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On the extremal structure of least upper bound norms and their dual

E. Marques de Sá [a,1], Virgínia Santos [b,*,2]

[a] *Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal*
[b] *Departamento de Matemática, Universidade de Aveiro, 3800 Aveiro, Portugal*

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Abstract

Given finite dimensional real or complex Banach spaces, E and F, with norms $\nu : E \to \mathbb{R}$ and $\mu : F \to \mathbb{R}$, we denote by $N_{\mu\nu}$ the least upper bound norm induced on $\mathscr{L}(E, F)$. Some results are given on the extremal structures of $\mathfrak{B}$, the unit ball of $N_{\mu\nu}$, of its polar $\mathfrak{B}^\circ$, and of $\mathfrak{B}'$, which is the polar of the unit ball of the least upper bound norm $N_{\mu^\circ\nu^\circ}$.

The exposed faces, the extreme points, and a large family of other faces of $\mathfrak{B}^\circ$ and $\mathfrak{B}'$ are presented. It turns out that $\mathfrak{B}'$ is a subset of $\mathfrak{B}$; the set of tangency points of the surfaces of $\mathfrak{B}$ and $\mathfrak{B}'$ is completely determined and represented as the union of the exposed faces of $\mathfrak{B}'$ which are normal to rank-one mappings. We determine sharp bounds on the ranks of mappings in these exposed faces.

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* Corresponding author. Fax: +351 234 382 014.
E-mail addresses: emsa@mat.uc.pt (E.M. de Sá), virginia@ua.pt (V. Santos).
[1] Member of the research unit CMUC/FCT (Centro de Matemática da Universidade de Coimbra/Fundação para a Ciência e a Tecnologia). Partially supported by Praxis XXI Mat/485/94, and Project 574/94 of Fundação Luso-Americana para o Desenvolvimento. Fax: +351 239 832 568.
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1. Introduction

We let E be an n-dimensional vector space over $\mathbb{K}$ (where $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$) endowed with an inner product $\langle\cdot|\cdot\rangle$, and a norm ν (we assume norms are *strictly homogeneous*, i.e., $\nu(\lambda x) = |\lambda|\nu(x)$ for all $\lambda \in \mathbb{K}$). In the complex case, E has an underlying *real* vector space structure, of dimension $2n$, and the functional $(x, y) \mapsto \Re\langle x|y\rangle$ ($\Re$ means *real part*) endows E with the structure of a real inner product space; as a consequence we have two kinds of orthogonality: if $\langle x|y\rangle = 0$ we say x and y are $\mathbb{C}$-*orthogonal*; if $\Re\langle x|y\rangle = 0$ we say x and y are $\mathbb{R}$-*orthogonal*. The concepts of "polar" and "subdifferential" to be given below refer, in the complex case, to the underlying real inner product space structure.

The symbols $\mathfrak{B}(\nu)$ and $\mathscr{S}(\nu)$ [often shortened to $\mathfrak{B}_\nu$ and $\mathscr{S}_\nu$] denote, respectively, the closed unit ball and the unit sphere of ν (so $\mathscr{S}(\nu)$ is the boundary of $\mathfrak{B}(\nu)$). The mapping $\nu^\circ : E \to \mathbb{R}$, given by $\nu^\circ(y) := \sup\{\Re\langle x|y\rangle : x \in \mathfrak{B}_\nu\}$, is called the *polar* of ν. It is well known that ν° is a norm, $\nu^{\circ\circ} = \nu$ and

$$\Re\langle a|u\rangle \leqslant \nu(a)\nu^\circ(u). \tag{1}$$

A second vector space F is given, of dimension m over $\mathbb{K}$, endowed with a norm μ and an inner product also denoted $\langle\cdot|\cdot\rangle$. Given A in $\mathscr{L}(E, F)$, the space of $\mathbb{K}$-linear mappings of E into F, $A^* \in \mathscr{L}(F, E)$ denotes the *adjoint of* A, uniquely determined by the condition $\langle x|A^*y\rangle = \langle Ax|y\rangle$, for all $x \in E$ and $y \in F$.

We have chosen to workout the whole paper in a coordinate free context, to stress the fact that our results are of a geometrical nature, i.e., they do not depend on the choice of bases in E and F. Of course the whole thing may be done in a matrix setting; so the reader may well think of x and y as column vectors and of A as an $m \times n$ matrix; then A^* is the conjugate transpose of A. In case $E = F$, the trace, the determinant, as well as any other similarity invariant of A are well defined. The composition of linear mappings, A and B (in appropriate spaces), is denoted by AB.

For nonzero $u \in E$ and $w \in F$, the *rank-one tensor* $u \otimes w$ is the element of $\mathscr{L}(E, F)$ given by $(u \otimes w)x := \langle x|u\rangle w$. Any $A \in \mathscr{L}(E, F)$ may be expressed as a sum of rank-one tensors; the minimum number of terms in such a sum is the rank of A. For any $U \subseteq E$ and $W \subseteq F$, the set $\{u \otimes w : u \in U, w \in W\}$ is denoted by $U \otimes W$.

In $\mathscr{L}(E, F)$ we consider the standard inner product $\langle A|B\rangle := \mathrm{tr}(B^*A)$. For any norm N on $\mathscr{L}(E, F)$, the polar of N is given by

$$N^\circ(A) = \sup\{\Re(\mathrm{tr}(A^*\theta)) : \theta \in \mathfrak{B}_N\}.$$

We define $N^*(B) := N(B^*)$, for any B in $\mathscr{L}(F, E)$. N^* is clearly a norm in $\mathscr{L}(F, E)$. In the sequel, we shall be concerned with the least upper bound norm $N_{\mu\nu}$ on $\mathscr{L}(E, F)$, given by

$$N_{\mu\nu}(A) := \sup\{\mu(Ax) : x \in \mathfrak{B}_\nu\}.$$

A *face* of a convex subset $K \subseteq E$ is a convex subset Φ of K such that, for any $x, y \in K$, the condition $]x, y[\cap\Phi \neq \varnothing$ implies $x, y \in \Phi$; *exposed faces* of K are the special faces obtained by intersecting K with its supporting hyperplanes (for these and other concepts of convex analysis, see [14]).

