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On the corners of certain determinantal ranges

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Abstract

Let A be a complex $n \times n$ matrix and let $\mathrm{SO}(n)$ be the group of real orthogonal matrices of determinant one. Define $\Delta(A) = \{\det(A \circ Q) : Q \in \mathrm{SO}(n)\}$, where $\circ$ denotes the Hadamard product of matrices. For a permutation σ on $\{1, \ldots, n\}$, define $z_\sigma = d_\sigma(A) = \prod_{i=1}^{n} a_{i\sigma(i)}$. It is shown that if the equation $z_\sigma = \det(A \circ Q)$ has in $\mathrm{SO}(n)$ only the obvious solutions ($Q = (\varepsilon_i \delta_{\sigma i,j})$, $\varepsilon_i = \pm 1$ such that $\varepsilon_1 \cdots \varepsilon_n = \mathrm{sgn}\sigma$), then the local shape of $\Delta(A)$ in a vicinity of z_σ resembles a truncated cone whose opening angle equals $z_{\sigma_1} \widehat{z_\sigma} z_{\sigma_2}$, where σ_1, σ_2 differ from σ by transpositions. This lends further credibility to the well known de Oliveira Marcus Conjecture (OMC) concerning the determinant of the sum of normal $n \times n$ matrices. We deduce the mentioned fact from a general result concerning multivariate power series and also use some elementary algebraic topology.

AMS classification: 15A15

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1. Introduction

1.1. Notation

Our notation is standard where advisable. Here are listed in telegram style the notations and definitions that may need clarification.

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$\mathbb{R}_{\geqslant 0}$, $\mathbb{R}^n_{>0}$, $\dot{\mathbb{R}}$, etc.	reals$\geqslant 0$, $(\mathbb{R}_{>0})^n$, extended reals: $\mathbb{R} \cup \{\infty\}$, etc.
$S_n, \mathscr{T}, i \in \tau$	symmetric group on $\{1, \ldots, n\}$, set $\mathscr{T} = \{(i, j) : 1 \leqslant i < j \leqslant n\}$ often identified with the set of transpositions in S_n; $i \in \tau = \langle k, l\rangle \in \mathscr{T}$ means $i = k$ or $i = l$
$\mathrm{so}(n), \mathrm{su}(n)$	the Lie-algebras of (real) skew-symmetric and (complex) skew-hermitian $n \times n$ matrices of trace 0
$\mathrm{SO}(n), \mathrm{SU}(n)$	Lie-groups of orthogonal and unitary $n \times n$ matrices of determinant 1
$A; Q$	an arbitrary $n \times n$ complex matrix mostly fixed, a matrix in $\mathrm{SO}(n)$ respectively
$d_\sigma(M), z_\sigma, z_{id}$	the diagonal product of matrix M associated to permutation σ. $d_\sigma(M) = \prod_{i=1}^n m_{i\sigma(i)}$, in particular $d_{id}(M) = m_{11}m_{22}\cdots m_{nn}$. For the particular matrix A mentioned before, we sometimes use $z_\sigma := d_\sigma(A)$
$\|u\|$	mostly the norm of an element u in a normed space; $\mathbb{R}^n$, $\mathbb{C}$ carry euclidean norm
$B(z, \rho),\ B(\underline{x}, \rho)$	open balls of radius $\rho > 0$ centers z or $\underline{x}$, in $\mathbb{C}$ or $\mathbb{R}^n$ respectively
$\|B\|; P_\sigma; \mathscr{P}_\sigma$	the matrix $(\|b_{ij}\|)$; for $\sigma \in S_n$ the matrix $(\delta_{\sigma i, j})$; the set $\{Q \in \mathrm{SO}(n) : \|Q\| = P_\sigma\}$.
$A \circ B$	the Hadamard product of matrices A, B of same size: $(A \circ B)_{ij} = a_{ij}b_{ij}$
lhs(.), rhs(.), mid(.)	left hand side, right hand side, mid of an expression
l^+, px^+, px	a ray; for points p, x, the ray with origin p containing x; segment joining p to x
$f \simeq g; X \approx Y$	homotope maps; homoeomorphic spaces
$\mathrm{cl}X$, or $\overline{X}$	the topological closure of a subset X of the plane
S^1	the 1-sphere (unit circle) in $\mathbb{R}^2$
diameter(U)	for $U \subseteq \mathbb{R}^n$ the supremum $\sup\{\|u - u'\| : u, u' \in U\}$
$p, x, 0; \underline{x}, \underline{0}$	points $p, x, 0$ in the complex plane; a point in $\mathbb{R}^n$, dimension n will follow from context; the zero of $\mathbb{R}^n$
$\min \underline{b}$; $\max \underline{b}$	minimum/maximum of entries of real n-tuple $\underline{b} = (b_1, \ldots, b_n)$
[9, p45c-3]	example of reference to book or article: see [9] page 45, about 3cm from last text row.
cone$Z$; co$Z$	for a set $Z \subseteq \mathbb{C}$, the set (cone) $\{\sum_{i=1}^k r_i z_i : k \in \mathbb{Z}_{\geqslant 1}, r_i \geqslant 0, z_i \in Z\}$; the similarly constructed set (convex hull) with additional restriction $\sum_i r_i = 1$
monomial $c_{\underline{i}}\underline{x}^{\underline{i}}$; $\|\underline{i}\|$	an expression of the form $c_{i_1 i_2 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$.$\|\underline{i}\| = i_1 + \cdots + i_n$ is its *degree*
powerseries	a sum of possibly infinitely many monomials formally summed in any order

1.2. *Content and outline of results*

Let $A = (a_{ij})$ be a complex $n \times n$-matrix. Since $\mathrm{SO}(n) =$ Lie group of unitary $n \times n$ matrices of determinant 1 is a compact connected set [9, pp.104c-4, 147c-1], the region $\Delta(A) = \{\det(A \circ Q) : Q \in \mathrm{SO}(n)\}$ is a compact connected set in the complex plane. Let $z_\sigma = z_\sigma(A) = \prod_{i=1}^n a_{i\sigma i}$ be the (unsigned) diagonal product of A associated to $\sigma \in S_n$. The following formulation of a slightly weakened form of the Oliveira Marcus Conjecture [2] appears first implicitly in [6]; OMC itself claims the same thing to be true even if $\Delta(A)$ is defined using $\mathrm{SU}(n)$ instead of $\mathrm{SO}(n)$.

