_0, x_2)$, we conclude that

$$\inf_{x \in f^{-1}(y_1)} \big(a(x_0, x) \vee v_0\big) + \big(a(x, x_2) \vee v_1\big) = v_0 + v_1 = a(x_0, x_2) \vee (v_0 + v_1).$$

If $v_0 + v_1 < a(x_0, x_2)$, let $u_0 = v_0$ and $u_1 = a(x_0, x_2) - v_0$ in $(**)$. Then

$$\begin{aligned} &\inf_{x \in f^{-1}(y_1)} \big(a(x_0, x) \vee v_0\big) + \big(a(x, x_2) \vee v_1\big) \\ &\quad \leqslant \inf_{x \in f^{-1}(y_1)} \big(a(x_0, x) \vee u_0\big) + \big(a(x, x_2) \vee u_1\big) \\ &\quad \leqslant a(x_0, x_2) = a(x_0, x_2) \vee (v_0 + v_1). \qquad \square \end{aligned}$$

We observe that, defining, for a premetric space (X, a) and $X' \subseteq X$,

$$\downarrow S := \{x \in X;\ \exists x' \in X\colon\ d(x', x) < \infty\} \quad \text{and}$$
$$\uparrow S := \{x \in X;\ \exists x' \in X\colon\ d(x, x') < \infty\},$$

one has:

Corollary A. An embedding $f : (X, a) \to (Y, b)$ in **PMet** is exponentiable if and only if

$$\downarrow f(X) \cap \uparrow f(X) = f(X).$$

From the Theorem one derives also the following characterization of exponentiable objects in **PMet**:

Corollary B. A premetric space (X, a) is exponentiable in **PMet** if and only if, for each $x_0, x_2 \in X$ and $u_0, u_1 \in \overline{\mathbb{R}}_+$ such that $u_0 + u_1 = a(x_0, x_2)$,

$$\forall \varepsilon > 0\ \exists x_1 \in X\colon\ a(x_0, x_1) < u_0 + \varepsilon \text{ and } a(x_1, x_2) < u_1 + \varepsilon.$$

Using this characterization it is obvious that:

- $\mathbb{Q}^+$ with the hom structure (inherited from $\overline{\mathbb{R}}_+$), and $\mathbb{Q}$ and $\mathbb{Q}^+$ with the usual Euclidean metric, are exponentiable in **PMet**;
- finite premetric spaces with non-trivial premetric (i.e. having points whose distance differs from 0 and ∞) are not exponentiable in **PMet**.

These examples give an interesting contrast with the situation in **Top**: with their induced topology the former ones are not exponentiable in **Top** while the latter ones are.

4.3. Exponentiation in **Met**

We finally study the situation in the category **Met** of (symmetric, separated, with non-infinite distances) metric spaces and non-expansive maps.

Proposition. *Let (X, a), (Y, b) and (Z, c) be metric spaces and let $f : (X, a) \to (Y, b)$ be a non-expansive map.*

(a) *If there exists the partial product (P, d) of (Z, c) over f in* **PMet**, *then the premetric d is symmetric and separated.*
(b) *If the partial products of (Z, c) over f exist in both* **PMet** *and* **Met**, *then they coincide.*

Proof. (a) From the description of d above, it is obvious that symmetry of d is inherited by the symmetry of a, b and c. Moreover, $d((s, y), (s', y')) = 0$ means that $b(y, y') = 0$, hence $y = y'$, and, for every $x, x' \in f^{-1}(y)$, $a(x, x') \geqslant c(s(x), s'(x'))$. In particular, for $x = x'$ it follows that $c(s(x), s'(x)) = 0$, hence $s = s'$.

(b) Let (P, d) and (P', d') be the partial products of (Z, c) over f in **PMet** and **Met**, respectively. To show that they coincide, it is enough to show that d cannot take the value ∞. By the universal property of (P,d), there exists a non-expansive bijection $t:(P',d')\to(P,d)$. Hence, $d'\geqslant d$ and the result follows. □

Theorem. *A non-expansive map $f:(X,a)\to(Y,b)$ between metric spaces is exponentiable in* **Met** *if and only if it satisfies condition* $(**)$ *and has bounded fibres.*

Proof. We first show that boundedness of the fibres of f, together with $(**)$, is sufficient for exponentiability.

By the proposition above, it remains to be shown that d does not take the value ∞. For each $(s,y),(s',y')\in P$, boundedness of $f^{-1}(y)$ and $f^{-1}(y')$ guarantees that $s(f^{-1}(y))$ and $s'(f^{-1}(y'))$ are bounded subsets of (Z,c), hence $\{c(s(x),s'(x'));\ x\in f^{-1}(y),\ x'\in f^{-1}(y')\}$ is bounded as well; let $u(s,s')$ be its supremum. Then $d((s,y),(s',y'))\leqslant b(y,y')\vee u(s,s')$, hence d is a metric.