The problem of the determination of the extreme structure of the unit ball of an arbitrary least upper bound norm and of its polar is an interesting and important one, and it is still far from being solved. Contributions to this matter appeared in the papers [21,22,20,24,23]; the present paper gives some steps further.

The central problem we are dealing here is the characterization of the faces and exposed faces of the convex bodies $\mathfrak{B}(N_{\mu\nu})$ and $\mathfrak{B}(N_{\mu\nu}{}^{\circ})$, and, in particular, their extreme and exposed points. These concepts naturally occur in approximation theory (e.g., [18,8]): for example, if $\mathscr{H}$ is a real hyperplane of E and d is the distance (with respect to ν) of a point $u \in E$ to $\mathscr{H}$, then the best approximation subset of $\mathscr{H}$ is of the form $u + d\Phi$, where Φ is an exposed face of $\mathfrak{B}_\nu$, and all exposed faces of $\mathfrak{B}_\nu$ occur as best approximation subsets of hyperplanes. This is a special case of a well-known result of Singer [18, Theorems 1.1 and 1.2] often used in the literature to describe best approximation sets in concrete cases (e.g., [18,7,23,25,27]).

Complete characterizations of the exposed faces of the unit balls of unitarily invariant norms and of dual least upper bound norms appeared in the literature in various degrees of generality (see [21,4,5,19,23,26,15,16,17]). The papers [9,10] contain far-reaching developments of this kind of results.

We may get information on the group of isometries of two normed spaces, say E and F, using the extremal structure of their unit balls. As a matter of fact, an isometry $f : E \to F$ maps each (exposed) face of $\mathfrak{B}_\nu$ onto an (exposed) face of $\mathfrak{B}_\mu$, it preserves the faces dimensions, and induces an isomorphism between the lattice of (exposed) faces of $\mathfrak{B}_\nu$ into the lattice of (exposed) faces of $\mathfrak{B}_\mu$. In one way or another, this kind of property is frequently used to study isometries of Banach spaces (see [1] and references therein). The closely related problem of characterizing the linear mappings between normed matrix spaces that preserve given norms has extensively been considered in the literature; in many cases, the main techniques involve the determination of the extreme matrices with respect to the norms given (see, e.g., [3,2,6,11–13]).

We define the *dual of* $N_{\mu\nu}$ as the functional $N^D_{\mu\nu} : \mathscr{L}(F, E) \to \mathbb{R}$, given by $N^D_{\mu\nu}(B) := N_{\mu\nu}{}^{\circ}(B^*)$. The papers [20,24] consider dual matrix norms in the case $E = F$ and $\nu = \mu$.

In Section 2 we extend some results of [20] to the norms $N_{\mu\nu}$ and $N^D_{\nu\mu}$. In Section 3, the facial structure of $\mathfrak{B}(N_{\mu\nu}{}^{\circ})$ is considered in detail: as in [21,23], the exposed faces are easy to describe; we characterize the extreme points, and give a large family of (in general non exposed) faces of $\mathfrak{B}(N_{\mu\nu}{}^{\circ})$, but a complete description of such faces is left open. The unit ball of $N^D_{\nu\mu}$ is contained in the unit ball of $N_{\mu\nu}$, and the rank one tensors, suitably normalized, are tangency points of the surfaces of those two unit balls. In Section 4 we determine the set of all other tangency points, which is represented as the union of the exposed faces of $\mathfrak{B}(N^D_{\nu\mu})$ which are normal to rank-one tensors. In Section 5 we determine sharp bounds on the ranks of mappings in these exposed faces.

2. Preliminary results

Given $\mathscr{K} \subseteq \mathscr{L}(E, F)$, we define the sets $\mathscr{K}^* := \{A^* : A \in \mathscr{K}\}$, $\mathscr{K}^D := \{B \in \mathscr{L}(F, E) : \Re(\operatorname{tr}(B\kappa)) \leqslant 1 \ \forall \kappa \in \mathscr{K}\}$, and $\mathscr{K}^{\circ} := \{A \in \mathscr{L}(E, F) : \Re\langle A|\kappa\rangle \leqslant 1 \ \forall \kappa \in \mathscr{K}\}$. Each element $\kappa \in \mathscr{K}$ determines a *real* half-space of $\mathscr{L}(E, F)$, namely

$$\mathscr{H}_\kappa := \{A : \Re\langle A|\kappa\rangle \leqslant 1\}. \tag{2}$$

This half-space is bounded by a *real* hyperplane, and κ is $\mathbb{R}$-orthogonal to this hyperplane. Of course $\mathscr{K}^{\circ}$ is the intersection of all half-spaces $\mathscr{H}_\kappa$, with $\kappa \in \mathscr{K}$. Clearly $\mathscr{K}^D = (\mathscr{K}^{\circ})^* = (\mathscr{K}^*)^{\circ}$ and $\mathscr{K}^{DD} = \mathscr{K}^{\circ\circ} = \overline{\operatorname{conv}}(\mathscr{K} \cup \{0\})$. Moreover $\mathfrak{B}(N^D_{\mu\nu}) = \mathfrak{B}(N_{\mu\nu})^D$. In the sequel $\mathscr{R}_{\mu\nu}$ denotes the following compact subset of $\mathscr{L}(E, F)$:

$$\begin{aligned}\mathscr{R}_{\mu\nu} &:= \{x \otimes y : x \in E, y \in F, N_{\mu\nu}(x \otimes y) = 1\}\\ &= \{x \otimes y : \nu^{\circ}(x) = \mu(y) = 1\}.\end{aligned}$$