Conjecture (*OMC for* SO(n)). *If A is a rank* 2 *matrix, then*

$$\Delta(A) \subseteq \text{co}\{z_\sigma(A) : \sigma \in S_n\}.$$

Example. Although experiments indicate that the inclusion seems to remain true in many cases in which rank$A > 2$, this is not so in general: consider the case $A = \text{diag}(1, 1, 1)$ and choose Q as the matrix

$$Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

In this article we prove a result, see Theorem 11, related to the shape of $\Delta(A)$ near points $z_\sigma(A) \in \mathbb{C}$.

In Section 2 we compute the first terms of the power series $\det(A \circ \exp S)$ in the real and imaginary parts of the entries of $S \in \text{su}(n)$ around the zero matrix. The salient feature is that the nontrivial homogeneous component of lowest degree of this series is a linear combination of the squares of these parts with coefficients that are simple expressions in the $d_\sigma(A)$. Section 3 defines the concept of a corner of a region in the plane. An archetypical corner is a disk-sector of angle measure $< \pi$. We show that under natural restrictions a set valued map defined on such a sector and deviating from the identity by small enough a quantity as its argument approaches its vertex has as image region approximately the sector. The proof employs some elementary algebraic topology. Section 4 gives a lemma on power series of the type encountered for $\det(A \circ \exp S)$. It assures that such power series defines in a natural manner a set valued map of the type considered previously. This is used to deduce the main result, Theorem 11, in Section 5. We end with some remarks.

2. A power series

Recall that $\text{so}(n) =$ Lie-algebra of real skew-symmetric $n \times n$ matrices S is associated to SO(n) via the exponential map: indeed, by [9, p147c-2] (or [1, p165c4]), every $Q \in \text{SO}(n)$ can be written $Q = \exp(S)$ for some $S \in \text{so}(n)$. Hence

$$\Delta(A) = \{\det(A \circ Q) : Q \in \text{SO}(n)\} = \{\det(A \circ \exp S) : S \in \text{so}(n)\}.$$

For the proper understanding of the theory of absolutely summable series in a Banach space, and in particular function spaces and power series, as referred below, see [3, pp. 94–95, 127–128, 193–197]. For the formal background to these (of lesser importance here), see [10].

Note that the matrices $S \in \text{su}(n)$ are precisely the matrices of the form $S = A + \text{i}B$ where A is a real skew symmetric and B is real symmetric of trace 0. Hence there enter $(n^2 - n)/2 + (n^2 - n)/2 + (n - 1) = n^2 - 1$ real variables. By a polynomial in the entries of S, we mean a polynomial in these real variables; in particular the square of the modulus of such entries is a polynomial of degree 2 in these variables. Finally recall that if $\tau = \langle i, j\rangle \in \mathscr{T}$, then we permit s_τ as a shorthand for s_{ij}, $i < j$.

Theorem 1. *Let A be a complex $n \times n$ matrix and let S be a matrix in* su(n). *For $\tau \in \mathscr{T}$ put $\tilde{d}_\tau(A) = d_\tau(A) - d_{\text{id}}(A)$. Then we have a development*

$$\det(A \circ \exp(S)) = d_{id}(A) + \sum_{\tau \in \mathscr{T}} \tilde{d}_\tau(A)|s_\tau|^2 + \sum_{k \geqslant 3} p_k(S).$$

Here each $p_k(S)$ as well as $|s_\tau|^2$ is either 0 *or a homogeneous polynomial of degree k respectively* 2, *in $\leqslant n^2-1$ real variables. There is for any neighbourhood U_0 of the zero* (*matrix*) *in* $\mathrm{su}(n)\approx\mathbb{R}^{n^2-1}$, *a constant M, so that for every monomial $m(\cdot)$ occuring in this power series, and every $S\in U_0$, there holds $|m(S)|\leqslant M$.*

Proof. Since the matrix $S=(s_{ij})$, satisfies for all $i,j\in\{1,\dots,n\}$, the relations $s_{ij}=-\overline{s}_{ji}$, in particular $s_{ii}\in\sqrt{-1}\mathbb{R}$, we find that the (i,i)-entry of S^2 is given by

$$\sum_{\nu=1}^{n}s_{i\nu}s_{\nu i}=-|s_{ii}|^2-\sum_{\tau:i\in\tau}|s_\tau|^2.$$

Since $\exp S=I+S+\frac{1}{2}S^2+\cdots$, and since the nonzero entries of S^k are homogeneous polynomials of degree k in the s_{ij}, we find

$$(\exp S)_{ij}=\begin{cases}1+s_{ii}-\frac{1}{2}|s_{ii}|^2-\frac{1}{2}\sum\limits_{\tau:i\in\tau}|s_\tau|^2+p_{ii}(S), & \text{if } i=j,\\ s_{ij}+p_{ij}(S), & \text{if } i\neq j,\end{cases}$$

where the power series $p_{ii}(S)$ has under-degree $\geqslant 3$, while for $i\neq j$, $p_{ij}(S)$ has under-degree $\geqslant 2$. From this we extract information about the diagonal products $d_\sigma(\exp S)$. First, using $\sum_i s_{ii}=0$, and hence also $0=(\sum_i s_{ii})^2=2\sum_{l<k}s_{ll}s_{kk}-\sum_i|s_{ii}|^2$, we find

$$\begin{aligned}d_{id}(\exp S)&=\prod_{i=1}^{n}\left(1+s_{ii}-\frac{1}{2}|s_{ii}|^2-\frac{1}{2}\sum_{\tau:i\in\tau}|s_\tau|^2+p_{ii}(S)\right)\\&=1+\sum_i s_{ii}+\sum_{i<j}s_{ii}s_{jj}-\frac{1}{2}\sum_i|s_{ii}|^2-\frac{1}{2}\sum_i\sum_{\tau:i\in\tau}|s_\tau|^2+p_{id}(S)\\&=1-\frac{1}{2}\sum_i\sum_{\tau:i\in\tau}|s_\tau|^2+p_{id}(S)\\&=1-\sum_{\tau\in\mathscr{T}}|s_\tau|^2+p_{id}(S),\end{aligned}$$

where the power series $p_{id}(S)$ has under-degree $\geqslant 3$. The diagonal products corresponding to transpositions are given as follows.

