NIJENHUIS FORMS ON LIE-INFINITY ALGEBRAS ASSOCIATED TO LIE ALGEBROIDS

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Abstract. Introducing Nijenhuis forms on L_∞-algebras gives a general frame to understand deformations of the latter. We give here a Nijenhuis interpretation of a deformation of an arbitrary Lie algebroid into an L_∞-algebra. Then we show that Nijenhuis forms on L_∞-algebras also give a short and efficient manner to understand Poisson-Nijenhuis structures and, more generally, the so-called exact Poisson quasi-Nijenhuis structures with background.

Keywords: L_∞-algebra, Nijenhuis, Gerstenhaber algebra, Lie algebroid.

Introduction

The present article is a continuation of [4] by the same authors, where Nijenhuis forms on Lie-infinity algebras [13] (from now on referred to as L_∞-algebras) were introduced and various examples were given. Its purpose is to show that Nijenhuis forms on L_∞-algebras are an efficient unification tool, which appears in contexts which are not obviously related neither between themselves, nor obviously related to Nijenhuis structures or higher structures.

Nijenhuis tensors arise naturally in complex geometry (a complex structure being a Nijenhuis tensor squaring to $-\mathrm{id}$), but they also appear while studying some integrable systems, in particular those which are bihamiltonian, with one of the two Poisson structures being symplectic [14]. These so-called Poisson-Nijenhuis structures can be defined on an arbitrary Lie algebroid [12, 10], give hierachies of Poisson structures, and may even describe entirely the initial integrable system [6]. Several authors [8, 11, 3, 2] have also studied Nijenhuis tensors on Loday algebras and Courant algebroids. These extensions share a common idea: a tensor N is defined to be Nijenhuis for the structure X (with X = Poisson, Lie algebroid, Courant algebroid and so on) when deforming X twice by N is like deforming X by N^2. They also share a common feature: when N is Nijenhuis, the structure X_N obtained by deforming X by N is still of the same type as X (i.e., X_N is Poisson when X is Poisson, X_N is a Lie algebroid/Courant algebroid when X is a Lie algebroid/Courant algebroid and so on) and N is still Nijenhuis for X_N. This last property allows to repeat the procedure to get a hierarchy of structures of the same type as X, structures that can be shown to be compatible (i.e., any linear combinations of those are still of the same type as the structure X).

Interesting "unifications" happen while generalizing the notion of Nijenhuis to Courant algebroids [3]. For instance, a Poisson structure on a Lie algebroid A, being a tensor from A^* to A, becomes itself a tensor on the Courant algebroid $A \oplus A^*$ which can be shown to be Nijenhuis. Hence, both Poisson structures and Nijenhuis tensors become, eventually, Nijenhuis tensors, and so do closed 2-forms, ΩN and Poisson-Nijenhuis structures [3].

Having in mind that Courant algebroids are particular cases of L_∞-algebras [15], the authors of the present article have introduced in [4] the notion of a Nijenhuis deformation on an arbitrary L_∞-algebra, defined on a graded vector space E. To reach this level of generality, however, one has to accept Nijenhuis tensors which are not just tensors, i.e. linear endomorphisms, but collections of graded symmetric multilinear endomorphisms of E, exactly as the brackets of an L_∞-algebra are a collection of n-ary brackets for an arbitrary n. These collections are referred to as (graded symmetric) vector valued forms and come equipped with a natural bracket: the Richardson-Nijenhuis bracket, which corresponds to the bracket of the coderivations of the graded symmetric algebra $S(E)$. L_∞-algebras are precisely forms of degree $+1$ commuting with themselves.

Nijenhuis forms on L_∞-algebras can however not be simply defined as vector valued forms $\mathcal{N}$ of degree 0 such that deforming the L_∞-algebra twice by $\mathcal{N}$ (i.e., taking the Richardson-Nijenhuis bracket with $\mathcal{N}$) is like deforming it by the square of $\mathcal{N}$. The difficulty is that the square of a vector valued form can not be defined anymore. However, the equivalent of these squares are often quite natural, and [4] has suggested a solution which consists in defining altogether Nijenhuis forms $\mathcal{N}$ and their "square" $\mathcal{K}$, with $\mathcal{K}$ a vector valued form of degree 0. By a square, we simply mean an other vector valued form such that deforming the L_∞-structure twice by $\mathcal{N}$ is like deforming it by $\mathcal{K}$. When this square $\mathcal{K}$ commutes with $\mathcal{N}$, a hierarchy of compatible L_∞-algebras arises naturally by deforming by $\mathcal{N}$ several times [12, 2]. The purpose of this generalization, of course, is to obtain even more "unifications" in the process.

The present article shows that two different kind of deformations can be explained by using Nijenhuis forms on L_∞-algebras defined in [4]. The first one, detailed in Section 2, is inspired by Delgado [7], who gave an explicit construction showing that, for all Lie algebroid A, the graded space of sections of $\wedge A$ carries not only the Gerstenhaber bracket, but also an intriguing L_∞-structure whose 2-ary bracket is the Gerstenhaber bracket, the 3-ary bracket is given by:

$$(P, Q, R) \mapsto [P, Q] \wedge R + \circlearrowleft_{P,Q,R}, \text{ for all } P, Q, R \in \Gamma(\wedge A),$$

and the n-ary brackets are given by similar formulas. We show in the present article that the wedge product of sections:

$$(P, Q) \mapsto P \wedge Q, \text{ for all } P, Q \in \Gamma(\wedge A),$$

which is a graded symmetric 2-form of degree 0 on $\Gamma(\wedge A)$, is co-boundary Nijenhuis (i.e., slightly more general than being Nijenhuis) with respect to the Gerstenhaber bracket. The square is, up to a coefficient, the graded symmetric 3-form of degree 0 given by the wedge product of three sections $(P, Q, R) \mapsto P \wedge Q \wedge R$. The reader used to higher structures maybe will not be surprised, but considering the wedge product as an analogous of a Nijenhuis tensor on a manifold may seem quite strange at first. By (a slightly modified version of) the general theory of [4], a pencil (indexed by $\mathbb{N}$) of L_∞-algebras can be derived: the L_∞-algebra of Delgado [7] is among this pencil. In presence of a Nijenhuis tensor N, in the usual sense, of the Lie algebroid, the L_∞-structures on this pencil can themselves be deformed by N.

