Non-Fickian convection–diffusion models in porous media

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Abstract In this paper we propose a numerical scheme to approximate the solution of a non-Fickian coupled model that describes, e.g., miscible transport in porous media. The model is defined by a system of a quasilinear elliptic equation, which governs the fluid pressure, and a quasilinear integro-differential equation, which models the convection–diffusion transport process. The numerical scheme is based on a conforming piecewise linear finite element method for the discretization in space. The fully discrete approximations is obtained with an implicit–explicit method. Estimates for the continuous in time and the fully discrete methods are derived, showing that the numerical approximation for the concentrations and the pressure are second order convergent in a discrete L^2-norm and in a discrete H^1-norm, respectively.

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1 Introduction

Transport processes in porous media are usually modeled by the classical convection–diffusion equation

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (vc) + \nabla \cdot J = q_1 \quad \text{in } \Omega \times (0, T], \tag{1}$$

where Ω represents the spatial domain, ϕ is the porosity of the medium, c is the concentration of the injected fluid, v its velocity, and J designates the mass flux defined by Fick's law

$$J = -D_v \nabla c. \tag{2}$$

In (2), D_v denotes the dispersion–diffusion tensor that depends on the velocity v and is given by

$$D_v = d_m \phi I + \alpha_t \|v\| I + (\alpha_\ell - \alpha_t) \frac{1}{\|v\|} v v^T, \tag{3}$$

where $\|.\|$ denotes the euclidian norm, I is the two dimensional identity matrix, d_m is the molecular diffusion coefficient, and α_ℓ and α_t are the transversal and the longitudinal dispersivities, respectively.

The parabolic equation defined by (1), (2) is usually coupled with the elliptic pressure equation

$$-\nabla \cdot \left(\frac{K}{\mu} \nabla p \right) = q_2 \quad \text{in } \Omega \times (0, T], \tag{4}$$

where the permeability tensor K and/or the viscosity μ can depend on the concentration. The velocity v in Eq. (1) depends on the pressure p through Darcy's law

$$v = -\frac{K}{\mu} \nabla p \quad \text{in } \Omega \times (0, T]. \tag{5}$$

In (1) and (4), q_1 and q_2 represent source and sink terms.

Despite the popularity of this model, gaps between experimental data and simulation results were observed in several scenarios. Without being exhaustive we mention [5–10,20,22,29]. To overcome the limitations of traditional Fickian transport models, several non-Fickian models were proposed in the literature. For instance, in [26,31,33], hyperbolic equations were introduced to replace the classical diffusion equations. Continuous time random walks models were tested, e.g., in [5–7,10]. In the present paper, we consider an integro-differential model identical to those proposed in [8, 9,23], and that have been extensively studied in the literature. We refer [3,27] for overviews on non-Fickian models for transport in porous media. It should be noted that integro-differential models have been also used to describe diffusion in viscoelastic materials ([11,12,16,18]).

In what follows the convection–diffusion equation (1), (2), with $\phi = 1$, is replaced by the following integro-differential equation

$$\frac{\partial c}{\partial t} - \nabla \cdot (D\nabla c) + \nabla \cdot (Bc) = \int_0^t K_{er}(t-s)\nabla \cdot (E\nabla c)(s)\, ds + q_1 \quad \text{in } \Omega \times (0, T], \tag{6}$$

where $K_{er}(s)$ denotes a memory kernel. In this work we use the following notation: if $w : \overline{\Omega} \times [0, T] \to \mathbb{R}$ then by $w(t)$ we denote the function $w(t) : \overline{\Omega} \to \mathbb{R}$ such that $w(t)(x) = w(x, t)$. Equation (6) is established using the mass conservation law (1) with the mass flux J given by

$$J = J_F + J_{nF} + J_{ad}, \tag{7}$$

where J_{ad} stands for the advective mass flux, while J_F and J_{nF} represent the Fickian and non-Fickian dispersive mass fluxes, respectively. In (7), J_F is defined by (2), $J_{ad} = Bc$, and J_{nF} is the nonlocal in time operator

$$J_{nF}(t) = \int_0^t K_{er}(t-s)(E\nabla c)(s)\, ds,$$

where E depends on the velocity v and eventually on the concentration c. Equation (6) for the concentration is coupled with the elliptic equation

$$-\nabla \cdot (A\nabla p) = q_2 \quad \text{in } \Omega \times (0, T], \tag{8}$$

where A is a diagonal tensor. We observe that (8) coincides with (4) for the particular choice $A = \frac{K}{\mu}$.

As already mentioned, the classical convection–diffusion equation is not able to capture the behavior of transport processes in porous media. For illustration, we reproduce in Fig. 1 some of the results presented in [20]. In that figure, the results of two laboratory tracer experiments described in [29] (left image) and [5] (right image) are compared with the best-fit curves obtained with the classical convection–diffusion equation (1), (2), and the integro-differential equation (6) with $K_{er}(s) = \frac{1}{\tau}e^{-\frac{s}{\tau}}$. The measured concentration values are represented by dots, and we observe that the integro-differential model (green line), unlike the classical model (blue line), accurately describes the experimental data in particular the late-time tails. More details about Fig. 1 can be found in [20].

The development of efficient and accurate numerical methods to solve the integro-differential equation (6) has attracted the attention of researchers during the last decades. A significative number of contributions can be found in the literature. Without being exhaustive we mention [24,25,32,35], for the study of semi-discrete finite element approximations [28], for the analysis of semi-discrete lumped mass approximations, [13,14,30], for the construction of semi-discrete finite volume approximations, and [1,2,4,21], for finite difference methods presenting the same

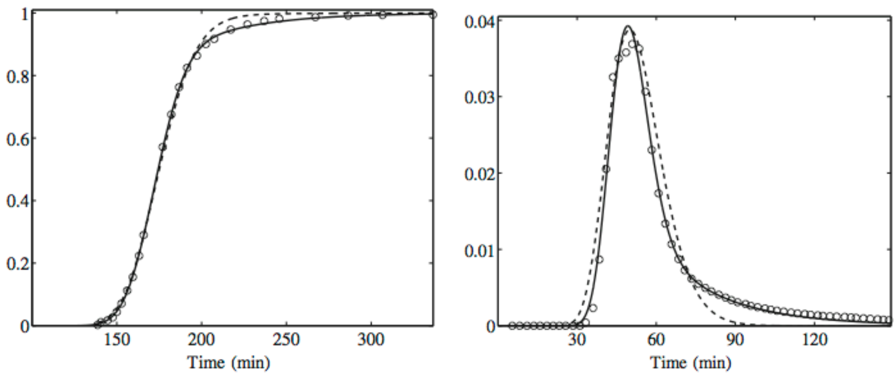

Fig. 1 Time evolution of the concentration at a specific point of the domain, as given by (6) (solid line) and by (1), (2) (dash line). The experimental data are represented by dots

qualitative behavior as the continuous integro-differential initial boundary value problems. We note that the finite difference methods studied in this last group of papers can be seen as piecewise linear finite element methods with convenient quadrature rules. To the best of our knowledge the numerical discretization of the non-Fickian coupled problem (6), (8) was not yet analysed. In this paper we introduce finite difference methods for the approximation of the pressure and the concentration whose errors are second order convergent in discrete H^1 and L^2 norms, respectively. From these results we conclude that the numerical velocity is also a second order approximation. In this way, we extend to non-Fickian coupled problems, the results presented in [15,19] for piecewise linear finite element methods. We point out that these results are somehow unexpected in the context of finite difference methods as well as finite element methods. In fact, the truncation errors induced by the spatial discretizations that we consider are only of first order when non-uniform grids are used and it is also well known that piecewise linear finite element methods are first order convergent with respect to the H^1-norm. Moreover we note that the analysis in this paper differs from the one used in [15,19], which is based on the definition of a convenient auxiliary problem and was introduced by Wheeler in [34]. Here, we apply the approach of [21].

The remaining of the paper is organized as follows. Section 2 is devoted to the construction of the semi-discrete approximation for the solution of the coupled system (6), (8). In this section we also introduce the variational formulation and the finite difference scheme. The convergence analysis of the semi-discrete approximation for the pressure and the concentration is presented in Sect. 3. The main result of this section is Theorem 1 which establishes the second order convergence rate of the numerical scheme for the pressure and the concentration with respect to discrete versions of the H^1-norm and L^2-norm, respectively. An implicit–explicit (IMEX) method to compute the fully discrete approximations (in time and space) for the pressure and concentration is studied in Sect. 4. In Sect. 5, some numerical experiments are included and in Sect. 6 we present some conclusions.

2 Space discretization

2.1 Notation and definitions

Let $\Omega = (0,1) \times (0,1)$. We consider the coupled system (6), (8) with Dirichlet boundary conditions

$$p = p_b \quad \text{on } \partial\Omega \times (0,T], c = 0 \quad \text{on } \partial\Omega \times (0,T], \tag{9}$$

and known initial concentration

$$c(0) = c_0 \quad \text{in } \Omega. \tag{10}$$

In (6), (8) the coefficient functions A, D, and E are second order diagonal square matrices with entries a_i, d_i, and e_i, $i = 1, 2$, respectively, where a_i depends on c while d_1, e_1 and d_2, e_2 depend on $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$, respectively, and also on c and eventually on the time and space variables. The two dimensional vector B has entries b_1 and b_2 which depend on $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$, respectively, and both depend also on c.

Remark 1 The dependence of A on c comes directly from the pressure equation (4). On the other hand, the dependence of D, B, and E on $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}$, and c comes naturally from Eqs. (1)–(3) and Darcy's law (5). Note also that, for simplicity, K, D, and E were assumed diagonal. However, the results given in this paper can be extended to the 2×2 matrix case provided that these terms are treated properly. See some details for such approximations in [17,21].

By $L^2(\Omega)$, $L^2(\partial\Omega)$, $H^1(\Omega)$, and $H_0^1(\Omega)$ we denote the usual Hilbert spaces. In $L^2(\Omega)$ we consider the usual inner product $(.,.)$ and the induced norm represented by $\|\cdot\|$. By $H^{1/2}(\partial\Omega)$ we represent the usual Sobolev space. By $[V]^2$ we represent the usual cartesian product of the space V.

Assuming that $q_1(t), q_2(t), c_0 \in L^2(\Omega)$, and $p_b \in H^{1/2}(\partial\Omega)$, the weak solution of the system (6)–(10) can be obtained by solving the following variational problem: find $p(t) \in H^1(\Omega)$ and $c(t) \in H_0^1(\Omega)$, with $p(t) = p_b(t)$ on $\partial\Omega$ and $\frac{dc}{dt}(t) \in L^2(\Omega)$, such that

$$(A(c(t))\nabla p(t), \nabla u) = (q_2(t), u), \quad \forall u \in H_0^1(\Omega), \tag{11}$$

$$\left(\frac{dc}{dt}(t), w\right) + (D(c(t), \nabla p(t))\nabla c(t), \nabla w) - (B(c(t), \nabla p(t))c(t), \nabla w)$$
$$+ \int_0^t K_{er}(t-s)(E(c(s), \nabla p(s))\nabla c(s), \nabla w)ds = (q_1(t), w), \quad \forall w \in H_0^1(\Omega), \tag{12}$$

for $t \in (0, T]$, and where (10) holds in L^2-sense. In the above formulation the kernel function can be defined by $K_{er}(s) = \dfrac{1}{\tau}e^{-\frac{s}{\tau}}$, but it is not limited to that case. We remark that the inner product in $[L^2(\Omega)]^2$ is also denoted by $(.,.)$.

In what follows we derive the semi-discrete approximation for the pressure and concentration defined by the coupled variational problem (11), (12). We start by introducing some basic definitions and notations.

In $\overline{\Omega}$ we introduce a non-uniform rectangular grid which is the cartesian product of two 1D non-uniform grids $\{x_i, i = 0, \dots, N_x\}$, $\{y_j, j = 0, \dots, N_y\}$. Let $h = (h_1, \dots, h_{N_x})$ and $k = (k_1, \dots, k_{N_y})$ be vectors of positive entries such that $\sum_{i=1}^{N_x} h_i = \sum_{j=1}^{N_y} k_j = 1$. Let $x_i = x_{i-1} + h_i$, $i = 1, \dots, N_x$, with $x_0 = 0$ and let $y_j = y_{j-1} + k_j$, $j = 1, \dots, N_y$, with $y_0 = 0$. In $\overline{\Omega}$ we define the grid

$$\overline{\Omega}_H = \left\{(x_i, y_j),\ i = 0, \dots, N_x;\ j = 0, \dots, N_y\right\}.$$

We also introduce the set of grid points $\Omega_H = \overline{\Omega}_H \cap \Omega$, $\partial\Omega_H = \overline{\Omega}_H \cap \partial\Omega$.

