Portfolio Choice with High Frequency Data: CRRA Preferences and the Liquidity Effect[*]

R. P. Brito[†] H. Sebastião[‡] P. Godinho[§]

Abstract

This paper suggests a new approach for portfolio choice. In this framework, the investor, with CRRA preferences, has two objectives: the maximization of the expected utility and the minimization of the portfolio expected illiquidity. The CRRA utility is measured using the portfolio realized volatility, realized skewness and realized kurtosis, while the portfolio illiquidity is measured using the well-known Amihud illiquidity ratio. Therefore, the investor is able to make her choices directly in the expected utility/liquidity (EU/L) bi-dimensional space. We conduct an empirical analysis in a set of fourteen stocks of the CAC 40 stock market index, using high frequency data for the time span from January 1999 to December 2005 (seven years). The robustness of the proposed model is checked according to the out-of-sample performance of different EU/L portfolios relative to the minimum variance and equally weighted portfolios. For different risk aversion levels, the EU/L portfolios are quite competitive and in several cases consistently outperform those benchmarks, in terms of utility, liquidity and certainty equivalent.

JEL Classification: C44; C55; C58; C61; C63; C88; G11

Keywords: portfolio choice, high frequency data, realized moments, Amihud illiquidity ratio, CRRA preferences

1 Introduction

The traditional mean-variance approach of Markowitz (1952) suggests that the optimal strategy is to allocate the investor's wealth bearing in mind the trade-off between the mean and the variance of the portfolio's return. However, the classical mean-variance portfolios tend to be very unstable, as the portfolio weights are quite sensitive to the input information set. Additionally, the poor out-of-sample performance of these portfolios has been widely documented in the literature (e.g., Michaud, 1989; Jagannathan and Ma, 2003; DeMiguel et al., 2009). Among a wide stream of literature proposing different ways of improving the properties of the classical portfolio choice model, some studies suggest the usefulness of incorporating more than just the two first moments of the returns' distribution (Chunhachinda et al., 1997; Athayde and Flôres, 2004; Maringer and Parpas, 2009; Mencia and Sentana, 2009; Harvey et al., 2010).

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The availability of high frequency financial databases has increased exponentially in recent years, which has opened new fields of research both in financial economics and financial econometrics. Although Merton (1980) has observed that the variance can be accurately estimated as the sum of the realizations of the squared intraday returns, the researchers' attention has been devoted to ARCH type (Engle, 1982; Bollerslev, 1986; Nelson, 1991) and stochastic volatility models (Taylor, 1986), at least until the end of the '90s. Nevertheless, based on the work of Schwert (1989) and Hsieh (1991), authors such as Andersen et al. (2001) and Barndorff-Nielsen and Shephard (2002) began to use intraday data to estimate the variance as the sum of squared returns, sampled at very short intraday intervals. This new approach, known in the literature as realized volatility, has a straightforward reasoning: since the sample path of the variance is continuous, the accuracy of the variance estimates increases with the sampling frequency. Moreover, it is quite appealing as it is a model-free and an error-free measure of volatility (because it is observable) which converges to the quadratic variation (Andersen et al., 2006). Several useful surveys on realized volatility can be found in the literature (see, e.g., Barndorff-Nielsen and Shephard, 2005; Andersen et al., 2006; McAleer and Medeiros, 2008; Meddahi et al., 2011). Papers like Andersen et al. (2001), Andersen et al. (2001), Areal and Taylor (2002), and Koopman et al. (2005) use the realized volatility in univariate frameworks. Other papers, like Andersen et al. (2003), Flemming et al. (2003), Barndorff-Nielsen and Shephard (2004), Liu (2009), Fan et al. (2012), and Hautsch et al. (2012) extend the approach to multivariate cases. With the remarkable growth of related literature, special attention has been given to two ubiquitous problems when dealing with high frequency financial data: market microstructure noise and asynchronous price observations. The most common response to the presence of microstructure noise has been to reduce the sampling frequency to some arbitrary level, say 5-minutes or 30-minutes (Andersen et al., 2001; Hansen and Lunde, 2006). Another possibility is to use all the available high frequency data (seconds, milliseconds) and taking explicitly into account the microstructure noise in the volatility estimation (Aït-Sahalia et al., 2005a;b; 2011). Regarding the non-synchronous price observations, Barndorff-Nielsen et al. (2011) have proposed the well-known realized kernel estimator for the multivariate case, which has the advantage of being a positive-definite estimator, quite suitable for dealing with asynchronous data.

The approaches originally defined for realized volatility can be extended, with equal interest and potential, to higher moments. Neuberger (2012) introduced the realized skewness as the sum of the 3rd power of returns, while Amaya et al. (2015) defined the realized kurtosis as the sum of the 4th power of returns. These authors also found significant negative and positive effects of skewness and kurtosis on weekly stock returns, respectively.

The majority of studies in portfolio choice assume that securities' returns are the only source of information. However, the subprime crisis, which led to a worldwide market liquidity crisis, has highlighted the importance of liquidity not only for each particular investor but also for the achievement of the allocative rationale inherent to financial markets. Liquidity is the easiness to trade a security. Although quite simple to enunciate, liquidity is an elusive concept, and in fact, one may enumerate three main dimensions of a liquid market: depth (high quantities available for sale or purchase away from the current market price), breadth (large number of market participants) and resiliency (price impacts caused by trading are small and transitory). Some authors have found a positive relationship between stock returns and alternative proxies for liquidity: Amihud and Mendelsen (1986) and Datar et al. (1998) used as a liquidity proxy the bid-ask spread; Brennan and Subrahmanyam (1996) used price impacts; Easley et al. (2002) used the probability of informed trading (PIN). Other papers found the existence of commonality and predictability in liquidity (Chordia et al., 2001; Hasbrouck and Seppi, 2001; Huberman and Halka, 2001; Amihud, 2002; Pástor and Stambaugh, 2003).

It is known that liquidity is a direct function of the implicit and explicit trading costs. However, the quantification of these costs is not a trivial task, not only due to its conceptual vagueness but also, sometimes, due to the lack of information. Thus, there are several proxies to measure liquidity: bid-ask spread, trading volume, turnover, quote size and price impact. Goyenko et al. (2009) compared different liquidity measures and found that the Amihud illiquidity ratio (Amihud, 2002) is one of the best liquidity proxies, having a strong correlation with several other liquidity measures.

In this paper we suggest the construction of expected utility-liquidity portfolios. By solving the proposed expected utility-liquidity (EU/L) problem, the investor will be able to identify the optimal portfolios, which have the maximum expected utility among all that provide at least a certain expected liquidity level. We assume that the investor has a CRRA (constant relative risk aversion) utility, as it is often assumed in the literature (Aït-Sahalia and Brandt, 2001). The reason behind the use of this particular utility function is clearly stated by Brandt et al. (2009, p. 3421): "the advantage of CRRA utility is that it incorporates preferences toward higher moments in a parsimonious manner. In addition, the utility function is twice continuously differentiable, which allows us to use more efficient numerical optimization algorithms that make use of the analytic gradient and Hessian of the objective function". However, it is worth noticing that the proposed methodology is applicable to any other type of utility function. In this study, we consider the fourth order Taylor expansion of the expected utility, around the portfolio expected return. Thus the expected utility is a function of the portfolio expected return, variance, skewness and kurtosis. Relying on intraday transaction data, we use the daily estimates of the portfolio's moments as inputs for the optimization model, using as estimators the portfolio realized variance, realized skewness and realized kurtosis. In addition, since we are interested on the relationship between liquidity and the behaviour of stock prices, following Goyenko et al. (2009) and Chiang and Zheng (2015), the daily illiquidity level is measured by the intraday Amihud illiquidity ratio (Amihud, 2002).

In summary, this paper presents a new methodology for portfolio choice in the expected utility-liquidity space. The proposed EU/L model allows the investor to identify the optimal portfolios, which have the maximum expected utility, computed with higher moments, among all that provide at least a certain expected level of liquidity. This paper also considers high frequency data by using realized estimators as the inputs of the optimization model.