Theorem 2.1. *For any $a \in E$, $b \in F$ and $A \in \mathscr{L}(E, F)$, the following hold*:

(a) $N_{\mu\nu}(a \otimes b) = N^D_{\nu\mu}(a \otimes b) = \nu^\circ(a)\mu(b)$.
(b) $\mathscr{R}^*_{\mu\nu} = \mathscr{R}_{\nu^\circ\mu^\circ}$.
(c) $N^D_{\nu\mu}(A) \geqslant N_{\mu\nu}(A)$.
(d) $N_{\mu\nu}(A) = \sup\{\Re\mathrm{tr}(A\rho) : \rho \in \mathscr{R}_{\nu\mu}\}$.
(e) $\mathfrak{B}(N_{\mu\nu}) = \mathscr{R}^D_{\nu\mu}$ *and* $\mathfrak{B}(N^D_{\nu\mu}) = \mathrm{conv}(\mathscr{R}_{\mu\nu})$.
(f) $N^D_{\nu\mu} = N_{\mu^\circ\nu^\circ}{}^\circ$ *and* $N^*_{\nu\mu} = N_{\mu^\circ\nu^\circ}$.
(g) $N^D_{\nu\mu}(A) = \inf\{\sum_{i=1}^s \lambda_i : s \in \mathbb{N}, \lambda_i \geqslant 0, p_i \in \mathscr{R}_{\mu\nu}, \sum_{i=1}^s \lambda_i p_i = A\}$.

Proof. We follow the ideas of [20, Theorem 1], but there are items here with no counterpart in [20], namely (b) and (f). Note that $\mathscr{R}_{\nu\mu}$ plays the same role here as the role played by P in [20]. Property (b) follows easily from (a). The proof of (d), which is one of the main properties of $\mathscr{R}_{\mu\nu}$, may be done in the same manner as [20, Eq. (5)]. The first identity of (e) follows from (d), and the second one is the dual of the first. To prove the first identity of (f), use (b) and (e) to obtain:

$$\begin{aligned}\mathfrak{B}(N^D_{\nu\mu}) &= \mathscr{R}^{DD}_{\mu\nu} = ((\mathscr{R}^*_{\mu\nu})^D)^\circ \\ &= (\mathscr{R}_{\nu^\circ\mu^\circ}{}^D)^\circ = \mathfrak{B}(N_{\mu^\circ\nu^\circ})^\circ = \mathfrak{B}(N_{\mu^\circ\nu^\circ}{}^\circ).\end{aligned} \tag{3}$$

The second identity of (f) is the ‘polar’ of the first one. □

Using definition (2), the first identity in (e) may be rephrased as $\mathfrak{B}(N_{\mu\nu})$ is the intersection of the half-spaces $\mathscr{H}_\kappa$ for $\kappa \in \mathscr{R}^*_{\nu\mu}$. Taking (b) into account, we obtain

$$\mathfrak{B}(N_{\mu\nu}) = \bigcap\{\mathscr{H}_{x\otimes y} : \nu(x) = \mu^\circ(y) = 1\}. \tag{4}$$

We may get other properties by simple formal manipulation as we did in (3). An easy one, to be used in the sequel, is the following:

$$\mathfrak{B}(N_{\mu\nu}{}^\circ) = \mathrm{conv}(\mathscr{R}_{\mu^\circ\nu^\circ}). \tag{5}$$

Another one is that $N^D_{\nu\mu}$, $N_{\nu\mu}{}^{\circ *}$, $N_{\nu\mu}{}^{*\circ}$ and $N_{\mu^\circ\nu^\circ}{}^\circ$ are the same norm. So one may argue that the concept of “dual” is superfluous and might have been removed. However, there are several reasons why the operator ^{D} deserves to be considered on its own. Firstly, it shares with the polar very interesting spectral properties, as stressed in [20,24]. Moreover, the inclusion $\mathfrak{B}(N^D_{\nu\mu}) \subseteq \mathfrak{B}(N_{\mu\nu})$ is quite interesting from a geometrical viewpoint and deserves a further analysis to be developed in Sections 4 and 5. On the other hand, the dual operator is the origin of interesting, highly nontrivial problems, as the following one: describe the pairs (μ, ν) for which $N_{\mu\nu}$ is auto-dual, i.e., $N^D_{\mu\nu} = N_{\mu\nu}$. The corresponding problem for the polar is more or less trivial.

3. Faces of $\mathfrak{B}(N_{\mu\nu}{}^\circ)$

In the special case of a norm ν, the *subdifferential* of ν at $a \in E$ is given by

$$\partial\nu(a) = \{u \in E : \nu(x) - \nu(a) \geqslant \Re\langle u|x - a\rangle,\ \text{all } x \in E\} \tag{6}$$

(For the general concept of *subdifferential*, we refer [14].) It is easily seen that: (i) $\partial\nu(a)$ is the set of all $u \in \mathscr{S}(\nu^\circ)$ for which equality holds in (1); in such case, $\langle a|u\rangle$ is real nonnegative; (ii) $\partial\nu(0)$ is the unit ball of ν°; (iii) for any nonzero $a \in E$, $\partial\nu(a)$ is the exposed face of $\mathfrak{B}_{\nu^\circ}$ having a as outwards normal vector (here, in the case E is a complex space, the concept "outwards normal" is referred to the underlying real inner product space structure); (iv) each *nontrivial* (i.e., proper and nonempty) exposed face of $\mathfrak{B}_{\nu^\circ}$ equals $\partial\nu(a)$ for some nonzero a (see, e.g. [17, Section 2]).

