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The structure of matrices with a maximum multiplicity eigenvalue

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Abstract

There is remarkable and distinctive structure among Hermitian matrices, whose graph is a given tree T and that have an eigenvalue of multiplicity that is a maximum for T. Among such structure, we give several new results: (1) no vertex of T may be "neutral"; (2) neutral vertices may occur if the largest multiplicity is less than the maximum; (3) every Parter vertex has at least two downer branches; (4) removal of a Parter vertex changes the status of no other vertex; and (5) every set of Parter vertices forms a Parter set. Statements (3), (4) and (5) are also not generally true when the multiplicity is less than the maximum. Some of our results are used to give further insights into prior results, and both the review of necessary background and the development of new structural lemmas may be of independent interest.

Keywords: Hermitian matrices; Eigenvalues; Multiplicities; Maximum multiplicity; Trees; Path cover number; Parter vertices

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1. Introduction

For a simple (labelled) graph G, let $\mathcal{S}(G)$ denote the set of all Hermitian matrices $A = (a_{ij})$ whose graph is G, i.e., for $i \neq j$, $a_{ij} \neq 0$ if and only if $\{i, j\}$ is an edge of G. No restrictions is placed upon the diagonal entries of A (other than reality) by G. Since all that we consider about A is permutation similarity invariant, the labelling of G is incidental (for reference only), and we generally consider G to be unlabelled.

We are interested in $\sigma(A)$, $A \in \mathcal{S}(G)$, especially the multiplicities of the eigenvalues, and, particularly, the case in which $G = T$, a tree. We use conventional submatrix, subgraph notation and often move informally between the two. If α is a subset of the indices $N = \{1, \ldots, n\}$ of A (resp. vertices of G), $A(\alpha)$ (resp. $G - \alpha$) denotes the principal submatrix of A (resp. induced subgraph of G) resulting from deletion of the indices (resp. vertices) α. $A(\{i\})$ (resp. $G - \{i\}$) is abbreviated by $A(i)$ (resp. $G - i$). $A[\alpha]$ or $G[\alpha]$ denotes the principal submatrix or induced subgraph resulting from keeping only the indices or vertices α. If $G' = G[\alpha]$ we often write $A[G']$, meaning the principal submatrix $A[\alpha]$. In case T is a tree and v is a vertex of degree k, $T - v$ is a forest consisting of k branches (trees) at T: $T_1, \ldots, T_k$. We often assume this branch notation without further explanation.

Let $m_A(\lambda)$ denote the multiplicity of λ as a root of the characteristic polynomial of the n-by-n matrix A. We allow $m_A(\lambda) = 0$. Since we consider Hermitian matrices A, there is no ambiguity between algebraic and geometric multiplicity. The vertex v of a tree T is called *Parter* for λ in $A \in \mathcal{S}(T)$ (Parter, for short) if

$$m_{A(v)}(\lambda) = m_A(\lambda) + 1.$$

In the only two other possibilities (because of the interlacing inequalities [2])

$$m_{A(v)}(\lambda) = m_A(\lambda) \quad \text{and} \quad m_{A(v)}(\lambda) = m_A(\lambda) - 1,$$

the vertex v is called *neutral* or a *downer*, respectively. In general, each of these can easily occur and the theory is well developed in [6,9], with a summary of known results in [4]. A set of vertices α of cardinality k is called a *Parter set* if

$$m_{A(\alpha)}(\lambda) = m_A(\lambda) + k.$$

In general a set of Parter vertices, each of which is Parter for λ, A and T is *not* a Parter set. (Examples will be discussed later.) By the *status* of a vertex, relative to a particular λ, A, T, we mean its classification as Parter, neutral or a downer.

Our purpose here is to more deeply understand the special structure occurring when an Hermitian $A \in \mathcal{S}(T)$ has an eigenvalue λ such that $m_A(\lambda) = M(T)$, the maximum possible multiplicity for an eigenvalue among matrices in $\mathcal{S}(T)$.

Our main results appear in Section 4 and include: (1) no vertex is neutral if $m_A(\lambda) = M(T)$; (2) for any T, if $m < M(T)$, then matrices A occur in $\mathcal{S}(T)$ for which $m_A(\lambda) = m$ and for which neutral vertices are present; (3) $m_A(\lambda) = M(T)$ implies that for each Parter vertex at least two adjacent vertices are downer vertices in their branches; (4) the status of no other vertex changes with the removal of a vertex Parter for λ when $m_A(\lambda) = M(T)$; and (5) any set of Parter vertices for λ is a Parter set for λ if $m_A(\lambda) = M(T)$. Further observations are made in discussion involving the main results, and several examples, that refine our results, are given in Section 5. In the next two sections, we review necessary background and develop some useful lemmas that may be of independent interest.