For the empirical application we use intraday data on fourteen French stocks for seven years (from January 1999 to December 2005, which corresponds to 14.5GB of raw data). These data were provided by EUROFIDAI (European Financial Data Institute), and were not subjected to any kind of sample selection. These fourteen stocks are constituents of the French Stock Market Index (CAC 40) at current date (March, 2017). In-sample we compute the EU/L Pareto frontier for a moderately risk averse investor. The EU/L Pareto frontier shows the existence of a positive relationship between the expected utility and the expected illiquidity. Out-of-sample, we compute three different EU/L optimal portfolios (according to three different, pre-established, illiquidity levels) and compare their performance with two hard to beat benchmark portfolios: the minimum variance and the equally weighted portfolios. The EU/L optimal portfolios are very competitive and in most cases are able to consistently beat the benchmarks. This is observable in terms of out-of-sample utility, liquidity and certainty equivalent. These results hold for different risk aversion levels, which indicates that the proposed EU/L model is quite robust.

The paper is organized as follows. Section 2 introduces some notation and presents the expected utility-liquidity problem (EU/L problem). Section 3 describes the data, the procedures and the main results from the empirical application. Finally, Section 4 presents the main conclusions and some directions for future work.

2 The Expected Utility-Liquidity Optimization Problem

2.1 The Portfolio Choice with Higher Moments

In this subsection, in order to formulate the investor's problem with higher moments, we follow Brito et al. (ming). Let us then begin by considering that an investor has a given wealth to invest in a universe of N stocks. The investor wants to maximize her expected utility, $E_t\left[u\left(r_{p,t+1}\right)\right]$, where $r_{p,t+1}$ represents the portfolio's return at day $t+1$, in the feasible space, X. Thus, the standard utility maximization investor's problem can be formulated as

$$\begin{aligned} \max_{x\in\mathbb{R}^N} \quad & E_t\left[u\left(r_{p,t+1}\right)\right] = E_t\left[u\left(\sum_{i=1}^{N} x_i r_{i,t+1}\right)\right] \\ \text{such that} \quad & x \in X, \end{aligned} \tag{1}$$

where x is the $N\times 1$ vector of weights, $r_{i,t+1}$ represents the return of stock i, with $i \in \{1,\ldots,N\}$, at day $t+1$, and $X = \{x \in \mathbb{R}^N : \sum_{i=1}^{N} x_i = 1,\ x_i \geq 0,\ i = 1,\ldots,n\}$ is a polyhedral set. Note that the feasible set excludes short-selling, which in some markets is total or circumstantially forbidden.

Assuming that the investor has CRRA preferences, then

$$u\left(r_{p,t+1}\right) = \begin{cases} \dfrac{\left(1+r_{p,t+1}\right)^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma > 1, \\ \\ \log(1+r_{p,t+1}) & \text{if} \quad \gamma = 1, \end{cases} \tag{2}$$

where γ represents the relative risk aversion coefficient (the higher the value of γ the more risk averse is the investor). Let us denote

$$\begin{aligned} V_t\left(r_{p,t+1}\right) &= E_t\left[r_{p,t+1} - E_t\left(r_{p,t+1}\right)\right]^2, \\ S_t\left(r_{p,t+1}\right) &= E_t\left[r_{p,t+1} - E_t\left(r_{p,t+1}\right)\right]^3, \\ K_t\left(r_{p,t+1}\right) &= E_t\left[r_{p,t+1} - E_t\left(r_{p,t+1}\right)\right]^4, \end{aligned} \tag{3}$$

as the portfolio variance, skewness and kurtosis at day $t+1$, respectively. Then, the fourth order Taylor expansion of the expected utility, $E_t\left[u\left(r_{p,t+1}\right)\right]$, around the expected return of the portfolio, $E_t\left(r_{p,t+1}\right)$, is given by

$$
\begin{aligned}
E_t\left[u\left(r_{p,t+1}\right)\right] &\approx u\left[E_t\left(r_{p,t+1}\right)\right] + \frac{1}{2!}u''\left[E_t\left(r_{p,t+1}\right)\right]V_t\left(r_{p,t+1}\right) \\
&+ \frac{1}{3!}u'''\left[E_t\left(r_{p,t+1}\right)\right]S_t\left(r_{p,t+1}\right) \\
&+ \frac{1}{4!}u''''\left[E_t\left(r_{p,t+1}\right)\right]K_t\left(r_{p,t+1}\right).
\end{aligned} \tag{4}
$$

Defining

$$
\begin{aligned}
a &= u\left[E_t\left(r_{p,t+1}\right)\right], \qquad b = -\frac{u''\left[E_t\left(r_{p,t+1}\right)\right]}{2}, \\
c &= \frac{u'''\left[E_t\left(r_{p,t+1}\right)\right]}{6}, \quad d = -\frac{u''''\left[E_t\left(r_{p,t+1}\right)\right]}{24},
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
u\left[E_t\left(r_{p,t+1}\right)\right] &= \begin{cases} \dfrac{\left[1+E_t\left(r_{p,t+1}\right)\right]^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma > 1, \\ \log\left[1+E_t\left(r_{p,t+1}\right)\right] & \text{if} \quad \gamma = 1, \end{cases} \\
u''\left[E_t\left(r_{p,t+1}\right)\right] &= \begin{cases} -\gamma\left[1+E_t\left(r_{p,t+1}\right)\right]^{-(\gamma+1)} & \text{if} \quad \gamma > 1, \\ -\dfrac{1}{\left[1+E_t\left(r_{p,t+1}\right)\right]^2} & \text{if} \quad \gamma = 1, \end{cases} \\
u'''\left[E_t\left(r_{p,t+1}\right)\right] &= \begin{cases} \gamma(\gamma+1)\left[1+E_t\left(r_{p,t+1}\right)\right]^{-(\gamma+2)} & \text{if} \quad \gamma > 1, \\ \dfrac{2}{\left[1+E_t\left(r_{p,t+1}\right)\right]^3} & \text{if} \quad \gamma = 1, \end{cases} \\
u''''\left[E_t\left(r_{p,t+1}\right)\right] &= \begin{cases} -\gamma(\gamma+1)(\gamma+2)\left[1+E_t\left(r_{p,t+1}\right)\right]^{-(\gamma+3)} & \text{if} \quad > 1, \\ -\dfrac{6}{\left[1+E_t\left(r_{p,t+1}\right)\right]^4} & \text{if} \quad \gamma = 1, \end{cases}
\end{aligned} \tag{6}
$$

we can rewrite Equation (4) as

$$
\begin{aligned}
E_t\left[u\left(r_{p,t+1}\right)\right] &\approx a\left(r_{p,t+1}\right) - b\left(r_{p,t+1}\right)V_t\left(r_{p,t+1}\right) \\
&+ c\left(r_{p,t+1}\right)S_t\left(r_{p,t+1}\right) - d\left(r_{p,t+1}\right)K_t\left(r_{p,t+1}\right)
\end{aligned} \tag{7}
$$

Thus the investor's problem with higher moments can be stated as

$$\begin{aligned} \max_{x\in\mathbb{R}^N} \quad & a\left(r_{p,t+1}\right) - b\left(r_{p,t+1}\right) V_t\left(r_{p,t+1}\right) \\ & + c\left(r_{p,t+1}\right) S_t\left(r_{p,t+1}\right) - d\left(r_{p,t+1}\right) K_t\left(r_{p,t+1}\right) \\ \text{such that} \quad & x \in X. \end{aligned} \tag{8}$$

2.2 The Realized Higher Moments

When the available intraday trading data suffer from nonsynchronous trading effects, this induces potentially serious biases in the moments and co-moments of returns (Campbell et al., 1997, p. 84-98). In such case, we should adopt some procedure in order to synchronize the data. One possible way to accomplish this, is using the all refresh-time method (Barndorff-Nielsen et al., 2011).