Clearly, if Φ is a face of $\mathfrak{B}_\nu$, then $\alpha\Phi$ is also a face of $\mathfrak{B}_\nu$, for $\alpha \in \mathbb{K}$, $|\alpha| = 1$. It is an easy exercise to prove that, if Φ is a nontrivial face and $\alpha \in \mathbb{K}$:

$$\alpha\Phi \cap \Phi \neq \varnothing \Rightarrow \alpha = 1. \tag{7}$$

This implies, in case $\mathbb{K} = \mathbb{C}$, that $\mathfrak{B}_\nu$has an infinite number of faces, i.e., $\mathfrak{B}_\nu$ is not a polytope.

Let $\mathfrak{F}$ be a proper, nonempty face of $\mathfrak{B}(N_{\mu\nu}{}^\circ)$. From (5) it follows that $\mathfrak{F}$ is the convex hull of a subset of $\mathscr{R}_{\mu^\circ\nu^\circ}$, namely

$$\mathfrak{F} = \operatorname{conv}(\mathscr{R}_{\mu^\circ\nu^\circ} \cap \mathfrak{F}). \tag{8}$$

So the concrete problem emerging here is the determination of the intersections of $\mathscr{R}_{\mu^\circ\nu^\circ}$ with the faces of $\mathfrak{B}(N_{\mu\nu}{}^\circ)$.

Definition 3.1. We keep the face $\mathfrak{F}$ fixed, and define the binary relation T afforded by $\mathfrak{F}$ in the following way: xTy means that $\nu(x) = 1$, $\mu^\circ(y) = 1$ and $x \otimes y \in \mathfrak{F}$. We denote by xT [by Ty] the set of all y [resp. of all x] such that xTy. For any $M \in \mathscr{S}(N_{\mu\nu})$, we let $\mathfrak{F}_M$ be the set $\partial N_{\mu\nu}(M)$, i.e., the exposed face of $\mathfrak{B}(N_{\mu\nu}{}^\circ)$ with outwards normal M. Finally, T_M denotes the binary relation afforded by $\mathfrak{F}_M$.

Note that $\mathfrak{F}^*(=\{A^* : A \in \mathfrak{F}\})$ is a face of $\mathfrak{B}(N_{\mu\nu}{}^\circ)^* = \mathfrak{B}(N_{\nu^\circ\mu^\circ}{}^\circ)$; hence, the binary relation afforded by $\mathfrak{F}^*$, call it T^*, satisfies xTy iff yT^*x. So, theorems about $\mathfrak{F}$ and T automatically transfer, by taking adjoints, to theorems about $\mathfrak{F}^*$ and T^*; the pair (ν, μ) is then replaced by (μ°, ν°). In the following theorem we collect some properties to be used later on. Item (b) has been proved in [21, p. 183] (see also [23]) for mappings on real vector spaces.

Theorem 3.2

(a) *For any $M \in \mathscr{S}(N_{\mu\nu})$, the exposed face $\mathfrak{F}_M$ is given by*

$$\mathfrak{F}_M = \operatorname{conv}\{x \otimes y : \nu(x) = \mu(Mx) = 1,\ y \in \partial\mu(Mx)\}. \tag{9}$$

(b) *For any $u \in \mathscr{S}_\nu$ and $w \in \mathscr{S}_{\mu^\circ}$, uT and Tw are proper faces of $\mathfrak{B}_{\mu^\circ}$ and $\mathfrak{B}_\nu$, respectively.*

(c) *If $\mathfrak{F}$ is an exposed face of $\mathfrak{B}(N_{\mu\nu}{}^\circ)$, then, for all $u \in \mathscr{S}_\nu$ and $w \in \mathscr{S}_{\mu^\circ}$, uT and Tw are exposed faces of $\mathfrak{B}_{\mu^\circ}$ and $\mathfrak{B}_\nu$, respectively.*

Proof. (a) We just apply (8) to the face $\mathfrak{F}_M$, and check the explicit expression in (9) under 'conv' for the set $\mathscr{R}_{\mu^\circ\nu^\circ} \cap \mathfrak{F}_M$.

(b) Let $y, y' \in \mathfrak{B}_{\mu^\circ}$ and $\lambda > 0$ satisfy $\lambda y + (1-\lambda)y' \in uT$. Then $\lambda u \otimes y + (1-\lambda)u \otimes y' \in \mathfrak{F}$. As $\mathfrak{F}$ is a face, $u \otimes y$ and $u \otimes y'$ lie in $\mathfrak{F}$. Then $y, y' \in uT$, and therefore uT is a face of $\mathfrak{B}_{\mu^\circ}$. uT is clearly proper, otherwise $0 \in uT$, and then $\mathfrak{F}$ would be improper as well. The case Tw is dealt with in the same manner (or by considering $\mathfrak{F}^*$).

(c) We only treat the case when $\mathfrak{F}$ is a proper, nonempty exposed face. Then $\mathfrak{F} = \mathfrak{F}_M$ for some $M \in \mathscr{S}(N_{\mu\nu})$. It is not difficult to see that

$$\mathfrak{F}_M^* = \partial N_{\nu^\circ\mu^\circ}(M^*).$$

We now combine this with (9) to get the following: $uT_M = \partial\mu(Mu)$, in case $\mu(Mu) = 1$, otherwise uT_M is empty; $T_Mw = \partial\nu^\circ(M^*w)$, in case $\nu^\circ(M^*w) = 1$, otherwise T_Mw is empty. □

We conjecture that the converse of Theorem 3.2(c) is also true. Now consider sets of the form

$$\mathfrak{F} = \operatorname{conv}(U \otimes W), \tag{10}$$

where $U \subseteq \mathscr{S}_\nu$ and $W \subseteq \mathscr{S}_{\mu^\circ}$. We shall always assume that the real affine variety spanned by $U \otimes W$ does not contain the origin, i.e., that both U and W span real affine varieties that do not contain the origin.