We consider a sequence of grids Ω_H such that the maximal mesh-size $H_{max} = \max\{h_i, k_j,\ i = 1, \dots, N_x;\ j = 1, \dots, N_y\}$ tends to zero. We use the symbol "Λ" for the sequence of mesh-size vectors and write "$H \in \Lambda$" for the convergence when $H_{max} \to 0$ and with respect to H running through this sequence. By W_H we represent the space of grid functions defined in $\overline{\Omega}_H$ and by $W_{H,0}$ the subspace of W_H of grid functions vanishing on $\partial\Omega_H$. By R_H we denote the operator of pointwise restriction to the grid $\overline{\Omega}_H$. Let $\mathcal{T}_H$ be a triangulation of $\overline{\Omega}$ using the set $\overline{\Omega}_H$ as vertices. We denote by diam Δ the diameter of the triangle $\Delta \in \mathcal{T}_H$. By $P_H v_H$ we denote the continuous piecewise linear interpolant of v_H with respect to $\mathcal{T}_H$.

In $W_{H,0}$ we introduce the inner product

$$(v_H, w_H)_H = \sum_{(x_i, y_j) \in \overline{\Omega}_H} |\square_{i,j}| v_H(x_i, y_j) w_H(x_i, y_j), \quad w_H, v_H \in W_{H,0},$$

where $|\square_{i,j}|$ denotes the area of $\square_{i,j}$ with $\square_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}) \cap \Omega$ for $x_{i+1/2} = x_i + \dfrac{h_{i+1}}{2}$, $x_{i-1/2} = x_i - \dfrac{h_i}{2}$, and $y_{j\pm 1/2}$ is defined analogously. By $\|\cdot\|_H$ we denote the norm induced by this inner product.

For $v_H = (v_{1,H}, v_{2,H})$, $w_H = (w_{1,H}, w_{2,H}) \in [W_H]^2$, we use the notation

$$(v_H, w_H)_{H,+} = (v_{1,H}, w_{1,H})_{H,x} + (v_{2,H}, w_{2,H})_{H,y},$$

where

$$(v_H, w_H)_{H,x} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y - 1} h_i k_{j+1/2} v_H(x_i, y_j) w_H(x_i, y_j), \quad v_H, w_H \in W_H,$$

$$(v_H, w_H)_{H,y} = \sum_{i=1}^{N_x - 1} \sum_{j=1}^{N_y} h_{i+1/2} k_j v_H(x_i, y_j) w_H(x_i, y_j), \quad v_H, w_H \in W_H.$$

We define $\|w_H\|_{H,x} = \sqrt{(w_H, w_H)_{H,x}}$ and $\|w_H\|_{H,y} = \sqrt{(w_H, w_H)_{H,y}}$.

Let D_{-x} and D_{-y} be the usual backward finite difference operators with respect to the variables x and y, respectively,

$$D_{-x}w_H(x_i, y_j) = \frac{w_H(x_i, y_j) - w_H(x_{i-1}, y_j)}{h_i}$$

and

$$D_{-y}w_H(x_i, y_j) = \frac{w_H(x_i, y_j) - w_H(x_i, y_{j-1})}{k_j},$$

and let ∇_H be the discrete version of the gradient operator ∇ defined by $\nabla_H w_H = (D_{-x}w_H, D_{-y}w_H)$. We introduce the following discrete version of the H^1-norm

$$\|w_H\|_{1,H} = \left(\|w_H\|_H^2 + \|\nabla_H w_H\|_{H,+}^2\right)^{1/2},$$

where

$$\|\nabla_H w_H\|_{H,+}^2 = \|D_{-x}w_H\|_{H,x}^2 + \|D_{-y}w_H\|_{H,y}^2.$$

With these definitions holds the following discrete Poincaré–Friedrichs inequality

$$\|w_H\|_H^2 \leq \frac{1}{2}\|\nabla_H w_H\|_{H,+}^2, \quad \forall w_H \in W_{H,0}.$$

In order to define the discrete approximations in space $c_H(t)$ and $p_H(t)$ we introduce the following notation:

$$\begin{aligned}
M_h(w_H)(x_i, y_j) &= \frac{w_H(x_i, y_j) + w_H(x_{i-1}, y_j)}{2},\\
M_k(w_H)(x_i, y_j) &= \frac{w_H(x_i, y_j) + w_H(x_i, y_{j-1})}{2},\\
(M_H w_H, \nabla_H v_H)_{H,+} &= (M_h(w_H), D_{-x}w_H)_{x,+} + (M_k(w_H), D_{-y}w_H)_{y,+},
\end{aligned}$$

and

$$D_h w_H(x_i, y_j) = \frac{h_i D_{-x}w_H(x_{i+1}, y_j) + h_{i+1}D_{-x}w_H(x_i, y_j)}{h_i + h_{i+1}},$$

being the finite difference operator D_k defined analogously with respect to the variable y. To approximate the coefficient functions, we introduce the diagonal matrices $A_H(t)$, $D_H(t)$ and $E_H(t)$ whose entries $a_{\ell,H}(t)$, $d_{\ell,H}(t)$ and $e_{\ell,H}(t)$, $\ell = 1, 2$, respectively, depend on the numerical concentration $c_H(t)$ and pressure $p_H(t)$, that are given by

$$a_{1,H}(t) = a_1(M_h(c_H(t))), \quad d_{1,H}(t) = d_1(M_h(c_H(t)), D_{-x}p_H(t)),$$

and

$$e_{1,H}(t) = e_1(M_h(c_H(t)), D_{-x}p_H(t)).$$

The vector $B_H(t)$ depends on $c_H(t)$ and $p_H(t)$

$$b_{1,H}(t) = b_1(c_H(t), D_h p_H(t)).$$

The entries $a_{2,H}(t)$, $d_{2,H}(t)$, $e_{2,H}(t)$ and $b_{2,H}(t)$ are defined analogously.

2.2 Numerical scheme

We now define the semi-discrete approximations $c_H(t)$ and $p_H(t)$ for the solution of (11), (12): find $p_H(t) \in W_H$ and $c_H(t) \in W_{H,0}$, with $p_H(t) = R_H p_b(t)$ on $\partial\Omega_H$ and $\dfrac{dc_H}{dt}(t) \in W_{H,0}$, such that

$$(A_H(t)\nabla_H p_H(t), \nabla_H u_H)_{H,+} = (q_{2,H}(t), u_H)_H, \tag{13}$$

for all $u_H \in W_{H,0}$,

$$\left(\frac{dc_H}{dt}(t), w_H\right)_H + a_H(c_H(t), w_H) + \int_0^t b_H(s, t, c_H(s), w_H)\, ds = (q_{1,H}(t), w_H)_H, \tag{14}$$

for all $w_H \in W_{H,0}$, and

$$c_H(0) = R_H c_0 \quad \text{in } \Omega_H, \tag{15}$$

for $t \in (0, T]$.

In (14), $a_H(c_H(t), w_H)$ and $b_H(s, t, c_H(s), w_H)$ are given by

$$\begin{aligned} a_H(c_H(t), w_H) &= (D_H(t)\nabla_H c_H(t), \nabla_H w_H)_{H,+} \\ &\quad - (M_H(B_H(t)c_H(t)), \nabla_H w_H)_{H,+}, \\ b_H(s, t, c_H(s), w_H) &= K_{er}(t-s)(E_H(s)\nabla_H c_H(s), \nabla_H w_H)_{H,+} \end{aligned}$$

and in (13), (14)

$$q_{\ell,H}(x_i, y_j, t) = \frac{1}{|\Box_{i,j}|}\int_{\Box_{i,j}} q_\ell(x, y, t)dxdy,\ (x_i, y_j) \in \Omega_H, \ell = 1, 2. \tag{16}$$

We observe that (13), (14) can also be obtained from the finite element coupled variational equations

$$(A(P_H c_H(t))\nabla P_H p_H(t), \nabla P_H u_H) = (q_2(t), P_H u_H), \quad \forall u_H \in W_{H,0}, \tag{17}$$

$$\left(\frac{d}{dt}P_Hc_H(t), P_Hw_H\right) + (D(P_Hc_H(t), \nabla P_Hp_H(t))\nabla P_Hc_H(t), \nabla P_Hw_H)$$
$$- (B(P_Hc_H(t), \nabla P_Hp_H(t))P_Hc_H(t), \nabla P_Hw_H)$$
$$+ \int_0^t K_{er}(t-s)(E(P_Hc_H(s), \nabla P_Hp_H(s))\nabla P_Hc_H(s), \nabla P_Hw_H)\, ds$$
$$= (q_1(t), P_Hw_H), \tag{18}$$

for all $w_H \in W_{H,0}$, using suitable quadrature rules (see [17]).

Remark 2 The discrete in space coupled variational problem (13), (14) is equivalent to the following finite difference method

$$-\nabla_H^* \cdot (A_H(t)\nabla_H p_H(t)) = q_{2,H}(t), \tag{19}$$

$$\frac{dc_H}{dt}(t) - \nabla_H^*.(D_H(t)\nabla_H c_H(t)) + \nabla_{c,H}^* \cdot (B_H(t)c_H(t))$$
$$= \int_0^t K_{er}(t-s)\nabla_H^* \cdot (E_H(s)\nabla_H c_H(s))ds + q_{1,H}(t), \tag{20}$$

complemented by the initial conditions (15) and the boundary conditions $c_H(t) = 0$ and $p_H(t) = R_H p_b(t)$ on $\partial\Omega$. Here, $\nabla_H^* w_H = (D_x w_H, D_y w_H)$ and $\nabla_{c,H}^* w_H = (D_{c,x} w_H, D_{c,y} w_H)$, where

$$D_x w_H(x_i, y_j) = \frac{w_H(x_{i+1}, y_j) - w_H(x_i, y_j)}{h_{i+1/2}},$$
$$D_{c,x} w_H(x_i, y_j) = \frac{w_H(x_{i+1}, y_j) - w_H(x_{i-1}, y_j)}{h_i + h_{i+1}},$$

and the corresponding operators in y-dimension, D_y and $D_{c,y}$, are defined analogously.

In the following section we show that the solutions $p_H(t)$ and $c_H(t)$ of the finite difference problem (19), (20), or equivalently, the fully discrete in space piecewise linear finite element solutions of the variational problem (17), (18), are second order convergent approximations for the pressure $p(t)$ and concentration $c(t)$.

3 Convergence analysis

This section is dedicated to derive error estimates for the numerical solutions of our finite difference scheme, namely,

$$e_{H,p}(t) = R_H p(t) - p_H(t) \quad \text{and} \quad e_{H,c}(t) = R_H c(t) - c_H(t).$$

We observe that these two error functions are solutions of the following initial boundary value problem

$$(A_H(t)e_{H,p}(t), \nabla_H u_H)_{H,+} = \tau_1(u_H), \tag{21}$$

$$\begin{aligned}&\left(\frac{de_{H,c}}{dt}(t), w_H\right)_H + (D_H(t)\nabla_H e_{H,c}(t), \nabla_H w_H)_{H,+}\\ &\quad - (M_H(B_H(t)e_{H,c}(t)), \nabla_H w_H)_{H,+}\\ &= -\int_0^t K_{er}(t-s)(E_H(s)\nabla_H e_{H,c}(s), \nabla_H w_H)_{H,+}ds + \tau_2(w_H),\end{aligned} \tag{22}$$

for all $u_H, w_H \in W_{H,0}$, and

$$\begin{aligned}&e_{H,p}(t) = e_{H,c}(t) = 0 \quad \text{on } \partial\Omega_H\\ &e_{H,c}(0) = 0 \quad \text{in } \Omega_H.\end{aligned}$$

In (21) and (22), the error terms $\tau_1(u_H)$ and $\tau_2(w_H)$ are defined by

$$\tau_1(u_H) = (A_H(t)\nabla_H R_H p(t), \nabla_H u_H)_H - (q_{2,H}(t), u_H)_H$$

and

$$\begin{aligned}\tau_2(w_H) &= \left(R_H\frac{dc}{dt}(t), w_H\right)_H + (D_H(t)\nabla_H R_H c(t), \nabla_H w_H)_{H,+}\\ &\quad - (M_H(B_H(t)R_H c(t)), \nabla_H w_H)_{H,+}\\ &\quad + \int_0^t K_{er}(t-s)(E_H(s)\nabla_H R_H c(s), \nabla_H w_H)_{H,+}ds - (q_{1,H}(t), w_H)_H.\end{aligned} \tag{23}$$

A possible approach could be to follow the procedure introduced by Wheeler in [34] and used, e.g., in [19]. However, this technique requires that the sequence of spatial grids is quasi-uniform in the sense that

$$\frac{H_{max}}{H_{min}} \le C, \text{ for } H \in \Lambda.$$

Here we propose a type of analysis that avoids the above smoothness assumption on the spatial grids. In addition, our approach is less restrictive regarding the regularity of the solutions p and c.

