

Corrigendum to *Cubic polynomials on Lie groups: reduction of the Hamiltonian system*, J. Phys. A: Math. Theor. 44 (2011) 355203*

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The purpose of this note is to replace Lemma 6 in page 13 of the paper, to guarantee the accuracy of other results derived from it, in particular, the discussion after Remark 4 in page 15. In the original version, the result we prove does not allow to conclude, as we claim, that the set of constants of the motion we identify can be used with Lie-Cartan theorem.

The formulation of the lemma is misleading. Besides, we need the additional hypothesis that G is semisimple to be able to prove the correct statement. Therefore, both the statement and the proof should be replaced by the following:

Lemma 1 *If the Lie group G is semisimple, then $\{l_j : j = 1, \dots, n+1\}$ is a set of functionally independent functions on an open dense subset of $\mathcal{O}_\eta \times \mathfrak{g} \times \mathfrak{g}^*$.*

PROOF. In the proof and for the sake of simplicity, we identify $\eta \in \mathfrak{g}^*$ with an element of \mathfrak{g} via the Riemannian metric. We shall also consider \mathcal{O}_η to be the adjoint orbit defined by a regular element η in a Cartan subalgebra \mathfrak{t} of \mathfrak{g} and r be the rank of \mathfrak{g} .

Consider the coordinate expression for the invariants, with respect to the natural basis taken from the orthonormal basis $\{A_i\}_{i=1, \dots, n}$ of the Lie algebra \mathfrak{g} :

$$l_1 = \sum_{j=1}^n y^j \theta_j(\nu_1, \dots, \nu_{2m}) + \frac{1}{2} \sum_{j=1}^n (\xi_j)^2$$
$$l_{i+1} = \theta_i(\nu_1, \dots, \nu_{2m}) + \sum_{j,k=1}^n C_{ji}^k y^j \xi_k, \quad i = 1, \dots, n,$$

where ν_1, \dots, ν_{2m} are the variables in the orbit \mathcal{O}_η . The differentials of the invariants can be written as

$$dl_1 = \sum_{\alpha=1}^{2m} \sum_{j=1}^n y^j \frac{\partial \theta_j}{\partial \nu_\alpha} d\nu_\alpha + \sum_{j=1}^n \theta_j dy^j + \sum_{j=1}^n \xi_j d\xi_j$$

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$$dl_{i+1} = \sum_{\alpha=1}^{2m} \frac{\partial \theta_i}{\partial \nu_\alpha} d\nu_\alpha + \sum_{j,k=1}^n C_{ji}^k \xi_k dy^j + \sum_{j,k=1}^n C_{ji}^k y^j d\xi_k, \quad i = 1, \dots, n.$$

We shall prove that $dl_1 \wedge dl_2 \wedge \dots \wedge dl_{n+1} \neq 0$ on an open dense subset of $\mathcal{O}_\eta \times \mathfrak{g} \times \mathfrak{g}^*$. The coefficients of the above exterior product corresponding to the elements $d\nu_1 \wedge d\nu_2 \wedge \dots \wedge d\nu_{2m} \wedge d\xi_{i_1} \wedge \dots \wedge d\xi_{i_{r+1}}$ are sums containing $2m$ terms, not depending on the variables ξ_i , and $r+1$ terms, each one depending linearly on a different variable ξ_i . The $r+1$ terms are given by minors of order n of the matrix representing the linear map F from $T_\theta \mathcal{O}_\eta \times \mathfrak{g}$ into \mathfrak{g} that applies (Z, W) to $i_{*\theta}(Z) - ad_Y W$, where i is the inclusion of \mathcal{O}_η into \mathfrak{g} . If we prove that the map F has full rank in an open dense subset of $\mathcal{O}_\eta \times \mathfrak{g}$, then the corresponding minor of order n of the matrix representation gives the non-vanishing term we are looking for.

In order to do so, let us recall the standard root space decomposition (see for instance [1]) for the complexified algebra $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha^{\mathbb{C}} \right)$$

with respect to a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ (i.e., $\mathfrak{g}_0^{\mathbb{C}}$ corresponds to the centralizer of $\mathfrak{t}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$ which is equal to $\mathfrak{t}^{\mathbb{C}}$ if the algebra is semisimple). The related vectors $X_\alpha, Y_\alpha \in \mathfrak{g}$ such that $[T, X_\alpha] = \alpha(T)Y_\alpha$ and $[T, Y_\alpha] = -\alpha(T)X_\alpha$, for all $T \in \mathfrak{t}$ and for each root $\alpha \in \Delta$, induce the decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\sum_{\alpha \in \Delta_+} \mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha \right)$$

and give a basis $B_{\mathfrak{g}}^1$ of \mathfrak{g} . Let us consider the tangent space $T_\theta \mathcal{O}_\eta = \{[\theta, A], A \in \mathfrak{g}\}$, for each $\theta \in \mathcal{O}_\eta$. Using the basis $B_{\mathfrak{g}}^1$, it is possible to check that there exists an open dense subset of \mathcal{O}_η defined by elements θ such that $T_\theta \mathcal{O}_\eta \cap \mathfrak{t} = \{0\}$. Under this condition, it is possible to extend a basis $B_{T_\theta \mathcal{O}_\eta}$ of $T_\theta \mathcal{O}_\eta$, using a basis of \mathfrak{t} , in order to obtain a basis $B_{\mathfrak{g}}^2$ of \mathfrak{g} . Now, we consider the basis $B_{T_\theta \mathcal{O}_\eta} \times B_{\mathfrak{g}}^1$ of $T_\theta \mathcal{O}_\eta \times \mathfrak{g}$ and the basis $B_{\mathfrak{g}}^2$ of \mathfrak{g} . It is clear that the matrix of the map F relatively to these basis has full rank for all $\theta \in \mathcal{O}_\eta$ such that $T_\theta \mathcal{O}_\eta \cap \mathfrak{t} = \{0\}$ and for all $Y = T + \sum_{\alpha \in \Delta_+} (b_\alpha X_\alpha + c_\alpha Y_\alpha)$ with no null coefficients b_α and c_α , for each $\alpha \in \Delta_+$. Therefore, we proved that the map F has full rank in an open dense subset of $\mathcal{O}_\eta \times \mathfrak{g}$. This implies that there is an open dense subset of $\mathcal{O}_\eta \times \mathfrak{g} \times \mathfrak{g}^*$ where the functions $\{l_1, \dots, l_{n+1}\}$ are functionally independent.

This new version of the Lemma guarantees the accuracy of the results contained in the last part of the original paper.

References

- [1] Helgason, S. (1978), *Differential geometry, Lie groups, and symmetric spaces*, Academic Press (New York)