2. Necessary background and motivation

Here, we summarize some necessary background, organized into subsections for easy reference. As elsewhere, we benignly blur the distinction between tree and matrix and between subgraph and submatrix. In all statements, there is an implicit identified eigenvalue and matrix relative to a given tree. For more thorough explanations or further background, see [6,4].

2.1. Maximum multiplicity, path covers and residual path maximizing sets

For a given tree T, the maximum possible multiplicity $M(T)$ may be characterized in two combinatorial ways [3]. A *path cover* of a tree is a collection of vertex disjoint, induced paths of T that cover the vertices of the tree. The least number of paths in a path cover is the *path cover number* $P(T)$. A *residual path maximizing* (RPM) set for T is a collection of q vertices of T, whose removal from T leaves a forest of p paths in such a way that $p - q$ is a maximum. The maximum multiplicity $M(T)$ is equal to both $P(T)$ and this maximum $p - q$. A matrix $A \in \mathscr{S}(T)$ achieving $m_A(\lambda) = M(T)$ may be constructed by assigning λ as smallest eigenvalue to each submatrix of A corresponding to a path after removal of an RPM set for T. The vertices of this RPM set will then form a Parter set (see Section 2.3) for λ in A.

2.2. Parter vertices and downer branches

The survey [6], which contains original material, describes the relation between a Parter vertex v for an eigenvalue λ of $A \in \mathscr{S}(T)$ and neighbors of v that are downer vertices for λ in their branches. For each eigenvalue λ such that $m_A(\lambda) \geqslant 2$, there exist Parter vertices (and they may exist for eigenvalues of multiplicity 0 or 1). If v is a Parter vertex (for λ in $A \in \mathscr{S}(T)$), then there is a neighbor u_i of v that is a downer vertex for λ in its branch T_i, i.e., in $A[T_i]$. We call such a branch T_i a *downer branch* for the Parter vertex v, and, conversely, if v has a downer branch, then v is Parter for λ.

This downer branch mechanism for identifying Parter vertices is subtle and very important. In general, there may be only one downer branch at a Parter vertex, but we show that if λ attains maximum multiplicity, there must be at least two, a strong structural distinction in the maximum multiplicity case.

Whenever $\lambda \in \sigma(A(v))$ for any vertex v of T such that $A \in \mathscr{S}(T)$, there will be at least one Parter vertex for λ (not necessarily v), even if $m_A(\lambda) = 0$ or $m_A(\lambda) = 1$. In this event, the above downer branch mechanism is still in place.

2.3. Parter sets and fragmenting Parter sets

For an identified eigenvalue λ, removal of a Parter set of k vertices increases the multiplicity of λ by k. By interlacing, each vertex in a Parter set is, initially, individually Parter and each is Parter in its subtree after removal of any subset of the others. On the other hand, a set of initially Parter vertices need not be a Parter set [5], but, in the event the relevant eigenvalue attains $M(T)$, a set of Parter vertices will be a Parter set, another strong structural distinction (Corollary 9 below). We call a Parter set *fragmenting* (a “fragmenting Parter set” or FPS, for short) if its removal from T leaves a forest in which the multiplicity of λ is at most 1 in the submatrix of A associated with each tree of the forest. Of course, FPS's always exist and any maximal (wrt inclusion) Parter set will be one. An FPS is a "fully fragmenting Parter set" if no vertex of any of the trees that remain is Parter, i.e., none of the trees may be further broken down.

In general, the trees that remain after an FPS or fully FPS is removed may be rather arbitrary, but, we will see that if λ attains $M(T)$, removal of any Parter vertex leaves trees, in each of which maximum multiplicity is attained by λ. Thus, if λ attains $M(T)$, removal of an FPS leaves only paths, and, in fact, any FPS is an RPM, another structural distinction.

2.4. *Facts about paths*

In all of this, facts about paths are especially basic and important. A rather complete theory was developed in [6].

It has long been known that if T is a path, $M(T) = 1$, i.e., the eigenvalues of $A \in \mathcal{S}(T)$ are distinct. Moreover, paths are the only graphs for which maximum multiplicity is 1 [1], and this is obvious among trees. It is also known that removal of a pendent vertex from a path results in strict interlacing of eigenvalues, which need not be the case among trees that are not paths. A pendent vertex of a path is a downer vertex for any eigenvalue occurring in that path. Removal of an interior vertex from a path may not result in strict interlacing, but if λ appears in both T and $T - v$, then v is Parter for λ and *both* branches at v are downer branches at v. Of course, if λ appears in T (i.e., as an eigenvalue of $A \in \mathcal{S}(T)$), then λ attains $M(T) = 1$, and we will see that the occurrence of two downer branches generalizes to all cases of maximum multiplicity.