$$\begin{aligned}d_{\langle i,j\rangle}(\exp S)&=\left(\prod_{l\neq i,j}^{n}\left(1+s_{ll}-\frac{1}{2}|s_{ll}|^2-\frac{1}{2}\sum_{\tau:l\in\tau}|s_\tau|^2+p_{ll}(S)\right)\right)\\&\quad\times(s_{ij}+p_{ij}(S))(-\overline{s}_{ij}+p_{ji}(S))\\&=-|s_{ij}|^2+p'_{ij}(S),\end{aligned}$$

where $p'_{ij}(S)$ has under-degree $\geqslant 3$. Finally, what concerns the diagonal products corresponding to $\sigma\notin\{id\}\cup\mathscr{T}$, the set $\{i:\sigma(i)\neq i\}$ contains at least three elements. It follows that an associated diagonal product yields a power series of under-degree $\geqslant 3$. Consequently

$$\det(A \circ \exp S) = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma\, d_\sigma(A) d_\sigma(\exp S)$$
$$= d_{id}(A)\left(1 - \sum_{\tau} |s_\tau|^2 + p_{id}(S)) - \sum_{\tau \in \mathscr{T}} d_\tau(A)(-|s_\tau|^2 + p'_\tau(S)\right)$$
$$+ \sum_{\sigma \notin \mathscr{T} \cup \{id\}} \operatorname{sgn}\sigma\, d_\sigma(A) d_\sigma(\exp S).$$

This formula and the degree properties of $p_{id}(S)$, $p'_\tau(S)$, $d_\sigma(\exp S)$ imply the formal expression given for $\det(A \circ \exp S)$. Now each of the n^2 functions $\mathrm{su}(n) \ni S \mapsto (\exp S)_{ij}$, $i, j = 1, \ldots, n$, is a power series of complex coefficients in $n^2 - 1$ real variables. Since the exponential series converges absolutely on U_0 [9, p. 25], the family of monomials in these variables occuring in the power series $(\exp S)_{ij}$ is absolutely (or normally) summable on U_0 in the sense of [3, p95c7, p128]. Since $\det(.)$ is a polynomial in the entries of a matrix, the claim concerning $m(S)$ is easily inferred. □

3. A set valued map

Definition 2

(a) Call a cone in the sense of the notation section *degenerate* if it is one of these: the plane $\mathbb{C}$, a half plane, a ray, or a straight line.
(b) A closed (convex) non-degenerate cone will be called a *cnd-cone*, for short. It is an exercise in plane geometry to show that a cnd-cone can be uniquely written in the form $C = \operatorname{cone}\{e^{i\theta_1}, e^{i\theta_2}\}$ with $\theta_1, \theta_2 \in]-\pi, \pi]$, satisfying $0 < \alpha = \min\{2\pi - |\theta_1 - \theta_2|, |\theta_1 - \theta_2|\} < \pi$. The real α is the usual measure of the angle the cone defines.
(c) An *angular region* (*or cone*) *at* z is a set given by $\mathrm{ar} = z + C$, with C a cnd-cone.
(d) The (disk-)sector of radius ρ given by this ar is $S(\mathrm{ar}, \rho) = \mathrm{ar} \cap B(z, \rho)$.
(e) Let ar be a (nondegenerate) angular region at z with angle $\alpha > 0$ and let $\varepsilon > 0$ be such that $0 < \alpha - 2\varepsilon < \alpha < \alpha + 2\varepsilon < \pi$. We call the two angular regions with the same *vertex* z and bissector as ar, but by a small angle $2\varepsilon > 0$ smaller/wider than α the *ε-contraction* $\mathrm{ar}_{-\varepsilon}$/ *$\varepsilon$-extension* $\mathrm{ar}_{+\varepsilon}$ of ar.

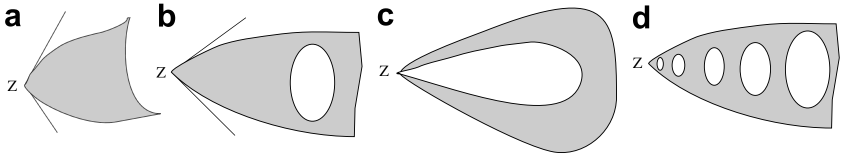
The central definition for this paper is that of a corner of a subset of the plane.

Definition 3. Let Δ be a subset of $\mathbb{C}$, and let $z \in \Delta$. The point z is called a *corner* of Δ, if there exists a nondegenerate angular region ar at z such that:

for every small $\varepsilon > 0$ there existsa $\delta > 0$ so that $S(\mathrm{ar}_{-\varepsilon}, \delta) \subseteq \Delta \cap B(z, \delta) \subseteq S(\mathrm{ar}_{+\varepsilon}, \delta)$.

In this case we also may say *Δ has in z the corner* ar.

Example 4. The idea of what a corner is, can be gleaned from the following series of pictures: the shaded regions (a) and (b) have in z corners whose angular regions ar are indicated by tangent lines. The region (c) has in z no corner. Similarly region (d) has in z no corner, since it has a sequence of 'holes' converging towards z. Assume a boundary curve of Δ near z exists. If it is strictly convex ('inward bounded') then as $\varepsilon \to 0$, δ has to go to 0 to satisfy the second inclusion, while if it is concave, $\delta \to 0$ is required to satisfy the first inclusion.



Observation 5. Let Δ, Δ', Δ'' be subsets of the plane.

(a) If $\Delta \subseteq \Delta' \subseteq \Delta''$ and Δ and Δ'' have in z the corner ar then Δ' has in z the corner ar.
(b) Δ has in z the corner ar iff $\Delta \cap B(z, r)$ has for some small $r > 0$ the corner ar.
(c) If Δ has in z the corner ar, then $u + \Delta$ has in $u + z$ the corner $u +$ ar.

Proof. The simple considerations necessary are left to the reader. □

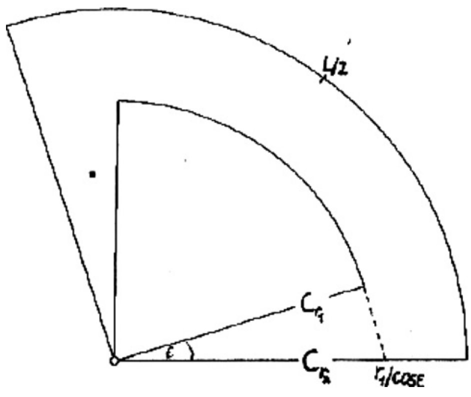
Let $\mathscr{P}(\mathbb{R}^2)$ = family of subsets (i.e. powerset) of $\mathbb{R}^2$.

Theorem 6. *Let $S = S(\text{ar}, \rho)$ be a disk sector with vertex in* 0 *and let $F : S \to \mathscr{P}(\mathbb{R}^2)$ be a set valued map with the following further properties*:

(i) *For some function $r : S \to \dot{\mathbb{R}}_{\geqslant 0}$, satisfying* $\lim_{x \to 0} r(x)/|x| = 0$ *and* $r(0) = 0$, *there holds* $F(x) \subseteq B(x, r(x))$ *for all* $x \in S$.
(ii) *There exists a continuous selection* $S \ni x \mapsto f(x) \in F(x)$.

Then for all small $r' > 0$, *the set* $F(S(\text{ar}, r'))$ *has* ar *as a corner at* 0.

