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Inverse mass expansion of the one-loop effective action

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Abstract

A method is described for the development of the one-loop effective action expansion as an asymptotic series in inverse powers of the fermion mass. The method is based on the Schwinger–DeWitt proper-time technique, which allows for loop particles with nondegenerate masses. The case with $SU(2) \times SU(2)$ as the symmetry group is considered. The obtained novel series generalizes the well-known Schwinger–DeWitt inverse mass expansion for equal masses, and is chiral invariant at each order. We calculate the asymptotic coefficients up to the fifth order and clarify their relationship with the standard Seeley–DeWitt coefficients. © 2001 Elsevier Science B.V. All rights reserved.

Quite often in physics one has to extract the dominant contribution of the short distance effects on the large distance behavior. This is the case where usually effective field theories (EFT) come into play [1]. Chiral perturbation theory (CHPT) in QCD is a typical example [2]. It is an effective theory approach in which the low-energy QCD Lagrangian is constructed as a combined derivative and light quark mass expansion. The expansions for the effective action accumulate the most important features of the short distance physics, i.e., for instance, the quantum fluctuations of the light particles (pions) in CHPT, or the low-energy effects resulting from the presence of heavy particles in the fundamental theory [3]. Other examples are given by models of the Nambu–Jona-Lasinio (NJL) type [4] where the action of the respective low-energy EFT results from bosonization of the quark–antiquark interaction [5]. In all the above cases and in many others arises the necessity of calculating the determinant of the positive definite elliptic operator which governs quadratic fluctuations of quantum fields in the presence of a definite background and which contains in compact form all information about the one-loop contribution of quantum fields. This determinant may be defined using the Schwinger proper-time representation [6,7] in terms of its heat kernel. The heat kernel admits at this stage an asymptotic expansion in powers of proper-time with coefficients which are known as Seeley–DeWitt coefficients [7,8]. In this way one can finally arrive at the expansion for the effective action at large distances. It is a very powerful method which is known as the background field method [9] and which allows the construction of the low-energy EFT action in the single-closed-loop approximation, when the underlying high energy theory (which in particular might be an EFT of some fundamental theory) is known. Many aspects of this approach in relation to chiral gauge theories are reviewed in [10].

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In a recent study [11] of the low-energy structure of the NJL model with the linear realization of explicitly broken $SU(2) \times SU(2)$ chiral symmetry on the basis of the Schwinger–DeWitt proper-time formalism we came to the necessity of performing systematic resummations inside the proper-time expansion. These resummations occur when the one loop diagrams of the proper-time Green’s function involve particles with different but still comparable masses. There we have obtained the first three terms in the asymptotic expansion of the corresponding heat kernel, however, at that time we could not suggest the general resummation procedure without which the method cannot be considered as complete. The aim of this Letter is to present a general method for the construction of the heat kernel asymptotic expansion for such nondegenerate cases. The algorithm for resummations, Eq. (23), is formulated on a purely algebraic basis and is completely novel. It leads us to an expansion which can be classified as a series in inverse powers of mass with coefficient functions generalizing the standard Seeley–DeWitt coefficients. Let us clarify this place. The incorporation of nontrivial mass matrices in the heat kernel expansion is by no means a new, or unsolved problem. It has been considered in detail, for example, in [3]. However, without the resummation procedure the result cannot be cast in a chiral invariant form. We consider this symmetry property of asymptotic coefficients to be a crucial condition on any generalization of the Schwinger–DeWitt result, which as it is well known fulfills this requirement. The resummations come into play only after performing fully the integrations over the proper time. It is important to note that in our approach at no instance do we recur to the proper-time expansion. We do not expand in powers of proper-time the mass dependent part of the heat kernel, for example, by absorbing the mass term in the background fields. It is this feature which makes our approach differ essentially from the ones (see, for example, [12,13]) where authors study the proper-time asymptotics of heat kernels with arbitrary matrix-valued scalar potentials, and thus nontrivial mass matrices in particular.

Our starting point for calculations is the modulus of the functional fermion determinant for the one-loop effective action, given by the proper-time integral

$$W[Y] = -\ln |\det D| = \frac{1}{2} \int_0^\infty \frac{dT}{T} \rho(T, \Lambda^2) \text{Tr}(e^{-TD^\dagger D}), \quad (1)$$

which can always be regularized by using, for example, the Pauli–Villars cutoff [14] incorporated through the kernel $\rho(T, \Lambda^2)$. We do not need the explicit form of $\rho(T, \Lambda^2)$ in the following. The calculation will be performed in Euclidean space. The elliptic operator $D^\dagger D$ has the form:

$$D^\dagger D = m^2 + B, \quad B = -\partial^2 + Y, \quad (2)$$

where Y is a matrix-valued function of scalar and pseudoscalar background fields. In the most general case the mass term m^2 does not commute with Y . The first step is the evaluation of the heat kernel in a fictitious Hilbert space. We shall use here the formalism developed by Fujikawa [15]. As a result we have