In the context of general trees, we say that a path is *pendent* at a vertex v, not in the path, if the only vertex of the path that is adjacent to v is a degree one vertex.

3. New structural lemmas

Lemma 1. *Let T be a tree on n vertices, $A \in \mathcal{S}(T)$, and suppose that there is an eigenvalue λ of A of multiplicity $M(T)$. If v is a Parter vertex for λ we have the following.*

(1) *The degree of v in T is at least* 2.
(2) *If $T_1, \ldots, T_k$ are the branches of T at v, then $m_{A[T_i]}(\lambda) = M(T_i)$, $i = 1, \ldots, k$.*

Proof. For (1), it suffices to note that any Parter vertex for an eigenvalue belongs to an FPS and, in case the multiplicity is $M(T)$, each FPS is an RPM set of T. Since a pendent vertex cannot belong to an RPM set (because $\max[p - q]$ cannot be achieved when a pendent vertex is among the q removed vertices from T in order to leave p components) we conclude that a Parter vertex for an eigenvalue of multiplicity $M(T)$ must have, at least, degree 2. This also follows from the fact that $P(T - v) \leqslant P(T)$ when v is pendant in T.

For (2), note that since $A(v) = A[T_1] \oplus \cdots \oplus A[T_k]$ and v is Parter for λ, we have that $m_{A(v)}(\lambda) = M(T) + 1 = m_{A[T_1]}(\lambda) + \cdots + m_{A[T_k]}(\lambda)$. In order to obtain a contradiction, we suppose that $m_{A[T_i]}(\lambda) < M(T_i)$ for a particular branch of T at v. If we consider a matrix $B \in \mathcal{S}(T)$ such that $m_{B[T_i]}(\lambda) = M(T_i)$, $i = 1, \ldots, k$, we would have $m_{B(v)}(\lambda) = M(T_1) + \cdots + M(T_k) > m_{A(v)}(\lambda) = M(T) + 1$ and, by the interlacing inequalities, we obtain $m_B(\lambda) > M(T)$, which is a contradiction because $M(T)$ is the maximum possible. $\square$

Lemma 2. *Let T be a tree on n vertices, $A \in \mathscr{S}(T)$ and λ be an eigenvalue of A. Let u_1 and u_2 be two adjacent Parter vertices for λ relative to A. We denote by T_1 the component of $T - u_2$ containing u_1 and by T_2 the component of $T - u_1$ containing u_2. Then we have the following*:

(1) *u_1 is a Parter vertex for λ relative to $A[T_1]$ if and only if u_2 is a Parter vertex for λ relative to $A[T_2]$.*
(2) *u_1 is a downer vertex for λ relative to $A[T_1]$ if and only if u_2 is a downer vertex for λ relative to $A[T_2]$.*
(3) *Neither u_1 is a neutral vertex for λ relative to $A[T_1]$ nor is u_2 a neutral vertex for λ relative to $A[T_2]$.*

Proof. First note that $A(u_1) = A[T_1 - u_1] \oplus A[T_2]$ and $A(u_2) = A[T_1] \oplus A[T_2 - u_2]$ and, since u_1 and u_2 are Parter vertices for λ relative to A, we have $m_{A(u_1)}(\lambda) = m_{A(u_2)}(\lambda) = m_A(\lambda) + 1$.

If u_1 is Parter for λ relative to A, then u_2 being Parter for λ relative to $A[T_2]$ implies that $m_{A(\{u_1,u_2\})}(\lambda) = m_A(\lambda) + 2$. Thus, the initial removal of u_2 means that u_1 is Parter for λ relative to $A[T_1]$. Similarly, u_1 being Parter for λ relative to $A[T_1]$ implies that u_2 is Parter for λ relative to $A[T_2]$, which verifies (1). (Note that this argument does not depend upon the adjacency of u_1 and u_2.)

For (2), we suppose without loss of generality that u_1 is a downer vertex for λ relative to $A[T_1]$. Since

$$\begin{aligned} m_{A[T_1]}(\lambda) + m_{A[T_2-u_2]}(\lambda) &= m_{A(u_2)}(\lambda) \\ &= m_{A(u_1)}(\lambda) \\ &= m_{A[T_1-u_1]}(\lambda) + m_{A[T_2]}(\lambda) \\ &= m_{A[T_1]}(\lambda) - 1 + m_{A[T_2]}(\lambda) \end{aligned}$$

we have $m_{A[T_2-u_2]}(\lambda) = m_{A[T_2]}(\lambda) - 1$, i.e., u_2 is a downer vertex for λ relative to $A[T_2]$.