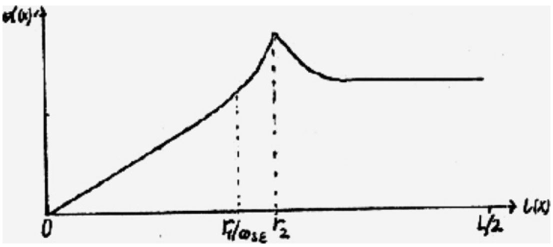
Proof



The figure shows the boundaries C_{r_1}, C_{r_2} of two disk-sectors which we think of being $\bar{I}_{r_1} = \text{cl}S(\text{ar}_{-\varepsilon}, r_1)$, $\bar{I}_{r_2} = \text{cl}S(\text{ar}, r_2)$. Of ε, r_1, r_2 we require in the moment only that ε be small enough so that $\text{ar}_{-\varepsilon}$ is nontrivial, and that the radii are assumed to satisfy $0 < r_1/\cos\varepsilon < r_2 \leqslant \rho$. We dispense with proving that C_{r_1}, C_{r_2} are rectifiable curves; that the Jordan curve theorem [7, p31] applies to them; that their respective Jordan-interiors [7, p36c-1; Enc. 93B&K] I_{r_1}, I_{r_2}, as well as $\bar{I}_{r_1}, \bar{I}_{r_2}$ are (convex) disk sectors; that $C_{r_2} \setminus \{0\}$ lies in the Jordan-exterior of C_{r_1}; and that we have a homeomorphism $\bar{I}_{r_2} \approx$ closed unit disc, which induces a homeomorphism $C_{r_2} \approx S^1$.

Let $L =$ perimeter of C_{r_2} and parametrize C_{r_2} by traversing it counterclockwise from 0 to 0 and defining $l : C_{r_2} \to [0, L[$ by $l(x) =$ arc-length from 0 to x; also let $d(x) =$ distance from $x \in C_{r_2}$ to C_{r_1}. Note that l is a continuous bijection. Simple geometry, in particular the cosine theorem, yields the following:

$$d(x) = \begin{cases} l(x)\sin(\varepsilon) & \text{for } l(x) \in [0, r_1/\cos\varepsilon], \\ \sqrt{l(x)^2 + r_1^2 - 2l(x)r_1\cos\varepsilon} & \text{for } l(x) \in [r_1/\cos\varepsilon, r_2], \\ \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(1+\varepsilon-(l(x)/r_2))} & \text{for } l(x) \in [r_2, r_2(1+\varepsilon)], \\ r_2 - r_1 & \text{for } l(x) \in [r_2(1+\varepsilon), \frac{L}{2}], \\ d(l^{-1}(L - l(x))) & \text{for } l(x) \in [\frac{L}{2}, L[. \end{cases}$$



The graph $l(x)$-versus-$d(x)$ for the example shown above is the figure at the left for $l(x) \leqslant L/2$. The requirement $r_1/\cos\varepsilon < r_2$ (instead of simply $r_1 < r_2$) was made to simplify analysability of $d(x)$.

We define the function $[0, \rho] \ni t \mapsto \tilde{r}(t) := \sup\{r(x) : x \in S, |x| = t\} \in \dot{\mathbb{R}}_{\geqslant 0}$. From the hypothesis on r we get $*_1 : \lim_{t\downarrow 0} \tilde{r}(t)/t = 0$. Now fix an ε satisfying $0 < \varepsilon \leqslant \min\{0.9, \alpha/2, (\pi - \alpha)/2\}$.

Fact 1. *For small r_2, there exists r_1 with $0 < r_1/\cos\varepsilon < r_2$ so that for $x \in C_{r_2} \setminus \{0\}, r(x) < d(x)$.*

$\triangleright$ By $*_1$ we find for small $r_2 \leqslant \rho$ that for all $0 < t \leqslant r_2$, $\tilde{r}(t) < \frac{\sin\varepsilon}{1+\sin\varepsilon} t$. Choose such an r_2 and put $r_1 = r_2/(1 + \sin\varepsilon)$. Then from the hypothesis on ε one checks that we have $r_2 > r_1/\cos\varepsilon > r_1$. Note that for $x \in C_{r_2}$, $|x| = \min\{l(x), r_2\} \leqslant r_2$. Then from the formulae for $d(x)$ one finds by routine checks for $x \in C_{r_2} \setminus \{0\}$, that $r(x) \leqslant \tilde{r}(|x|) < \frac{\sin\varepsilon}{1+\sin\varepsilon}|x| \leqslant d(x)$. $\triangleleft$

Let $r_1 < r_2$ be as in Fact 1; it implies for $x \in C_{r_2} \setminus \{0\}$, that $F(x) \cap C_{r_1} = \emptyset$. Since, when connecting x by a segment to a point $p \in I_{r_1}$ we cross C_{r_1}, it follows that $|x - p| > d(x)$. So $p \notin F(x)$. This shows $*_2 : \bar{I}_{r_1} \cap F(C_{r_2}) = \{0\}$.

Fact 2. *Every point in $\bar{I}_{r_1} \setminus \{0\}$ lies in the image of I_{r_2} under F: $\bar{I}_{r_1} \setminus \{0\} \subseteq F(I_{r_2})$.*

$\triangleright$ Assume there exists a point $p \in \bar{I}_{r_1} \setminus \{0\}$ so that $p \notin F(I_{r_2})$. Then $p \neq f(x)$ for all $x \in I_{r_2}$. It is also clear by $*_2$ that $p \notin f(C_{r_2})$. So we have a continuous map $f|\bar{I}_{r_2} : \bar{I}_{r_2} \xrightarrow{f} \mathbb{R}^2 \setminus \{p\}$. Let $\beta : \mathbb{R}^2 \setminus \{p\} \to C_{r_2}$ be the standard retraction map that carries each $x \in \mathbb{R}^2 \setminus \{p\}$ to the unique intersection of the ray px^+ with $C_{r_2} : \beta(x) = px^+ \cap C_{r_2}$. Then we get a continuous map $\beta \circ f|\bar{I}_{r_2} : \bar{I}_{r_2} \to C_{r_2}$ extending $\beta \circ f|C_{r_2} : C_{r_2} \to C_{r_2}$. By Spanier [8, p27] this means that $\beta \circ f|C_{r_2}$ is nullhomotopic. Note that we can write $f(x) = x + e(x)$ for some continuous map $e(x)$ satisfying $|e(x)| \leqslant r(x)$. Since for $t \in [0, 1]$, $|te(x)| \leqslant |e(x)|$, by Fact 1 we have a homotopy $C_{r_2} \times [0, 1] \ni (x, t) \overset{H}{\mapsto} x + te(x) \in \mathbb{R}^2 \setminus \{p\}$ showing $id_{C_{r_2}} \simeq f|C_{r_2}$ as $t : 0 \nearrow 1$. But since $C_{r_2} \approx S^1$ and id_{S^1} is not nullhomotopic (as follows from the observations [8, pp25c-7, 56c4, 59c5, 23c6]), we get that $id_{C_{r_2}}$ is not nullhomotopic. Now $\beta \circ H$ yields a homotopy $id_{C_{r_2}} = \beta \circ id_{C_{r_2}} \simeq \beta \circ f|C_{r_2}$; so we get a contradiction, proving the claim. ◁