To show the reverse implication, we check first that boundedness of the fibres is necessary for exponentiability of f. Let f be exponentiable. For any $y\in Y$, consider $(Z,c)=(f^{-1}(y),a)$, the identity $s:(Z,c)\to(Z,c)$ and a constant map $s':(Z,c)\to(Z,c)$, taking every point x into a fixed $x_0\in f^{-1}(y)$. Then

$$\begin{aligned}\infty > d\big((s,y),(s',y)\big) &\geqslant \min\big\{v;\ v\geqslant b(y,y)\ \&\ \forall x\in f^{-1}(y)\\ &\qquad a(x,x)\vee v\geqslant c\big(s(x),s'(x)\big)\big\}\\ &=\min\big\{v;\ \forall x\in f^{-1}(y)\ v\geqslant a(x,x_0)\big\},\end{aligned}$$

hence $f^{-1}(y)$ is a bounded subset of X.

It remains to be shown that condition $(**)$ is necessary for exponentiability. First we observe that $(**)$ is trivially verified when $X=\emptyset$. For $X\neq\emptyset$, we first check that, if f is exponentiable in **Met**, then f is surjective. Assume that $y_1\in Y\setminus f(X)$. Consider $x_0\in X$, $y_0=f(x_0)$, and

$$v=\sup\big\{a(x,x');\ x,x'\in f^{-1}(y_0)\big\}.$$

Then $v<\infty$ because f has bounded fibres. Let $Z:=\{0,1\}$, with $c(0,1)=\max\{v,b(y_0,y_1)+b(y_1,y_0)\}+1$. Define the constant maps $s:f^{-1}(y_0)\to Z$, $s'':f^{-1}(y_0)\to Z$, which send every x to 0 and 1, respectively, and let $s':f^{-1}(y_1)=\emptyset\to Z$. Then it is easy to check that

$$\begin{aligned}&d\big((s,y_0),(s',y_1)\big)=b(y_0,y_1),\\ &d\big((s',y_1),(s'',y_0)\big)=b(y_1,y_0),\\ &d\big((s,y_0),(s'',y_0)\big)=c(0,1),\end{aligned}$$

which shows that d is not transitive, since

$$\begin{aligned}d\big((s,y_0),(s'',y_0)\big)&=c(0,1)>b(y_0,y_1)+b(y_1,y_0)\\ &=d\big((s,y_0),(s',y_1)\big)+d\big((s',y_1),(s'',y_0)\big).\end{aligned}$$

Finally the result follows from an easy adaptation of the argumentation of the proof of Theorem 3.4, using the maps s_0,s_1,s_2, replacing the hom structure in $\overline{\mathbb{R}}_+$ by the usual metric structure in $[0,\infty[$. Continuity of these maps follows from the continuity of s_0,s_1,s_2

of Theorem 3.4 since the metric structure is just the symmetrization of the hom structure of $\overline{\mathbb{R}}_+$. The only detail to check is that, in the definition of s_2,

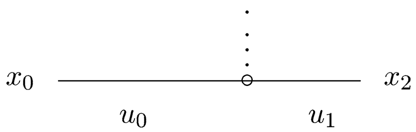
$$s_2 : f^{-1}(y_2) \to [0, \infty[$$
$$x \mapsto \inf_{x_1 \in f^{-1}(y_1)} \big(a(x_0, x_1) \vee v_0\big) + \big(a(x_1, x) \vee v_1\big),$$

the meet is non-empty, but this follows immediately from the surjectivity of f. □

In the particular case of metric spaces, we conclude directly the following

Corollary A. A metric space (X, a) is exponentiable in **Met** if and only if it is bounded and, for each $x_0, x_2 \in X$ and $u_0, u_1 \in \overline{\mathbb{R}}_+$ such that $u_0 + u_1 = a(x_2, x_0)$:

$$\forall \varepsilon > 0\ \exists x_1 \in X\colon\ a(x_0, x_1) < u_0 + \varepsilon \text{ and } a(x_1, x_2) < u_1 + \varepsilon:$$



Metric spaces satisfying the condition above have been studied in different contexts: in [1], as *almost* 3*-hyperconvex* metric spaces; in [2], under the name *almost convex* spaces, where this condition is used in the study of topologies in the hyperspace of closed subsets of X. In fact, exponentiability of f is equivalent to the "composition of" (closed) balls (see [2, Proposition 4.1.4]), as stated below. We denote by $S_u(x_0)$ ($\overline{S}_u(x_0)$) the open (closed) ball with center x_0 and radius u.

Corollary B. For a bounded metric space (X, a), the following conditions are equivalent:

(i) (X, a) is exponentiable;
(ii) $\forall x_0 \in X\ \forall u, v \in\,]0, \infty[\ S_u(S_v(x_0)) = S_{u+v}(x_0)$;
(iii) $\forall x_0 \in X\ \forall u, v \in\,]0, \infty[\ \overline{S}_u(\overline{S}_v(x_0)) = \overline{S}_{u+v}(x_0)$.

Finally, from the Theorem we obtain an interesting characterization of exponentiable complete metric spaces.

Corollary C. For a complete metric space (X, a), the following conditions are equivalent:

(i) (X, a) is exponentiable;
(ii) (X, a) is bounded and totally convex, i.e.

$$\forall x_0, x_2 \in X\ \forall u_0, u_1 \in [0, \infty[\ u_0 + u_1 = a(x_0, x_2)$$
$$\Longrightarrow \exists x_1 \in X\colon\ a(x_0, x_1) = u_0\ \&\ a(x_1, x_2) = u_1. \qquad \square$$

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