For (3), in order to obtain a contradiction, we suppose that u_1 is a neutral vertex for λ relative to $A[T_1]$, i.e., $m_{A[T_1-u_1]}(\lambda) = m_{A[T_1]}(\lambda)$. Because u_2 is Parter for λ relative to A, we conclude that T_1 is not a downer branch for λ at u_2 relative to A and, thus, u_2 must be Parter for λ relative to $A[T_2]$. By (1), the vertex u_1 must be Parter for λ relative to $A[T_1]$, which gives a contradiction. □

Corollary 3. *Let T be a tree on n vertices, $A \in \mathscr{S}(T)$ and λ be an eigenvalue of A. Let u_1 and u_2 be two adjacent Parter vertices for λ relative to A. We denote by T_1 the component of $T - u_2$ containing u_1 and by T_2 the component of $T - u_1$ containing u_2. Then u_1 and u_2 form a Parter set for λ relative to A if and only if u_1 and u_2 are Parter vertices for λ relative to $A[T_1]$ and $A[T_2]$, respectively.*

Lemma 4. *Let T be a tree on n vertices, $A \in \mathscr{S}(T)$ and λ be an eigenvalue of A of multiplicity $M(T)$. Let u_1 and u_2 be two adjacent Parter vertices for λ relative to A. We denote by T_1 the component of $T - u_2$ containing u_1 and by T_2 the component of $T - u_1$ containing u_2. Then, u_1 is a Parter vertex for λ relative to $A[T_1]$ and u_2 is a Parter vertex for λ relative to $A[T_2]$, i.e., u_1 and u_2 form a Parter set for λ relative to A.*

Proof. Because u_1 and u_2 are individually Parter vertices for λ, by (2) of Lemma 1, we have that $m_{A[T_i]}(\lambda) = M(T_i)$, $i = 1, 2$. In order to obtain a contradiction, we suppose that u_1 and u_2 do not form a Parter set. By Lemma 2 and Corollary 3, this assumption implies that u_i is a downer vertex for λ relative to $A[T_i]$, i.e., $m_{A[T_i-u_i]}(\lambda) = m_{A[T_i]}(\lambda) - 1 = M(T_i) - 1$, $i = 1, 2$. We first note that neither T_1 nor T_2 may be a path. If, for example, T_2 were a path, any component of $T_2 - u_2$ would be a path (having λ as an eigenvalue by Lemma 1, part (2)) and, hence, these paths should still be downer branches for λ at u_2 relative to $A[T - u_1]$, which implies that u_1 and u_2 form a Parter set. Therefore, we may assume that $m_{A[T_i]}(\lambda) > 1$, $i = 1, 2$.

Since u_1 is Parter for λ, we have that $m_{A(u_1)}(\lambda) = M(T) + 1$. Because $A(u_1) = A[T_1 - u_1] \oplus A[T_2]$ and under the hypothesis that u_1 and u_2 do not form a Parter set implies that u_1 is a downer vertex for λ relative to $A[T_1]$, it follows that $M(T) + 1 = M(T_1) - 1 + M(T_2)$, i.e., $M(T) = M(T_1) + M(T_2) - 2$. Since $m_{A[T_i]}(\lambda) > 1$, $i = 1, 2$, there must exist a FPS of k_i vertices of T_i whose removal from T_i leaves $M(T_i) + k_i$ paths in which, each of the corresponding direct summands of A has λ as an eigenvalue of multiplicity 1. By hypothesis, u_i is not Parter for λ relative to $A[T_i]$, $i = 1, 2$, so we may conclude that u_1 is a vertex of one path T' among the $M(T_1) + k_1$ paths and u_2 is a vertex of one path T'' among the $M(T_2) + k_2$ paths. If we connect T' and T'' by the edge $\{u_1, u_2\}$ we obtain a subtree T''' of T. Now, consider a matrix $B''' \in \mathscr{S}(T''')$ such that λ is an eigenvalue of B''' and change $A[T''']$ to B''' in A to obtain $B \in \mathscr{S}(T)$. By construction, we have $M(T_1) + M(T_2) + k_1 + k_2 - 1$ direct summands of B having λ as an eigenvalue, after the removal of $k_1 + k_2$ vertices and, by the interlacing inequalities, it implies that $m_B(\lambda) \geqslant M(T_1) + M(T_2) - 1$, i.e., $m_B(\lambda) > M(T)$, which is a contradiction. □