Fact 3. *For all small* $r_2 > 0$ *there exists* $r_1 > 0$ *so that*

$$*_3 : S(\mathrm{ar}_{-\varepsilon}, r_1) \subseteq F(S(\mathrm{ar}, r_2)) \cap B(0, r_1) \subseteq S(\mathrm{ar}_{+\varepsilon}, r_1).$$

▷ Recall that $\bar{I}_{r_1} = \mathrm{cl}S(\mathrm{ar}_{-\varepsilon}, r_1)$. Also, by i, $F(0) = \{0\}$. So for given ε, as above, Facts 1 and 2 yield that for all small r_2 there exists an $r_1 > 0$, so that $S(\mathrm{ar}_{-\varepsilon}, r_1) \subseteq F(S(\mathrm{ar}, r_2))$. Intersecting both sides with $B(0, r_1)$ yields the left of the inclusions. Next let $u \in\!| \ (*_3)$. Then $u \in F(x)$ for some $x \in S(\mathrm{ar}, r_2)$. As in the proof of Fact 1 we have observed that this means $r(x) \leqslant \frac{\sin\varepsilon}{1+\sin\varepsilon}|x| < |x| \sin\varepsilon$. Consequently $u \in B(x, |x| \sin\varepsilon)$. Suppose $u \notin \mathrm{ar}_{+\varepsilon}$. Since $x \in \mathrm{ar} \subseteq \mathrm{ar}_{+\varepsilon}$, $u \notin \mathrm{ar}$. It follows that the segment ux has to contain a point in a side of ar and another in a side of $\mathrm{ar}_{+\varepsilon}$. These two sides define an angle $\geqslant \varepsilon$ with vertex 0. Consequently $|u - x| \geqslant |x| \sin\varepsilon$. Contradiction. Hence $u \in \mathrm{ar}_{+\varepsilon}$. Since also $|u| \leqslant r_1$, we get $u \in \mathrm{rhs}(*_3)$. ◁

With Fact 3 the theorem is proved. □

4. A lemma on power series

Lemma 7. *Let* $f(\underline{x}) = \sum_{k\geqslant 2} f_k(\underline{x})$ *be a power series over* $\mathbb{C}$ *where every* f_k *is either* 0 *or a homogeneous polynomial of degree* k. *Assume that*

(i) $f_2(\underline{x}) = \sum_{i=1}^{n} c_i x_i^2$, *with coefficients satisfying* $0 \notin \mathrm{co}\{c_i : i = 1, \ldots, n\}$;
(ii) *there exist* $M > 0$, *and* $\underline{b} \in \mathbb{R}^n_{>0}$, *so that* $|c_{\underline{i}} \underline{b}^{\underline{i}}| < M$ *for all monomials* $c_{\underline{i}} \underline{x}^{\underline{i}}$ *of* $f(\underline{x})$.

For any real positive $r < \min \underline{b}$, *we have a continuous function* $[-r, r]^n \ni \underline{x} \mapsto f(\underline{x}) \in \mathbb{C}$. *Furthermore*, $|f_2(\underline{x})| \to 0, \underline{x} \in [-r, r]^n$, *implies* $\sum_{k\geqslant 3} f_k(\underline{x})/|f_2(\underline{x})| \to 0$.

Proof. That f defines in the closed cube $[-r, r]^n$ a continuous function is a consequence of [3, p194c1..5]. From i we get that there exist $0 < \rho_1 < \rho_2 = \max\{|c_i| : i = 1, \ldots, n\}$ such that

$$\rho_1 \leqslant \left| \sum_{j=1}^{n} c_j \frac{x_j^2}{x_1^2 + \cdots + x_n^2} \right|$$

$$\text{and hence : } \rho_1(x_1^2 + \cdots + x_n^2) \leqslant |f_2(\underline{x})| \leqslant \rho_2(x_1^2 + \cdots + x_n^2) \qquad (*)$$

for the set of values the expression $\sum \ldots$ assumes as $\underline{x}$ varies over any neighbourhood of $\underline{0}$ is just the convex hull of $c_1, \ldots, c_n$. Henceforth, we assume $f_k(\underline{x}) = \sum_{|\underline{i}|=k} c_{\underline{i}} \underline{x}^{\underline{i}}$, $k = 3, \ldots$

We put

$$L_k = \{\underline{i} : |\underline{i}| = k, i_\nu \leqslant 1 \text{ for all } \nu\}, \qquad Q_k = \{\underline{i} : |\underline{i}| = k, i_\nu \geqslant 2 \text{ for some } \nu\}.$$

Case $\underline{i} \in L_k$. Then exactly k of the i_νs are 1, say $i_{\nu_1} = \cdots = i_{\nu_k} = 1$. We have the estimates

$$x_{i_{\nu_1}} \cdots x_{i_{\nu_k}} \leqslant \frac{1}{k}(|x_{i_{\nu_1}}|^k + \cdots + |x_{i_{\nu_k}}|^k); \quad \text{and} \quad \frac{|x_i|^k}{x_1^2 + \cdots + x_n^2} \leqslant |x_i|^{k-2},$$

$i = 1, \ldots, n$, the first following from the arithmetic geometric mean inequality, the second being trivial. These inequalities imply

$$\left| c_{\underline{i}} \frac{\underline{x}^{\underline{i}}}{x_1^2 + \cdots + x_n^2} \right| \leqslant \frac{1}{k} \sum\nolimits_{\nu : i_\nu = 1} |c_{\underline{i}}||x_{i_\nu}|^{k-2}.$$

Case $\underline{i} \in Q_k$. Then, for a definite choice, we can define $j = j(\underline{i}) = \min\{\nu : i_\nu \geqslant 2\}$, and find

$$\begin{aligned} \left| c_{\underline{i}} \frac{\underline{x}^{\underline{i}}}{x_1^2 + \cdots + x_n^2} \right| &= |c_{\underline{i}}| \frac{|x_j|^2}{x_1^2 + \cdots + x_n^2} |x_1|^{i_1} \cdots |x_j|^{i_j - 2} \cdots |x_n|^{i_n} \\ &\leqslant |c_{\underline{i}}||x_1|^{i_1} \cdots |x_j|^{i_j - 2} \cdots |x_n|^{i_n}. \end{aligned}$$