Nevertheless, the convergence analysis that we present still requires some regularity conditions on p and c as well as on the coefficient functions of the model. For the coefficient functions we assume the following:

$$\begin{aligned}&a_\ell \in C_B^1(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}), \quad d_\ell,\, b_\ell,\, e_\ell \in C_B^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2),\\ &0 < A_{min} \le a_\ell \quad \text{in } \mathbb{R}, \quad \text{and} \quad 0 < D_{min} \le d_\ell \quad \text{in } \mathbb{R}^2,\ \ell = 1, 2.\end{aligned}$$

Here, $C_B^1(\mathbb{R})$ and $C_B^1(\mathbb{R}^2)$ represent the space of functions defined in $\mathbb{R}$ and $\mathbb{R}^2$, respectively, with bounded first order partial derivatives. By $W^{2,\infty}(\mathbb{R})$ and $W^{2,\infty}(\mathbb{R}^2)$ we denote the usual Sobolev spaces. To simplify the proofs of the convergence results, we start by introducing some notation. Let $\tilde{A}_H$ and A_H^* be defined as A, replacing a_1, a_2 by $\tilde{a}_1$, $\tilde{a}_2$ and a_1^*, a_2^*, respectively, with

$$\begin{aligned}
\tilde{a}_1(x_i, y_j, t) &= a_1(c(x_{i-1/2}, y_j, t)),\\
\tilde{a}_2(x_i, y_j, t) &= a_2(c(x_i, y_{j-1/2}, t)),\\
a_1^*(x_i, y_j, t) &= a_1\left(\frac{1}{2}(c(x_{i-1}, y_j, t) + c(x_i, y_j, t))\right),\\
a_2^*(x_i, y_j, t) &= a_2\left(\frac{1}{2}(c(x_i, y_{j-1}, t) + c(x_i, y_j, t))\right).
\end{aligned}$$

$\tilde{D}_H$ and D_H^* are defined in a corresponding way.

In the following, $\|.\|_{C^q}$ denotes the usual norm in $C^q(\overline{\Omega})$, $q \in \mathbb{N}_0$. We start by estimating the error term $e_{H,p}(t)$.

Proposition 1 *If $p(t) \in H^3(\Omega)$, $c(t) \in H^2(\Omega)$, then there exists a positive constant C such that*

$$\|\nabla_H e_{H,p}(t)\|_{H,+} \le C\Big(\|p(t)\|_{C^1}\|e_{H,c}(t)\|_H + \tau_p(t)\Big), \tag{24}$$

where

$$\begin{aligned}
\tau_p(t) = \Big(&\sum_{\Delta\in\mathcal{T}_H} (\text{diam}\,\Delta)^4\big(\|c(t)\|^2_{H^2(\Delta)} + \|c(t)\|^4_{H^1(\Delta)}\big)\Big)^{1/2}\\
&+ \big(\|p(t)\|_{C^1} + 1\big)\Big(\sum_{\Delta\in\mathcal{T}_H} (\text{diam}\,\Delta)^4\|p(t)\|^2_{H^3(\Delta)}\Big)^{1/2}\Big).
\end{aligned} \tag{25}$$

Proof As we have successively

$$\begin{aligned}
&(A_H(t)\nabla_H e_{H,p}(t), \nabla_H e_{H,p}(t))_{H,+}\\
&\quad= (A_H(t)\nabla_H R_H p(t), \nabla_H e_{H,p}(t))_{H,+} - (q_{2,H}(t), e_{H,p}(t))_H\\
&\quad= (A_H(t)\nabla_H R_H p(t), \nabla_H e_{H,p}(t))_{H,+} + \Big(\Big(\nabla\cdot\Big(A(t)\nabla p(t)\Big)\Big)_H, e_{H,p}(t)\Big)_H\\
&\quad= (A_H(t)\nabla_H R_H p(t), \nabla_H e_{H,p}(t))_{H,+}\\
&\qquad- \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i \int_{y_{j-1/2}}^{y_{j+1/2}} a_1(x_{i-1/2}, y, t)\frac{\partial p}{\partial x}(x_{i-1/2}, y, t)dy D_{-x}e_{H,p}(x_i, y_j, t)\\
&\qquad- \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} k_j \int_{x_{i-1/2}}^{x_{i+1/2}} a_2(x, y_{j-1/2}, t)\frac{\partial p}{\partial y}(x, y_{j-1/2}, t)dx D_{-y}e_{H,p}(x_i, y_j, t),
\end{aligned}$$

where $\left(\nabla\cdot\left(A(t)\nabla p(t)\right)\right)_H$ is defined by (16) with q_ℓ replaced by $\nabla\cdot\left(A(t)\nabla p(t)\right)$, we conclude the following error equation

$$(A_H(t)\nabla_H e_{H,p}(t), \nabla_H e_{H,p}(t))_{H,+} = \tau_1(e_{H,p}(t)),$$

where $\tau_1(e_{H,p}(t))$ admits now the representation

$$\tau_1(e_{H,p}(t)) = \sum_{i=1}^{3}\tau_A^{(i)}(t)$$

with

$$\tau_A^{(1)}(t) = ((A_H(t) - A_H^*(t))\nabla_H R_H p, \nabla_H e_{H,p}(t))_{H,+},$$
$$\tau_A^{(2)}(t) = ((A_H^*(t) - \tilde{A}_H(t))\nabla_H R_H p, \nabla_H e_{H,p}(t))_{H,+},$$

and

$$\begin{aligned}\tau_A^{(3)}(t) = {} & (\tilde{A}_H(t)\nabla_H R_H p, \nabla_H e_{H,p}(t))_{H,+}\\ & - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i\int_{y_{j-1/2}}^{y_{j+1/2}} a_1(x_{i-1/2}, y, t)\frac{\partial p}{\partial x}(x_{i-1/2}, y, t)dy D_{-x}e_{H,p}(x_i, y_j, t)\\ & - \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} k_j\int_{x_{i-1/2}}^{x_{i+1/2}} a_2(x, y_{j-1/2}, t)\frac{\partial p}{\partial y}(x, y_{j-1/2}, t)dx D_{-y}e_{H,p}(x_i, y_j, t),\end{aligned}$$

with $a_1(x_{i-1/2}, y, t) = a_1(c(x_{i-1/2}, y, t))$, $a_2(x, y_{j-1/2}, t) = a_2(c(x, y_{j-1/2}))$ to shorten notation. Since $p(t) \in H^3(\Omega)$ and $H^3(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$, holds $\|\nabla_H R_H p(t)\|_{H,+} \le \|\nabla p(t)\|_{C^0}$, and then

$$|\tau_A^{(1)}(t)| \le C\|p(t)\|_{C^1}\|e_{H,c}(t)\|_H\|\nabla_H e_{H,p}(t)\|_{H,+}.$$

To obtain an estimate for $\tau_A^{(2)}(t)$ we observe that, for $g(c(x_i, y_j, t)) = M_h c(x_i, y_j, t) - c(x_{i-1/2}, y_j, t)$, one gets

$$\begin{aligned}k_{j+1/2}|g(c(x_i, y_j, t))| \le {} & k_{j+1/2}\int_{y_{j-1/2}}^{y_{j+1/2}} |g\left(\frac{\partial c}{\partial y}(x_i, y, t)\right)|\,dy\\ & + \int_{y_{j-1/2}}^{y_{j+1/2}} |g(c(x_i, y, t))|\,dy\\ \le {} & k_{j+1/2}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1}}^{x_i} |\frac{\partial^2 c}{\partial x\partial y, t}|\,dxdy\\ & + h_i\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1}}^{x_i} |\frac{\partial^2 c}{\partial x^2}|\,dxdy,\end{aligned}$$

where the last inequality was derived using the Bramble–Hilbert Lemma. We note that

$$h_{i+1/2}M_k c(x_i, y_j, t) - c(x_i, y_{j-1/2}, t)$$

can be estimated in a similar way,

$$|\tau_A^{(2)}(t)| \leq C\|p(t)\|_{C^1}\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\|c(t)\|^2_{H^2(\Delta)}\Big)^{1/2}\|\nabla_H e_{H,p}(t)\|_{H,+}.$$

To conclude the proof we observe that Lemma 5.1 of [17] can be applied to establish the estimate,

$$\begin{aligned}|\tau_A^{(3)}(t)| \leq C\Big(\sum_{\Delta\in\mathcal{T}_H}&(\text{diam }\Delta)^4\big(\|p(t)\|^2_{C^1}\big(\|c(t)\|^4_{H^1(\Delta)} + \|c(t)\|^2_{H^2(\Delta)}\big)\\ &+ \|c(t)\|^2_{H^1(\Delta)}\|p(t)\|^2_{H^2(\Delta)} + \|p(t)\|^2_{H^3(\Delta)}\big)\Big)^{1/2}\|\nabla_H e_{H,p}(t)\|_{H,+}.\end{aligned}$$

□

In what follows we establish an upper bound for $\tau_2(w_H)$ defined by (23). We start by remarking that this error can be rewritten in the following equivalent form

$$\tau_2(w_H) = \tau_D(w_H) + \tau_B(w_H) + \tau_E(w_H) + \tau_d(w_H),$$

with

$$\begin{aligned}\tau_D(w_H) &= (D_H(t)\nabla_H R_H c(t), \nabla_H w_H)_{H,+} + \Big(\Big(\nabla.\Big(D(t)\nabla c(t)\Big)\Big)_H, w_H\Big)_H\\ &= (D_H(t)\nabla_H R_H c(t), \nabla_H w_H)_{H,+}\\ &\quad - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i\int_{y_{j-1/2}}^{y_{j+1/2}} d_1(x_{i-1/2}, y, t)\frac{\partial c}{\partial x}(x_{i-1/2}, y, t)dy D_{-x}w_H(x_i, y_j)\\ &\quad - \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} k_j\int_{x_{i-1/2}}^{x_{i+1/2}} d_2(x, y_{j-1/2}, t)\frac{\partial c}{\partial y}(x, y_{j-1/2}, t)dx D_{-y}w_H(x_i, y_j),\end{aligned}$$

where $\Big(\Big(\nabla\cdot\Big(D(t)\nabla c(t)\Big)\Big)_H, w_H\Big)_H$ is defined by (16) with q_ℓ replaced by $\nabla\cdot(D(t)\nabla c(t))$ and, to simplify, the following notations were used

$$\begin{aligned}d_1(x_{i-1/2}, y, t) &= d_1(c(x_{i-1/2}, y, t), \frac{\partial p}{\partial x}(x_{i-1/2}, y, t)),\\ d_2(x, y_{j-1/2}, t) &= d_2(c(x, y_{j-1/2}, t), \frac{\partial p}{\partial y}(x, y_{j-1/2}, t)),\end{aligned} \tag{26}$$

$$
\begin{aligned}
\tau_E(w_H) &= \int_0^t K_{er}(t-s)(E_H(s)\nabla_H R_H c(s), \nabla_H w_H)_{H,+} ds \\
&\quad + \int_0^t K_{er}(t-s)\Big(\Big(\nabla\cdot\Big(E(t)\nabla c(s)\Big)\Big)_H, w_H\Big)_H ds \\
&= \int_0^t K_{er}(t-s)(E_H(s)\nabla_H R_H c(s), \nabla_H w_H)_{H,+} ds \\
&\quad - \int_0^t K_{er}(t-s)\sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i \int_{y_{j-1/2}}^{y_{j+1/2}} e_1(x_{i-1/2}, y, s) \\
&\quad \times \frac{\partial c}{\partial x}(x_{i-1/2}, y, s)\, dy ds\, D_{-x} w_H(x_i, y_j) \\
&\quad - \int_0^t K_{er}(t-s)\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} k_j \int_{x_{i-1/2}}^{x_{i+1/2}} e_2(x, y_{j-1/2}, s) \\
&\quad \times \frac{\partial c}{\partial y}(x, y_{j-1/2}, s)\, dx ds\, D_{-y} w_H(x_i, y_j),
\end{aligned}
$$

where $e_\ell,\ \ell = 1, 2$, are defined by (26) with d_ℓ replaced by e_ℓ,

$$
\begin{aligned}
\tau_B(w_H) &= -(M_H(B_H(t)R_H c(t)), \nabla_H w_H)_{H,+} - \Big(\Big(\nabla\cdot\Big(B(t)c(t)\Big)\Big)_H, w_H\Big)_H \\
&= -(M_H(B_H(t)R_H c(t)), \nabla_H w_H)_{H,+} \\
&\quad + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i \int_{y_{j-1/2}}^{y_{j+1/2}} b_1(x_{i-1/2}, y, t) c(x_{i-1/2}, y, t) dy D_{-x} w_H(x_i, y_j) \\
&\quad + \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} k_j \int_{x_{i-1/2}}^{x_{i+1/2}} b_2(x, y_{j-1/2}, t) c(x, y_{j-1/2}, t) dx D_{-y} w_H(x_i, y_j)
\end{aligned}
$$

with

$$
\begin{aligned}
b_1(x_{i-1/2}, y, t) &= b_1\Big(c(x_{i-1/2}, y, t), \frac{\partial p}{\partial x}(x_{i-1/2}, y, t)\Big), \\
b_2(x, y_{j-1/2}, t) &= b_2\Big(c(x, y_{j-1/2}, t), \frac{\partial p}{\partial y}(x, y_{j-1/2}, t)\Big),
\end{aligned} \tag{27}
$$

and $\Big(\nabla\cdot\Big(B(t)c(t)\Big)\Big)_H$ is defined by (16) with q_ℓ replaced by $\nabla\cdot\Big(B(t)c(t)\Big)$,

$$
\tau_d(t) = \Big(R_H \frac{dc}{dt}(t), w_H\Big)_H - \Big(\Big(\frac{dc}{dt}(t)\Big)_H, w_H\Big)_H,
$$

where $\Big(\frac{dc}{dt}(t)\Big)_H$ is defined by (16) with q_ℓ replaced by $\frac{dc}{dt}(t)$.