Supposing that for each stock, $i \in \{1, \ldots, N\}$, we have K synchronized intraday price observations, in day $t+1$ we have $P_{t+(k/K)}$, with $k \in \{1, \ldots, K\}$, price observations. Note that the closing price of day $t+1$ is given by $P_{t+(K/K)} = P_{t+1}$. In this setting, the daily realized variance (Andersen et al., 2001) at day $t+1$, for each individual stock i, is given by

$$RV_{i,t+1}^{K} = \sum_{k=1}^{K} r_{i,t+(k/K)}^{2}, \tag{9}$$

where $r_{i,t+(k/K)}$ is the return of stock i in the intraday period k. For each pair of stocks, i and j, with $i, j \in \{1, \ldots, N\}$, the corresponding daily realized covariance, at day $t+1$, is given by

$$RCOV_{i,j,t+1}^{K} = \sum_{k=1}^{K} r_{i,t+(k/K)} r_{j,t+(k/K)}. \tag{10}$$

The daily portfolio realized variance can thus be computed as

$$R\Sigma(x) = x^{\top} RM_2 x, \tag{11}$$

where RM_2 represents the realized covariance matrix. Each entry, $c_{ij,t+1}$, of the RM_2 matrix is given by

$$c_{ij,t+1} = \sum_{k=1}^{K} r_{i,t+(k/K)} r_{j,t+(k/K)}. \tag{12}$$

Analogously to the realized variance approach, the daily realized skewness (Neuberger, 2012) at day $t+1$, for each individual stock i, can be defined as

$$RS_{i,t+1}^{K} = \sum_{k=1}^{K} r_{i,t+(k/K)}^{3}. \tag{13}$$

The realized coskewness matrix can be computed as a $N \times N^2$ matrix (as described in detail by Athayde and Flôres, 2004). According to this procedure, the daily portfolio realized skewness can be computed as

$$R\Phi(x) = x^\top RM_3(x \otimes x), \tag{14}$$

where RM_3 is the realized coskewness matrix and $\otimes$ represents the Kronecker product. The realized coskewness matrix corresponds to N matrixes $A_{ijl,t+1}$ of dimension $N \times N$ such that

$$RM_3 = \left[\begin{array}{cccc} A_{1jl,t+1} & A_{2jl,t+1} & \cdots & A_{Njl,t+1} \end{array} \right], \tag{15}$$

where each element, $a_{ijl,t+1}$, is given by

$$a_{ijl,t+1} = \sum_{k=1}^{K} r_{i,t+(k/K)} r_{j,t+(k/K)} r_{l,t+(k/K)}. \tag{16}$$

Finally, the daily realized kurtosis (Amaya et al., 2015) at day $t+1$, for each individual stock i, can be defined as

$$RK_{i,t+1}^{K} = \sum_{k=1}^{K} r_{i,t+(k/K)}^4. \tag{17}$$

The daily portfolio realized kurtosis, can be obtained by computing the following products

$$R\Psi(x) = x^\top RM_4(x \otimes x \otimes x), \tag{18}$$

where RM_4 represents the realized cokurtosis matrix. The RM_4 matrix corresponds to N^2 matrixes $B_{ijlm,t+1}$ of dimension $N \times N$ such that

$$RM_4 = \left[\begin{array}{cccc|cccc|c|cccc} B_{11lm,t+1} & B_{12lm,t+1} & \cdots & B_{1Nlm,t+1} & B_{21lm,t+1} & B_{22lm,t+1} & \cdots & B_{2Nlm,t+1} & \cdots & B_{N1lm,t+1} & B_{N2lm,t+1} & \cdots & B_{NNlm,t+1} \end{array} \right], \tag{19}$$

where each element, $b_{ijlm,t+1}$, is given by

$$b_{ijlm,t+1} = \sum_{k=1}^{K} r_{i,t+(k/K)} r_{j,t+(k/K)} r_{l,t+(k/K)} r_{m,t+(k/K)}. \tag{20}$$

As discussed before, we propose the use of intraday data to compute the realized moments as inputs of Problem (8):

$$\begin{aligned} V_t\left(r_{p,t+1}\right) &\approx R\Sigma(x) = x^\top RM_2 x, \\ S_t\left(r_{p,t+1}\right) &\approx R\Phi(x) = x^\top RM_3(x \otimes x), \\ K_t\left(r_{p,t+1}\right) &\approx R\Psi(x) = x^\top RM_4(x \otimes x \otimes x). \end{aligned} \tag{21}$$

Hence, Problem (8) can be reformulated as

$$\begin{aligned} \max_{x \in \mathbb{R}^N} \quad & a(x) - b(x)R\Sigma(x) + c(x)R\Phi(x) - d(x)R\Psi(x) \\ \text{such that} \quad & x \in X. \end{aligned} \tag{22}$$

Notice that in estimating the daily return, $r_{i,t+1}$, of each stock i (with $i \in \{1, \ldots, N\}$), from high-frequency data, only the first and last price observations will matter:

$$
\begin{aligned}
r_{i,t+1} &= \sum_{k=1}^{K}\left[\ln(P_{i,t+(k/K)}) - \ln(P_{i,t+(k-1/K)})\right] \\
&= \left[\ln(P_{i,t+(1/K)}) - \ln(P_{i,t})\right] + \left[\ln(P_{i,t+(2/K)}) - \ln(P_{i,t+(1/K)})\right] + \dots \\
&+ \left[\ln(P_{i,t+1}) - \ln(P_{i,t+(K-1/K)})\right] \\
&= \left[\ln(P_{i,t+1}) - \ln(P_{i,t})\right]
\end{aligned}
\tag{23}
$$

Consequently, the estimation of the daily portfolio mean, $E_t\left(r_{p,t+1}\right)$, is given by

$$
E_t\left(r_{p,t+1}\right) \quad \approx \quad \mu(x) = m^{\top}x, \tag{24}
$$

where m represents the vector of dimension $N \times 1$ where each entry corresponds to the daily mean return of each stock i.

2.3 The EU/L Problem

As referred by Goyenko et al. (2009) and Chiang and Zheng (2015), the Amihud illiquidity ratio is one of the best proxies to measure stock liquidity since it has a strong correlation with several other measures of liquidity. The Amihud illiquidity ratio for stock i at day $t+1$, $A_{i,t+1}$, is given by

$$
A_{i,t+1} = \frac{1}{K}\sum_{k=1}^{K}\frac{\left|r_{i,t+(k/K)}\right|}{v_{i,t+(k/K)}}, \tag{25}
$$

where $r_{i,t+(k/K)}$ represents the return of stock i in the intraday period k, and $v_{i,t+(k/K)}$ the corresponding trading volume in euros.

In this paper, we suggest the construction of efficient portfolios, where the investors maximise their expected utility while taking into account the liquidity level associated to those portfolios. Thereby, motivated by Problem (22) and using the Amihud illiquidity ratio as a liquidity measure, we propose the following expected utility-liquidity (EU/L) problem

$$
\begin{aligned}
\max_{x\in\mathbb{R}^N} \quad & a(x) - b(x)R\Sigma(x) + c(x)R\Phi(x) - d(x)R\Psi(x) \\
\text{such that} \quad & I^{\top}x \ \leq \ I_{target}, \\
& x \in X,
\end{aligned}
\tag{26}
$$

where I represents the vector of dimension $N \times 1$, with elements equal to the expected Amihud illiquidity ratio (computed according to Equation (25)) of each stock $i \in \{1,\dots,N\}$, and I_{target} is a given illiquidity upper limit. The objective function of the EU/L problem (Problem (26)), $f(x) = a(x) - b(x)R\Sigma(x) + c(x)R\Phi(x) - d(x)R\Psi(x)$, is as continuous nonlinear but smooth function, all constraints are linear and the feasible space is compact (it is a bounded and closed space). Given these properties, the existence of a maximum for the EU/L problem is guaranteed by the well-known Weierstrass theorem. By solving Problem (26), which can be done by using any standard nonlinear optimization software for constrained optimization, one can identify the efficient EU/L frontier, i.e. those portfolios which have the maximum expected utility among all feasible portfolios that provide at least a certain level of expected liquidity.