Now put $m(\underline{x}) = \max\{|x_1|, \ldots, |x_n|\}$. Then

$$\begin{aligned} \left| \sum_{k \geqslant 3} f_k(\underline{x}) / f_2(\underline{x}) \right| &\leqslant \frac{1}{\rho_1} \sum_{k \geqslant 3} |f_k(\underline{x})| / (x_1^2 + \cdots + x_n^2) \\ &\leqslant \frac{1}{\rho_1} \sum_{k \geqslant 3} \left(\sum_{\underline{i} \in L_k} \frac{1}{k} \sum_{\nu : i_\nu = 1} |c_{\underline{i}}||x_{i_\nu}|^{k-2} + \sum_{\underline{i} \in Q_k} |c_{\underline{i}}||x_1|^{i_1} \cdots |x_{j(\underline{i})}|^{i_{j(\underline{i})} - 2} \cdots |x_n|^{i_n} \right) \\ &\leqslant \frac{1}{\rho_1} \sum_{k \geqslant 3} \sum_{\underline{i} : |\underline{i}| = k} |c_{\underline{i}}| (\max\{|x_1|, \ldots, |x_n|\})^{k-2} \\ &= \frac{1}{\rho_1} \sum_{k \geqslant 3} \sum_{\underline{i} : |\underline{i}| = k} |c_{\underline{i}}| m(\underline{x})^{k-2} = \frac{1}{\rho_1} \sum_{\underline{i} : |\underline{i}| \geqslant 3} |c_{\underline{i}}| m(\underline{x})^{|\underline{i}| - 2}. \end{aligned}$$

The last equality sign is justified as follows: let $b = \min\{b_1, \ldots, b_n\}$. By hypothesis (ii) we know $|c_{\underline{i}}| b^{|\underline{i}| - 2} \leqslant M / b^2$. Put $q = r/b$. For all $\underline{x} \in]-r, r[^n$, $m(\underline{x})/b \leqslant q$, and so

$$|c_{\underline{i}}| m(\underline{x})^{|\underline{i}| - 2} \leqslant |c_{\underline{i}}| q^{|\underline{i}| - 2} b^{|\underline{i}| - 2} \leqslant M / b^2 q^{|\underline{i}| - 2}.$$

Now

$$\sum_{\underline{i} : |\underline{i}| \geqslant 3} q^{|\underline{i}| - 2} \leqslant 1/q^2 \sum_{\underline{i} \in \mathbb{Z}^n_{\geqslant 0}} q^{|\underline{i}|} = (1 - q)^{-n-2}.$$

Therefore, by [3, p95c4..8], the denumerable family $(|c_{\underline{i}}| m(\underline{x})^{|\underline{i}| - 2})_{\underline{i} : |\underline{i}| \geqslant 3}$ of bounded continuous functions on polycylinder $]-r, r[^n$ is absolutely summable. Furthermore, by [3, pp 128c7,129c3] it is continuous. Since $m(\underline{0}) = 0$, we have that, as $\underline{x} \to 0$, the right hand side converges to 0. This proves the lemma. □

Example 8. Consider the polynomial $f(x, y) = x^2 + y^3$ as a power series in x, y. Here, $f_2(\underline{x}) \to 0$ does not imply $f_3(\underline{x}) \to 0$. So hypothesis (i) of Lemma 7 cannot be weakened to $0 \notin \text{co}\{c_i : c_i \neq 0, i = 1, \ldots, n\}$.

Note that if Lemma 7 holds for a certain $r > 0$, then it holds also when formulated with a neighbourhood $U \subseteq [-r, r]^n$ of $\underline{0}$ instead of $[-r, r]^n$.

Corollary 9. *Assume the hypotheses and notation of Lemma* 7 *in force and additionally that the* c_i *are not collinear. Then for all small neighbourhoods* U *of* $\underline{0} \in \mathbb{R}^n$, $f(U)$ *has in* 0 *the angular region* $\mathrm{ar} = \mathrm{cone}\{c_1, \ldots, c_n\}$ *as a corner.*

Proof. The noncollinearity condition, ensures that ar obeys the nondegeneracy condition implicit in Definition 2. We prove next two general facts.

Fact 1. *For every neighbourhood* U *of* $\underline{0} \in \mathbb{R}^n$ *we can find* $0 < r_1 = r_1(U)$ *and* $0 < r_2 = r_2(U)$ *such that* $S(\mathrm{ar}, r_1) \subseteq f_2(U) \subseteq S(\mathrm{ar}, r_2)$ *and so that diameter*$(U) \to 0$ *implies* $r_2(U) \to 0$.

$\triangleright$ Recall that according to inequality $(*)$ in the proof of Lemma 7 there exist two constants $0 < \rho_1 < \rho_2$ so that $\rho_1|\underline{x}|^2 \leqslant |f_2(\underline{x})| \leqslant \rho_2|\underline{x}|^2$. Choose balls $B(\underline{0}, \rho) \subseteq U \subseteq B(\underline{0}, \rho')$ with $\rho' = \mathrm{diameter}(U) \in \dot{\mathbb{R}}$. Define $r_1 = \rho_1\rho^2, r_2 = \rho_2\rho'^2$. Let $x \in S(\mathrm{ar}, r_1)$. Since from the very definition of a cone it follows that $f_2(\mathbb{R}^n) = \mathrm{ar}$, there is an $\underline{x} \in \mathbb{R}^n$ so that $x = f_2(\underline{x})$. Hence $\rho_1|\underline{x}|^2 \leqslant |x| \leqslant r_1$. Consequently $|\underline{x}|^2 \leqslant \rho^2$. This shows $S(\mathrm{ar}, r_1) \subseteq f_2(B(\underline{0}, \rho)) \subseteq f_2(U)$. Next, assume $x \in f_2(U)$. Then there exists $\underline{x} \in U$, hence $|\underline{x}| \leqslant \rho'$, so that $x = f_2(\underline{x})$. So $|x| \leqslant \rho_2\rho'^2 = r_2$ and so we have $f_2(U) \subseteq S(\mathrm{ar}, r_2)$. The remaining claim follows from the definitions of r_2, ρ'. $\triangleleft$

Now we define for any neighbourhood U of $\underline{0} \in \mathbb{R}^n$ with $U \subseteq]-r, r[^n$, for $x \in f_2(U)$:

$$C(x) = \{\underline{x} \in U : f_2(\underline{x}) = x\}, \quad S(x) = \left\{\sum_{k \geqslant 3} f_k(\underline{x}) : \underline{x} \in C(x)\right\}, \quad \text{and} \quad F(x) = x + S(x).$$

Fact 2. $f(U) = F(f_2(U))$.