In the following we estimate each error term individually.

Proposition 2 *If $p(t),\ c(t) \in H^3(\Omega)$ and $\dfrac{\partial^2 p}{\partial x \partial y}(t) = \dfrac{\partial^2 p}{\partial y \partial x}(t)$ in Ω, then there exists a positive constant C such that, for H_{max} small enough and $w_H \in W_{H,0}$, we have*

$$
\begin{aligned}
|\tau_D(w_H)| \le C\Big(& \|c(t)\|_{C^1}\big(1 + \|p(t)\|_{C^1}\big)\Big(\|e_{H,c}(t)\|_H \\
&+ \Big(\sum_{\Delta \in \mathcal{T}_H} (\operatorname{diam} \Delta)^4 \|p(t)\|^2_{H^3(\Delta)}\Big)^{1/2}\Big) \\
&+ (1 + \|c(t)\|_{C^1})^3 \Big(\sum_{\Delta \in \mathcal{T}_H} (\operatorname{diam} \Delta)^4 \Big(\|c(t)\|^2_{H^3(\Delta)} + \|c(t)\|^4_{H^2(\Delta)} \\
&+ \|p(t)\|^2_{H^3(\Delta)} + \|p(t)\|^4_{H^2(\Delta)}\Big)\Big)^{1/2}\Big) \|\nabla_H w_H\|_{H,+}.
\end{aligned}
$$

Proof For $\tau_D(w_H)$ holds the decomposition

$$
\tau_D(w_H) = \sum_{i=1}^{3} \tau_D^{(i)}(w_H),
$$

with $\tau_D^{(1)}(w_H)$, $\tau_D^{(2)}(w_H)$, and $\tau_D^{(3)}(w_H)$ defined by

$$
\begin{aligned}
\tau_D^{(1)}(w_H) &= (\big(D_H(t) - D_H^*(t)\big)\nabla_H R_H c(t), \nabla_H w_H)_{H,+}, \\
\tau_D^{(2)}(w_H) &= (\big(D_H^*(t) - \tilde{D}_H(t)\big)\nabla_H R_H c(t), \nabla_H w_H)_{H,+},
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_D^{(3)}(w_H) = (&\tilde{D}_H(t)\nabla_H R_H c, \nabla_H w_H)_{H,+} \\
&- \sum_{i=1}^{N_x} \sum_{j=1}^{N_y - 1} h_i \int_{y_{j-1/2}}^{y_{j+1/2}} d_1(x_{i-1/2}, y, t) \frac{\partial c}{\partial x}(x_{i-1/2}, y) dy D_{-x} w_H(x_i, y_j) \\
&- \sum_{i=1}^{N_x - 1} \sum_{j=1}^{N_y} k_j \int_{x_{i-1/2}}^{x_{i+1/2}} d_2(x, y_{j-1/2}, t) \frac{\partial c}{\partial y}(x, y_{j-1/2}, t) dx D_{-y} w_H(x_i, y_j).
\end{aligned}
$$

For $\tau_D^{(1)}(w_H)$ we can easily establish the estimate

$$
|\tau_D^{(1)}(w_H)| \le C\|c(t)\|_{C^1}\Big(\|e_{H,c}(t)\|_H + \|e_{H,p}(t)\|_{1,H}\Big)\|\nabla_H w_H\|_{H,+}.
$$

For $\tau_D^{(2)}(w_H)$ we have $\tau_D^{(2)}(w_H) = \tau_D^{(2,1)}(w_H) + \tau_D^{(2,2)}(w_H)$ with

$$\tau_D^{(2,1)}(w_H) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i k_{j+1/2}\frac{\partial d_1}{\partial x} g_1(c(x_i,y_j,t))D_{-x}c(x_i,y_j,t)D_{-x}w_H(x_i,y_j)$$
$$+\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} h_{i+1/2}k_j\frac{\partial d_2}{\partial x} g_2(c(x_i,y_j,t))D_{-y}c(x_i,y_j,t)D_{-y}w_H(x_i,y_j),$$

where $g_1(c(x_i,y_j,t) = M_h c(x_i,y_j,t) - c(x_{i-1/2},y_j,t)$, $g_2(c(x_i,y_j,t) = M_k c(x_i,y_j,t) - c(x_i,y_{j-1/2},t)$, and

$$\tau_D^{(2,2)}(w_H) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i k_{j+1/2}\frac{\partial d_1}{\partial y} g_1(p(x_i,y_j,t))D_{-x}c(x_i,y_j,t)D_{-x}w_H(x_i,y_j)$$
$$+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_{i+1/2}k_j\frac{\partial d_2}{\partial y} g_2(p(x_i,y_j,t))D_{-y}c(x_i,y_j,t)D_{-y}w_H(x_i,y_j),$$

with $g_1(p(x_i,y_j,t)) = D_{-x}p(x_i,y_j,t) - \frac{\partial p}{\partial x}(x_{i-1/2},y_j,t)$ and $g_2(p(x_i,y_j,t)) = D_{-y}p(x_i,y_j,t) - \frac{\partial p}{\partial y}(x_i,y_{j-1/2},t)$. In $\tau_D^{(2,\ell)}(w_H)$, $\ell = 1,2$, the partial derivatives of d_1 and d_2 are evaluated at convenient points.

Following the steps used to estimate $\tau_A^{(2)}(e_{H,p}(t))$ on the proof of Proposition 1, it can be shown that

$$|\tau_D^{(2,1)}(w_H)| \le C\|c(t)\|_{C^1}\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\|c(t)\|^2_{H^2(\Delta)}\Big)^{1/2}\|\nabla_H w_H\|_{H,+}.$$

To estimate $\tau_D^{(2,2)}(w_H)$ we observe that if $p(t)\in H^3(\Omega)$ then $\dfrac{\partial p}{\partial x}\in H^2(\Omega)$. Under the previous assumptions for g_1, we get

$$k_{j+1/2}|g_1(p(x_i,y_j,t))| \le k_{j+1/2}\int_{y_{j-1/2}}^{y_{j+1/2}}|g_1\left(\frac{\partial p}{\partial y}(x_i,y,t)\right)|dy$$
$$+\int_{y_{j-1/2}}^{y_{j+1/2}}|g_1(p(x_i,y,t))|dy$$
$$\le \int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1}}^{x_i}\Big(k_{j+1/2}|\frac{\partial^3 p}{\partial x^2\partial y}(t))| + h_i|\frac{\partial^3 p}{\partial x^3}(t)|\Big)dxdy,$$

being the last upper bound obtained using the Bramble–Hilbert Lemma. For g_2 holds a similar result, and we obtain

$$|\tau_D^{(2,2)}(w_H)| \le C\|c(t)\|_{C^1}\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\|p(t)\|^2_{H^3(\Delta)}\Big)^{1/2}\|\nabla_H w_H\|_{H,+}.$$

Lemma 5.1 of [17] allows us to deduce the upper bound for $\tau_D^{(3)}(w_H)$,

$$|\tau_D^{(3)}(w_H)| \leq C(1+\|c(t)\|_{C^1})^3\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|c(t)\|^2_{H^3(\Delta)}+\|c(t)\|^4_{H^2(\Delta)}$$
$$+\|p(t)\|^2_{H^3(\Delta)}+\|p(t)\|^4_{H^2(\Delta)}\Big)\Big)^{1/2}\|\nabla_H w_H\|_{H,+}.$$

Finally, taking into account (24) we conclude the proof. □

Under the assumptions of Proposition 2 and following its proof we can derive the next result.

Proposition 3 *Under the assumptions of Proposition* 2, there exists a positive constant C such that, for H_{max} small enough, we have

$$|\tau_E(w_H)| \leq \int_0^t |K_{er}(t-s)|\tau_{E,e}(w_H)ds,$$

where

$$\tau_{E,e}(w_H) = C\Big(\|c(s)\|_{C^1}\big(1+\|p(s)\|_{C^1}\big)\Big(\|e_{H,c}(s)\|_H$$
$$+\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\|p(s)\|^2_{H^3(\Delta)}\Big)^{1/2}\Big)$$
$$+(1+\|c(s)\|_{C^1})^3\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|c(s)\|^2_{H^3(\Delta)}+\|c(s)\|^4_{H^2(\Delta)}$$
$$+\|p(s)\|^2_{H^3(\Delta)}+\|p(s)\|^4_{H^2(\Delta)}\Big)\Big)^{1/2}\Big)\|\nabla_H w_H\|_{H,+}.$$

Proposition 4 *If* $p(t)\in H^3(\Omega)$, $\dfrac{\partial^2 p}{\partial x\partial y}(t)=\dfrac{\partial^2 p}{\partial y\partial x}(t)$ *in* Ω, $c(t)\in H^2(\Omega)$, *and the spatial grid satisfies*

$$\frac{k_j}{k_{j+1}}\leq C,\quad \frac{h_i}{h_{i+1}}\leq C, \tag{28}$$

then there exists a positive constant C *such that, for* H_{max} *small enough and* $w_H\in W_{H,0}$, *we have*

$$|\tau_B(w_H)| \leq C\Big(\|c(t)\|_{C^0}\big(1+\|p(t)\|_{C^1}\big)\|e_{H,c}(t)\|_H$$
$$+\big(\|c(t)\|_{C^0}\big(1+\|p(t)\|_{C^1}\big)+1\big)\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\big(\|c(t)\|^4_{H^1(\Delta)}$$
$$+\|c(t)\|^2_{H^2(\Delta)}\|p(t)\|^4_{H^2(\Delta)}+\|p(t)\|^2_{H^3(\Delta)}\big)\Big)^{1/2}\Big)\|\nabla_H w_H\|_{H,+}.$$

Proof Let $B^*_H(t)$ and $\tilde{B}_H(t)$ be the vectors with entries $b_1(c(x_i, y_j, t), D_h p(x_i, y_j, t))$, $b_2(c(x_i, y_j, t), D_k p(x_i, y_j, t))$ and $b_1(c(x_i, y_j, t), \frac{\partial p}{\partial x}(x_i, y_j, t))$, $b_2(c(x_i, y_j, t), \frac{\partial p}{\partial y}(x_i, y_j, t))$, respectively. Using these vectors, it is easily established for $\tau_B(w_H)$ the representation

$$\tau_B(w_H) = \sum_{i=1}^{3} \tau_B^{(i)}(w_H),$$

where

$$\tau_B^{(1)}(w_H) = (M_H((B^*_H(t) - B_H(t))R_H c(t)), \nabla_H w_H)_{H,+},$$
$$\tau_B^{(2)}(w_H) = (M_H((\tilde{B}_H(t) - B^*_H(t))R_H c(t)), \nabla_H w_H)_{H,+},$$

and

$$\begin{aligned}\tau_B^{(3)}(w_H) = &-(M_H(\tilde{B}_H(t)R_H c), \nabla_H w_H)_{H,+}\\ &+ \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i \int_{y_{j-1/2}}^{y_{j+1/2}} b_1(x_{i-1/2}, y, t)c(x_{i-1/2}, y, t)dy D_{-x} w_H(x_i, y_j)\\ &+ \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} k_j \int_{x_{i-1/2}}^{x_{i+1/2}} b_2(x, y_{j-1/2}, t)c(x, y_{j-1/2}, t)dx D_{-y} w_H(x_i, y_j),\end{aligned}$$

For $\tau_B^{(1)}(w_H)$, we can easily establish the following estimate

$$|\tau_B^{(1)}(w_H)| \leq C\|c(t)\|_{C^0}\Big(\|e_{H,c}(t)\|_H + \|\nabla_H e_{H,p}(t)\|_{H,+}\Big)\|\nabla_H w_H\|_{H,+}.$$

An estimate for $\tau_B^{(2)}(w_H)$ can be obtained following the approach used to estimate $\tau_D^{(2)}(w_H)$ in the proof of Proposition 2. For this, we must replace g_i, $i = 1, 2$, (introduced in the construction of the upper bound for $\tau_D^{(2,2)}(w_H)$) by $g_1(x_i, y, t) = D_h p(x_i, y_j, t) - \frac{\partial p}{\partial x}(x_i, y_j, t)$ and $g_2(x_i, y, t) = D_k p(x_i, y_j, t) - \frac{\partial p}{\partial y}(x_i, y_j, t)$, respectively. In this case we obtain

$$\begin{aligned}|\tau_B^{(2)}(w_H)| \leq C\|c(t)\|_{C^0}\bigg(&\Big(\sum_{\Delta\in\mathcal{T}_H} (\text{diam } \Delta)^4 \|c(t)\|^2_{H^2(\Delta)}\Big)^{1/2}\\ &+ \Big(\sum_{\Delta\in\mathcal{T}_H} (\text{diam } \Delta)^4 \|p(t)\|^2_{H^3(\Delta)}\Big)^{1/2}\bigg)\|\nabla_H w_H\|_{H,+}.\end{aligned}$$

Considering now Lemma 5.5 of [17] we obtain for $\tau_B^{(3)}(w_H)$ the estimate

$$|\tau_B^{(3)}(w_H)| \le C\big(\|c(t)\|_{C^0}+1\big)\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\big(\|c(t)\|^4_{H^1(\Delta)}+\|c(t)\|^2_{H^2(\Delta)}$$
$$+\|p(t)\|^4_{H^2(\Delta)}+\|p(t)\|^2_{H^3(\Delta)}\big)\Big)^{1/2}\|\nabla_H w_H\|_{H,+}.$$

Finally, combining the upper bound for $|\tau_B^{(1)}(w_H)|$ with Proposition 1 we conclude the proof. □

Lemma 5.7 of [17] allows us to derive the next proposition.