Lemma 5. *Let T be a tree on n vertices and $A \in \mathscr{S}(T)$. Suppose that v is a vertex of T and that u is a neighbor of v whose branch at v, T', satisfies $m_{A[T']}(\lambda) = M(T')$. Then, either u is Parter for λ in T' or v is Parter for λ in T (or both).*

Proof. Suppose that u is *not* Parter for λ in T'. If T' is a path, then u is a downer in T', either because u is pendent in T' or because u is interior in T' and not Parter. Since u is a downer for v, v must be Parter for λ. If T' is not a path, then $M(T') > 1$ and there is a fully FPS for λ in T' that does not contain u. This is a Parter set for λ in T'; call it Q. Q is also a Parter set in T because, by Lemma 1, every Parter vertex must have a downer branch not in the direction of u. The path of T' in which u occurs (and in which λ must occur) is pendent at v after removal of the set Q from T to produce a tree T'', which again means that v is Parter for λ in T'' and that $Q \cup \{v\}$ is a Parter set for λ in T. Thus, v is Parter for λ in T. □

4. Main results

Theorem 6. *Suppose that T is a tree on n vertices and that $A \in \mathscr{S}(T)$ and λ is an eigenvalue of A of multiplicity $M(T)$. Then, no vertex of T is neutral for λ, A.*

Proof. Suppose that $\lambda \in \sigma(A)$, $A \in \mathscr{S}(T)$ and $m_A(\lambda) = M(T)$. Then, suppose that v is a neutral vertex in T for λ, A and that $T_1, \ldots, T_k$ are the branches of T at v and u_i is the neighbor of v in T_i, $i = 1, \ldots, k$. Then,

$$M(T) = m_A(\lambda) = m_{A(v)}(\lambda) = \sum_{i=1}^{k} m_{A[T_i]}(\lambda) \leqslant \sum_{i=1}^{k} M[T_i].$$

By interlacing, either

$$\sum_{i=1}^{k} M[T_i] = M(T) \quad \text{(Case 1)} \quad \text{or} \quad \sum_{i=1}^{k} M[T_i] = M(T) + 1 \quad \text{(Case 2)}.$$

In Case 1, we reach a contradiction as follows. By Lemma 5 u_i is Parter for λ in T_i, $i = 1, \ldots, k$ (as $m_{A[T_i]}(\lambda) = M(T_i)$, $i = 1, \ldots, k$) because v is not Parter for λ in T. Thus, since each u_i has a downer branch for λ in T_i, each has a downer branch for λ in $T - v$ and, thus, $\{u_1, \ldots, u_k\}$ is a Parter set for λ in $T - v$. Because v is neutral, we have

$$m_{A(\{u_1,\ldots,u_k,v\})}(\lambda) = M(T) + k = m_{A(\{u_1,\ldots,u_k\})}(\lambda).$$

Since $T - \{u_1, \ldots, u_k\}$ contains v, we may define another matrix $B \in \mathscr{S}(T)$ such that $B[v] = \lambda$, and, otherwise, B agrees with A. Now,

$$m_{B(\{u_1,\ldots,u_k\})}(\lambda) = \sum_{i=1}^{k} m_{A[T_i]}(\lambda) + k + 1 = \sum_{i=1}^{k} M(T_i) + k + 1 = M(T) + k + 1,$$

which implies $m_B(\lambda) \geqslant M(T) + 1$, by interlacing, a contradiction, as $B \in \mathscr{S}(T)$.

In Case 2, we consider two possibilities: $k = 1$, i.e., v is pendent (Case 2a) and $k > 1$ (Case 2b).

In Case 2a, $T - v = T_1$, and, as v is neutral, $m_{A(v)}(\lambda) = M(T) = M(T_1) - 1$. Since the path cover number cannot increase with the deletion of a pendent vertex, we then have

$$M(T_1) = 1 + M(T) > M(T) = P(T) \geqslant P(T_1) = M(T_1),$$

a contradiction showing that Case 2a cannot occur.