In the meanwhile, we check that the wedge product of n-terms generate graded symmetric vector valued forms on $\Gamma(\wedge A)$ that, when equipped with the Richardson-Nijenhuis bracket, form a Lie algebra isomorphic to the Lie algebra of formal vector fields, which is an interesting point of its own.

Section 3 gives an other occurrence of a Nijenhuis deformation of an L_∞-algebra which is natural in Poisson geometry. Poisson structures, Poisson-Nijenhuis, ΩN structures and $P\Omega$ structures and their counterpart with background can be all unified under the notion of exact Poisson quasi-Nijenhuis structures with background [3], a special case of Poisson quasi-Nijenhuis structures with background on Lie algebroids, introduced by Antunes in [1], which, in turn, are generalizations of Poisson quasi-Nijenhuis structures on manifolds [17] and on Lie algebroids [5]. Exact Poisson quasi-Nijenhuis structures with background are made of a bivector π, a 2-form ω and a $\mathcal{C}^\infty$-linear bundle map N on a Lie algebroid A assumed to satisfy several relations, in relation with some 3-form (the background). We show that these relations precisely mean that there is a vector valued form of degree zero on the graded vector space of sections of $\wedge A$, constructed out of π, ω and N, which is a co-boundary Nijenhuis form with a certain square for the L_∞-algebra of sections of $\wedge A$ equipped with the usual Gerstenhaber bracket together with some background 3-form. We also explain how to recover the notion of Poisson quasi-Nijenhuis manifolds of Stiénon and Xu [17] in terms of Nijenhuis forms on L_∞-algebras.

1. Richardson-Nijenhuis bracket

Richardson-Nijenhuis bracket and L_∞-algebras. In this section we recall from [4] the notion of Richardson-Nijenhuis bracket of graded symmetric vector valued forms on graded vector spaces and the notion of L_∞-algebra. We also recall that it characterizes L_∞-structures on graded vector spaces. We start by fixing some notations on graded vector spaces.

Let $E = \oplus_{i\in\mathbb{Z}} E_i$ be a graded vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For a given $i \in \mathbb{Z}$, the vector space E_i is called the component of degree i, elements of E_i are called *homogeneous elements of degree* i, and elements in the union $\cup_{i\in\mathbb{Z}} E_i$ are called the *homogeneous elements*. We denote by $|X|$ the degree of a non-zero homogeneous element X. Given a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ and an integer p, $E[p]$ is the graded vector space whose component of degree i is E_{i+p}.

We denote by $S(E)$ the symmetric space of E. For a given $k \geq 0$, $S^k(E)$ is the image of $\otimes^k E$ through the quotient map $\otimes E \to S(E)$. One has the following natural decomposition

$$S(E) = \oplus_{k\geq 0} S^k(E),$$

where $S^0(E)$ is simply the field $\mathbb{K}$.

Definition 1.1. A *symmetric vector valued k-form*, $k \geq 0$, on a graded vector space E is a graded symmetric E-valued k-linear map on E.

Equivalently, a symmetric vector valued k-form on E can be seen as a linear map from the space $S^k(E)$ to E. We denote by $\mathcal{S}^k(E)$ the space of symmetric vector valued k-forms on E, while $\mathcal{S}(E)$ stands for $\oplus_{k\geq 0}\mathcal{S}^k(E)$,

$$\mathcal{S}(E) = \oplus_{k\geq 0}\mathcal{S}^k(E).$$

Notice that $\mathcal{S}^0(E)$, the space of vector valued zero-forms, is isomorphic to E.

Let E be a graded vector space, $E = \oplus_{i\in\mathbb{Z}} E_i$. *The insertion operator* of a symmetric vector valued k-form K is an operator

$$\iota_K : \mathcal{S}(E) \to \mathcal{S}(E)$$

defined by

$$\iota_K L(X_1, ..., X_{k+l-1}) = \sum_{\sigma\in Sh(k,l-1)} \epsilon(\sigma) L(K(X_{\sigma(1)}, ..., X_{\sigma(k)}), ..., X_{\sigma(k+l-1)}),$$

for all $L \in \mathcal{S}^l(E)$, $l \geq 0$ and $X_1, \cdots, X_{k+l-1} \in E$, where $Sh(i, j-1)$ stands for the set of $(i, j-1)$-unshuffles and $\epsilon(\sigma)$ is the Koszul sign which is defined as follows

$$X_{\sigma(1)} \odot \cdots \odot X_{\sigma(n)} = \epsilon(\sigma) X_1 \odot \cdots \odot X_n,$$

for all $X_1, \cdots, X_n \in E$, with $\odot$ the symmetric product.