Proposition 5 *If $\frac{dc}{dt}(t)\in H^2(\Omega)$, then there exists a positive constant C such that, for H_{max} small enough, we have*

$$|\tau_d(w_H)|\le C\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\|\frac{dc}{dt}(t)\|^2_{H^2(\Delta)}\Big)^{1/2}\|\nabla_H w_H\|_{H,+}$$

for all $w_H\in W_{H,0}$.

From Propositions 1–5, with the aid of Gronwall's Lemma, we conclude the next convergence result.

Theorem 1 *If $p,\ c\in L^\infty(0,T,H^3(\Omega))$, $p(t)$ is such that $\frac{\partial^2 p}{\partial x\partial y}(t)=\frac{\partial^2 p}{\partial y\partial x}(t)$ in Ω, the spatial grids $\overline{\Omega}_H$, $H\in\Lambda$, satisfy (28), then there exists a positive constant C such that*

$$\|e_{H,c}(t)\|^2_H+\int_0^t\|\nabla_H e_{H,c}(s)\|^2_{H,+}ds\le CH^4_{max},\quad t\in[0,T], \tag{29}$$

and

$$\|e_{H,p}(t)\|_{1,H}\le CH^2_{max},\quad t\in[0,T], \tag{30}$$

Proof Taking in (21) and (22) $u_H=e_{H,p}(t)$ and $w_H=e_{H,c}(t)$, respectively, we obtain

$$(A_H(t)e_{H,p}(t),\nabla_H e_{H,p})_{H,+}=\tau_1(e_{H,p}(t)), \tag{31}$$

$$\begin{aligned}&\frac{d}{dt}\|e_{H,c}(t)\|^2_H+2(D_H(t)\nabla_H e_{H,c}(t),\nabla_H e_{H,c}(t))_{H,+}\\&\quad-2(M_H(B_H(t)e_{H,c}(t)),\nabla_H e_{H,c}(t))_{H,+}\\&\quad=-2\int_0^t K_{er}(t-s)(E_H(s)\nabla_H R_H c(s),\nabla_H e_{H,c}(t))_{H,+}ds+2\tau_2(e_{H,c}(t)),\end{aligned} \tag{32}$$

where $\tau_1(e_{H,p}(t))$ and $\tau_2(e_{H,c}(t))$ were estimated in Propositions 1 and 2–5, respectively.

From Proposition 2 we get

$$
\begin{aligned}
|\tau_D(e_{H,c}(t))| \leq{}& \frac{C}{4\epsilon^2}\|c(t)\|_{C^1}^2(1+\|p(t)\|_{C^1})^2\|e_{H,c}(t)\|_H^2 + 2\epsilon^2\|\nabla_H e_{H,c}(t)\|_{H,+}^2 \\
&+ \frac{C}{4\epsilon^2}\Big(\|c(t)\|_{C^1}(1+\|p(t)\|_{C^1}) + (1+\|c(t)\|_{C^1})^3\Big)^2 \\
&\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|c(t)\|_{H^2(\Delta)}^4 + \|c(t)\|_{H^3(\Delta)}^2 + \|p(t)\|_{H^2(\Delta)}^4 + \|p(t)\|_{H^3(\Delta)}^2\Big),
\end{aligned} \tag{33}
$$

where $\epsilon \neq 0$. Proposition 3 allows us to establish the following upper bounds for $\tau_E(e_{H,c}(t))$,

$$
\begin{aligned}
|\tau_E(e_{H,c}(t))| \leq{}& \frac{C}{4\epsilon^2}\|K_{er}\|_{L^2}^2\int_0^t \|c(s)\|_{C^1}^2(1+\|p(s)\|_{C^1})^2\|e_{H,c}(s)\|_H^2 ds \\
&+ \frac{C}{4\epsilon^2}\|K_{er}\|_{L^2}^2\int_0^t \Big(\|c(s)\|_{C^1}(1+\|p(s)\|_{C^1}) + (1+\|c(s)\|_{C^1})^3\Big)^2 \\
&\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|c(s)\|_{H^2(\Delta)}^4 + \|c(s)\|_{H^3(\Delta)}^2 + \|p(t)\|_{H^2(\Delta)}^4 + \|p(s)\|_{H^3(\Delta)}^2\Big)ds \\
&+ 2\epsilon^2\|\nabla_H e_{H,c}(t)\|_{H,+}^2 \\
\leq{}& \frac{C}{4\epsilon^2}\|K_{er}\|_{L^2}^2\|c\|_{L^\infty(C^1)}^2(1+\|p\|_{L^\infty(C^1)})^2\int_0^t \|e_{H,c}(s)\|_H^2 ds \\
&+ \frac{C}{4\epsilon^2}\|K_{er}\|_{L^2}^2\Big(\|c\|_{L^\infty(C^1)}(1+\|p\|_{L^\infty(C^1)}) + (1+\|c\|_{L^\infty(C^1)})^3\Big)^2 \\
&\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|c\|_{L^4(H^2(\Delta))}^4 + \|c\|_{L^2(H^3(\Delta))}^2 + \|p\|_{L^4(H^2(\Delta))}^4 + \|p\|_{L^2(H^3(\Delta))}^2\Big) \\
&+ 2\epsilon^2\|\nabla_H e_{H,c}(t)\|_{H,+}^2,
\end{aligned} \tag{34}
$$

where $\|.\|_{L^\infty(C^m)}$, $m \in \mathbb{N}$, and $\|.\|_{L^r(H^n(\Delta)}$, $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, denote the usual norms in the usual spaces $L^\infty(0, T, C^m(\Omega))$ and $L^r(0, T, H^n(\Delta))$, respectively.

For $\tau_B(e_{H,c}(t)$ it can be easily shown that

$$
\begin{aligned}
|\tau_B(e_{H,c}(t))| \leq{}& \frac{C}{4\epsilon^2}\|c(t)\|_{C^0}^2(1+\|p(t)\|_{C^1})^2\|e_{H,c}(t)\|_H^2 + 2\epsilon^2\|\nabla_H e_{H,c}(t)\|_{H,+}^2 \\
&+ \frac{C}{4\epsilon^2}\Big(\|c(t)\|_{C^0}(1+\|p(t)\|_{C^1}) + 1\Big)^2 \\
&\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|c(t)\|_{H^1(\Delta)}^4 + \|c(t)\|_{H^2(\Delta)}^2 + \|p(t)\|_{H^2(\Delta)}^4 + \|p(t)\|_{H^3(\Delta)}^2\Big).
\end{aligned} \tag{35}
$$

Finally, from Proposition 5 the following upper bound for $\tau_d(e_{H,c}(t))$ is easily obtained

$$|\tau_d(e_{H,c}(t))| \le \frac{C}{4\epsilon^2}\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\|\frac{dc}{dt}(t)\|^2_{H^2(\Delta)} + \epsilon^2\|\nabla_H e_{H,c}(t)\|^2_{H,+}. \quad (36)$$

Combining the upper bounds (33)–(36) with (32) and attending that

$$|(M_H(B_H(t)e_{H,c}(t)), \nabla_H e_{H,c}(t))_{H,+}| \le \frac{C}{4\epsilon^2}\|e_{H,c}(t)\|^2_H + \epsilon^2\|\nabla_H e_{H,c}(t)\|^2_{H,+},$$

and

$$\begin{aligned}&|\int_0^t K_{er}(t-s)(E_H(s)\nabla_H R_H c(s), \nabla_H e_{H,c}(t))_{H,+}ds|\\ &\quad\le \frac{C}{4\epsilon^2}\|K_{er}\|^2_{L^2}\int_0^t\|\nabla_H e_{H,c}(s)\|^2_{H,+}ds + \epsilon^2\|\nabla_H e_{H,c}(t)\|^2_{H,+},\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{d}{dt}\|e_{H,c}(t)\|^2_H + 2(D_{min}-10\epsilon^2)\|\nabla_H e_{H,c}(t)\|^2_{H,+}\\ &\quad\le \frac{C}{\epsilon^2}\Big(\|c(t)\|^2_{C^1}(1+\|p(t)\|_{C^1})^2+1\Big)\|e_{H,c}(t)\|^2_H\\ &\quad + \frac{C}{2\epsilon^2}\|K_{er}\|^2_{L^2}\|c\|^2_{L^\infty(C^1)}(1+\|p\|_{L^\infty(C^1)})^2\int_0^t\|e_{H,c}(s)\|^2_H ds\\ &\quad + \frac{C}{2\epsilon^2}\|K_{er}\|^2_{L^2}\Big(\|c\|^2_{L^\infty(C^1)}(1+\|p\|_{L^\infty(C^1)})^2+1\Big)\int_0^t\|\nabla_H e_{H,c}(s)\|^2_{H,+}ds\\ &\quad + \tau_{e,c}(t), \end{aligned}\quad (37)$$

where

$$\begin{aligned}\tau_{e,c}(t) = {}& \frac{C}{\epsilon^2}\Big(\|c(t)\|_{C^1}(1+\|p(t)\|_{C^1})+(1+\|c(t)\|_{C^1})\Big)^2\\ &\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|c(t)\|^4_{H^2(\Delta)}+\|c(t)\|^2_{H^3(\Delta)}+\|p(t)\|^4_{H^2(\Delta)}+\|p(t)\|^2_{H^3(\Delta)}\Big)\\ &+\frac{C}{2\epsilon^2}\|K_{er}\|^2_{L^2}\Big(\|c\|_{L^\infty(C^1)}(1+\|p\|_{L^\infty(C^1)})+(1+\|c\|_{L^\infty(C^1)})^3\Big)^2\\ &\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|c\|^4_{L^4(H^2(\Delta))}+\|c\|^2_{L^2(H^3(\Delta))}+\|p\|^4_{L^4(H^2(\Delta))}+\|p\|^2_{L^2(H^3(\Delta))}\Big)\\ &+\frac{C}{2\epsilon^2}\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\|\frac{dc}{dt}(t)\|^2_{H^2(\Delta)}.\end{aligned}$$

Inequality (37) leads to

$$
\begin{aligned}
&\|e_{H,c}(t)\|_H^2 + 2(D_{min} - 10\epsilon^2) \int_0^t \|\nabla_H e_{H,c}(s)\|_{H,+}^2 ds \\
&\quad \le \|e_{H,c}(0)\|_H^2 + \frac{C}{\epsilon^2}\Big(\|c\|_{L^\infty(C^1)}^2(1+\|p\|_{L^\infty(C^1)})^2 + 1\Big) \int_0^t \|e_{H,c}(s)\|_H^2 ds \\
&\quad + \frac{C}{2\epsilon^2}\|K_{er}\|_{L^2}^2 \|c\|_{L^\infty(C^1)}^2 (1+\|p\|_{L^\infty(C^1)})^2 \int_0^t \int_0^s \|e_{H,c}(\mu)\|_H^2 d\mu ds \\
&\quad + \frac{C}{2\epsilon^2}\|K_{er}\|_{L^2}^2 \Big(\|c\|_{L^\infty(C^1)}^2(1+\|p\|_{L^\infty(C^1)})^2 + 1\Big) \int_0^t \int_0^s \|\nabla_H e_{H,c}(\mu)\|_{H,+}^2 d\mu ds \\
&\quad + \int_0^t \tau_{e,c}(s) ds.
\end{aligned} \tag{38}
$$

Let ϵ be such that $D_{min} - 10\epsilon^2 > 0$. Under the smoothness assumptions on c and p it can be shown that $\int_0^t \tau_{e,c}(s)\, ds \le CH_{max}^4$. As $e_{H,c}(0) = 0$ and considering the discrete Poincaré–Friedrichs inequality (2.1) and Gronwall's Lemma we conclude (29).

From Proposition 1 we conclude the error estimate (30) for the pressure $p_H(t)$. □

Theorem 1 is the main result of this paper and it establishes the second order convergence rate of the finite difference scheme (19), (20), or equivalently, of the piecewise linear finite element method (17), (18).

4 Time discretization

Our goal in this section is to propose an IMEX method for the coupled non-Fickian problem (6), (8). The method is obtained by integrating in time the ordinary differential equation (20) or equivalently the discrete variational equation (14).