$$W[Y] = \frac{1}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} \int_0^\infty \frac{dT}{T^3} \rho(T, \Lambda^2) e^{-p^2} \text{tr}(e^{-T(m^2+A)}). \quad (3)$$

where $A = B - 2ip\partial/\sqrt{T}$ and tr is trace on the internal space. To simplify our consideration and make ideas more transparent let us choose the $SU(2) \times SU(2)$ group as a group of chiral transformations acting on background fields. In this case the most general expression for the square of the mass matrix is a sum

$$m^2 = K \tau_0 + M \tau_3, \quad K = \frac{1}{2}(m_u^2 + m_d^2), \quad M = \frac{1}{2}(m_u^2 - m_d^2), \quad (4)$$

with $\tau_0 = 1$, τ_i , ($i = 1, 2, 3$) being the Pauli matrices for isospin. Since $[m^2, Y] \neq 0$, we shall use the following operator identity, which is well-known in quantum mechanics, to factorize the mass matrix from the heat kernel in Eq. (3):

$$\text{tr}(e^{-T(M\tau_3+A)}) = \text{tr}\left(e^{-TM\tau_3}\left[1 + \sum_{n=1}^{\infty}(-1)^n f_n(T, A)\right]\right). \quad (5)$$

Here $f_n(T, A)$ is equal to

$$f_n(T, A) = \int_0^T ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n A(s_1)A(s_2)\cdots A(s_n), \quad (6)$$

where $A(s) = e^{sM\tau_3} A e^{-sM\tau_3}$. If one takes into account the permutation property of the trace operation in Eq. (5), the expressions for $f_n(T, A)$ can be simplified. We find in this way

$$f_1(T, A) = TA, \quad (7)$$

$$f_2(T, A) = \frac{T^2}{4}[A^2 + A\tau_3 A\tau_3 + c^{(2)}(T)(A^2 - A\tau_3 A\tau_3)], \quad (8)$$

$$f_3(T, A) = \frac{T^3}{8}\left[\frac{A}{3}\{A, \tau_3\}^2 + c_1^{(3)}(T)A\{A, \tau_3\}[\tau_3, A] + c_2^{(3)}(T)(A^3 - A\tau_3 A\tau_3 A)\right], \quad (9)$$

$$f_4(T, A) = \frac{T^4}{128}\left(\frac{1}{3}\{A, \tau_3\}^4 - c_1^{(4)}(T)[\tau_3, A]^2\{A, \tau_3\}^2 - c_2^{(4)}(T)[\tau_3, A]\{A, \tau_3\}^2[\tau_3, A] + c_3^{(4)}(T)(2[\tau_3, A]\{A, \tau_3\}[\tau_3, A]\{A, \tau_3\} + [\tau_3, A]^4)\right). \quad (10)$$

For the functions $c_j^{(i)}(T)$ we have

$$c^{(2)}(T) = \frac{1}{2T^2 M^2}(e^{2TM\tau_3} - 1 - 2TM\tau_3), \quad (11)$$

$$c_1^{(3)}(T) = \frac{\tau_3}{2T^3 M^3}[1 + TM\tau_3 + (TM\tau_3 - 1)e^{2TM\tau_3}], \quad (12)$$

$$c_2^{(3)}(T) = \frac{\tau_3}{2T^3 M^3}(e^{2TM\tau_3} - 1 - 2TM\tau_3 - 2T^2 M^2), \quad (13)$$

$$c_1^{(4)}(T) = \frac{3}{2T^4 M^4}(e^{2TM\tau_3} - 1 - 2TM\tau_3 - 2T^2 M^2 - \frac{4}{3}T^3 M^3 \tau_3), \quad (14)$$

$$c_2^{(4)}(T) = \frac{1}{2T^4 M^4}[(3 - 4TM\tau_3 + 2T^2 M^2)e^{2TM\tau_3} - 3 - 2TM\tau_3], \quad (15)$$

$$c_3^{(4)}(T) = \frac{1}{2T^4 M^4}[(2TM\tau_3 - 3)e^{2TM\tau_3} + 3 + 4TM\tau_3 + 2T^2 M^2]. \quad (16)$$

Now one can integrate over momentum in Eq. (3), which is a standard procedure [10]. Up to total derivatives, which can be omitted in the effective action, we obtain the expansion

$$W[Y] = \int \frac{d^4x}{32\pi^2} \int_0^\infty \frac{dT}{T^3} \rho(T, \Lambda^2) e^{-TK} \text{tr}(e^{-TM\tau_3}[1 - f_1(T, B) + f_2(T, B) - \tilde{f}_3(T, B) + \cdots]), \quad (17)$$

where the last term, $\tilde{f}_3(T, B)$, contains the additional contributions of order $\sim T^3$, coming from $f_4(T, A)$:

$$\tilde{f}_3(T, B) = f_3(T, B) + \frac{T^3}{48}\left(\{B, \tau_3\}\{\partial^2 B, \tau_3\} - 3c_2^{(4)}(T)[\tau_3, B][\tau_3, \partial^2 B]\right). \quad (18)$$