3 Empirical Application

3.1 The Data

The empirical application is conducted on fourteen French stocks[1]. These stocks were traded during all the sample period in the French Stock Market (Euronext Paris) and belonged to the CAC 40 Index at least once (but not necessarily always). All stocks are currently constituents of the CAC 40 Index (March, 2017). For each stock, we have access to intraday data gathered during each trading session (from 09:00 a.m. to 17:30 p.m., local time) for a total of 1777 trading days, from January 1999 to December 2005. The database was provided by the European Financial Institute (EUROFIDAI). In these files, for each stock, among many other information, we just retained the transactions timestamps, the stock trading prices and the traded number of securities.

The available intraday price observations, for the fourteen stocks, were not synchronized. Refresh-time methods used for synchronizing intraday trading among stocks include the pairwise refresh-time method and the all refresh-time method. The pairwise refresh-time method synchronizes the trading for each pair of stocks separately, allowing us to retain more data points (compared to the all refresh-time method); however, the resulting covariance matrix is not necessarily positive definite. In turn, the all refresh-time method synchronizes all stocks simultaneously and ensures that the resulting covariance matrix is positive definite (see Barndorff-Nielsen et al. (2011), for further details).

To ensure the positive-definiteness of the covariance matrix, in this paper we chose the all refresh-time method (Barndorff-Nielsen et al., 2011). This method was implemented through a `C++` routine. Briefly, this method can be described as follows:

- Let τ_1 be the first intraday period at day $t+1$ where all the available stocks have changed their price at least once since the market opening;
- Let τ_2 be the first intraday period at day $t+1$ where all the available stocks have changed their price at least once since τ_1;
- Proceeding in this way, allows the sequential definition of timestamps τ_k, with $k \in \{1, \ldots, K\}$, until τ_K is defined, corresponding to the market closure;
- Then we can compute the intraday returns for each stock $i \in \{1, \ldots, N\}$, in irregularly spaced but perfectly synchronous intervals
$$r_{i,t+\tau_k} = \ln(P_{i,t+\tau_k}) - \ln(P_{i,t+\tau_{k-1}}), \text{ with } k = 2, \ldots, K. \tag{27}$$

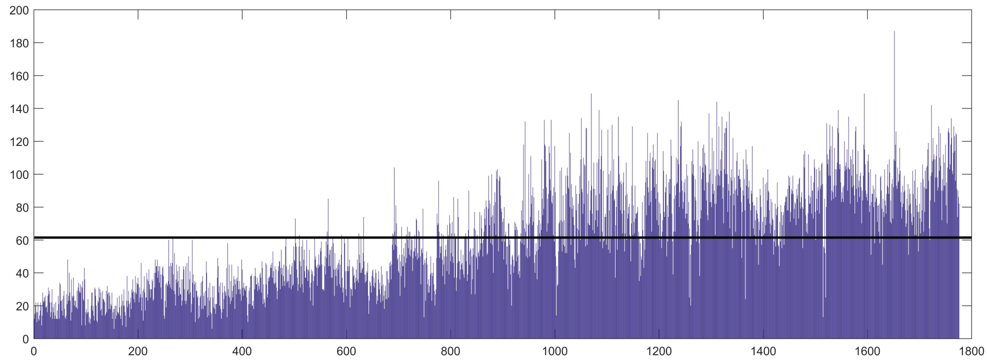
After overcoming the nonsynchronous trading problem, we obtained an average of about 61 synchronized price changes per day, which corresponds to an average duration of around 8-minutes (see Figure 1). From this figure it is also visible that the trading intensity has increased on average about five times during the period under scrutiny. Hereafter, when estimating the realized moments, it is assumed that microstructure noise does not exist.

The proposed methodology (described in Section 2.3) was implemented through `MATLAB`. By default, `MATLAB` has a 16 digit precision, which ensures the inexistence of relevant rounding

[1] In alphabetic order, the stocks are: AIR LIQUIDE; AXA; CARREFOUR; DANONE; ESSILOR INTL; FRANCE TELECOM; L'OREAL; LVMH; MICHELIN; PERNOD RICARD; SAINT GOBAIN; SANOFI AVENTIS; TOTAL; UNIBAIL

errors when computing the realized moments and co-moments[2]. Additionally, we have computed the condition number of the realized covariance (Equation (12)), realized coskewness (Equation (15)) and realized cokurtosis (Equation (19)) matrixes (for all the sample period) and we have obtained the values of 8.56, 10.26 and 22.88, respectively. This suggests that the estimates of the realized moments and co-moments are relatively stable.

Figure 1: Averaged number of intraday price changes



This figure reports the averaged (over the fourteen stocks) number of changes in the intraday price observations for each day. The horizontal axis corresponds to the number of trading days. In the vertical axis are the respective averaged intraday price changes. The horizontal line represents the averaged (on the overall sample) number of price changes per day (equal to 61.4347 price changes).

We can look to the EU/L problem (Problem (26)) as a biobjective problem

$$\begin{array}{rl} \max\limits_{x \in \mathbb{R}^N} & a(x) - b(x)R\Sigma(x) + c(x)R\Phi(x) - d(x)R\Psi(x) \\ \min\limits_{x \in \mathbb{R}^N} & I^\top x. \\ \text{such that} & x \in X \end{array} \tag{28}$$

The solution of Problem (28) is given in the form of a Pareto frontier in the expected utility-illiquidity space, allowing the investor to directly analyse the efficient trade-off between these two dimensions. Problem (28) can be solved using a multiobjective algorithm. Motivated by previous works (see Brito et al., 2016; ming) and since the first objective, $a(x) - b(x)R\Sigma(x) + c(x)R\Phi(x) - d(x)R\Psi(x)$, is a highly nonlinear function, we have decided to use a derivative-free solver based on direct multisearch (see Custódio et al., 2011, for a detailed description of the direct multisearch algorithm).

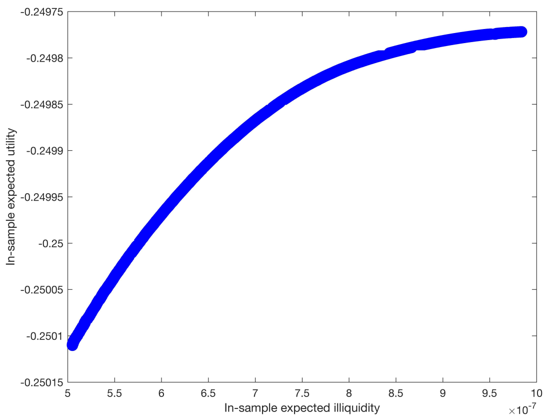
Setting the in-sample period equal to all the available time window (January 1999 to December 2005), we applied the solver `dms`[3] (version 0.3) to determine the EU/L Pareto frontier.

[2]The most critical co-moment, in terms of possible rounding errors, is cokurtosis, since it involves summing returns raised to the fourth power. Concerning this co-moment, we start by noticing that, in the dataset, we have double digit stock prices (no higher). With double digit stocks prices and ticks of 1c, high frequency returns can be as low as 10^{-4} and their fourth power can be of the 10^{-16} order. In turn, the highest value that the realized cokurtosis takes is of the 10^{-4} order. Therefore, since we work with a 16 digit precision, in the computation of the realized cokurtosis at least 4 significant digits of the fourth power of the high frequency returns are preserved.

[3]This solver is public and available by request at `http://www.mat.uc.pt/dms/`.

Figure 2 contains the plot of the EU/L Pareto frontier for an investor with a constant relative risk aversion parameter equal to 5 (see Equation (31)).