In the temporal domain $[0, T]$, let us introduce the uniform time grid $\{t_m = m\Delta t,\ m = 0, \ldots, M\}$, with $t_M = T$, and where Δt is a fixed time step. By p_H^m and c_H^m we represent the numerical approximations for $p(t_m)$ and $c(t_m)$, respectively, defined by the IMEX method

$$
-\nabla_H^* \cdot (A_H^m \nabla_H p_H^{m+1}) = (q_2)_H^{m+1}, \quad \text{in } \Omega_H, \tag{39}
$$

$$
\begin{aligned}
c_H^{m+1} &= c_H^m + \Delta t \nabla_H^* \cdot (D_H^{m,m+1} \nabla_H c_H^{m+1}) - \Delta t \nabla_{c,H} \cdot (B_H^{m,m+1} c_H^m) \\
&\quad + \Delta t^2 \sum_{\ell=0}^{m} K_{er}(t_{m+1} - t_\ell) \nabla_H^* \cdot (E_H^{\ell,\ell+1} \nabla_H c_H^\ell) + \Delta t (q_1)_H^{m+1}, \quad \text{in } \Omega_H,
\end{aligned} \tag{40}
$$

for $m = 0, \ldots, M-1$, and with the initial conditions

$$
c_H^0 = R_H c_0 \quad \text{in } \Omega_H, \tag{41}
$$

and boundary conditions

$$
c_H^\ell = 0,\ p_H^\ell = R_H p_b(t_\ell) \quad \text{on } \partial\Omega_H,\ \ell = 1, \ldots, M. \tag{42}
$$

Here we used the following notation: the non-null entries of A_H^m are given by $a_1(M_h c_H^m)$, $a_2(M_k c_H^m)$, the non-null entries of $D_H^{m,m+1}$ are given by $d_1(M_h c_H^m, D_{-x} p_H^{m+1})$, $d_2(M_k c_H^m, D_{-y} p_H^{m+1})$, being $B_H^{m,m+1}$ and $E_H^{m,m+1}$, $m = 0, \ldots, M$, defined analogously. By D_{-t} we denote the first order backward finite difference operator with respect to the time variable. We observe that (39), (40) is equivalent to the coupled discrete variational problem

$$(A_H^m \nabla_H p_H^{m+1}, \nabla_H w_H)_{H,+} = ((q_2)_H^{m+1}, w_H)_H, \quad \text{for all } w_H \in W_{H,0}, \tag{43}$$

$$\begin{aligned}(D_{-t} c_H^{m+1}, w_H)_H &= -(D_H^{m,m+1} \nabla_H c_H^{m+1}, \nabla_H w_H)_{H,+} \\ &\quad + (M_H(B_H^{m,m+1} c_H^m), \nabla_H w_H)_{H,+} \\ &\quad + \Delta t \sum_{\ell=0}^{m} K_{er}(t_{m+1} - t_\ell)(E_H^{\ell,\ell+1} \nabla_H c_H^\ell, \nabla_H w_H)_{H,+} \\ &\quad + ((q_1)_H^{m+1}, w_H)_H, \quad \text{for all } w_H \in W_{H,0}.\end{aligned} \tag{44}$$

Remark 3 From (44) it seems that we would need to save all previous solutions to compute the solution at the current time level. This would be computationally very demanding. However, we remark that our scheme can be, in certain cases, rewritten as a three-time-level method following the approach introduced in [20]. Therefore, using that formulation, there is no need to store all the previous solutions. In the numerical simulations presented in this paper we have followed that approach.

In what follows we establish bounds for the errors

$$e_{H,p}^m = R_H p(t_m) - p_H^m \quad \text{and} \quad e_{H,c}^m = R_H c(t_m) - c_H^m.$$

Following the proof of Proposition 1, it can be shown that

$$\begin{aligned}\|\nabla_H e_{H,p}^{m+1}\|_{H,+} \le C\Big(&\|p(t_{m+1})\|_{C^1} \|e_{H,c}^m\|_H + \tau_p(t_{m+1}) \\ &+ \|p(t_{m+1})\|_{C^1} \int_{t_m}^{t_{m+1}} \|R_H \frac{dc}{dt}(s)\|_H \, ds\Big),\end{aligned} \tag{45}$$

where $\tau_p(t_{m+1})$ is given by (25) with $t = t_{m+1}$.

We deduce in what follows several estimates needed to compute an upper bound for $\|e_{H,c}^{m+1}\|_H$. We observe that

$$(D_{-t} c_H^{m+1} - \Big(\frac{dc}{dt}\Big)_H(t_{m+1}), e_{H,c}^{m+1})_H = -(D_{-t} e_{H,c}^{m+1}, e_{H,c}^{m+1})_H + \tau_{d,IE}(e_{H,c}^{m+1}), \tag{46}$$

where

$$\begin{aligned}|\tau_{d,IE}(e_{H,c}^{m+1})| \le C\Big(&\int_{t_m}^{t_{m+1}} \|R_H \frac{d^2 c}{dt^2}(s)\|_H \, ds \|e_{H,c}^{m+1}\|_H \\ &+ \Big(\sum_{\Delta \in \mathcal{T}_H} (\text{diam } \Delta)^4 \|\frac{dc}{dt}(t_{m+1})\|_{H^2(\Delta)}^2\Big)^{1/2} \|\nabla_H e_{H,c}^{m+1}\|_{H,+}\Big).\end{aligned}$$

For

$$\tau_{D,d}(e_{H,c}^{m+1}) = (D_H^{m,m+1}\nabla_H c_H^{m+1}, \nabla_H e_{H,c}^{m+1})_{H,+}$$
$$- \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i \int_{y_{j-1/2}}^{y_{j+1/2}} d_1(x_i, y, t_{m+1})$$
$$\times \frac{\partial c}{\partial x}(x_{i-1/2}, y, t_{m+1})\, dy D_{-x}e_{H,c}^{m+1}(x_i, y_j)$$
$$- \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} k_j \int_{x_{i-1/2}}^{x_{i+1/2}} d_2(x, y_j, t_{m+1})$$
$$\times \frac{\partial c}{\partial y}(x, y_{j-1/2}, t_{m+1})\, dx D_{-y}e_{H,c}^{m+1}(x_i, y_j), \tag{47}$$

with $d_\ell(x_i, y, t_{m+1})$, $\ell = 1, 2$, defined by (26) with $t = t_{m+1}$, we have

$$\tau_{D,d}(e_{H,c}^{m+1}) = -(D_H^{m,m+1}\nabla_H e_{H,c}^{m+1}, \nabla_H e_{H,c}^{m+1})_{H,+} + \tau_{D,IE}(e_{H,c}^{m+1}), \tag{48}$$

where

$$|\tau_{D,IE}(e_{H,c}^{m+1})| \le \tau_{D,e}(e_{H,c}^{m+1})$$
$$+ C\|c(t_{m+1})\|_{C^1}\big(1 + \|p(t_{m+1})\|_{C^1}\big)\int_{t_m}^{t_{m+1}} \|R_H$$
$$\times \frac{dc}{dt}(s)\|_H ds \|\nabla_H e_{H,c}^{m+1}\|_{H,+},$$

being $\tau_{D,e}(e_{H,c}^{m+1})$ given by

$$\tau_{D,e}(e_{H,c}^{m+1}) = C\Big(\|c(t_{m+1})\|_{C^1}\big(1 + \|p(t_{m+1})\|_{C^1}\big)\Big(\|e_{H,c}^m\|_H$$
$$+ \Big(\sum_{\Delta\in\mathcal{T}_H} (\mathrm{diam}\ \Delta)^4 \|p(t_{m+1})\|_{H^3(\Delta)}^2\Big)^{1/2}\Big)$$
$$+ (1 + \|c(t_{m+1})\|_{C^1})^3\Big(\sum_{\Delta\in\mathcal{T}_H} (\mathrm{diam}\ \Delta)^4 \sum_{f\in\{c,p\}} \Big(\|f(t_{m+1})\|_{H^3(\Delta)}^2$$
$$+ \|f(t_{m+1})\|_{H^2(\Delta)}^4\Big)\Big)^{1/2}\Big)\|\nabla_H e_{H,c}^{m+1}\|_{H,+}.$$

For

$$\tau_{B,d}(e_{H,c}^{m+1}) = -(M_H(B_H^{m,m+1}c_H^m), \nabla_H e_{H,c}^{m+1})_{H,+}$$
$$+ \sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i \int_{y_{j-1/2}}^{y_{j+1/2}} b_1(x_i, y, t_{m+1}) c(x_{i-1/2}, y, t_{m+1})\, dy$$

$$\times\ D_{-x}e_{H,c}^{m+1}(x_i,y_j)$$
$$+\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y}k_j\int_{x_{i-1/2}}^{x_{i+1/2}}b_2(x,y_j,t_{m+1})c(x,y_{j-1/2},t)\,dx$$
$$\times\ D_{-y}e_{H,c}^{m+1}(x_i,y_j), \tag{49}$$

with b_ℓ, $\ell=1,2$, given by (27) and $t=t_{m+1}$, assuming (28), we can prove that,

$$\tau_{B,d}(e_{H,c}^{m+1})\leq(M_H(B_H^{m,m+1}e_H^m),\nabla_He_{H,c}^{m+1})_{H,+}+\tau_{B,IE}(e_{H,c}^{m+1}), \tag{50}$$

where

$$|\tau_{B,IE}(e_{H,c}^{m+1})|\leq\tau_{B,e}(e_{H,c}^{m+1})$$
$$+\ C\|c(t_{m+1})\|_{C^1}\big(1+\|p(t_{m+1})\|_{C^1}\big)\int_{t_m}^{t_{m+1}}\|R_H$$
$$\frac{dc}{dt}(s)\|_Hds\|\nabla_He_{H,c}^{m+1}\|_{H,+}$$

being $\tau_{B,e}(e_{H,c}^{m+1})$ equal to

$$\tau_{B,e}(e_{H,c}^{m+1})=C\Big(\|c(t_{m+1})\|_{C^0}\big(1+\|p(t_{m+1})\|_{C^1}\big)\|e_{H,c}^m\|_H$$
$$+\ \big(\|c(t_{m+1})\|_{C^0}\big(1+\|p(t_{m+1})\|_{C^1}\big)+1\big)$$
$$\times\Big(\sum_{\Delta\in\mathcal{T}_H}(\mathrm{diam}\,\Delta)^4\big(\|c(t_{m+1})\|_{H^1(\Delta)}^4$$
$$+\ \|c(t_{m+1})\|_{H^2(\Delta)}^2+\|p(t_{m+1})\|_{H^2(\Delta)}^4$$
$$+\ \|p(t_{m+1})\|_{H^3(\Delta)}^2\big)\Big)^{1/2}\Big)\|\nabla_He_{H,c}^{m+1}\|_{H,+}.$$

Finally, we establish an estimate for

$$\tau_{E,d}(e_{H,c}^{m+1})=-\Delta t\sum_{\ell=0}^{m}K_{er}^{m+1,\ell}(E_H^{\ell,\ell+1}\nabla_Hc_H^\ell,\nabla_He_{H,c}^{m+1})_{H,+}$$
$$+\int_0^{t_{m+1}}K_{er}(t_{m+1}-s)\sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1}h_i\int_{y_{j-1/2}}^{y_{j+1/2}}e_1(x_i,y,s)$$
$$\times\ \frac{\partial c}{\partial x}(x_{i-1/2},y,s)dyds\,D_{-x}e_{H,c}^{m+1}(x_i,y_j)$$
$$+\int_0^{t_{m+1}}K_{er}(t_{m+1}-s)\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y}k_j\int_{x_{i-1/2}}^{x_{i+1/2}}e_2(x,y_j,s)$$

$$\times \frac{\partial c}{\partial y}(x, y_{j-1/2}, s) dx ds D_{-y} e_{H,c}^{m+1}(x_i, y_j), \tag{51}$$

where $K_{er}^{m+1,\ell} = K_{er}(t_{m+1} - t_\ell)$. Using the decomposition

$$\tau_{E,d}(e_{H,c}^{m+1}) = \sum_{\ell=1}^{6} \tau_{E,i}, \tag{52}$$

with

$$\tau_{E,1} = \Delta t \sum_{\ell=0}^{m} K_{er}^{m+1,\ell} (E_H^{\ell,\ell+1} \nabla_H e_{H,c}^{\ell}, \nabla_H e_{H,c}^{m+1})_{H,+}, \tag{53}$$

$$\tau_{E,2} = \Delta t \sum_{\ell=0}^{m} K_{er}^{m+1,\ell} ((E_H^*(t_\ell, t_{\ell+1}) - E_H^{\ell,\ell+1}) \nabla_H R_H c(t_\ell), \nabla_H e_{H,c}^{m+1})_{H,+}, \tag{54}$$

with $E_H^*(t_\ell, t_{\ell+1})$ defined as $E_H^*(t_\ell)$ but considering the concentration and the pressure at time levels t_ℓ and $t_{\ell+1}$, respectively,

$$\tau_{E,3} = \Delta t \sum_{\ell=0}^{m} K_{er}^{m+1,\ell} ((\tilde{E}_H(t_\ell, t_{\ell+1}) - E_H^*(t_\ell, t_{\ell+1})) \nabla_H R_H c(t_\ell), \nabla_H e_{H,c}^{m+1})_{H,+}, \tag{55}$$

with $\tilde{E}_H(t_\ell, t_{\ell+1})$ defined as $\tilde{E}_H(t_\ell)$ but considering the concentration and the pressure at time levels t_ℓ and $t_{\ell+1}$, respectively,

$$\tau_{E,4} = \Delta t \sum_{\ell=0}^{m} K_{er}^{m+1,\ell} ((\tilde{E}_H(t_\ell, t_\ell) - \tilde{E}_H(t_\ell, t_{\ell+1})) \nabla_H R_H c(t_\ell), \nabla_H e_{H,c}^{m+1})_{H,+}, \tag{56}$$

$$\tau_{E,5} = \int_0^{t_{m+1}} K_{er}(t_{m+1} - s)(E_H(s) \nabla_H R_H c(s) ds, \nabla_H e_{H,c}^{m+1})_{H,+}$$
$$- \Delta t \sum_{\ell=0}^{m} K_{er}^{m+1,\ell} (\tilde{E}_H(t_\ell, t_\ell) \nabla_H R_H c(t_\ell), \nabla_H e_{H,c}^{m+1})_{H,+}, \tag{57}$$

and

$$\begin{aligned}\tau_{E,6} = &-\int_0^{t_{m+1}} K_{er}(t_{m+1}-s)(E_H(s)\nabla_H R_H c(s)ds, \nabla_H e_{H,c}^{m+1})_{H,+}\\ &+\int_0^{t_{m+1}} K_{er}(t_{m+1}-s)\sum_{i=1}^{N_x}\sum_{j=1}^{N_y-1} h_i \int_{y_{j-1/2}}^{y_{j+1/2}} e_1(x_i, y, s)\\ &\times \frac{\partial c}{\partial x}(x_{i-1/2}, y, s)dyds D_{-x}e_{H,c}^{m+1}(x_i, y_j)\\ &+\int_0^{t_{m+1}} K_{er}(t_{m+1}-s)\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y} k_j \int_{x_{i-1/2}}^{x_{i+1/2}} e_2(x, y_j, s)\\ &\times \frac{\partial c}{\partial y}(x, y_{j-1/2}, s)dxds D_{-y}e_{H,c}^{m+1}(x_i, y_j). \qquad (58)\end{aligned}$$