In Case 2b ($k > 1$), after an appropriate numbering, we have

$$m_{A[T_i]}(\lambda) = M(T_i), \quad i = 1, \ldots, k - 1,$$

and

$$m_{A[T_k]}(\lambda) = M(T_k) - 1.$$

By Lemma 5, u_i is Parter for λ in T_i, $i = 1, \ldots, k - 1$, or else v is Parter for λ in T (and, thus, not neutral). Since u_i is Parter in T_i for λ, $i = 1, \ldots, k - 1$, $\{u_1, \ldots, u_{k-1}\}$ is a Parter set for λ in T, as verified by the same downer branches (which are branches of T at u_i, $i = 1, \ldots, k - 1$). Based upon this, we may calculate

$$\begin{aligned} m_{A(\{u_1,\ldots,u_{k-1}\})}(\lambda) &= M(T) + k - 1 \\ &= [M(T_1) + \cdots + M(T_{k-1})] + M(T_k) - 1 + k - 1. \end{aligned}$$

But, also, definitionally,

$$\begin{aligned} m_{A(\{u_1,\ldots,u_{k-1}\})}(\lambda) &= m_{A[T_1]}(\lambda) + \cdots + m_{A[T_{k-1}]}(\lambda) + m_{A[T_k+v]}(\lambda) + k - 1 \\ &= M(T_1) + \cdots + M(T_{k-1}) + M(T_k + v) + k - 1, \end{aligned}$$

the second equality following from Lemma 1. Comparing the two expressions for $m_{A(\{u_1,\ldots,u_{k-1}\})}(\lambda)$, we conclude that

$$M(T_k + v) = M(T_k) - 1,$$

a contradiction, as $P(T_k)$ cannot exceed $P(T_k + v)$. $\square$

Theorem 7. *Suppose that T is a tree on n vertices and that $A \in \mathscr{S}(T)$ and λ is an eigenvalue of A of multiplicity $M(T)$. Then, any Parter vertex for λ has at least two downer branches.*

Proof. Suppose to the contrary that v is a Parter vertex, of degree k, for λ with only one downer branch T_1 with vertex u_1 adjacent to v. By (1) of Lemma 1, $k \geqslant 2$. Consider the other branches T_i at v with u_i adjacent to v, $i = 2, \ldots, k$.

By (2) of Lemma 1, λ is an eigenvalue of each $A[T_i]$ with multiplicity $M(T_i)$. Thus, by Theorem 6, each u_i is a downer or Parter for λ in T_i. Under our hypothesis, for $i = 2, \ldots, k$, u_i is not a downer and so is Parter in T_i. Since u_i is Parter in T_i, $i = 2, \ldots, k$, it is Parter in T; in fact $\{u_2, \ldots, u_k\}$ is a Parter set in T. Let T' be the component of $T - \{u_2, \ldots, u_k\}$ that includes v. There, v is pendent and still Parter, as T_1 is a downer branch, and λ is an eigenvalue of $A[T']$ of multiplicity $M(T')$. By Lemma 1, this is a contradiction. □

Theorem 8. *Suppose that T is a tree on n vertices and that $A \in \mathscr{S}(T)$ and λ is an eigenvalue of A of multiplicity $M(T)$. Then, upon removal of a Parter vertex for λ, A in T, no other vertex changes status.*

Proof. Call the removed Parter vertex v. By (2) of Lemma 1, each branch at v must display maximum multiplicity for λ. Thus, by Theorem 6, no branch contains a neutral vertex for λ. Thus, it suffices that any vertex u of T other than v is Parter after removal of v if and only if it was Parter before. The implication "Parter after" implies "Parter before" is straightforward. If u is Parter after removal of v, then $\{u, v\}$ forms a Parter set in T. It follows that u must have been initially Parter (by counting based upon the interlacing inequalities).

The implication "Parter before" implies "Parter after" is more subtle and, generally, requires that we are working with maximum multiplicity. Suppose that $u \neq v$ is Parter before removal of the Parter vertex v. Since u was Parter before removal of v, by Theorem 7, it has at least two downer branches. Thus, at least one of them does not include v and remains after the removal of v. Thus, u (by virtue of having a downer branch) remains Parter after removal of v, completing the proof. □

Note that in the context of the above proof, it is possible that u have a downer branch containing v, so that having additional downer branches is important.

Corollary 9. *If T is a tree on n vertices and $A \in \mathscr{S}(T)$ and λ is an eigenvalue of A of multiplicity $M(T)$, then the set of all (initially) Parter vertices for λ, A in T is a Parter set of vertices for λ, A in T.*

Proof. As Parter vertices are removed from T, all initially Parter vertices remain Parter and may be removed to further increase the multiplicity of λ. □

Lemma 10. *In any tree on $n \geqslant 3$ vertices, there exists a vertex with at least two pendent paths.*

Proof. Let T be a tree on $n \geqslant 3$ vertices. For purposes of this proof, we may assume, without loss of generality, that T has no vertices of degree 2. The claim is obviously true for a path, and any other tree may be "compressed" (via reverse edge-subdivision) to one without degree 2 vertices and some vertices of degree at least 3 in such a way that the occurrence of pendent paths (now pendent vertices) or the degrees of remaining vertices is not changed.