Let us replace in Eq. (17) the operator B by its expression in terms of Y (see Eq. (2)). The result is

$$W[Y] = \int \frac{d^4x}{32\pi^2} \int_0^\infty \frac{dT}{T^3} \rho(T, \Lambda^2) e^{-TK} \text{tr} \left(e^{-TM\tau_3} [1 - f_1(T, Y) + f_2(T, Y) - \bar{f}_3(T, Y) + \dots] \right), \quad (19)$$

where $\bar{f}_3(T, Y)$ is equal to

$$\bar{f}_3(T, Y) = f_3(T, Y) + \frac{T^3}{8} \left[\left(\frac{1}{3} + 2c_1^{(3)} - c_2^{(4)} \right) (\partial Y)^2 + \left(\frac{1}{3} - 2c_1^{(3)} + c_2^{(4)} \right) (\tau_3 \partial Y)^2 \right]. \quad (20)$$

The last step is to integrate over the proper-time T in Eq. (19). The integrals over T can be reduced to combinations of some set of elementary integrals $J_n(\mu^2)$

$$J_n(\mu^2) = \int_0^\infty \frac{dT}{T^{2-n}} e^{-T\mu^2} \rho(T, \Lambda^2), \quad (21)$$

where n is integer. Our task now is to find the algorithm which will automatically give a chiral invariant grouping for the background fields as well as the mass dependent coefficients before them. To this end, it is necessary to reorganize the asymptotic series, given by Eq. (19), in the form

$$W[Y] = \int \frac{d^4x}{32\pi^2} \sum_{i=0}^\infty I_{i-1} \text{tr}(a_i), \quad (22)$$

where $2I_i \equiv J_i(K - M) + J_i(K + M)$. The necessary resummations inside the starting expansion (19) are determined by the recursion relations

$$J_i(K - M) - J_i(K + M) = \sum_{n=1}^\infty \frac{M^n}{n!} [J_{i+n}(K + M) - (-1)^n J_{i+n}(K - M)]. \quad (23)$$

One can prove this useful identity for any integer i , calculating the first order derivative in M in both sides of this equation. Passing from Eq. (19) to Eq. (22) corresponds to choosing $J_i(K - M) + J_i(K + M)$ as the mass dependent factor in the asymptotic expansion, instead of $J_i(\mu^2)$ in the standard case of the Schwinger–DeWitt series. The difference $J_i(K - M) - J_i(K + M)$ can be always expressed through the infinite sum of I_n with $n > i$, as it follows from Eq. (23). As a result of these manipulations one can find the coefficients a_i in Eq. (22). The first five of them are equal to

$$\begin{aligned} a_0 &= 1, & a_1 &= -Y, & a_2 &= \frac{Y^2}{2} + M\tau_3 Y, & a_3 &= -\frac{Y^3}{3!} - \frac{M}{2}\tau_3 Y^2 - \frac{1}{12}(\partial Y)^2, \\ a_4 &= \frac{Y^4}{4!} + \frac{M}{6}\tau_3 Y^3 - \frac{M^2}{12}(Y^2 - Y\tau_3 Y\tau_3) - \frac{M^3}{3}\tau_3 Y + \frac{1}{12}(Y + M\tau_3)(\partial Y)^2 + \frac{1}{120}(\partial^2 Y)^2. \end{aligned} \quad (24)$$

At this stage it can be verified that if the operator $D^\dagger D$ is defined to transform in the adjoint representation $\delta(D^\dagger D) = i[\omega, D^\dagger D]$, the coefficient functions a_i are invariant under the global infinitesimal chiral transformations with parameters $\omega = \alpha + \gamma\beta$. This property extremely simplifies calculations, since it gives an alternative way to obtain a_i . Indeed, to find coefficient functions in Eq. (22) one can simply integrate the equation $\delta a_i = 0$, using the corresponding Seeley–DeWitt coefficient as a starting point and constructing the necessary counterterms in such a way as to satisfy the aforementioned equation.

The present result is in agreement with the standard Schwinger–DeWitt expansion, when $M = 0$. For the case with $M \neq 0$ our formula (22) is a new asymptotic series which can be used to construct the low-energy EFT action when the local vertices are induced by one-loop diagrams involving particles with different masses. It is a

direct extension of the DeWitt WKB expansion constructed on the basis of the Schwinger proper-time method. Our approach can be used for a wide range of interesting applications, such as, for instance, the operator-product expansion [16], or derivative expansions [17], or heat-kernel one-loop renormalizations [18]. It is not difficult to extend our method to the cases with more complicated symmetry groups. For instance, in the sequel to this Letter we have already obtained asymptotic coefficients a_i in the case of $SU(3) \times SU(3)$ chiral symmetry group [19]. The other direction for development is to include minimal coupling of the loop particles to gauge fields.

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