For $\tau_{E,1}$ we easily establish the upper bounds

$$\begin{aligned}|\tau_{E,1}| &\leq C\Big(\Delta t\sum_{\ell=0}^{m}(K_{er}^{m+1,\ell})^2\Big)^{1/2}\Big(\Delta t\sum_{\ell=0}^{m}\|\nabla_H e_{H,c}^{\ell}\|_H^2\Big)^{1/2}\|\nabla_H e_{H,c}^{m+1}\|_{H,+}\\ &\leq C\Big(\|K_{er}\|_{L^2(0,T)}^2 + T\Delta t\|K_{er}'\|_{L^2(0,T)}^2\Big)^{1/2}\\ &\quad\times\Big(\Delta t\sum_{\ell=0}^{m}\|\nabla_H e_{H,c}^{\ell}\|_H^2\Big)^{1/2}\|\nabla_H e_{H,c}^{m+1}\|_{H,+}\\ &\leq C\sqrt{1+\Delta t}\|K_{er}\|_{H^1(0,T)}\Big(\Delta t\sum_{\ell=0}^{m}\|\nabla_H e_{H,c}^{\ell}\|_H^2\Big)^{1/2}\|\nabla_H e_{H,c}^{m+1}\|_{H,+}. \qquad (59)\end{aligned}$$

Using (45), it can be shown that

$$\begin{aligned}|\tau_{E,2}| &\leq C\Big(\Delta t\sum_{\ell=0}^{m}(K_{er}^{m+1,\ell})^2\Big)^{1/2}\Big(\Delta t\sum_{\ell=0}^{m}\|c(t_\ell)\|_{C^1}^2\big(1+\|p(t_{\ell+1})\|_{C^1}\big)^2\|e_{H,c}^{\ell}\|_H^2\\ &\quad+\Delta t\sum_{\ell=0}^{m}\|c(t_\ell)\|_{C^1}^2\big(\tau_p(t_{\ell+1})^2+\Delta t\int_{t_\ell}^{t_{\ell+1}}\|R_H\frac{dc}{dt}(s)\|_H^2\,ds\big)\Big)^{1/2}\|\nabla_H e_{H,c}^{m+1}\|_{H,+}\\ &\leq C\Big(\sqrt{1+\Delta t}\|K_{er}\|_{H^1(0,T)}\Big(\|c\|_{C^0(C^1)}^2(1+\|p\|_{C^0(C^1)})^2\Delta t\sum_{\ell=0}^{m}\|e_{H,c}^{\ell}\|_H^2\\ &\quad+\|c\|_{C^0(C^1)}^2\Delta t\sum_{\ell=0}^{m}\tau_p(t_{\ell+1})^2+\Delta t\int_{t_\ell}^{t_{\ell+1}}\|R_H\frac{dc}{dt}(s)\|_H^2\,ds\Big)^{1/2}\|\nabla_H e_{H,c}^{m+1}\|_{H,+},\end{aligned} \qquad (60)$$

where $\|.\|_{C^q(C^r)}$ denotes the usual norm in $C^q(0, T, C^r(\overline{\Omega}))$, $q, r \in \mathbb{N}_0$.

Following the steps used in the proof of Proposition 2 to estimate $\tau_D^{(2)}(w_H)$, we obtain

$$
\begin{aligned}
|\tau_{E,3}| &\leq C\Big(\Delta t\sum_{\ell=0}^{m}(K_{er}^{m+1,\ell})^2\Big)^{1/2}\Big(\Delta t\sum_{\ell=0}^{m}\|c(t_\ell)\|_{C^1}\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\big(\|c(t_\ell)\|^2_{H^2(\Delta)}\\
&\quad+\|p(t_{\ell+1})\|^2_{H^2(\Delta)}\big)\Big)\Big)^{1/2}\|\nabla_H e_{H,c}^{m+1}\|_{H,+}\\
&\leq C\sqrt{1+\Delta t}\|K_{er}\|_{H^1(0,T)}\|c\|_{C^0(C^1)}\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\big(\|c\|^2_{C^0(H^2)}\\
&\quad+\|p\|^2_{C^0(H^3)}\Big)^{1/2}\|\nabla_H e_{H,c}^{m+1}\|_{H,+}.
\end{aligned}
\tag{61}
$$

For $\tau_{E,4}$ we easily get

$$
\begin{aligned}
|\tau_{E,4}| &\leq C\Big(\Delta t\sum_{\ell=0}^{m}(K_{er}^{m+1,\ell})^2\Big)^{1/2}\Big(\Delta t^2\sum_{\ell=0}^{m}\|c(t_\ell)\|^2_{C^1}\int_{t_\ell}^{t_{\ell+1}}\|R_H\frac{\partial^2 c}{\partial t\partial x}(s)\|^2_H\,ds\Big)^{1/2}\\
&\quad\|\nabla_H e_{H,c}^{m+1}\|_{H,+}\\
&\leq C\Delta t\sqrt{1+\Delta t}\|K_{er}\|_{H^1(0,T)}\|c\|_{C^0(C^1)}\|c\|_{H^1(C^1)}\|\nabla_H e_{H,c}^{m+1}\|_{H,+},
\end{aligned}
\tag{62}
$$

where $\|.\|_{H^q(C^r)}$ denotes the usual norm in $H^q(0,T,C^r(\overline{\Omega}))$, $q,r\in\mathbb{N}_0$.

As $\tau_{E,5}$ represents the error of the left-rectangular rule, we deduce that

$$
\begin{aligned}
|\tau_{E,5}| &\leq C\Delta t\Big(\|K'_{er}\|_{L^2(0,T)}\|c\|_{L^2(C^1)}+\|K_{er}\|_{L^2(0,T)}\big(\|c\|_{C^0(C^1)}\big(\|c\|_{H^1(C^0)}\\
&\quad+\|p\|_{H^1(C^1)}\big)+\|c\|_{H^1(C^1)}\big)\Big)\|\nabla_H e_{H,c}^{m+1}\|_{H,+}\\
&\leq C\Delta t\|K_{er}\|_{H^1(0,T)}\Big(\|c\|_{C^0(C^1)}\big(1+\|c\|_{H^1(C^0)}+\|p\|_{H^1(C^1)}\big)+\|c\|_{H^1(C^1)}\Big)\\
&\quad\times\|\nabla_H e_{H,c}^{m+1}\|_{H,+}.
\end{aligned}
\tag{63}
$$

At last, for $\tau_{E,6}$ holds the following

$$
\begin{aligned}
|\tau_{E,6}| &\leq C\int_0^{t_{m+1}}K_{er}(t_{m+1}-s)(1+\|c(s)\|_{C^1})^3\\
&\quad\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\sum_{f\in\{c,p\}}\big(\|f(s)\|^4_{H^2(\Delta)}+\|f(s)\|^2_{H^3(\Delta)}\big)+1\Big)\Big)^{1/2}ds\\
&\quad\times\|\nabla_H e_{H,c}^{m+1}\|_{H,+}
\end{aligned}
$$

$$\leq C\|K_{er}\|_{L^2(0,T)}(1+\|c\|_{C^0(C^1)})^3$$
$$\Big(\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\sum_{f\in\{c,p\}}\Big(\|f\|^4_{L^4(H^2)}+\|f\|^2_{L^2(H^3)}\Big)+1\Big)\Big)^{1/2}ds$$
$$\times\|\nabla_H e^{m+1}_{H,c}\|_{H,+}. \tag{64}$$

Now we assume the following smoothness conditions: $c \in C^2(0,T,C^0(\overline{\Omega}))\cap H^1(0,T,H^3(\Omega))$, $p\in H^1(0,T,H^3(\Omega))$, and $K_{er}\in H^1(0,T)$.

From (44) to (64) it is a straightforward task to prove the existence of positive constants C_i, $i=1,2,3$, such that, for $m=0,\ldots,M-1$, the following holds

$$\|e^{m+1}_{H,c}\|^2_H + D_{min}\Delta t\|\nabla_H e^{m+1}_{H,c}\|^2_{H,+}\leq\|e^m_{H,c}\|^2_H$$
$$+\,C_1\Delta t\Big(\|e^m_{H,c}\|^2_H+\|e^{m+1}_{H,c}\|^2_H+\Delta t\sum_{\ell=0}^{m}\|\nabla_H e^{\ell}_{H,c}\|^2_{H,+}\Big)+\tau^{m+1}_{e,d}, \tag{65}$$

where

$$|\tau^{m+1}_{e,d}|\leq C_2\Delta t\Big(\Delta t\int_{t_m}^{t_{m+1}}\big(\|R_H\frac{d^2c}{dt^2}(s)\|^2_H+\|R_H\frac{dc}{dt}(s)\|^2_H\big)ds+\Delta t^2\Big)$$
$$+\,C_3\Delta t\sum_{\Delta\in\mathcal{T}_H}(\text{diam }\Delta)^4\Big(\|\frac{dc}{dt}(t_{m+1})\|^2_{H^2(\Delta)}$$
$$+\sum_{f\in\{c,p\}}\big(\|f(t_{m+1})\|^4_{H^2(\Delta)}+\|f(t_{m+1})\|^2_{H^3(\Delta)}\big)+1\Big).$$

Inequality (65) leads to

$$(1-C_1\Delta t)\|e^{m+1}_{H,c}\|^2_H+D_{min}\Delta t\sum_{\ell=0}^{m+1}\|\nabla_H e^{\ell}_{H,c}\|^2_{H,+}$$
$$\leq(1-C_1\Delta t)\|e^0_{H,c}\|^2_H+D_{min}\Delta t\|\nabla_H e^0_{H,c}\|_{H,+}$$
$$+\,C_1\Delta t\Big(\sum_{\ell=0}^{m}\|e^{\ell}_{H,c}\|^2_H+\Delta t\sum_{\ell=0}^{m}\sum_{j=0}^{\ell}\|\nabla_H e^{j}_{H,c}\|^2_{H,+}\Big)+\sum_{\ell=1}^{m+1}\tau^{\ell}_{e,d}.$$

Considering now the discrete Gronwall's Lemma we conclude that, for Δt such that $1-C_1\Delta t>0$,

$$\|e^{m+1}_{H,c}\|^2_H+\Delta t\sum_{\ell=0}^{m+1}\|\nabla_H e^{\ell}_{H,c}\|^2_{H,+}\leq\frac{1}{\min\{1-C_1\Delta t,D_{min}\}}\Big(\sum_{\ell=1}^{m+1}\tau^{\ell}_{e,d}$$
$$+\,C_2\Delta t\sum_{\ell=0}^{m}\sum_{j=1}^{\ell+1}\tau^{j}_{e,d}e^{C_2(m-\ell+1)\Delta t}\Big), \tag{66}$$

for $m = 0, \dots, M-1$ and with $C_2 = \dfrac{C_1}{\min\{1 - C_1 \Delta t, D_{min}\}}$. Finally, we remark that the error estimate (66) leads to

$$\|e_{H,c}^{m+1}\|_H^2 + \Delta t \sum_{\ell=0}^{m+1} \|\nabla_H e_{H,c}^{\ell}\|_{H,+}^2 \leq C\left(H_{max}^4 + \Delta t^2\right), \; m = 0, \dots, M-1, \tag{67}$$

while from (45) we get

$$\|\nabla_H e_{H,p}^{m+1}\|_{H,+} \leq C\left(H_{max}^4 + \Delta t^2\right), \quad m = 0, \dots, M-1. \tag{68}$$

5 Numerical results

This section is dedicated to some numerical experiments. We start by presenting two examples that illustrate the convergence result established in the previous section.