Figure 2: EU/L Pareto frontier



This figure reports the solution of the EU/L biobjective Problem (Problem (28)). The vertical axis corresponds to the first objective function (expected utility) and the horizontal axis represents the second objective function (expected illiquidity). This solution is for a moderate risk aversion parameter ($\gamma = 5$).

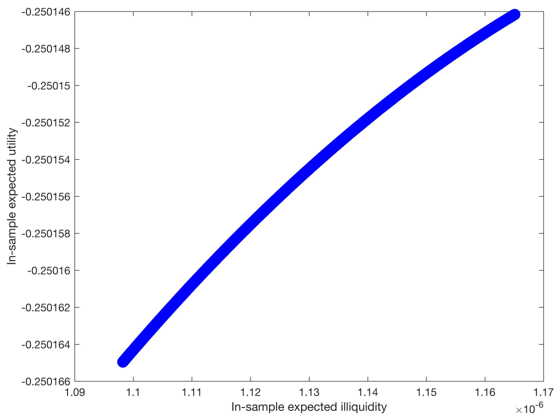
From the analysis of Figure 2, we can observe a positive relationship between the expected utility and the expected illiquidity[4].

Portfolio constraints commonly used in practice (e.g. turnover constraints, buy-in threshold constraints, cardinality constraints) can be introduced and explored in the proposed EU/L model. For example, if we introduce a turnover constraint[5] we obtain a different EU/L frontier (see Figure 3). From the comparison between the EU/L Pareto frontier (see Figure 2) and the EU/L Pareto frontier with a turnover constraint (see Figure 3), it is possible to verify that the inclusion of a turnover constraint leads to a deterioration of the portfolios in the two consider objectives (expected utility and expected illiquidity) although the frontier maintains its convexity.

[4]The same pattern was found for two different choices of the relative risk aversion parameter ($\gamma = 1$ and $\gamma = 10$).

[5]A turnover constraint can be formulated as $\sum_{i=1}^{N} |x_{i,t+1} - x_{i,t}^{0}| \leq h$, where $x_{i,t}^{0}$ is the reference portfolio and h is the turnover upper bound.

Figure 3: EU/L Pareto frontier with a turnover constraint



This figure reports the solution of the EU/L biobjective Problem (Problem (28)) with the inclusion of a turnover constraint (the reference portfolio was the equally weighted portfolio and the turnover upper bound was set to 5%). The vertical axis corresponds to the first objective function (expected utility) and the horizontal axis represents the second objective function (expected illiquidity). This solution is for a moderate risk aversion parameter ($\gamma = 5$).

3.2 Out-of-Sample Results and Sensitivity to Risk Aversion

In order to analyse the robustness of the efficient portfolios on the EU/L space, this subsection compares the out-of-sample performance of three different EU/L optimal portfolios in relation to two benchmark portfolios.

The three chosen EU/L optimal portfolios are: $x_{I_{10}}$, $x_{I_{50}}$ and $x_{I_{90}}$, the portfolios that correspond to the solution of Problem (26) for $I_{target} = I_{10}$, $I_{target} = I_{50}$ and $I_{target} = I_{90}$, respectively, where I_y represents the yth percentile of the expected illiquidity across all stocks. Hence, by construction, the aggregate level of liquidity of the resulting efficient portfolios diminishes with the increasing percentile.

Previous works, such as Brito et al. (2016) and DeMiguel et al. (2009), have showed the good out-of-sample performance of the well-known minimum variance portfolio, x_{mv}, and the equally weighted portfolio, x_{ew}. In this study, we have thus decided to use these two portfolios as benchmark portfolios. The x_{mv} portfolio corresponds to the solution of the following quadratic problem

$$\begin{array}{rl} \min\limits_{x \in \mathbb{R}^N} & x^\top M_2 x \\ \text{such that} & x \in X, \end{array} \tag{29}$$

where M_2 denotes the daily covariance matrix (estimated with daily returns). The minimum variance portfolio, x_{mv}, is commonly mentioned in the literature as a hard to beat benchmark portfolio. The explanation for the good out-of-sample performance of the minimum variance portfolio lies on the fact that the estimation error of the expected returns is absent by definition, while excluding short selling from the problem contributes to reducing the ill-conditioning of the covariance matrix (see Jagannathan and Ma, 2003, for further details).

The equally weighted portfolio, x_{ew}, corresponds to the adage "do not put all the eggs in the same basket", that is

$$x_i = \frac{1}{N},\ i = 1, \ldots, N. \tag{30}$$

In a provocative article, DeMiguel et al. (2009) showed the superior out-of-sample performance of the equally weighted portfolio regarding several portfolio optimization models. Arguably this puzzling result is due to the absence of estimation errors and to the intrinsic high level of diversification.

As it is usual in the literature (see, e.g., DeMiguel et al., 2009), we used a rolling-sample approach for the out-of-sample performance evaluation. We considered an estimation window of 1520 days, while the remaining 257 days are used for evaluating the out-of-sample performance measures. The first estimation window is from January 1999 to December 2004, and January 3, 2005 is the first day where we evaluate the out-of-sample performance. For each estimation window, we computed the two benchmark portfolios, x_{mv} and x_{ew}, using the daily returns and three optimal EU/L portfolios, $x_{I_{10}}$, $x_{I_{50}}$ and $x_{I_{90}}$, for each of three different levels of risk aversion ($\gamma = 1$, $\gamma = 5$ and $\gamma = 10$), using intraday data. Then each portfolio was held fixed and its daily returns were observed over the next day. The estimation window was then moved forward one day, and the daily returns were computed for the next day of the evaluation period. The process was thus repeated until exhausting the 257 trading days of 2005.

Table 1 presents some out-of-sample descriptive statistics. The returns of all portfolios present above normal kurtosis and are skewed. For $\gamma = 1$ and $\gamma = 5$, all the EU/L portfolios present a higher out-of-sample mean than the benchmark portfolios (with equally higher out-of-sample standard deviation). It is interesting to notice that the most liquid EU/L portfolio, $x_{I_{10}}$, has a higher out-of-sample mean than the benchmark portfolios, for two different risk aversion levels ($\gamma = 1$ and $\gamma = 5$).

Given the time series of daily out-of-sample returns for each portfolio (two benchmark portfolios and nine EU/L optimal portfolios), we computed three performance evaluation measures. The out-of-sample utility, $\hat{U}$, defined as

$$\hat{U} = \begin{cases} \dfrac{(1+\hat{\mu})^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma > 1, \\ \\ \log(1+\hat{\mu}) & \text{if} \quad \gamma = 1, \end{cases} \tag{31}$$

where $\hat{\mu}$ corresponds to the out-of-sample mean return. The out-of-sample illiquidity, $\hat{I}$, defined as the averaged out-of-sample portfolio illiquidity. And, the certainty equivalent, $\widehat{CE}$,

$$U(\widehat{CE}) = EU(R_{p,t+1}), \tag{32}$$

where $EU(.)$ represents the expected utility defined in Equation (7). The certainty equivalent can be interpreted as the risk-free rate that an investor is willing to accept in order to give up a particular risky portfolio.

The results for these three performance evaluation measures are presented in tables 2 to 4.