Lemma 3.3. *In the above conditions*:

(a) $\operatorname{conv}(U \otimes W) = \operatorname{conv}[(\operatorname{conv} U) \otimes (\operatorname{conv} W)]$.
(b) $x \otimes y \in \operatorname{conv}(U \otimes W) \Leftrightarrow x \otimes y \in (\operatorname{conv} U) \otimes (\operatorname{conv} W)$.
(c) $\operatorname{conv}(U \otimes W) = \operatorname{conv}(U' \otimes W')$ *if and only if* $(\operatorname{conv} U) \otimes (\operatorname{conv} W) = (\operatorname{conv} U') \otimes (\operatorname{conv} W')$.

Proof. Only the implication $\Rightarrow$ of (b) will be proved [note that (c) follows from (b)]. Let $x \otimes y = \sum_{i=1}^{s} \lambda_i a_i \otimes b_i$ be a convex combination with positive λ_i, $a_i \in U$ and $b_i \in W$. Our hypotheses imply the existence of $u \in E$ and $v \in F$ such that $\langle u|a_i\rangle = \langle v|b_i\rangle = 1$, for all i. As the affine variety spanned by W does not contain 0, $\langle u|x\rangle$ is nonzero; so we may assume that $\langle u|x\rangle = 1$. Computing $(x \otimes y)u$ gives $y = \sum_{i=1}^{s} \lambda_i b_i$, i.e., $y \in \operatorname{conv} W$; so $\langle v|y\rangle = 1$. Computing $(y \otimes x)v$ gives $x \in \operatorname{conv} U$. $\square$

This shows that, if (10) is a proper face of $\mathfrak{B}(N_{\mu\nu}{}^\circ)$, $a \in U$ and $b \in W$, then

$$aT = \operatorname{conv} W \quad \text{and} \quad Tb = \operatorname{conv} U. \tag{11}$$

If M is a rank-one tensor, (9) has a simple expression.

Proposition 3.4. *For $a \otimes b \in \mathscr{R}_{\mu\nu}$, we have*

$$\mathfrak{F}_{a\otimes b} = \operatorname{conv}[\partial\nu^\circ(a) \otimes \partial\mu(b)]. \tag{12}$$

Proof. We may assume $a \in \mathscr{S}_{\nu^\circ}$ and $b \in \mathscr{S}_\mu$. Any rank-one tensor ρ in $\mathfrak{F}_M \cup \partial\nu^\circ(a) \otimes \partial\mu(b)$ may be represented as $\rho = x \otimes y$ where $\nu(x) = \mu^\circ(y) = 1$ and $\langle a|x\rangle$ is real nonnegative. With such representation in mind, we have

$$\begin{aligned}
\rho \in \mathscr{R}_{\mu^\circ\nu^\circ} \cap \mathfrak{F}_{a\otimes b} &\Leftrightarrow \operatorname{tr}[(x \otimes y)(a \otimes b)^*] = 1\\
&\Leftrightarrow \langle a|x\rangle\langle y|b\rangle = 1 \Leftrightarrow \langle a|x\rangle = \langle y|b\rangle = 1\\
&\Leftrightarrow x \in \partial\nu^\circ(a) \wedge y \in \partial\mu(b)\\
&\Leftrightarrow \rho \in \partial\nu^\circ(a) \otimes \partial\mu(b).
\end{aligned}$$

So the result follows from (8). $\square$

Theorem 3.5. *For $U \subseteq \mathscr{S}_\nu$ and $W \subseteq \mathscr{S}_{\mu^\circ}$, the set* $\operatorname{conv}(U \otimes W)$ *is a nontrivial face of* $\mathfrak{B}(N_{\mu\nu}{}^\circ)$ *if and only if* conv U *and* conv W *are nontrivial faces of* $\mathfrak{B}_\nu$ and $\mathfrak{B}_{\mu^\circ}$, *respectively. If these conditions hold, the exposed face generated by* $\operatorname{conv}(U \otimes W)$ *is* $\operatorname{conv}(\widehat{U} \otimes \widehat{W})$, *where $\widehat{U}$ and $\widehat{W}$ are the exposed faces of* $\mathfrak{B}_\nu$ and $\mathfrak{B}_{\mu^\circ}$ *generated by U and W, respectively.*

Proof. The *only if* part follows from Theorem 3.2(b) and (11). Conversely, assume conv U and conv W are nontrivial faces of $\mathfrak{B}_\nu$ and $\mathfrak{B}_{\mu^\circ}$, respectively. There exist $u \in \mathscr{S}_{\nu^\circ}$ and $v \in \mathscr{S}_\mu$ such that $\langle u|a\rangle = \langle v|b\rangle = 1$, for any $a \in U$ and $b \in W$. Let $A = \sum_{i=1}^{k} \lambda_i \rho_i$ be a convex combination of rank-one tensors $\rho_i \in \mathscr{R}_{\mu^\circ\nu^\circ}$, with positive coefficients λ_i, such that $A \in \operatorname{conv}(U \otimes W)$. We write $\rho_i = x_i \otimes y_i$ where $\nu(x_i) = \mu^\circ(y_i) = 1$ and $\langle u|x_i\rangle$ is real nonnegative. We then get

$$Au = \sum_{i=1}^{k} \lambda_i \langle u|x_i\rangle y_i \in \operatorname{conv} W.$$

As conv W is a face of $\mathfrak{B}_{\mu^\circ}$, we have, for all i, $\langle u|x_i\rangle y_i \in \operatorname{conv} W$; as $\langle u|x_i\rangle \leqslant 1$, we must have $\langle u|x_i\rangle = 1$ and $y_i \in \operatorname{conv} W$. Computing A^*v and using the fact that conv U is a face of $\mathfrak{B}_\nu$, we easily obtain $x_i \in \operatorname{conv} U$ for all i. Therefore $\rho_i \in \operatorname{conv}(U \otimes W)$ for all i. This proves that $\operatorname{conv}(U \otimes W)$ is a (nontrivial) face of $\mathfrak{B}(N_{\mu\nu^\circ})$.