$\triangleright$ Choose any $\underline{x} \in U$. Put $x = f_2(\underline{x})$. Then $x \in f_2(U)$, $\underline{x} \in C(x)$, and $f(\underline{x}) = f_2(\underline{x}) + \sum_{k \geqslant 3} f_k(\underline{x}) \in x + S(x) = F(x)$. This shows $f(U) \subseteq F(f_2(U))$. Now choose any $x \in f_2(U)$. Next choose any $s \in S(x)$. Then $s = \sum_{k \geqslant 3} f_k(\underline{x})$ for some $\underline{x} \in C(x)$; so that $x = f_2(\underline{x})$. Hence $x + s = f_2(\underline{x}) + \sum_{k \geqslant 3} f_k(\underline{x}) = f(\underline{x})$. Since $\underline{x} \in U$, we have $x + s \in f(U)$. This shows $x + S(x) \subseteq f(U)$ and $F(f_2(U)) \subseteq f(U)$. $\triangleleft$

We emphasize that Facts 1 and 2 hold for an arbitrary neighbourhood U of $\underline{0} \in \mathbb{R}^n$ with $U \subseteq]-r, r[^n$ and $f_2(U)$, $S(x)$, $C(x)$, are conditioned by this choice.

We now fix U to be a neighbourhood satisfying $U \subseteq]-r, r[^n$, r being chosen as in Lemma 7. The set valued map F can by Fact 1 be restricted to a disc-sector D of type ar contained in $f_2(U)$: $*_1$: $D \subseteq f_2(U)$.

Fact 3. $F : D \to \mathscr{P}(\mathbb{R}^2)$ *satisfies the hypotheses of Theorem* 6.

$\triangleright$ Define for $x \in D$ the function $r(x) = 1.1 \cdot \sup\{|s| : s \in S(x)\}$. Then $S(x) \subseteq B(0, r(x))$. By lemma 7 we know that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f_2(x)| < \delta \to |\sum_{k \geqslant 3} f_k(\underline{x})| \leqslant \varepsilon|f_2(\underline{x})|$. Now fix an $\varepsilon > 0$, and choose an associated $\delta > 0$ accordingly. Let $x \in D$, $|x| < \delta$. By $*_1$, $x = f_2(\underline{x})$ for all $\underline{x} \in C(x)$. Hence $|\sum_{k \geqslant 3} f_k(\underline{x})| \leqslant \varepsilon|x|$ for all $\underline{x} \in C(x)$. This means $r(x) \leqslant \varepsilon|x|$. Since $\varepsilon > 0$ here is arbitrary, we have shown, $r(x)/|x| \to 0$ as $x \to 0$. Also, $S(0) = \{0\}$. Since $F(x) = x + S(x)$ we see $F(x) \subseteq B(x, r(x))$, so F satisfies hypothesis (i) of Theorem 6. To see (ii), we use that there exist two c_i, c_1 and c_2, say so that $\mathrm{ar} = \mathrm{cone}\{c_1, c_2\}$. We can then write each $x \in D$ in a unique way as $x = c_1x_1^2 + c_2x_2^2$. Clearly the coordinate functions $x_1 =$

$x_1(x), x_2 = x_2(x)$ depend continuously on x. So $D \ni x \mapsto f((x_1(x), x_2(x), 0_{n-2})) \in F(x)$ is a continuous selection, showing (ii). ◁

There exists, by Theorem 6, an $r_2 \leqslant$radius of D so that for all $0 < r' \leqslant r_2$ the set $F(S(\text{ar}, r'))$ has in 0 a corner of type ar. By (the arguments which proved) Fact 1, we can choose a neighbourhood $U' \subseteq U$ of $\underline{0}$, and an $r_1 > 0$ so that $S(\text{ar}, r_1) \subseteq f_2(U') \subseteq S(\text{ar}, r_2)$. Upon applying F, we get $F(S(\text{ar}, r_1)) \subseteq F(f_2(U')) \subseteq F(S(\text{ar}, r_2))$. The left and the right subsets of this inclusion are corners of type ar. Hence, by observation 5a, $F(f_2(U')) = f(U')$ also has ar as a corner in 0. This was to prove. □

5. The main result

Lemma 10. *Let A, Q, D, P_σ be $n \times n$ matrices, D diagonal, $\sigma, \rho \in S_n$, P_σ, P_ρ the associated permutation matrices. Then there hold the following computational rules.*

$$P_{\rho\sigma} = P_\sigma P_\rho,\ d_\sigma(P_\rho A) = d_{\rho^{-1}\sigma}(A),\ D(A \circ Q) = A \circ (DQ) = (DA) \circ Q,$$
$$P_\sigma(A \circ Q) = (P_\sigma A) \circ (P_\sigma Q),\ \det(A \circ P_\sigma) = \operatorname{sgn}\sigma\, d_\sigma(A).$$

Proof. The easy proofs are left to the reader; see also [5, p304]. □

Let $\mathscr{P}_\sigma = \{Q \in \mathrm{SO}(n) : |Q| = P_\sigma\}$. Clearly each $Q \in \mathscr{P}_\sigma$ can be written $Q = DP_\sigma$, with $D = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_i \in \{-1, +1\}$, $\det(D) = \operatorname{sgn}\sigma$. One consequence of Lemma 10 is that if $Q \in \mathscr{P}_\sigma$, then $\det(A \circ Q) = d_\sigma(A)$.

Theorem 11. *Let A be a complex $n \times n$ matrix, and let $\sigma \in S_n$. Assume that the only matrices $Q \in \mathrm{SO}(n)$ for which $\det(A \circ Q) = d_\sigma(A)$ are the matrices in $\mathscr{P}_\sigma$; and that the complex numbers $\tilde{d}_{\sigma\tau}(A) = d_{\sigma\tau}(A) - d_\sigma(A)$, $\tau \in \mathscr{T}$, lie in an open half plane whose support contains the origin, and that they are not all collinear with 0. Then $\Delta(A) = \{\det(A \circ Q) : Q \in \mathrm{SO}(n)\}$ has in $d_\sigma(A)$ the corner $d_\sigma(A) + \operatorname{cone}\{\tilde{d}_{\sigma\tau}(A) : \tau \in \mathscr{T}\}$.*

Proof. Case $\sigma = id$. The essentials lie in the proof for this case. By the theory of Lie-groups [9, pp31c5, 145c4] we can choose small open neighbourhoods, U_0 of $0 \in \mathrm{so}(n)$ and U_I of $I \in \mathrm{SO}(n)$ so that the map $U_0 \ni S \mapsto \exp(S) \in U_I$ delivers a bijection. Also, by [9, p91c-5], if $D = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n) \in \mathrm{SO}(n)$, then, $U_D = DU_I$ is a neighbourhood of D. Let $K = \mathrm{SO}(n) \setminus \bigcup\{U_D : D = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n) \in \mathrm{SO}(n)\}$. Then K is compact.