The three EU/L optimal portfolios ($x_{I_{10}}$, $x_{I_{50}}$ and $x_{I_{90}}$) consistently have a lower illiquidity level than the benchmark portfolios (x_{mv} and x_{ew}), for any of the considered risk aversion levels. A possible explanation for this result is that liquidity has a persistent nature, thus the

Table 1: Out-of-sample descriptive statistics

Descriptive statistics	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\xi}$	$\hat{\kappa}$
Benchmark portfolios				
x_{mv}	0.00072746	0.0068	-0.0407	3.3707
x_{ew}	0.00069979	0.0064	-0.1181	3.5758
EU/L portfolios				
$\gamma = 1$				
$x_{I_{10}}$	0.00077098	0.0085	0.2823	4.5101
$x_{I_{50}}$	0.00075402	0.0084	0.2350	4.5533
$x_{I_{90}}$	0.00077375	0.0085	0.2836	4.5155
$\gamma = 5$				
$x_{I_{10}}$	0.00074280	0.0075	-0.0281	3.4576
$x_{I_{50}}$	0.00074384	0.0075	-0.0260	3.4545
$x_{I_{90}}$	0.00074411	0.0075	-0.0258	3.4547
$\gamma = 10$				
$x_{I_{10}}$	0.00070052	0.0073	-0.2337	3.7339
$x_{I_{50}}$	0.00071437	0.0073	-0.2407	3.7289
$x_{I_{90}}$	0.00070040	0.0073	-0.2340	3.7342

This table reports the out-of-sample mean ($\hat{\mu}$), standard deviation ($\hat{\sigma}$), skewness ($\hat{\xi}$) and kurtosis ($\hat{\kappa}$) for each portfolio. The benchmark portfolios, x_{mv} and x_{ew}, refer to the minimum variance and equally weighted portfolios, computed using daily data. The three portfolios $x_{I_{10}}$, $x_{I_{50}}$ and $x_{I_{90}}$, denote the optimal expected utility/liquidity portfolios considering percentiles 10, 50 and 90 of the overall illiquidity spectrum across all stocks, and are computed using intraday data. For the computation of the EU/L optimal portfolios we considered three different levels of risk aversion: $\gamma = 1$, $\gamma = 5$ and $\gamma = 10$.

most liquid stocks *ex-ante* tend to be the most liquid ones *ex-post*. This result highlights the robustness of the EU/L model in building reliably liquid portfolios.

The robustness of the EU/L model is also reflected in the out-of-sample utility, $\hat{U}$, results. For $\gamma = 1$ and $\gamma = 5$, all the EU/L portfolios consistently show a significant (at a 5% significance level) higher utility than the benchmarks portfolios. In turn, when the investor is more sensitive to losses, the case of $\gamma = 10$, all the EU/L portfolios significantly underperform one of the benchmark portfolios (the x_{mv} portfolio) and slightly outperform the other (the x_{ew} portfolio).

In terms of out-of-sample certainty equivalent return, $\widehat{CE}$, for a low ($\gamma = 1$) and a moderate ($\gamma = 5$) risk aversion levels, the three chosen EU/L portfolios present a competitive certainty equivalent (compared with the benchmark portfolios). However, this pattern does not hold for $\gamma = 10$: here the EU/L portfolios clearly underperform the benchmark portfolios in terms of certainty equivalent return. We must note that both the utility and the certainty equivalent do not take the liquidity into account; since the EU/L portfolios must also take into account liquidity, it is not surprising that they underperform (or at least some of them underperform) the benchmark portfolios.

Table 2: Out-of-sample performance evaluation for $\gamma = 1$

Performance measures	$\hat{U}$	$\hat{I}$	$\widehat{CE}$
Benchmark portfolios			
x_{mv}	7.2719E-04	3.7452E-07	7.0416E-04
x_{ew}	6.9955E-04	4.7819E-07	6.7900E-04
EU/L portfolios			
$x_{I_{10}}$	7.7069E-04	2.7257E-07	7.3464E-04
	$(0.0079)^{mv}(0.0024)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0028)^{mv}(0.0017)^{ew}$
$x_{I_{50}}$	7.5374E-04	2.7632E-07	7.1865E-04
	$(0.0204)^{mv}(0.0021)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0049)^{mv}(0.0191)^{ew}$
$x_{I_{90}}$	7.7345E-04	2.7253E-07	7.3742E-04
	$(0.0011)^{mv}(0.0045)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0041)^{mv}(0.0054)^{ew}$

This table reports the out-of-sample utility ($\hat{U}$), illiquidity ($\hat{I}$) and certainty equivalent ($\widehat{CE}$), for each portfolio. In parenthesis are the bootstrap p-values of the difference between the respective performance measure of each EU/L portfolio and the benchmark portfolios, denoted by the superscripts mv and ew, respectively. These bootstrap p-values were computed according the classical methodology proposed by Efron and Tibshirani (1994).

In addition to the three aforementioned performance measures, we decided to include in this out-of-sample analysis the Sharpe ratio and net Sharpe ratio. We point out the need to be careful when using the Sharpe ratio when the returns are not normally distributed (the ratio does not account for skewness or kurtosis). However, as the Sharpe ratio is one of the most referenced out-of-sample performance evaluation measures in the literature and the net Sharpe ratio allows the analysis of the impact of transaction costs, we decided to investigate how the EU/L portfolios behave in these two performance evaluation measures comparatively to the benchmark portfolios. The out-of-sample Sharpe ratio, $\hat{S}$, is defined as

Table 3: Out-of-sample performance evaluation for $\gamma = 5$

Performance measures	$\hat{U}$	$\hat{I}$	$\widehat{CE}$
Benchmark portfolios			
x_{mv}	-0.2492739	3.7452E-07	6.1095E-04
x_{ew}	-0.2493014	4.7819E-07	5.9574E-04
EU/L portfolios			
$x_{I_{10}}$	-0.2492586	3.1594E-07	6.0192E-04
	$(0.0075)^{mv}(0.0030)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0524)^{mv}(0.0918)^{ew}$
$x_{I_{50}}$	-0.2492575	3.1595E-07	6.0298E-04
	$(0.0051)^{mv}(0.0000)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0599)^{mv}(0.0353)^{ew}$
$x_{I_{90}}$	-0.2492573	3.1596E-07	6.0325E-04
	$(0.0049)^{mv}(0.0000)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0847)^{mv}(0.0141)^{ew}$

This table reports the out-of-sample utility ($\hat{U}$), illiquidity ($\hat{I}$) and certainty equivalent ($\widehat{CE}$), for each portfolio. In parenthesis are the bootstrap p-values of the difference between the respective performance measure of each EU/L portfolio and the benchmark portfolios, denoted by the superscripts mv and ew, respectively. These bootstrap p-values were computed according the classical methodology proposed by Efron and Tibshirani (1994).

Table 4: Out-of-sample performance evaluation for $\gamma = 10$

Performance measures	$\hat{U}$	$\hat{I}$	$\widehat{CE}$
Benchmark portfolios			
x_{mv}	-0.1103863	3.7452E-07	4.9432E-04
x_{ew}	-0.1104138	4.7819E-07	4.9142E-04
EU/L portfolios			
$x_{I_{10}}$	-0.1104130	3.4059E-07	4.3180E-04
	$(0.0000)^{mv}(0.2212)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$
$x_{I_{50}}$	-0.1104131	3.2626E-07	4.3166E-04
	$(0.0000)^{mv}(0.3077)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$
$x_{I_{90}}$	-0.1104130	3.2626E-07	4.3171E-04
	$(0.0000)^{mv}(0.2402)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$	$(0.0000)^{mv}(0.0000)^{ew}$

This table reports the out-of-sample utility ($\hat{U}$), illiquidity ($\hat{I}$) and certainty equivalent ($\widehat{CE}$), for each portfolio. In parenthesis are the bootstrap p-values of the difference between the respective performance measure of each EU/L portfolio and the benchmark portfolios, denoted by the superscripts mv and ew, respectively. These bootstrap p-values were computed according the classical methodology proposed by Efron and Tibshirani (1994).