As $\widehat{U}$ and $\widehat{W}$ are nontrivial exposed faces of $\mathfrak{B}_\nu$ and $\mathfrak{B}_{\mu^\circ}$, there exists $a \otimes b \in \mathscr{R}_{\mu\nu}$ such that $\widehat{U} = \partial\nu^\circ(a)$ and $\widehat{W} = \partial\mu(b)$; by Proposition 3.4, $\operatorname{conv}(\widehat{U} \otimes \widehat{W})$ is an exposed face of $\mathfrak{B}(N_{\mu\nu^\circ})$; so it contains the exposed face generated by $\operatorname{conv}(U \otimes W)$. On the other hand, if T_1 denotes the binary relation afforded by the exposed face generated by $\operatorname{conv}(U \otimes W)$, and if we select any $\alpha \in U$ and $\beta \in W$, αT_1 and $T_1\beta$ are exposed faces of $\mathfrak{B}_{\mu^\circ}$ and $\mathfrak{B}_\nu$ (cf Theorem 3.2), that contain W and U, respectively; thus $\alpha T_1 \supset \widehat{W}$ and $T_1\beta \supset \widehat{U}$. Therefore, the exposed face generated by $\operatorname{conv}(U \otimes W)$ contains $\operatorname{conv}(\widehat{U} \otimes \widehat{W})$ $\square$.

Corollary 3.6. *The set* $\operatorname{conv}(U \otimes W)$ *is a nontrivial exposed face of* $\mathfrak{B}(N_{\mu\nu^\circ})$ *if and only if* $\operatorname{conv} U$ *and* $\operatorname{conv} W$ *are nontrivial exposed faces of* $\mathfrak{B}_\nu$ *and* $\mathfrak{B}_{\mu^\circ}$, *respectively.*

For real spaces and real linear mappings, [21, p. 159] gives the *only if* part for extreme points, and the *if* part for exposed points, of the following:

Corollary 3.7. $\rho \in \mathscr{L}(E, F)$ *is an extreme* [*exposed*] *point of* $\mathfrak{B}(N^D_{\nu\mu})$ *if and only if* $\rho = u \otimes w$, *where* u *and* w *are extreme* [*exposed*] *points of* $\mathfrak{B}_{\nu^\circ}$ *and* $\mathfrak{B}_\mu$, *respectively.*

4. Tangency points

Let $\mathfrak{B}$ and $\mathfrak{C}$ be two closed convex sets in a finite dimensional real vector space, such that $\mathfrak{B} \subseteq \mathfrak{C}$. Let us look at the *surfaces* of $\mathfrak{B}$ and $\mathfrak{C}$, i.e., their relative boundaries. Any point in the intersection of these surfaces is called a *tangency point*. We assume, without loss of generality, that $\mathfrak{C}$ has interior points. To keep away from triviality we assume $\mathfrak{B}$ is not contained in the surface of $\mathfrak{C}$. For any tangency point p consider a supporting hyperplane to $\mathfrak{C}$ at p; the intersection of this hyperplane with $\mathfrak{B}$ is a proper exposed face of $\mathfrak{B}$ whose points are tangency points of our surfaces. So the set of all tangency points is a union of proper exposed faces of $\mathfrak{B}$. In concrete cases it is only natural to look for the exposed faces of $\mathfrak{B}$ that are in contact with the surface of $\mathfrak{C}$.

For example, let $M \in \mathscr{L}(E, F)$ satisfy $N_{\mu\nu}(M) = 1$. As $M[\mathfrak{B}_\nu] \subseteq \mathfrak{B}_\mu$, we may apply our previous comments to $\mathfrak{B} := \mathfrak{B}_\nu$ and $\mathfrak{C} := M^{-1}[\mathfrak{B}_\mu]$; according to what has been said in between (9) and Theorem 3.2, the set of tangency points of the two surfaces $\mathscr{S}_\nu$ and $M^{-1}[\mathscr{S}_\mu]$ is the union of all exposed faces $T_M b$ of $\mathfrak{B}_\nu$. For the same M, we have the following "adjoint" assertion: $N_{\nu^\circ\mu^\circ}(M^*) = 1$ and the set of tangency points of the two surfaces $\mathscr{S}_{\mu^\circ}$ and $(M^*)^{-1}[\mathscr{S}_{\nu^\circ}]$ is the union of all exposed faces aT_M of $\mathfrak{B}_{\mu^\circ}$.

Items (c) and (a) of Theorem 2.1 tell us that the unit ball of $N^D_{\nu\mu}$ is contained in the unit ball of $N_{\mu\nu}$, and all elements of $\mathscr{R}_{\mu\nu}$ are tangency points of the surfaces of these unit balls. We now give a complete characterization of the set of tangency points of any rank.

Theorem 4.1. *The set of tangency points of surfaces $\mathscr{S}(N_{\mu\nu})$ and $\mathscr{S}(N^D_{\nu\mu})$ is the union of all exposed faces of $\mathfrak{B}(N^D_{\nu\mu})$ which are normal to rank-one tensors.*

Proof. Recall that $\mathfrak{B}(N_{\mu\nu})$ is the intersection of the half-spaces $\mathscr{H}_\kappa$ for $\kappa \in \mathscr{R}^*_{\nu\mu}$ [cf (4)]. Let A be any tangency point of the surfaces $\mathscr{S}(N_{\mu\nu})$ and $\mathscr{S}(N^D_{\nu\mu})$, and let $d(\kappa)$ be the (Euclidean) distance of A to the boundary of $\mathscr{H}_\kappa$. The infimum of $d(\kappa)$ is zero; by the compactness of $\mathscr{R}^*_{\nu\mu}$, there exists $\pi \in \mathscr{R}^*_{\nu\mu}$ such that $d(\pi) = 0$, i.e., the boundary of $\mathscr{H}_\pi$ is a supporting (real) hyperplane of $\mathfrak{B}(N_{\mu\nu})$ at A. This hyperplane intersects $\mathfrak{B}(N^D_{\nu\mu})$ along an exposed face containing A. $\square$