Now, we define a bracket on the space of symmetric vector-valued forms on E as follows. Given a symmetric vector valued k-form $K \in \mathcal{S}^k(E)$ and a symmetric vector valued l-form $L \in \mathcal{S}^l(E)$, the *Richardson-Nijenhuis bracket* of K and L is the symmetric vector valued $(k+l-1)$-form $[K, L]_{_{RN}}$, given by

$$[K, L]_{_{RN}} = \iota_K L - (-1)^{\bar{K}\bar{L}} \iota_L K,$$

where $\bar{K}$ is the degree of K as a graded map, that is $K(X_1, \cdots, X_k) \in E_{1+\cdots+k+\bar{K}}$, for all $X_i \in E_i$. For an element $X \in E$, $\bar{X} = |X|$, that is, the degree of a vector valued 0-form, as a graded map, is just its degree as an element of E. We need to consider also infinite sums, which is often referred in the literature as taking the completion of $\mathcal{S}(E)$. By a *formal sum*, we mean a sequence $\phi : \mathbb{N}\bigcup\{0\} \to \mathcal{S}(E)$ mapping an integer k to an element $a_k \in \mathcal{S}^k(E)$: we shall, by a slight abuse of notation, denote by $\sum_{k=0}^{\infty} a_k$ such an element. We denote the set of all formal sums by $\tilde{\mathcal{S}}(E)$. The algebra structure on $\mathcal{S}(E)$ extends in an unique manner to $\tilde{\mathcal{S}}(E)$. For two formal sums $a = \sum_{k=0}^{\infty} a_k$ and $b = \sum_{k=0}^{\infty} b_k$ we define $a + b$ to be $\sum_{k=0}^{\infty}(a_k + b_k)$, while the the Richardson-Nijenhuis bracket of a and b is the infinite sum $\sum_{k=0}^{\infty} c_k$ with $c_k = \sum_{i=0}^{k}[a_i, b_{k+1-i}]_{_{RN}}$.

If $K \in \mathcal{S}^k(E)$ is a vector valued k-form, an easy computation gives

$$K(X_1, \cdots, X_k) = [X_k, \cdots, [X_2, [X_1, K]_{_{RN}}]_{_{RN}} \cdots]_{_{RN}}, \tag{1}$$

for all $X_1, \cdots, X_k \in E$

Next, we recall the notion of L_∞-algebra, considering all brackets to be graded symmetric as in [9].

Definition 1.2. An L_∞*-algebra* is a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ together with a family of symmetric vector valued forms $(l_i)_{i\geq 1}$ of degree 1, with $l_i : \otimes^i E \to E$ satisfying the following relation:

$$\sum_{i+j=n+1} \sum_{\sigma\in Sh(i,j-1)} \epsilon(\sigma) l_j(l_i(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), \cdots, X_{\sigma(n)}) = 0,$$

for all $n \geq 1$ and all homogeneous $X_1, \cdots, X_n \in E$, where $\epsilon(\sigma)$ is the Koszul sign. The family of symmetric vector valued forms $(l_i)_{i\geq 1}$ is called an L_∞*-structure* on the graded vector space E. Usually, we denote this L_∞-structure by $\mu := \sum_{i\geq 1} l_i$.

Two L_∞-structures μ and μ' on a graded vector space are said to be *compatible* if $[\mu, \mu']_{_{RN}} = 0$. A *pencil* indexed by a set I is a family $(\mu_i)_{i\in I}$ of pairwise compatible L_∞-structures.

The Richardson-Nijenhuis bracket on graded vector spaces, introduced previously, is intimately related to L_∞-algebras. In the next theorem, that appears in an implicit form in [16], we use the Richardson-Nijenhuis bracket to characterize a L_∞-structure on a graded vector space E as a homological vector field on E.

Theorem 1.3. *Let $E = \oplus_{i\in\mathbb{Z}} E_i$ be a graded vector space and $(l_i)_{i\geq 1} : \otimes^i E \to E$ be a family of symmetric vector valued forms on E of degree 1. Set $\mu = \sum_{i\geq 1} l_i$. Then, μ is an L_∞-structure on E if and only if $[\mu, \mu]_{_{RN}} = 0$.*

Richardson-Nijenhuis bracket and Gerstenhaber algebras. Let $(A = \oplus_{i\in\mathbb{Z}} A^i, \wedge)$ be a graded commutative associative algebra. Recall that a graded Lie algebra structure on $A[1]$ is called a *Gerstenhaber algebra* bracket if, for each $P \in A^i[1]$, $[P, \cdot]$ is a derivation of degree i of $(A = \oplus_{i\in\mathbb{Z}} A^i, \wedge)$. A Gerstenhaber algebra will be denoted by $(A, [\cdot, \cdot], \wedge)$.

Remark 1.4. It follows from the definition of a Gerstenhaber algebra $(A, [\cdot, \cdot], \wedge)$ that, for all homogeneous elements $P, Q, R \in A$, the following identities hold:

$$[P, Q] = -(-1)^{(deg(P)-1)(deg(Q)-1)} [Q, P], \tag{2}$$

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(deg(P)-1)\,deg(Q)} Q \wedge [P, R]. \tag{3}$$

Let $(A, [\cdot,\cdot], \wedge)$ be a Gerstenhaber algebra. For all positive integers k, and all homogeneous elements $P_1, ..., P_k \in A$, set

$$E_{-k} := A^k, \quad \text{i.e.,} \quad |P| = -deg(P) = -k, \quad \text{if} \quad P \in A^k,$$

$$\mathcal{N}_k\left(P_1, \cdots, P_k\right) := P_1 \wedge \cdots \wedge P_k,$$

$$l_1 := 0, \qquad l_2(P_1, P_2) := (-1)^{|P_1|}\left[P_1, P_2\right]$$

and for $i > 2$,

$$\begin{aligned} l_i\left(P_1, \cdots, P_i\right) &:= \iota_{l_2}\mathcal{N}_{i-1}\left(P_1, \cdots, P_i\right) \\ &= \textstyle\sum_{\sigma \in Sh(2,i-2)} \epsilon(\sigma)(-1)^{|P_{\sigma(1)}|}\left[P_{\sigma(1)}, P_{\sigma(2)}\right] \wedge P_{\sigma(3)} \wedge \cdots \wedge P_{\sigma(i)}. \end{aligned}$$

By construction, $\mathcal{N}_k \in \mathcal{S}^k(E)$ has degree 0 and $l_k \in \mathcal{S}^k(E)$ has degree 1, for all $k \geq 1$.