$$\hat{S} = \frac{\hat{\mu}}{\hat{\sigma}}. \tag{33}$$

The out-of-sample net Sharpe ratio[6] (this is the out-of-sample Sharpe ratio after transaction costs), $\widehat{S_{net}}$, is given by

$$\widehat{S_{net}} = \frac{\hat{\mu}_{tc}}{\hat{\sigma}_{tc}}, \tag{34}$$

where $\hat{\mu}_{tc} = \hat{\mu} - tc$ is the average of the out-of-sample returns ($\hat{\mu}$) minus the proportional transaction costs (tc), and $\hat{\sigma}_{tc}$ is the standard deviation of the out-of-sample returns after transaction costs. We set the proportional transaction costs equal to 50 basis points per transaction (as commonly assumed in the literature). Thus the cost of a trade over all stocks is defined as

$$tc = \sum_{t=1}^{\#periods} 0.5\% \sum_{i=1}^{N} \left(\left| x_{i,t+1} - x_{i,t}^{h} \right| \right), \tag{35}$$

where $x_{i,t}^{h}$ and $x_{i,t+1}$ are the portfolio weights before and after rebalancing at day $t+1$, respectively. Thereby, $x_{i,t}^{h}$, can be computed as

$$x_{i,t}^{h} = x_{i,t-1} \frac{1 + r_{i,t}}{1 + r_{p,t}}. \tag{36}$$

Table 5 shows the results for these two additional performance evaluation measures, the Sharpe ratio ($\hat{S}$) and the net Sharpe ratio ($\widehat{S_{net}}$). Regarding the Sharpe ratios, we can observe that it is for a moderate risk aversion level ($\gamma = 5$) that the EU/L portfolios achieve the highest values. However, for the different risk aversion levels the benchmark portfolios always outperform the EU/L portfolios. As we noticed for the utility and the certainty equivalent measures, this is not surprising, since the EU/L portfolios must also take into account liquidity and higher moments.

When we take into account the transaction costs (by computing the net Sharpe ratio), we observe that, all the EU/L portfolios are not able to beat the two benchmark portfolios. This was somehow expected, since the EU/L portfolios present a higher turnover than the benchmarks (see Table 6). It is well-known in the literature that x_{mv} and x_{ew} present a much lower turnover than other alternative strategies (see, e.g, DeMiguel et al., 2009).

4 Conclusions and Future Research

This paper presents a new methodology for portfolio choice. We suggest that the investor may built her portfolios according to the utility maximization criteria, but, at the same time, taking into account a desired level of liquidity. Hence, in this framework, the investor will make her choices in the expected utility-liquidity space. Assuming that the investor has CRRA preferences, we have showed how to incorporate parsimoniously the portfolio's higher moments (skewness and kurtosis) in the expected utility function. The proposed EU/L model allows the investor to identify the portfolios which have the maximum expected utility among all that

[6]When the numerator was negative, the ratio was refined in order to achieve a correct rank of the portfolios. In this paper we used the methodology proposed by Israelsen (2005).

Table 5: Out-of-sample Sharpe ratios and net refined Sharpe ratios

	Sharpe ratios	Net refined Sharpe ratios
Benchmark portfolios		
x_{mv}	0.103175	-6.9389E-05
x_{ew}	0.105303	-5.1516E-05
EU/L portfolios		
$\gamma = 1$		
$x_{I_{10}}$	0.086226	-7.8601E-04
	$(0.2713)^{mv}(0.3219)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$
$x_{I_{50}}$	0.085486	-1.1954E-03
	$(0.2627)^{mv}(0.3189)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$
$x_{I_{90}}$	0.086568	-8.0057E-04
	$(0.2591)^{mv}(0.3291)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$
$\gamma = 5$		
$x_{I_{10}}$	0.095220	-2.8132E-04
	$(0.2855)^{mv}(0.5435)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$
$x_{I_{50}}$	0.095366	-2.8126E-04
	$(0.3095)^{mv}(0.5525)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$
$x_{I_{90}}$	0.095402	-2.8065E-04
	$(0.2933)^{mv}(0.5559)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$
$\gamma = 10$		
$x_{I_{10}}$	0.092137	-1.6135E-04
	$(0.2130)^{mv}(0.3673)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$
$x_{I_{50}}$	0.094073	-2.5132E-04
	$(0.2649)^{mv}(0.4453)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$
$x_{I_{90}}$	0.092108	-1.6095E-04
	$(0.2175)^{mv}(0.3703)^{ew}$	$(0.0010)^{mv}(0.0010)^{ew}$

This table reports the out-of-sample Sharpe ratio ($\hat{S}$) and net Sharpe ratio, ($\widehat{S_{net}}$), for each portfolio. Note that, for the net Sharpe ratios case, the presented values refer to the refined ratios according to the Israelsen (2005) methodology. In parenthesis are the bootstrap p-values of the difference between the respective performance measure of each EU/L portfolio and those of the benchmarks: the mv portfolio and the ew portfolio, in the first and second parenthesis, respectively. These p-values were computed according the robust methodology, developed specifically for the Sharpe ratio, of Ledoit and Wolf (2008).

Table 6: Portfolio turnover

	Turnover
Benchmark portfolios	
x_{mv}	0.008462
x_{ew}	0.006747
EU/L portfolios	
$\gamma = 1$	
$x_{I_{10}}$	0.072378
$x_{I_{50}}$	0.111241
$x_{I_{90}}$	0.073724
$\gamma = 5$	
$x_{I_{10}}$	0.029733
$x_{I_{50}}$	0.029729
$x_{I_{90}}$	0.029666
$\gamma = 10$	
$x_{I_{10}}$	0.017698
$x_{I_{50}}$	0.027296
$x_{I_{90}}$	0.017653

This table reports the portfolio turnover for each portfolio. The portfolio turnover is computed according the following equation: turnover $= \frac{1}{\#periods} \sum_{t=1}^{\#periods} \sum_{i=1}^{N} \left(\left| x_{i,t+1} - x_{i,t}^{h} \right| \right)$.

provide at least a certain expected level of liquidity. The investor can thus directly analyse the efficient trade-off between expected utility and expected liquidity.

In this paper we also consider high frequency data for the estimation of the inputs for the optimization model. Firstly, the transaction data was synchronized by the all refresh-time method, and then we computed the daily higher moments using realized estimators. The daily liquidity metrics were computed by applying the intraday Amihud illiquidity ratio.

The empirical application, based on fourteen stocks from the CAC 40 Index, showed a positive relationship between the expected utility and the expected illiquidity.

The analysis of the out-of-sample performance, for different levels of risk aversion, revealed that the EU/L portfolios are usually competitive with the minimum variance and equally weighted portfolios. These two benchmarks were always beaten in terms of liquidity, but they sometimes performed better in terms of utility and certainty equivalent, and they were always better in terms of Sharp ratio. This shows that, in order to achieve a higher liquidity, the EU/L portfolios must sometimes sacrifice other performance measures. Finally, the results for the net Sharpe ratio show that the EU/L portfolios tend to exhibit a higher turnover than the two benchmark portfolios considered.

The sample used in the paper already comprises a period of sharp decreasing prices (the dot-com bubble in 2000). As future work, it would be interesting to test the proposed model in face of more recent "black swan events" (the global financial crisis (2008-2009) and the sovereign debt crisis in Europe (2011)). Although it is not easy to have access to reliable intraday data, as future work we would like to test the informational impact of using high frequency data in comparison with using daily data, especially after these events. We are also interested in refining the computation of trading costs and to test the model's robustness to other utility functions.

References

Aït-Sahalia, Y. and M. W. Brandt (2001). Variable selection for portfolio choice. *The Journal of Finance 56*, 1297–1351.

Aït-Sahalia, Y., P. A. Mykland, and L. Zhang (2005a). How often to sample a continuous-time process in the presence of market microstructure noise. *The Review of Financial Studies 18*, 351–416.

Aït-Sahalia, Y., P. A. Mykland, and L. Zhang (2005b). A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *Journal of the American Statistical Association 100*, 1394–1411.

Aït-Sahalia, Y., P. A. Mykland, and L. Zhang (2011). Ultra high frequency volatility estimation with dependent microstructure noise. *Journal of Econometrics 160*, 160–175.

Amaya, D., P. Christoffersen, K. Jacobs, and A. Vasquez (2015). Does realized skewness predict the cross-section of equity returns? *Journal of Financial Economics 118*, 135–167.

Amihud, Y. (2002). Illiquidity and stock returns: cross-section and time-series effects. *Journal of Financial Markets 5*, 31–56.

Amihud, Y. and H. Mendelsen (1986). Asset pricing and the bid-ask spread. *Journal of Financial Economics 17*, 223–249.

