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On the eigenvalues of some tridiagonal matrices☆

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Abstract

A solution is given for a problem on eigenvalues of some symmetric tridiagonal matrices suggested by William Trench. The method presented can be generalizable to other problems.

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1. Inverse of a tridiagonal matrix

In [4], Trench proposed and solved the problem of finding eigenvalues and eigenvectors of the classes of symmetric matrices:

$$A = [\min\{i, j\}]_{i,j=1,\ldots,n}$$

and

$$B = [\min\{2i - 1, 2j - 1\}]_{i,j=1,\ldots,n}.$$

Later Kovačec presented a different proof of this problem [2]. Here a new proof is given. We show that solving this problem is equivalent to solving the eigenvalue problem for tridiagonal matrices with -1 on the 2 on the diagonal except for the (1, 1)-entry.

First note that these two matrices are in fact particular cases of a more general matrix:

$$C = [\min\{ai - b, aj - b\}]_{i,j=1,\ldots,n},$$

with $a > 0$ and $a \neq b$. It is very interesting that, under the above conditions, C is always invertible and its inverse is a tridiagonal matrix.

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Given an n-by-n nonsingular tridiagonal matrix T

$$T=\begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix},$$

Usmani [1,5,6] gave an elegant and concise formula for the inverse:

$$(T^{-1})_{ij}=\begin{cases}(-1)^{i+j}b_i\cdots b_{j-1}\theta_{i-1}\phi_{j+1}/\theta_n & \text{if } i\leqslant j,\\ (-1)^{i+j}c_j\cdots c_{i-1}\theta_{j-1}\phi_{i+1}/\theta_n & \text{if } i>j,\end{cases}$$

where θ_i's verify the recurrence relation $\theta_i=a_i\theta_{i-1}-b_{i-1}c_{i-1}\theta_{i-2}$, for $i=2,\ldots,n$, with initial conditions $\theta_0=1$ and $\theta_1=a_1$, and ϕ_i's verify the recurrence relation $\phi_i=a_i\phi_{i+1}-b_ic_i\phi_{i+2}$, for $i=n-1,\ldots,1$, with initial conditions $\phi_{n+1}=1$ and $\phi_n=a_n$. Observe that $\theta_n=\det T$. See also [3].

Proposition 1.1. *For $a>0$ and $a\neq b$, the tridiagonal matrix of order n*

$$T_n=\begin{bmatrix} 1+\dfrac{a}{a-b} & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{1.1}$$

is the inverse of $(1/a)C$.

Proof. We only have to observe that θ_i's verify the recurrence relation $\theta_i=2\theta_{i-1}-\theta_{i-2}$, for $i=2,\ldots,n-1$, and $\theta_n=\theta_{n-1}-\theta_{n-2}$, with initial conditions $\theta_0=1$ and $\theta_1=((2a-b)/(a-b))$, and ϕ_i's verify the recurrence relation $\phi_i=2\phi_{i+1}-\phi_{i+2}$, for $i=n-1,\ldots,2$, with initial conditions $\phi_{n+1}=1$ and $\phi_n=1$. $\square$

2. Eigenpairs of a particular tridiagonal matrix

According to the previous section, the problem of finding the eigenvalues of C is equivalent to describing the spectra of a tridiagonal matrix. Here, we give a general procedure to locate the eigenvalues of the matrix T_n (1.1).

Let us consider the set of polynomials $\{Q_k(x)\}$ defined by the recurrence relation given by $Q_0(x)=1$ and $Q_1(x)=(ax+1)Q_0(x)$,

$$Q_k(x)=(ax+2)Q_{k-1}(x)-Q_{k-2}(x)\quad \text{for } k=2,\ldots,n-1$$

and

$$Q_n(x)=\left(ax+\frac{2a-b}{a-b}\right)Q_{n-1}(x)-Q_{n-2}(x).$$

Note that each polynomial $Q_k(x)$, for $k=0,\ldots,n$, is of degree k. The last recurrence relation has the following matricial form:

$$x\begin{bmatrix}Q_{n-1}(x)\\ Q_{n-2}(x)\\ \vdots\\ Q_1(x)\\ Q_0(x)\end{bmatrix}=-\frac{1}{a}\begin{bmatrix} \dfrac{2a-b}{a-b} & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}\begin{bmatrix}Q_{n-1}(x)\\ Q_{n-2}(x)\\ \vdots\\ Q_1(x)\\ Q_0(x)\end{bmatrix}+Q_n(x)\begin{bmatrix}1\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}.$$

Since $Q_k(x) = U_k(ax/2+2) - U_{k-1}(ax/2+2)$, for $k = 0, \dots, n-1$, and

$$Q_n(x) = U_n\left(\frac{ax}{2}+1\right) - U_{n-1}\left(\frac{ax}{2}+1\right) - \left(1 - \frac{a}{a-b}\right)\left(U_{n-1}\left(\frac{ax}{2}+1\right) - U_{n-2}\left(\frac{ax}{2}+1\right)\right),$$

where $U_k(x)$, for $k = 0, \dots, n$, are the Chebyshev polynomials of second kind of degree k, the zeros of $Q_n(x)$ are exactly the eigenvalues of $-(1/a)C$, i.e., the (real) values which satisfy the equality

$$p_n(x) := \frac{U_n(ax/2+1) - U_{n-1}(ax/2+1)}{U_{n-1}(ax/2+1) - U_{n-2}(ax/2+1)} = 1 - \frac{a}{a-b}\ . \tag{2.1}$$

In general, (2.1) means that the eigenvalues of $-(1/a)C$ are the intersections of the graph of $p_n(x)$ with the line $y = 1 - a/(a-b)$.

As a first consequence, consider the case when $a = 1$ and $b = 0$. The eigenvalues of $-A$ are the solutions of the equation $U_n(x/2+1) - U_{n-1}(x/2+1) = 0$, which are, for $k = 0, \dots, n-1$,

$$\lambda_k = 2\cos\left(\frac{2k+1}{2n+1}\pi\right) - 2.$$

The value of an eigenvector associated to λ_k follows immediately:

$$[Q_{n-1}(\lambda_k) \ \cdots \ Q_1(\lambda_k) \ Q_0(\lambda_k)]^{\mathrm{t}}.$$

Hence we proved the following:

Theorem 2.1 (*Kovačec [2], Trench [4]*). *The matrix A of order n, $n \geqslant 3$, has the eigenpairs (λ_k, v_k)* given by

$$\lambda_k = \tfrac{1}{2}(1 - \cos(r_k))^{-1} \quad \textit{and} \quad v_k = [\sin(jr_k)]^t_{j=1,\dots,n},$$

where

$$r_k = \frac{2k+1}{2n+1}\pi,$$

for $k = 0, \dots, n-1$.

If $a = 2$ and $b = 1$, then the eigenvalues of $-(1/2)B$ are solutions of the equation $U_n(x+1) - U_{n-2}(x+1) = 0$, which are, for $k = 0, \dots, n-1$,

$$\cos\left(\frac{2k+1}{2n}\pi\right) - 1.$$

Remark 2.1. Analogously, for a positive integer n, the tridiagonal matrix of order n

$$M = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 + \dfrac{1}{n} \end{bmatrix}$$

is invertible and its inverse is $M^{-1} = [\max\{i, j\}]_{i,j=1,\dots,n}$. Details are left to the reader.

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