Lemma 1.5. *For any Gerstenhaber algebra $(A, [\cdot,\cdot], \wedge)$, the l_i's and $\mathcal{N}_i$'s introduced above satisfy:*

(a) $l_2(P \wedge Q, R) = (-1)^{|Q||R|}\, l_2(P, R) \wedge Q + (-1)^{|P|(|Q|+|R|)}\, l_2(Q, R) \wedge P$, *for all homogeneous $P, Q, R \in A$;*
(b) $[\mathcal{N}_i, \mathcal{N}_j]_{_{RN}} = \frac{(j-i)(i+j-1)!}{i!j!}\mathcal{N}_{i+j-1}$, *for all positive integers i, j;*
(c) $[\mathcal{N}_m, l_n]_{_{RN}} = \binom{m+n-2}{m} l_{m+n-1}$, *for all $m, n \geq 2$;*
(d) $[l_m, l_n]_{_{RN}} = 0$, *for all positive integers m, n.*

Proof. Equations (2) and (3) in Remark 1.4 and the definition of l_2 prove *(a)*. A direct computation proves *(b)*. Let us now prove *(c)*. By definition of the insertion operator, we have

$$\begin{aligned} &\iota_{\mathcal{N}_m} l_n(P_1, \cdots, P_{m+n-1}) = \\ &\textstyle\sum_{\sigma \in Sh(m,n-1)} \epsilon(\sigma)\, l_n\left(P_{\sigma(1)} \wedge \cdots \wedge P_{\sigma(m)}, P_{\sigma(m+1)}, \cdots, P_{\sigma(m+n-1)}\right). \end{aligned} \tag{4}$$

For a fixed $\sigma \in Sh\,(m, n-1)$ we have, by definition of l_n,

$$\begin{aligned} & l_n\left(P_{\sigma(1)} \wedge \cdots \wedge P_{\sigma(m)}, P_{\sigma(m+1)}, \cdots, P_{\sigma(m+n-1)}\right) \\ = & \textstyle\sum_{i=m+1}^{m+n-1} (-1)^{\alpha_i}\, l_2\left(P_{\sigma(1)} \wedge \cdots \wedge P_{\sigma(m)}, P_{\sigma(i)}\right) \wedge P_{\sigma(m+1)} \wedge \overset{\widehat{i}}{\cdots} \wedge P_{\sigma(m+n-1)} \\ + & \textstyle\sum_{m+1 \leq r < s \leq m+n-1} (-1)^{\alpha_{r,s}}\, l_2\left(P_{\sigma(r)}, P_{\sigma(s)}\right) \wedge P_{\sigma(1)} \wedge \overset{\widehat{r,s}}{\cdots} \wedge P_{\sigma(m+n-1)}, \end{aligned}$$

where

$$\alpha_i = |P_{\sigma(i)}|\left(|P_{\sigma(m+1)}| + \cdots + |P_{\sigma(i-1)}|\right)$$

and

$$\alpha_{r,s} = |P_{\sigma(s)}|\left(|P_{\sigma(1)}| + \cdots + |P_{\sigma(s-1)}|\right) + |P_{\sigma(r)}|\left(|P_{\sigma(s)}| + |P_{\sigma(1)}| + \cdots + |P_{\sigma(r-1)}|\right).$$

Hence, using (2) and (3), we get

$$\begin{aligned} & l_n\left(P_{\sigma(1)} \wedge \cdots \wedge P_{\sigma(m)}, P_{\sigma(m+1)}, \cdots, P_{\sigma(m+n-1)}\right) \\ = & \textstyle\sum_{i=m+1}^{m+n-1} (-1)^{\alpha_i} \sum_{j=1}^{m} (-1)^{\beta_{i,j}}\, l_2\left(P_{\sigma(j)}, P_{\sigma(i)}\right) \wedge P_{\sigma(1)} \wedge \overset{\widehat{j,i}}{\cdots} \wedge P_{\sigma(m+n-1)} \\ + & \textstyle\sum_{m+1 \leq r < s \leq m+n-1} (-1)^{\alpha_{r,s}}\, l_2\left(P_{\sigma(r)}, P_{\sigma(s)}\right) \wedge P_{\sigma(1)} \wedge \overset{\widehat{r,s}}{\cdots} \wedge P_{\sigma(m+n-1)}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \beta_{i,j} &= |P_{\sigma(i)}|\left(|P_{\sigma(1)}| + \cdots + |P_{\sigma(i-1)}|\right) + |P_{\sigma(j)}|\left(|P_{\sigma(1)}| + \cdots + |P_{\sigma(j-1)}|\right) \\ &\quad + |P_{\sigma(j)}||P_{\sigma(i)}|. \end{aligned}$$