On $\mathrm{so}(n)$ and $\mathrm{SO}(n)$, respectively, define the maps f, φ by

$$\mathrm{so}(n) \ni S \overset{f}{\mapsto} \det(A \circ \exp S) - d_{id}(A) \in \mathbb{C} \quad \text{and} \quad \mathrm{SO}(n) \ni Q \overset{\varphi}{\mapsto} \det(A \circ Q) \in \mathbb{C}.$$

From the hypothesis we find that φK is a compact set not containing $d_{id}(A)$. Since the distance between compact disjoint sets is positive [3, p61c-2], we can find a ball around $d_{id}(A)$ having with φK empty intersection. Now for every of the diagonal matrices D here present, and every $Q \in \mathrm{SO}(n)$, $\varphi(DQ) = \varphi(Q)$,

So

$$\begin{aligned}\Delta(A) &= \varphi(\mathrm{SO}(n)) = \varphi\Big(K \cup \bigcup_D U_D\Big) = \varphi K \cup \bigcup_D \varphi(DU_I)\\ &= \varphi K \cup \varphi U_I = \varphi K \cup (\varphi \circ \exp U_0)\\ &= \varphi K \cup (f(U_0) + d_{id}(A)).\end{aligned}$$

For small $r > 0$, we now have $\Delta(A) \cap B(d_{id}(A), r) = d_{\mathrm{id}}(A) + (f(U_0) \cap B(0, r))$. From Theorem 1 we know that for $S \in U_0$, $f(S) = \sum_{\tau\in\mathscr{T}} \tilde{d}_\tau(A)|s_\tau|^2 + \sum_{k\geqslant 3} p_k(S)$, and this can be rewritten as a real variable power series with complex coefficients, precisely in the form required in Lemma 7. This yields by Corollary 9 and the observation 5bc that Δ has in $d_{id}(A)$ the corner claimed.

Case $\sigma \in S_n$ arbitrary. As one may expect this case can be reduced to the previous one. Let $\tilde{A} = P_{\sigma^{-1}}A$ and let $Q \in SO(n)$. Choose a diagonal matrix D so that $DP_{\sigma^{-1}} \in \mathscr{P}_{\sigma^{-1}}$ and put $\tilde{Q} = DP_{\sigma^{-1}}Q$. Then $\det(\tilde{A} \circ \tilde{Q}) = \det(P_{\sigma^{-1}}A \circ (DP_{\sigma^{-1}}Q)) = \det(DP_{\sigma^{-1}})\det(A \circ Q) = \det(A \circ Q)$ and $d_\sigma(A) = d_{id}(\tilde{A})$. Now

$$
\begin{aligned}
\tilde{Q} \in \mathscr{P}_{id} \quad & \text{iff } Q \in \mathscr{P}_\sigma \quad \text{(easy)},\\
& \text{iff } \det(A \circ Q) = d_\sigma(A) \quad \text{(by hypotheses)},\\
& \text{iff } \det(\tilde{A} \circ \tilde{Q}) = d_{id}(\tilde{A}) \quad \text{(by the equations above)}.
\end{aligned}
$$

So we can apply the first case to the matrix $\tilde{A}$. So $\Delta(\tilde{A})$ has in $d_{id}(\tilde{A})$ the corner $\mathrm{ar} = d_{id}(\tilde{A}) + \mathrm{cone}\{\tilde{d}_\tau(\tilde{A}) : \tau \in S_n\}$. Now for any $Q \in \mathrm{SO}(n)$, $\det(\tilde{A} \circ Q) = \det((DP_{\sigma^{-1}}A) \circ Q) = \det(A \circ (P_\sigma DQ))$. Since $P_\sigma D\mathrm{SO}(n) = \mathrm{SO}(n)$, we can infer $\Delta(\tilde{A}) = \{\det(\tilde{A} \circ Q) : Q \in \mathrm{SO}(n)\} = \Delta(A)$. Furthermore $d_{id}(\tilde{A}) = d_\sigma(A)$, and $d_\tau(\tilde{A}) = d_\tau(P_{\sigma^{-1}}A) = d_{\sigma\tau}(A)$. From this we get $\mathrm{ar} = d_\sigma(A) + \mathrm{cone}\{d_{\sigma\tau}(A) - d_\sigma(A) : \tau \in \mathscr{T}\}$. The theorem is proved. □

We end with three remarks.

Remark 12

(a) For technical reasons (in particular what concerns the reasoning employed in Theorem 6, Fact 2) we have restricted the formulation of the main result to the case that the $\tilde{d}_\tau(A)$ are not all collinear with 0. It seems to us that with obvious modifications it will also hold without this restriction (and indeed the proof will be easier).

(b) For c, s reals satisfying $c^2 + s^2 = 1$, define $Q = Q(c, s) \in \mathrm{SO}(3)$, the matrix at the left. Then $\det(I \circ Q(c, s)) = 0 = d_\sigma(I)$ for all admissible c, s and $\sigma \neq id$. So the hypothesis of Theorem 11 usually is not satisfied.

$$
Q(c, s) = \begin{bmatrix} c & 0 & s \\ -s & 0 & c \\ 0 & -1 & 0 \end{bmatrix}.
$$

At the other hand, the condition of Theorem 11 is certainly not empty. For example $\det(I \circ Q) = 1$ will happen only if $Q \in \mathrm{SO}(n)$ is a signed identity matrix. Some proofs of the special cases of OMC already available provide more examples; see e.g. [4]. Indeed it seems to us that answering the question for which pairs $Q \in \mathrm{SO}(n)$, and permutations $\sigma \in S_n$ equations $\det(A \circ Q) = d_\sigma(A)$ can happen would mean – in case $\mathrm{rank}A = 2$ at least – to go a long way towards deciding OMC.

(c) The reader may well ask why we have not formulated Theorem 11 for $\mathrm{SU}(n)$. The reason is that the diagonal entries of an $S \in \mathrm{su}(n)$ do *not* enter in the homogeneus part of degree 2 in the real variable power series of complex coefficients, $f(S) = \det(A \circ \exp S)$. So in terms of Lemma 7, see also Example 8, we do not know whether $f_2(S) \to 0$ implies $\sum_{k\geqslant 3} f_k(S)/f_2(S) \to 0$; hence we cannot apply our reasoning to these cases.

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