Now, every vertex is either degree 1 (pendent) or high-degree (at least 3). Let p be the number of former and h the number of latter, so that $p + h = n$, the number of vertices of T.

Since the total degree of all vertices of T is $2(n-1)$, we have

$$3h + p \leqslant 2n - 2 = 2p + 2h - 2$$

from which we conclude that

$$h \leqslant p - 2,$$

which implies that

$$p > h.$$

Since the number of pendent vertices exceeds the number of high-degree vertices, there must be a high-degree vertex upon which at least two vertices are pendent. Thus, the original, uncompressed tree had two pendent paths at the same vertex, and the proof of the claim is complete. □

See also [10] for a different proof.

Lemma 11. *Let T be a tree on at least three vertices. Then there exists an RPM set whose vertices may be numbered $v_1, \ldots, v_q$ so that removal of v_{i+1} from $T_i = T - \{v_1, \ldots, v_i\}$ leaves paths not present in the forest T_i, $i = 0, \ldots, q-1$.*

Proof. Apply the previous Lemma 10 to obtain a vertex v_1 with at least two pendent paths. According to [7], this vertex may be removed on the way to maximizing $p - q$. Now, another application of Lemma 10 to any component of the resulting T_1 that is not a path produces v_2 and T_2. Continuing in this way until only paths remain, produces the $v_1, \ldots, v_q$ claimed in the lemma. □

Theorem 12. *Suppose that T is a tree and that $0 \leqslant m < M(T)$ is an integer. Given $\lambda \in \mathbb{R}$, there is an $A \in \mathscr{S}(T)$ such that $m_A(\lambda) = m$ and such that there are neutral vertices in T for λ, A.*

Proof. In case T is a path, even on one vertex, the conclusion is immediate by choosing A with the smallest eigenvalue greater than λ.

Otherwise, choose an RPM set of q vertices $v_1, \ldots, v_q$ of the type given in Lemma 11. From the "new" paths in T_i (previously pendent in T_{i-1}), choose and identify one as P_i, $i = 1, \ldots, q$. There is a total of $M(T) + q$ paths in T_q, including $P_1, \ldots, P_q$. Choose $m + q$ of them, including $P_1, \ldots, P_q$, to give $P_1, \ldots, P_q, P_{q+1}, \ldots, P_{q+m}$. This leaves $M(T) - m > 0$ paths. For each path P_i, $i = 1, \ldots, m+q$, construct a matrix $A_i \in \mathscr{S}(P_i)$ with λ as its smallest eigenvalue. By Perron–Frobenius applied to a diagonal similarity of $\alpha_i I - A_i$, the smallest eigenvalue of any proper principal submatrix of A_i is greater than λ. For the other $M(T) - m$ paths, choose matrices A_i, $i = m+q+1, \ldots, M(T)+q$, so that the smallest eigenvalue is greater than λ. Note that each vertex of P_i, $i = 1, \ldots, m+q$, is a downer for λ, A_i and, by the interlacing inequalities, for $i = m+q+1, \ldots, M(T)+q$, no principal submatrix of A_i has λ as an eigenvalue.

Now let A be any matrix in $\mathscr{S}(T)$ with principal submatrices $A_1, \ldots, A_{M(T)+q}$ in the appropriate positions. Then, in T, $\{v_1, \ldots, v_q\}$ is a Parter set for λ, because in T_{i-1}, P_i is a downer branch at v_i, $i = 1, \ldots, q$ ($T_0 = T$). Since $m_{A[T_q]}(\lambda) = m + q$, by design, $m_A(\lambda)$ must have been m.

Finally, we show that any vertex v that lies in one of the final paths P_i, other than $P_1, \ldots, P_{m+q}$, is neutral in T for λ, A. Delete one of these vertices from T to get T'. Since $\{v_1, \ldots, v_q\}$ is a Parter set in T' for $\lambda, A(v)$, for the same reason as before, and since $m_{A[T_q](v)}(\lambda) = m + q$, we conclude that $m_{A(v)}(\lambda) = m$ and, thus, that v is neutral in T for λ, A. □

Now, the characterization Theorem 13 is just the logical combination of Theorems 6 and 12.

Theorem 13. *Let m be a nonnegative integer and T a tree on n vertices. There is a matrix $A \in \mathscr{S}(T)$ and an eigenvalue $\lambda \in \sigma(A)$ such that $m_A(\lambda) = m$ and such that there are neutral vertices in T for λ, A if and only if $m < M(T)$.*

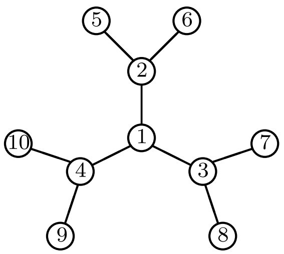
5. Examples

We give here several examples that serve various purposes: (1) to show how our ideas can be used to precisely classify some vertices; (2) to limit the generality of results that we have proven; and, as a by-product, (3) to illustrate our results.