Now, for two integers i and j, with $m+1 \leq i \leq m+n-1$ and $1 \leq j \leq m$, we introduce a new permutation, which depends on σ, i and j, denoted by $\tau_{\sigma ij}$, as follows:

$$\tau_{\sigma ij}(1) = \begin{cases} \sigma(i), & \text{if } \sigma(i) < \sigma(j) \\ \sigma(j), & \text{if } \sigma(j) < \sigma(i); \end{cases} \tag{6}$$

$$\tau_{\sigma ij}(2) = \begin{cases} \sigma(j), & \text{if } \sigma(i) < \sigma(j) \\ \sigma(i), & \text{if } \sigma(j) < \sigma(i); \end{cases} \tag{7}$$

(8)
$$\tau_{\sigma ij}(k+1) = \min\left(\{\sigma(1), \cdots, \sigma(m+n-1)\} - \{\tau_{\sigma ij}(1), \cdots, \tau_{\sigma ij}(k)\}\right), \; k \geq 2.$$

Next, for two integers r and s, with $m+1 \leq r < s \leq m+n-1$, we introduce another permutation, which depends on σ, r and s, denoted by $\varphi_{\sigma rs}$, as follows:

$$\varphi_{\sigma rs}(1) = \sigma(r), \quad \varphi_{\sigma rs}(2) = \sigma(s), \tag{9}$$

(10)
$$\varphi_{\sigma rs}(k+1) = \min\left(\{\sigma(1), \cdots, \sigma(m+n-1)\} - \{\varphi_{\sigma rs}(1), \cdots, \varphi_{\sigma rs}(k)\}\right), \; k \geq 2.$$

Then, we have

$$\tau_{\sigma ij}, \varphi_{\sigma rs} \in Sh\,(2, m+n-3)$$

and

(11)
$$\begin{array}{rl} & \epsilon(\sigma)\, l_n\left(P_{\sigma(1)} \wedge \cdots \wedge P_{\sigma(m)}, P_{\sigma(m+1)} \cdots, P_{\sigma(m+n-1)}\right) \\ = & \sum_{i=m+1}^{m+n-1} \sum_{j=1}^{m} \epsilon(\tau_{\sigma ij})\, l_2\left(P_{\tau_{\sigma ij}(1)}, P_{\tau_{\sigma ij}(2)}\right) \wedge P_{\tau_{\sigma ij}(3)} \wedge \cdots \wedge P_{\tau_{\sigma ij}(m+n-1)} \\ + & \sum_{m+1 \leq r < s \leq m+n-1} \epsilon(\varphi_{\sigma rs})\, l_2\left(P_{\varphi_{\sigma rs}(1)}, P_{\varphi_{\sigma rs}(2)}\right) \wedge P_{\varphi_{\sigma rs}(3)} \wedge \cdots \wedge P_{\varphi_{\sigma rs}(m+n-1)}. \end{array}$$

Note that $\epsilon(\tau_{\sigma ij}) = \epsilon(\sigma)(-1)^{\alpha_i}(-1)^{\beta_{i,j}}(-1)^{\beta}$ and $\epsilon(\varphi_{\sigma rs}) = \epsilon(\sigma)(-1)^{\alpha_{r,s}}(-1)^{\gamma}$, where $(-1)^{\beta}$ is the sign of the permutation

$$\begin{pmatrix} \sigma(i) & \sigma(j) & \sigma(1) & \sigma(2) & \overset{\widehat{i,j}}{\cdots} & \sigma(m+n-1) \\ \tau_{\sigma ij}(1) & \tau_{\sigma ij}(2) & \cdots & \cdots & \cdots & \tau_{\sigma ij}(m+n-1) \end{pmatrix}$$

and $(-1)^{\gamma}$ is the sign of the permutation

$$\begin{pmatrix} \sigma(r) & \sigma(s) & \sigma(1) & \sigma(2) & \overset{\widehat{r,s}}{\cdots} & \sigma(m+n-1) \\ \varphi_{\sigma rs}(1) & \varphi_{\sigma rs}(2) & \cdots & \cdots & \cdots & \varphi_{\sigma rs}(m+n-1) \end{pmatrix}.$$

Equation (11) shows that for each $\sigma \in Sh\,(m, n-1)$,

$$\epsilon(\sigma)\, l_n\left(P_{\sigma(1)} \wedge \cdots \wedge P_{\sigma(m)}, P_{\sigma(m+1)} \cdots, P_{\sigma(m+n-1)}\right)$$

has $m\,(n-1) + \binom{n-1}{2}$ terms of the form

$$\epsilon(\tau)\, l_2\left(P_{\tau(1)}, P_{\tau(2)}\right) \wedge P_{\tau(3)} \cdots, P_{\tau(i+k-1)}, \tag{12}$$

with $\tau \in Sh\,(2, m+n-3)$. [1] Therefore the right hand side of (4) has $A = \binom{m+n-1}{m}\left(m\,(n-1) + \binom{n-1}{2}\right)$ terms of the form (12). Note that

$$\begin{array}{rcl} A & = & \binom{m+n-1}{m}\left(\frac{(n-1)(n+2m-2)}{2}\right) \\ & = & \frac{(m+n-1)!}{2!(m+n-3)!} \times \frac{(m+n-3)!}{m!(n-2)!} \times (n+m-2+m) \\ & = & \binom{m+n-1}{2} \times \left(\binom{m+n-2}{m} + \binom{m+n-3}{m-1}\right). \end{array}$$

[1]Note that not all the permutations in $Sh\,(2, m+n-3)$ appear in (12) for a single σ. But, since for each unshuffle $\tau \in Sh\,(2, m+n-3)$ and $1 \leq j \leq m$, $m+1 \leq i \leq m+n-1$, there exists an unshuffle $\sigma \in Sh\,(m, n-1)$ such that $\tau(1) = \sigma(i)$ and $\tau(2) = \sigma(j)$, all elements in $Sh\,(2, m+n-3)$ can be obtained by Equations (6), (7), (8), (9) and (10).