By Theorem 2.1(f) and Proposition 3.4 the exposed face with outwards normal $a \otimes b \in \mathscr{R}^*_{\nu\mu}$ has the form (12), with ν and μ replaced by their polar. So the exposed faces of $\mathfrak{B}(N^D_{\nu\mu})$ in contact with the surface $\mathscr{S}(N_{\mu\nu})$ have the form

$$\operatorname{conv}(\partial\nu(a) \otimes \partial\mu^\circ(b)). \tag{13}$$

Corollary 4.2. *The following conditions are pairwise equivalent*:

(a) *All tangency points of the surfaces $\mathscr{S}(N_{\mu\nu})$ and $\mathscr{S}(N^D_{\nu\mu})$ are exposed points of $\mathfrak{B}(N^D_{\nu\mu})$*;
(b) *$N_{\mu^\circ\nu^\circ}$ is differentiable at any rank one A*;
(c) *ν is differentiable and μ is strictly convex.*

Proof. This follows because any of (a), (b) and (c) is equivalent to the fact that (13) is a singleton for all $a \otimes b \in \mathscr{R}^*_{\nu\mu}$. Recall that μ is strictly convex iff μ° is differentiable. $\square$

The following corollary generalizes [20, Theorem 5]. It also follows from Theorems 5.5 and 5.1 of the next section.

Corollary 4.3. *All contact points of the boundaries of $\mathfrak{B}(N_{\mu\nu})$ and $\mathfrak{B}(N^D_{\nu\mu})$ have rank one if and only if either ν is differentiable or μ is strictly convex.*

Proof. The *if* part is obvious. For the converse, assume there exists $a \otimes b \in \mathscr{R}^*_{\nu\mu}$ such that both $\partial\nu(a)$ and $\partial\mu^\circ(b)$ have more than one element, say: $x_1, x_2 \in \partial\nu(a)$ and $y_1, y_2 \in \partial\mu^\circ(b)$, $x_1 \neq x_2$ and $y_1 \neq y_2$. By (7) x_1 and x_2 [y_1 and y_2] are $\mathbb{K}$-linearly independent. Therefore $(x_1 \otimes y_1 + x_2 \otimes y_2)/2$ is a rank-two element of (13). $\square$

The exposed face (13) of $\mathfrak{B}(N^D_{\nu\mu})$ is obviously contained in the exposed face of $\mathfrak{B}(N_{\mu\nu})$ with outwards normal $a \otimes b$, but in general (13) is not an exposed face of $\mathfrak{B}(N_{\mu\nu})$. As a matter of fact, it is not difficult to prove that (13) is an exposed face of $\mathfrak{B}(N_{\mu\nu})$ for all nonzero $a \otimes b$ if and only if $N_{\mu\nu} = N^D_{\nu\mu}$, i.e., iff $N_{\mu\nu}$ is a *self-D-dual* norm. Except for the trivial case when E [or F] is a one dimensional $\mathbb{K}$-space, we did not find any self-D-dual norm. We conjecture they do not exist.

5. Ranks and dimensions

For any subset S of E we let $\langle S\rangle_\mathbb{R}$ be the real subspace of E spanned by S. If E is a complex space, $\langle S\rangle_\mathbb{C}$ denotes the complex subspace of E spanned by S; the dimension of $\langle S\rangle_\mathbb{K}$ over $\mathbb{K}$ is denoted by $\dim_\mathbb{K}(S)$. It is an easy exercise to show that, for an n-dimensional complex space E

and nonnegative integers r and c, the inequalities $c \leqslant r \leqslant 2c \leqslant 2n$ are equivalent to the existence of a real subspace $\Sigma \subseteq E$ such that $\dim_{\mathbb{R}}(\Sigma) = r$ and $\dim_{\mathbb{C}}(\Sigma) = c$.

The *dimension* of a convex subset $C \subseteq E$, denoted $\dim(C)$, is the dimension, over the *real field*, of the *real* affine variety spanned by C.

For nonempty sets $U \subseteq E$ and $W \subseteq F$, the rank of a given $A \in \operatorname{conv}(U \otimes W)$, i.e., the dimension over $\mathbb{K}$ of the image of A, is a nonnegative integer k that satisfies

$$k \leqslant \min\{\dim_{\mathbb{K}}(U), \dim_{\mathbb{K}}(W)\}. \tag{14}$$

It is also clear that any positive integer k satisfying the above condition is the rank of some $A \in \operatorname{conv}(U \otimes W)$. These considerations will apply to our contact faces (13), with U and W defined by

$$U := \partial\nu^\circ(a) \quad \text{and} \quad W := \partial\mu(b).$$

In case $\mathbb{K} = \mathbb{R}$ we have a nice expression for the upper bound in (14), because the current U spans a real subspace of dimension $\dim(U) + 1$. We have then

Theorem 5.1. *Let E and F be real spaces. An integer k is the rank of some element A of the contact face (13) if and only if*

$$1 \leqslant k \leqslant \min\{\dim(\partial\nu^\circ(a)), \dim(\partial\mu(b))\} + 1.$$

The complex case is not so neat. We first prove a lemma.