Among all matrices $A \in \mathscr{S}(T)$ that have an eigenvalue λ satisfying $m_A(\lambda) = M(T)$, some of the vertices must be downers for λ (including, at least, the pendent vertices), some (perhaps the empty set) must be Parter, and the rest (also, perhaps, the empty set) are ambiguous (downers for some such A and Parter for others).

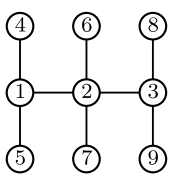
This is because of Theorem 6 and Theorem 7. In general, it is difficult to classify vertices of the first and second type (though it can be done), but often it may be done easily.

First, consider the tree

$T_1 =$ 

which was, early on, difficult for the determination of multiplicity lists. We have $M(T_1) = 4 = P(T_1)$. If $A \in \mathscr{S}(T_1)$ and $m_A(\lambda) = 4$, each of the 6 pendent vertices must be downers, and each of vertices 2, 3, 4 lies in every RPM set (there happens to be only the one: $\{2, 3, 4\}$), so that each must be Parter. Vertex 1, with no downer branches, cannot be Parter and, so, must be a downer. Thus, every vertex is unambiguously classified. Further, if the multiplicity list is 4, 2, 2, 1, 1 (which occurs [8]), no vertex can be neutral for *any* eigenvalue. The assignment must be λ to each vertex 1, 5, 6, 7, 8, 9, 10 to achieve $m_A(\lambda) = 4$ and μ and τ to each path $5 - 2 - 6$, $7 - 3 - 8$ and $9 - 4 - 10$ to achieve $m_A(\mu) = m_A(\tau) = 2$. The two multiplicities of 1 must correspond to the largest and smallest eigenvalues, via Perron–Frobenius [2], so that, for them, every vertex is a downer. Since every vertex is easily classified as Parter or downer for μ and τ, in this case, no vertex is neutral for any eigenvalue. This limits generalization of Theorem 6.

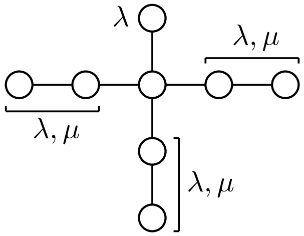
It may also happen that no vertex is unambiguously Parter, even when the maximum multiplicity is attained. Consider the tree

$T_2 =$ 

$M(T_2) = 3 = P(T_2)$, and if $m_A(\lambda) = 3$ for $A \in \mathscr{S}(T_2)$, each pendent vertex is a downer. However each of the subsets $\{2\}, \{1, 3\}, \{1, 2\}, \{2, 3\}$ or $\{1, 2, 3\}$ of $\{1, 2, 3\}$ may be Parter, so that no vertex is unambiguously Parter.

In contrast to T_1 it can happen that, even when $M(T)$ is attained and all other eigenvalues are as multiple as possible, there still may be a neutral vertex for a multiple eigenvalue (of multiplicity less than $M(T)$). Let

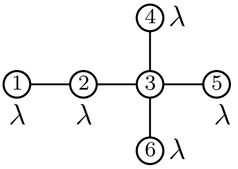
$T_3 =$



$M(T) = 3$, and for the assignment shown, the multiplicity list is 3, 2, 1, 1, 1, as concentrated as possible. Yet, the top vertex is neutral for the multiplicity 2 eigenvalue (and a downer for the multiplicity 3 eigenvalue).

Now let

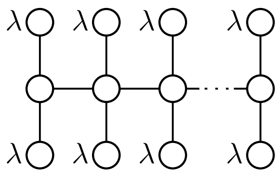
$T_4 =$



$M(T_4) = 3$, and $m_A(\lambda) = 2$ with the assignment shown. Vertices 1, 2 and 3 are all Parter for λ, but $\{1, 2, 3\}$ is not a Parter set, in contrast to Theorem 8.

We also note that it is possible to have arbitrarily many Parter vertices for an eigenvalue attaining $M(T)$ and have each one have only two downer branches. Consider

$T_5 =$



If there are $3k$ vertices, then $M(T) = P(T) = k$, $m_A(\lambda) = k$ and each of the k interior vertices is Parter. However, the two pendent vertices at each are the only downer branches for each Parter vertex.

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Further reading

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