This shows that

$$\iota_{\mathcal{N}_m} l_n = \left(\binom{m+n-2}{m} + \binom{m+n-3}{m-1} \right) l_{m+n-1}. \tag{13}$$

Similar computations show that

$$\iota_{l_n} \mathcal{N}_m = \binom{m+n-3}{m-1} l_{m+n-1}. \tag{14}$$

Equations (13) and (14) prove *(c)*.

We now prove *(d)*. Note that the proof below will be interpreted easily with the help of co-boundary Nijenhuis forms that appear in the next section.

Item *(c)* and the graded Jacobi identity of the Richardson-Nijenhuis yield:

$$\begin{array}{rcl} [l_m, l_n]_{RN} & = & \left[[\mathcal{N}_{m-1}, l_2]_{RN}, [\mathcal{N}_{n-1}, l_2]_{RN} \right]_{RN} \\ & = & \left[\mathcal{N}_{n-1}, \left[l_2, [\mathcal{N}_{m-1}, l_2]_{RN} \right]_{RN} \right]_{RN} + \left[l_2, \left[[\mathcal{N}_{m-1}, l_2]_{RN}, \mathcal{N}_{n-1} \right]_{RN} \right]_{RN} \\ & = & \left[l_2, [l_m, \mathcal{N}_{n-1}]_{RN} \right]_{RN} \\ & = & -\binom{m+n-3}{n-1} [l_2, l_{m+n-2}]_{RN} \\ & = & -\binom{m+n-3}{n-1} \left[l_2, [\mathcal{N}_{m+n-3}, l_2]_{RN} \right]_{RN} \\ & = & 0. \end{array}$$

□

We conclude this section with the following remark:

Remark 1.6. Let us interpret the Lie brackets computed in Lemma 1.5. The Lie algebra $\mathcal{V}$ of polynomial vector fields on $\mathbb{R}$ is generated by the vectors

$$\nu_i = x^i \frac{\partial}{\partial x}, \quad i \geq 0,$$

whose Lie bracket is given by $[\nu_i, \nu_j] = (j-i)\nu_{i+j-1}$. It acts on the space $\mathcal{P}$ of polynomials in one indeterminate, which is generated by $x_j := x^j$, as

$$\nu_i[x_j] = j\, x_{i+j-1},$$

allowing to endow the semi-direct product $\mathcal{V} \ltimes \mathcal{P}$ with the following Lie algebra bracket:

$$[\nu_i, \nu_j] = (j-i)\nu_{i+j-1},\ [\nu_i, x_j] = j\, x_{i+j-1},\ [x_i, x_j] = 0. \tag{15}$$

Consider E the vector space generated by $(\mathcal{N}_k)_{k\geq 1}$ and $(l_k)_{k\geq 1}$. Let us identify E as a vector space with $\mathcal{V} \ltimes \mathcal{P}$ through the isomorphism:

$$\begin{array}{rcl} \nu_i & \mapsto & i!\,\mathcal{N}_i \\ x_i & \mapsto & i!\, l_{i+1}. \end{array}$$

Lemma 1.5 means that the above isomorphism maps the semi-direct product Lie bracket given in (15) to the Schouten-Nijenhuis bracket. Equivalently, the vector space generated by $(\mathcal{N}_k)_{k\geq 1}$ and $(l_k)_{k\geq 1}$ is isomorphic, when equipped with the Schouten-Nijenhuis bracket, to the semi-direct product of the Lie algebra of polynomial vector fields on $\mathbb{R}$ with polynomial functions on $\mathbb{R}$.

Remark 1.7. We chose to prove item (c) in Lemma 1.5 by a direct computation, which was certainly complicated. Let us state, however, that it is quite clear that, for all positive integers i, j, $[\mathcal{N}_i, l_j]_{RN}$ is proportional to l_{i+j-1}. It suffices to realize that, while computing $[\mathcal{N}_i, l_j]_{RN}(P_1, \ldots, P_{i+j-1})$, all terms that will come out will be proportional to

$$[P_i, P_j] \wedge P_1 \wedge \stackrel{\widehat{j,i}}{\cdots} \wedge P_{i+j-1},$$

for some indices i and j. Using skew-symmetry, we see that the coefficient in front of the previous terms has to be proportional to the sign of the permutation mapping

i and j to the first and second position on the left. However, we need to be more precise and obtain this coefficient, and it seems there are no theoretical arguments giving it.

2. Pencils of L_∞-structures on Lie algebroids

In this section we recall from [4] the notion of Nijenhuis vector valued form with respect to a given vector valued form μ and deformation of μ by a Nijenhuis vector valued form. We then describe two examples.

Definition 2.1. Let E be a graded vector space and μ be a symmetric vector valued form on E of degree 1. A vector valued form $\mathcal{N}$ of degree zero is called

- *weak Nijenhuis* with respect to μ if
$$\Big[\mu, \big[\mathcal{N}, [\mathcal{N},\mu]_{_{RN}}\big]_{_{RN}}\Big]_{_{RN}} = 0,$$
- *co-boundary Nijenhuis* with respect to μ if there exists a vector valued form $\mathcal{K}$ of degree zero, such that
$$\big[\mathcal{N}, [\mathcal{N},\mu]_{_{RN}}\big]_{_{RN}} = [\mathcal{K},\mu]_{_{RN}},$$
- *Nijenhuis* with respect to μ if there exists a vector valued form $\mathcal{K}$ of degree zero, such that
$$\big[\mathcal{N}, [\mathcal{N},\mu]_{_{RN}}\big]_{_{RN}} = [\mathcal{K},\mu]_{_{RN}} \quad \text{and} \quad [\mathcal{N},\mathcal{K}]_{_{RN}} = 0.$$
Such a $\mathcal{K}$ is called a *square* of $\mathcal{N}$.