Lemma 5.2. *Let E be a complex space*, $\dim_{\mathbb{C}} E = n$, *and let r and c be nonnegative integers. There exists a norm ν in E whose unit ball has an exposed face $\mathfrak{F}$ such that* $\dim_{\mathbb{C}} \mathfrak{F} = c$ *and* $\dim_{\mathbb{R}} \mathfrak{F} = r$ *if and only if $c \leqslant r < 2c$ and $c \leqslant n$.*

Proof. Assume $\mathfrak{F}$ is an exposed face of $\mathfrak{B}_\nu$ with the appropriate dimensions. There exists a vector w, such that $\nu^\circ(w) = 1$ and $\partial\nu^\circ(w) = \mathfrak{F}$. Clearly $\langle w|x\rangle = 1$ for $x \in \mathfrak{F}$, and the same holds for any $x \in \mathscr{A}$, where $\mathscr{A}$ is the real affine variety spanned by $\mathfrak{F}$. We have $\mathscr{A} = a + S$, where $a \in \mathfrak{F}$ and S is the real subspace parallel to $\mathscr{A}$. Clearly $\dim_{\mathbb{R}}(S) = r - 1$. As $\langle w|s\rangle = 0$ for $s \in S$, a does not lie in $S + \mathrm{i}S = \langle S\rangle_{\mathbb{C}}$. The space $\langle \mathfrak{F}\rangle_{\mathbb{C}}$ is spanned (over $\mathbb{C}$) by $\{a\} \cup (S + \mathrm{i}S)$; therefore $c = \dim_{\mathbb{C}}(S + \mathrm{i}S) + 1$. We thus have

$$2(c - 1) = \dim_{\mathbb{R}}(S + \mathrm{i}S) = 2(r - 1) - \dim_{\mathbb{R}}(S \cap \mathrm{i}S).$$

The only if part of the lemma follows from $0 \leqslant \dim_{\mathbb{R}}(S \cap \mathrm{i}S) \leqslant r - 1$.

For the converse, assume $c \leqslant r \leqslant 2c - 1$ and $c \leqslant n$. We firstly consider the special case when $c = n$. Let $v_1, \ldots, v_n$ be an orthogonal basis of E and consider the set

$$V := v_n + \{0, v_1, \ldots, v_{n-1}, \mathrm{i}v_1, \ldots, \mathrm{i}v_{r-n}\}.$$

Let $\mathfrak{F}$ be the convex hull of V. Clearly $\dim_{\mathbb{C}} \mathfrak{F} = n$ and $\dim_{\mathbb{R}} \mathfrak{F} = r$. Define

$$W := \operatorname{conv}\{\mathrm{e}^{\mathrm{i}\theta} v : \theta \in \mathbb{R}, v \in V\}.$$

It is not difficult to prove that W is a convex, compact neighbourhood of the origin, and $\mathrm{e}^{\mathrm{i}\theta} W = W$; so W is the unit ball of a norm. Moreover, $\Re\langle v_n|\mathrm{e}^{\mathrm{i}\theta} v\rangle \leqslant 1$, for all real θ and $v \in V$, with equality iff $\mathrm{e}^{\mathrm{i}\theta} = 1$. Therefore, $\mathfrak{F}$ is an exposed face of W.

Now assume that $c < n$. Let $E = F \oplus G$ be a $\mathbb{C}$-orthogonal direct decomposition of E, where $\dim_{\mathbb{C}} F = c$. For any norms f in F and g in G, the direct sum $f \oplus g$, defined by $f \oplus g(x + y) := f(x) + g(y)$, for $x \in F$ and $y \in G$, is a norm on E such that

$$(f \oplus g)^\circ(x + y) = \max\{f^\circ(x), g^\circ(y)\},$$
$$\partial(f \oplus g)^\circ(x) = \partial f^\circ(x)$$

(we omit the details of proof). So the exposed face of $\mathfrak{B}_{f\oplus g}$ with outwards normal $x \in F\backslash\{0\}$ is the same as the corresponding one of $\mathfrak{B}_f$.

By the previous case, there exists a norm f in F, and a nonzero $u \in F$, such that the exposed face $\mathfrak{F} := \partial f(u)$ satisfies $\dim_{\mathbb{C}} \mathfrak{F} = c$ and $\dim_{\mathbb{R}} \mathfrak{F} = r$. This completes the proof. □

Remark 5.3. It is interesting to notice the influence that strict homogeneity (assumed all over the paper) has on the dimension of faces. In Lemma 5.2, $\dim(\mathfrak{F}) = r - 1$; therefore, in complex spaces, there is no face of affine dimension $2n - 1$. Now strict homogeneity is sometimes weakened to one of the forms

$$H_1 : \nu(\lambda x) = \lambda\nu(x) \quad \text{for real nonnegative } \lambda,$$
$$H_2 : \nu(\lambda x) = |\lambda|\nu(x) \quad \text{for real } \lambda.$$

With axiom H_1, which characterizes *pseudo-norms*, the unit ball may be any convex, compact neighbourhood of the origin; with H_2, the unit ball satisfies the additional property $-\mathfrak{B}_\nu = \mathfrak{B}_\nu$, while in the complex, strictly homogeneous case, we have $e^{i\theta}\mathfrak{B}_\nu = \mathfrak{B}_\nu$, for all real θ. In the latter case, the group of real isometries of $\mathfrak{B}_\nu$ is far from being trivial, while for pseudo-norms that group eventually reduces to the identity. With similar techniques to those used in Lemma 5.2, we may prove

Lemma 5.4. *Let E be a complex space*, $\dim_{\mathbb{C}} E = n$, *and let r and c be nonnegative integers. The following statements are equivalent*:

(i) *There exists $\mathfrak{B}$, a convex, compact neighbourhood of the origin* [*such that* $-\mathfrak{B} = \mathfrak{B}$], *having an exposed face $\mathfrak{F}$ such that* $\dim_{\mathbb{C}} \mathfrak{F} = c$ *and* $\dim_{\mathbb{R}} \mathfrak{F} = r$;
(ii) $c \leqslant r \leqslant 2c$ *and* $c \leqslant n$.

The next theorem is an immediate consequence of Lemma 5.2. The Lemma also shows that Theorem 5.5 is the best possible in an obvious sense.

Theorem 5.5. *Let E and F be complex spaces. Denote by $R(a, b)$ the maximum rank of the elements A of the contact face* (13). *Then*

$$R(a, b) \leqslant \min\{\dim(\partial\nu^\circ(a)), \dim(\partial\mu(b))\} + 1 < 2R(a, b).$$

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