Andersen, T. G., T. Bollerslev, P. F. Christoffersen, and F. X. Diebold (2006). *Volatility and Correlation Forecasting*, Volume 1 of *Handbook of Economic Forecasting*, Chapter 15, pp. 777–878. Amsterdam: North-Holland: Elsevier.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and H. Ebens (2001). The distribution of realized stock return volatility. *Journal of Financial Economics 61*, 43–76.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2001). The distribution of realized exchange rate volatility. *Journal of the American Statistical Association 96*, 42–55.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2003). Modeling and forecasting realized volatility. *Econometrica 71*, 579–625.

Areal, N. M. P. C. and S. J. Taylor (2002). The realized volatility of ftse-100 futures prices. *Journal of Futures Markets 22*, 627–648.

Athayde, G. M. and R. G. Flôres (2004). Finding a maximum skewness portfolio–a general solution to three-moments portfolio choice. *Journal of Economic Dynamics and Control 28*, 1335–1352.

Barndorff-Nielsen, O. E., P. R. Hansen, A. Lunde, and N. Shephard (2011). Multivariate realized kernels: Consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading. *Journal of Econometrics 162*, 149–169.

Barndorff-Nielsen, O. E. and N. Shephard (2002). Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *Journal of The Royal Statistical Society Series B 64*, 253–280.

Barndorff-Nielsen, O. E. and N. Shephard (2004). Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics. *Econometrica 72*, 885–925.

Barndorff-Nielsen, O. E. and N. Shephard (2005). Variation, jumps, market frictions and high frequency data in financial econometrics. Technical report, Oxford Financial Research Centre.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics 31*, 307–327.

Brandt, M. W., P. Santa-Clara, and R. Valkanov (2009). Parametric portfolio policies: Exploiting characteristics in the cross-section of equity returns. *The Review of Financial Studies 22*, 3411–3447.

Brennan, M. J. and A. Subrahmanyam (1996). Market microstructure and asset pricing: On the compensation for illiquidity in stock returns. *Journal of Financial Economics 41*, 441–464.

Brito, R. P., H. Sebastião, and P. Godinho (2016). Efficient skewness/semivariance portfolios. *Journal of Asset Management 17*, 331–346.

Brito, R. P., H. Sebastião, and P. Godinho (forthcoming). Portfolio management with higher moments: The cardinality impact. *International Transactions in Operational Research, DOI: 10.1111/itor.12404 (a preliminary version of this article is available as Brito RP, Sebastião H, and Godinho P (2015) Portfolio management with higher moments: The cardinality impact, Technical Report 2015-15, GEMF, Faculty of Economics, University of Coimbra).*

Campbell, J. Y., A. W. Lo, and A. G. MacKinlay (1997). *The Econometrics of Financial Markets.* Chichester: Princeton University Press.

Chiang, T. C. and D. Zheng (2015). Liquidity and stock returns: Evidence from international markets. *Global Finance Journal 27*, 73–97.

Chordia, T., R. Roll, and A. Subrahmanyam (2001). Market liquidity and trading activity. *The Journal of Finance 56*, 501–530.

Chunhachinda, P., K. Dandapani, S. Hamid, and A. J. Prakash (1997). Portfolio selection and skewness: Evidence from international stock markets. *Journal of Banking & Finance 21*, 143–167.

Custódio, A. L., J. F. A. Madeira, A. I. F. Vaz, and L. N. Vicente (2011). Direct multisearch for multiobjective optimization. *SIAM Journal on Optimization 21*, 1109–1140.

Datar, V. T., N. Y. Naik, and R. Radcliffe (1998). Liquidity and stock returns: An alternative test. *Journal of Financial Markets 1*, 203–219.

DeMiguel, V., L. Garlappi, and R. Uppal (2009). Optimal versus naive diversification: How inefficient is the $1/N$ portfolio strategy? *The Review of Financial Studies 22*, 1915–1953.

Easley, D., S. Hvidkjaer, and M. O'Hara (2002). Is information risk a determinant of asset returns? *The Journal of Finance 57*, 2185–2221.

Efron, B. and R. J. Tibshirani (1994). *An Introduction to the Bootstrap.* Chapman & Hall/CRC Monographs on Statistics & Applied Probability. Taylor & Francis.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica 50*, 987–1008.

Fan, J., Y. Li, and K. Yu (2012). Vast volatility matrix estimation using high-frequency data for porttolio selection. *Journal of The American Statistical Association 107*, 412–428.

Flemming, J., C. Kirby, and B. Ostdiek (2003). The economic value of volatility timing using "realized" volatility. *Journal of Financial Economics 67*, 473–509.

Goyenko, R. Y., C. W. Holden, and C. A. Trzcinka (2009). Do liquidity measures measure liquidity? *Journal of Financial Economics 92*, 153–181.

Hansen, P. R. and A. Lunde (2006). Realized variance and market microstructure noise. *Journal of Business & Economics Statistics 24*, 127–161.

Harvey, C. R., J. C. Liechty, M. W. Liechty, and P. Mueller (2010). Portfolio selection with higher moments. *Quantitative Finance 10*, 469–485.

Hasbrouck, J. and D. J. Seppi (2001). Common factors in prices, order flows, and liquidity. *Journal of Financial Economics 59*, 383–411.

Hautsch, N., L. M. Kyj, and R. C. A. Oomen (2012). A blocking and regularization approach to high-dimensional realized covariance estimation. *Journal of Applied Econometrics 27*, 625–645.

Hsieh, D. A. (1991). Chaos and nonlinear dynamics: Application to financial markets. *The Journal of Finance 46*, 1839–1877.

Huberman, G. and D. Halka (2001). Systematic liquidity. *The Journal of Financial Research 24*, 161–178.

Israelsen, C. L. (2005). A refinement to the sharpe ratio and information ratio. *Journal of Asset Management 5*, 423–427.

Jagannathan, R. and T. Ma (2003). Risk reduction in large portfolios: Why imposing the wrong constraints helps. *The Journal of Finance 58*, 1651–1684.

Koopman, S. J., B. Jungbacker, and E. Hol (2005). Forecasting daily variability of the s&p 100 stock index using historical, realized and implied volatility measurements. *Journal of Empirical Finance 12*, 445–475.

Ledoit, O. and M. Wolf (2008). Robust performance hypothesis testing with the sharpe ratio. *Journal of Empirical Finance 15*, 850–859.

Liu, Q. (2009). On portfolio optimization: How and when do we benefit from high-frequency data? *Journal of Applied Econometrics 24*, 560–582.

Maringer, D. and P. Parpas (2009). Global optimization of higher order moments in portfolio selection. *Journal of Global Optimization 43*, 219–230.

Markowitz, H. M. (1952). Portfolio selection. *The Journal of Finance 7*, 77–91.

McAleer, M. and M. Medeiros (2008). Realized volatility: A review. *Econometric Reviews 27*, 10–45.

Meddahi, N., P. Mykland, and N. Shephard (2011). Special issue on realized volatility. *Journal of Econometrics 160*, 1–288.

Mencia, J. and E. Sentana (2009). Multivariate location-scale mixtures of normals and mean-variance-skewness portfolio allocation. *Journal of Econometrics 153*, 105–121.

Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics 8*, 323–361.

Michaud, R. O. (1989). The markowitz optimization enigma: Is 'optimized' optimal? *Financial Analysts Journal 4*, 31–42.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica 59*, 347–370.

Neuberger, A. (2012). Realized skewness. *The Review of Financial Studies 25*, 3423–3455.

Pástor, L. and R. F. Stambaugh (2003). Liquidity risk and expected stock returns. *Journal of Political Economy 111*, 642–685.

Schwert, G. W. (1989). Why does stock market volatility change over time? *The Journal of Finance 44*, 1115–1153.

Taylor, S. J. (1986). *Modelling Financial Time Series*. Chichester, UK: Wiley.