Notice that $\mathcal{N}$ may contain an element of the underlying graded vector space.

Recall from [4] the following proposition:

Proposition 2.2. *Let μ be an L_∞-structure on a graded vector space E and let $\mathcal{N}$ be a vector valued form of degree zero. We have:*

(a) $\mathcal{N}$ is weak Nijenhuis with respect to μ if and only if $[\mathcal{N},\mu]_{_{RN}}$ is an L_∞-structure on E.

(b) If $\mathcal{N}$ is Nijenhuis with respect to μ, then the L_∞-structures $[\mathcal{N},\mu]_{_{RN}}$ and μ are compatible, i.e., $\big[[\mathcal{N},\mu]_{_{RN}},\mu\big]_{_{RN}} = 0$.

The L_∞-structure $[\mathcal{N},\mu]_{_{RN}}$ is called the *deformed structure* of μ by $\mathcal{N}$.

L_∞-structures mixing products and brackets on Gerstenhaber algebras. Next we give an example of co-boundary Nijenhuis vector valued forms with respect to L_∞-algebras obtained on a Gerstenhaber algebra.

Theorem 2.3. *Let $\left(A = \oplus_{k\in\mathbb{Z}} A^k, [\cdot,\cdot], \wedge\right)$ be a Gerstenhaber algebra and set $E = \oplus_{k\in\mathbb{Z}} E^k$, with $E_{-k} := A^k$. For all positive integers i and all homogeneous elements $P_1,\cdots,P_i \in A$, set $\mathcal{N}_i(P_1,\cdots,P_i) := P_1\wedge\cdots\wedge P_i$. Let $l_1 := 0$,*

$$l_2(P_1,P_2) := (-1)^{|P_1|}[P_1,P_2] \tag{16}$$

and for $i>2$,

$$\begin{aligned} l_i(P_1,\cdots,P_i) &:= \iota_{l_2}\mathcal{N}_{i-1}(P_1,\cdots,P_i) \\ &= \textstyle\sum_{\sigma\in Sh(2,i-2)} \epsilon(\sigma)(-1)^{|P_{\sigma(1)}|}\left[P_{\sigma(1)},P_{\sigma(2)}\right]\wedge P_{\sigma(3)}\wedge\cdots\wedge P_{\sigma(i)}. \end{aligned}$$

Then,

(a) for any $n\geq 1$, l_n is an L_∞-structure on E and for all $n,m\geq 1$, $\mathcal{N}_m$ is co-boundary Nijenhuis with respect to l_n and, up to a constant, l_{n+m-1} is the deformed structure of l_n by $\mathcal{N}_m$;

(b) the family $(l_n)_{n\geq 1}$ is a pencil of L_∞-algebras on E, therefore $\mu = \sum_{i\geq 1} a_i l_i$ is an L_∞-algebra on E, for all reals a_i;

(c) $\mathcal{N} := \sum_{i\geq 1} b_i \mathcal{N}_i$, *where* $b_i \in \mathbb{R}$ *for all positive integer* i, *is a co-boundary Nijenhuis vector valued form with respect to* l_n*;*

(d) $\mathcal{N}$ *is a weak Nijenhuis vector valued form with respect to* $\mu = \sum_{i\geq 1} a_i l_i$, *with* $a_i \in \mathbb{R}$.

Proof. First, notice that since $l_1 = 0$, $\mathcal{N}_m$ is co-boundary Nijenhuis with respect to l_1, for all $m \geq 1$. Also, because $\mathcal{N}_1$ is the identity on A, we have $[\mathcal{N}_1, l_n]_{RN} = (n-1)l_n$, for all $n \geq 1$, which implies that $\mathcal{N}_1$ is co-boundary Nijenhuis with respect to l_n, for all $n \geq 1$. By item *(c)* of Lemma 1.5, for all $m, n \geq 2$ there exists a constant $A_{n,m}$ such that

$$[\mathcal{N}_m, [\mathcal{N}_m, l_n]_{RN}]_{RN} \quad = \quad A_{n,m}\,[\mathcal{N}_{2m-1}, l_n]_{RN}\,. \tag{17}$$

This formula means that $\mathcal{N}_m$ is co-boundary Nijenhuis for l_2 for all $m \geq 2$. By item *(c)* of Lemma 1.5 specialized to $n = 2$, the deformed structure is, up to a constant, l_{m+1}. The latter is therefore by Proposition 2.2 an L_∞-structure for all $m \geq 1$, and is compatible with l_2. Relation (17) then also means that $\mathcal{N}_m$ is co-boundary Nijenhuis with respect to l_n for all $m, n \geq 2$. By item *(c)* of Lemma 1.5, the deformed structure is, up to a constant, l_{n+m-1}. By item *(d)* of Lemma 1.5, l_n and l_m are compatible for all $m, n \geq 1$, i.e. $(l_n)_{n\geq 1}$ is a pencil of L_∞-algebras. This proves the two first items.