For $e_{H,p}^m = R_H p(t_m) - p_H^m$ and $e_{H,c}^m = R_H c(t_m) - c_H^m$ we compute the error indicators

$$\text{Error}_p = \max_{m=1,\dots,M} \|e_{H,p}^m\|_{1,H},$$

and

$$\text{Error}_c = \max_{m=1,\dots,M} \left(\|e_{H,c}^m\|_H^2 + \Delta t \sum_{\ell=1}^{m} \|\nabla_H e_{H,c}^{\ell}\|_{H,+}^2\right)^{1/2},$$

where p and c are solutions of the coupled problem (6), (8) with boundary and initial conditions (9), (10), respectively, and where p_H^m and c_H^m are numerical solutions obtained with the IMEX method (39)–(42). To evaluate the convergence rate we use the formula

$$\text{Rate}_g = \frac{\log\left(\dfrac{\text{Error}_{g,1}}{\text{Error}_{g,2}}\right)}{\log\left(\dfrac{H_{1,max}}{H_{2,max}}\right)},$$

for $g = p, \; c$, and where H_1 and H_2 are two grid vectors with $\text{Error}_{g,1}$ and $\text{Error}_{g,2}$ the corresponding errors. The initial grid Ω_H is randomly generated. The new grids are defined considering the midpoints of the intervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$. Moreover, we fix $T = 0.01$ and $\Delta t = 10^{-7}$.

Table 1 Numerical errors and convergence rates for Example 1

H_{max}	Error_p	Rate_p	Error_c	Rate_c	N_x	N_y
1.316e–01	2.615e–04	1.981	2.841e–05	1.975	9	8
6.579e–02	6.625e–05	1.995	7.227e–06	1.993	18	16
3.290e–02	1.662e–05	1.999	1.815e–06	2.000	36	32
1.645e–02	4.158e–06	2.000	4.538e–07	2.009	72	64
8.224e–03	1.040e–06	2.000	1.127e–07	2.018	144	128
4.112e–03	2.600e–07	–	2.783e-08	–	288	256

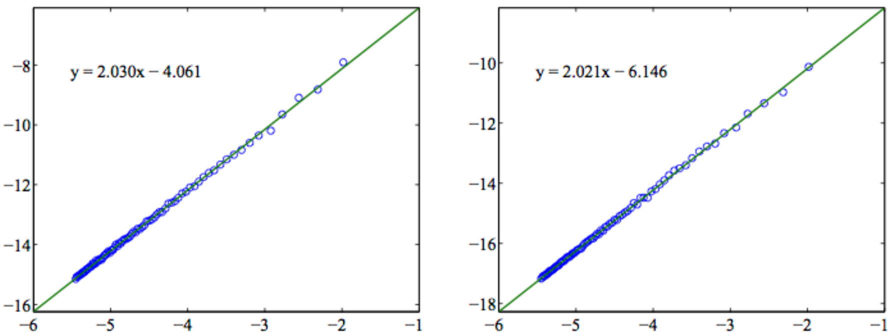

Fig. 2 From left to right: plot of $\log(\text{Error}_p)$ and $\log(\text{Error}_c)$ versus $\log(H_{max})$

Example 1 In this example, we consider the system (6), (8) with the following coefficients

$$A(c) = \begin{bmatrix} 1+c & 0 \\ 0 & 2+c \end{bmatrix}, \quad D(c, \nabla p) = \begin{bmatrix} 1+2c+\dfrac{\partial p}{\partial x} & 0 \\ 0 & 1+c+2\dfrac{\partial p}{\partial y} \end{bmatrix},$$

$$B(c, \nabla p) = \begin{bmatrix} c\dfrac{\partial p}{\partial x} \\ 3c\dfrac{\partial p}{\partial y} \end{bmatrix}, \quad E(c, \nabla p) = \begin{bmatrix} -\dfrac{\partial p}{\partial x} & 0 \\ 0 & -\dfrac{\partial p}{\partial y} \end{bmatrix}, \quad \text{and} \quad K_{er}(s) = e^{-s}.$$

We choose q_1, q_2, and the initial condition (10) so that the exact solution is

$$p(x, y, t) = e^t xy(x-1)(y-1)\sin(xy) \quad \text{and} \quad c(x, y, t) = e^t xy(x-1)(y-1).$$

In Table 1 we present the results of our simulation. We observe that the solutions p and c belong to $H_0^3(\Omega)$ and the numerical results illustrate the convergence estimates (67) and (68).

For further illustration, we solve Example 1 using a considerable number of randomly generated spatial grids. In Fig. 2 we plot the logarithmic norm of all errors

Table 2 Numerical errors and convergence rates for Example 2

H_{max}	Error_p	Rate_p	Error_c	Rate_c	N_x	N_y
1.381e–01	9.708e–03	1.050	7.783e–05	1.293	8	10
6.906e–02	4.688e–03	1.051	3.177e–05	1.121	16	20
3.453e–02	2.263e–03	1.079	1.461e–05	1.084	32	40
1.726e–02	1.071e–03	1.093	6.888e–06	1.086	64	80
8.632e–03	5.024e–04	1.009	3.246e–06	1.001	128	160
4.316e–03	2.497e–04	–	1.622e–06	–	256	320

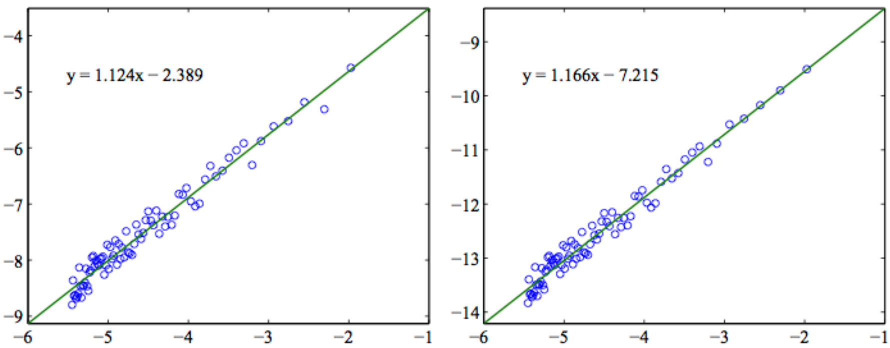

Fig. 3 From left to right: plot of $\log(\text{Error}_p)$ and $\log(\text{Error}_c)$ versus $\log(H_{max})$

Error_g, $g = p, c$, versus the logarithmic norm of all H_{max}. The slope of the least-square straight line (shown in green) is an approximation of the convergence order, and the values obtained, which are also displayed in Fig. 2, again confirm the convergence estimates (67) and (68).

In the next example we consider that $p(t) \in H_0^2(\Omega)$ but it doesn't belong to $H_0^3(\Omega)$. Following the lines of Theorem 1 and using the results of [17], we anticipate that the second order convergence rate will be lost for both $p(t)$ and $c(t)$.

Example 2 We now consider system (6), (8) with the coefficient functions used in Example 1 but we choose q_1, q_2, and the initial condition (10) so that the exact solution is

$$p(x, y, t) = e^t 2xy(x^2 - 1)(y^2 - 1)|x - 0.5|^{2.1} \quad \text{and} \quad c(x, y, t) = e^t xy(x - 1)(y - 1).$$

The numerical results presented in Table 2 agree with expectations, since the convergence rate for both p and c is of order $O(H_{max})$.

In Fig. 3, we repeat the same type of experiments plotted on Fig. 2. The slope of the least-square straight line, close to one, again confirms the first order convergence rate for Example 2.

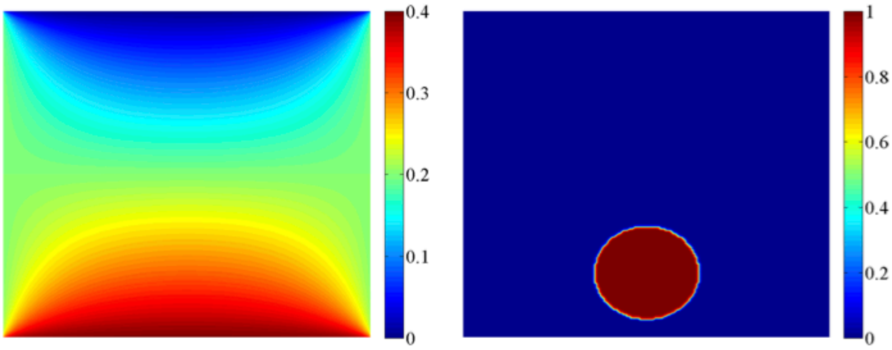

Fig. 4 From left to right: pressure and initial concentration

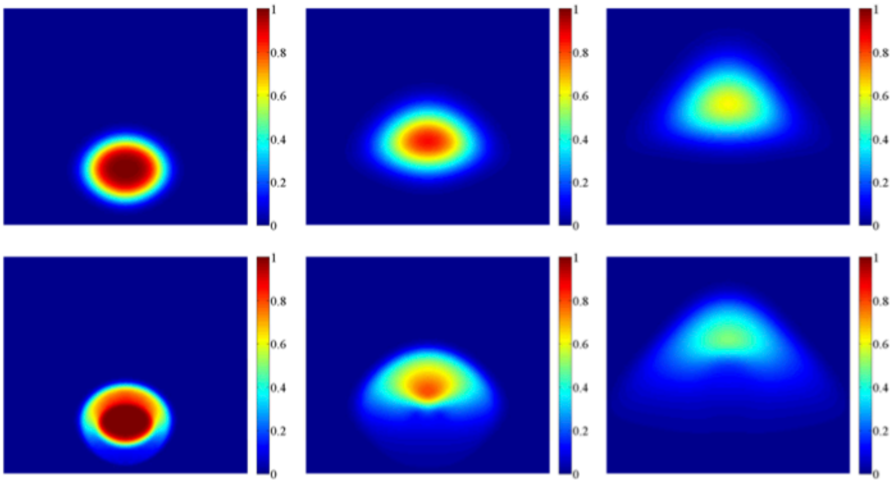

Fig. 5 From left to right: Fickian concentration (first row) and non-Fickian concentration (second row) at time 0.15, 0.5, and 1

In what follows we present one example that intent to illustrate not only the differences between Fickian and non-Fickian models, in the case of miscible displacement in porous media, but also the fact the non-Fickian model can replicate key properties observed in real world experiments.

Example 3 Let us consider the miscible displacement problem in porous media. We suppose that the resident fluid and the injected fluid are fully miscible and flow together as a unique fluid. We assume that there are no source or sink terms, i.e., $q_1 = q_2 = 0$, and that the initial distribution of the injected fluid is as given in Fig. 4 (on the right). In the pressure equation (4) we take $K = I$, $\mu = 1$, and the Dirichlet boundary conditions: 0.4 (bottom boundary), 0.2 (left and right boundaries) and 0 (top boundary). We also consider a uniform spatial mesh with size 0.004 and the time step $\Delta t = 10^{-3}$. The obtained pressure field is shown in Fig. 4 (on the left). Let c represent the concentration of the injected fluid. For both Fickian and non-Fickian model we define the diffusion tensor $D = d_m \phi I$ with $d_m = 5 \times 10^{-3}$ and $\phi = 1$, meaning that the longitudinal (α_ℓ) and transversal (α_t) dispersivity coefficients are zero. For the non-Fickian model we also take $\tau = 10^{-1}$ and $E = d_{m,nF}I$ with $d_{m,nF} = 10^{-2}$. The coupled problem is complemented with Dirichlet homogeneous boundary conditions for the concentration. This is equivalent to assume that the fluid is removed when it reaches the boundary.

In Fig. 5 we show the evolution of the concentration in the Fickian and non-Fickian case. As can be seen from the figures, the non-Fickian model is able to reproduce key features reported in experimental studies, such as highly asymmetric plumes with steep fronts and long and low concentration tails. Note that, for simplicity, in this example we omit physical units.

6 Conclusion

This paper deals with the numerical approximation of a coupled initial boundary value problem formed by the elliptic equation (8) and the integro-differential equation of Volterra type (6). This system can be used to describe, e.g., miscible transport in porous media where a memory effect in time is present.

To solve the coupled system (8), (6) we proposed the IMEX method (39), (40) which can be seen as a fully discrete in time and space piecewise linear finite element method (43), (44). The convergence properties of the method were studied. We proved in particular that the numerical pressure and concentration are second order convergent in space with respect to a discrete H^1-norm and L^2-norm, respectively. The convergence estimates (67) and (68) are somehow unexpected because (39), (40) is a finite difference method with first order truncation error with respect to the L^∞-norm. Moreover, we also proved that the IMEX method (39), (40) is first order accurate in time.

We point out that the convergence analysis was made avoiding the usual approach, introduced by Wheeler in [34], and where the error is split into two terms with the aid of an auxiliary stationary problem. This alternate technique relies on the analysis of a convenient error equation and allows to relax the smoothness assumptions required by the technique in [34].

Numerical experiments were also performed. The results of Example 1 illustrate the error estimates (67) and (68), while Example 2 shows the sharpness of these estimates, since the reduction of the smoothness of the solutions p, c imply losing the second order convergence rate. At last, in Example 3, we used the problem of miscible displacement in porous media to highlight the differences between Fickian and non-Fickian model.

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