Observe that, using Lemma 1.5 *(c)* , we have

$$[\mathcal{N}, l_n]_{RN} = \left[\sum_{i\geq 1} b_i \mathcal{N}_i, l_n\right]_{RN} = \sum_{i\geq 1} b_i \binom{i+n-2}{i} l_{i+n-1}. \tag{18}$$

Hence,

$$\begin{aligned}
[\mathcal{N}, [\mathcal{N}, l_n]_{RN}]_{RN} &= \textstyle\sum_{i\geq 1} b_i \binom{i+n-2}{i} \left[\sum_{j\geq 1} b_j \mathcal{N}_j, l_{n+i-1}\right]_{RN} \\
&= \textstyle\sum_{i,j\geq 1} b_i b_j \binom{i+n-2}{i}\binom{j+i+n-3}{j} l_{n+i+j-2} \\
&= \textstyle\sum_{i,j\geq 1} b_i b_j \frac{\binom{i+n-2}{i}\binom{j+i+n-1}{j}}{\binom{j+i+n-3}{i+j-1}} [\mathcal{N}_{i+j-1}, l_n]_{RN} \\
&= \left[\textstyle\sum_{i,j\geq 1} b_i b_j \frac{\binom{i+n-2}{i}\binom{j+i+n-3}{j}}{\binom{j+i+n-3}{i+j-1}} \mathcal{N}_{i+j-1}, l_n\right]_{RN}.
\end{aligned}$$

This proves that $\mathcal{N}$ is co-boundary Nijenhuis with respect to the L_∞-structure l_n, with square

$$\sum_{i,j\geq 1} b_i b_j \frac{\binom{i+n-2}{i}\binom{j+i+n-3}{j}}{\binom{j+i+n-3}{i+j-1}} \mathcal{N}_{i+j-1},$$

which proves *(c)*. By Equation (18), there is a family of reals $(c_i)_{i\geq 1}$ such that $[\mathcal{N}, \mu]_{RN} = \sum_{i\geq 1} c_i l_i$. From item *(b)*, $[\mathcal{N}, \mu]_{RN}$ is an L_∞-structure on the graded vector space $E = \oplus_{k\in\mathbb{Z}} E_k$. By Proposition 2.2, this proves that $\mathcal{N}$ is weak Nijenhuis with respect to μ, which proves *(d)*. □

Remark 2.4. Theorem 2.1.1 in Delgado [7], which was our starting point, is a particular case of item *(b)* of Theorem 2.3, by putting $a_i = 1$, for all positive integers i.

L_∞-structures mixing products and brackets in the presence of Nijenhuis $\mathcal{C}^\infty$-linear bundle maps on Lie algebroids. Theorem 2.3 in particular holds true for the Gestenhaber algebra of a Lie algebroid.

Given a Lie algebroid $(A, [.,.], \rho)$ over a manifold M, we denote by $[.,.]_{SN}$ the *Schouten-Nijenhuis* bracket on the space of multivectors of A and by d^A (or simply d, if there is no risk of confusion) the differential of A. Set $A'_i := \Gamma(\wedge^{i+1}A)$ and $A' = \oplus_{i\geq -1}A'_i$, with $A'_{-1} = \Gamma(\wedge^0 A) = \mathcal{C}^\infty(M)$. It is well known ([12]) that the Schouten-Nijenhuis bracket is a graded skew-symmetric bracket of degree zero on $A' = \oplus_{i\geq -1}A'_i$ that defines a graded Lie algebra bracket on $A' = \Gamma(\wedge A)[1]$ and that the differential d^A is a derivation of $\Gamma(\wedge A^*)$ that squares to zero. It is also well known that $\left(A' = \Gamma(\wedge A)[1], [\cdot,\cdot]_{SN}, \wedge\right)$ is a Gerstenhaber algebra.

Given a $\mathcal{C}^\infty$-linear bundle map $N : A \to A$ on a Lie algebroid $(A, [\cdot,\cdot], \rho)$, we define a linear map $\underline{N}$ on the graded vector space $\Gamma(\wedge A)[2]$, by setting

$$\underline{N}(f) := 0,$$

for all $f \in \mathcal{C}^\infty(M)$, and

$$\underline{N}(P) := \sum_{i=1}^{p} P_1 \wedge \cdots \wedge P_{i-1} \wedge N(P_i) \wedge P_{i+1} \wedge \cdots \wedge P_p,$$

for all monomial multi-sections $P = P_1 \wedge \cdots \wedge P_p \in \Gamma(\wedge^p A)[2]$. The map $\underline{N}$ is called the *extension of N by derivation* on the graded vector space $\Gamma(\wedge A)[2]$. It is a derivation on the graded vector space $\Gamma(\wedge A)[2]$, hence a symmetric vector valued 1-form on $\Gamma(\wedge A)[2]$, and it has degree zero.

A straightforward computation leads to the following:

Lemma 2.5. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, $N : A \to A$ a $\mathcal{C}^\infty$-linear bundle map and $\underline{N}$ its extension by derivation on the space of multi-sections $\Gamma(\wedge A)$. Then, for all integers $k \geq 1$, we have*

$$[\underline{N}, \mathcal{N}_k]_{RN} = 0, \tag{19}$$

where $\mathcal{N}_k(P_1, \cdots, P_k) := P_1 \wedge \cdots \wedge P_k$, for all homogeneous elements $P_1, \cdots, P_k \in \Gamma(A)$.

Let us now recall other notions and results that will be used in the sequel. Let $(A, \mu = [\cdot,\cdot], \rho)$ be a Lie algebroid and $N : A \to A$ a $\mathcal{C}^\infty$-linear bundle map. We may define a deformed bracket $\mu^N = [\cdot,\cdot]_N$ by setting

$$[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y]. \tag{20}$$

We denote by $[\cdot,\cdot]^N_{SN}$ the Schouten-Nijenhuis bracket on $\Gamma(\wedge A)$, associated to μ^N. Set

$$l_2^N(P, Q) = (-1)^{p-1}[P, Q]^N_{SN}, \ P \in \Gamma(\wedge^p A), Q \in \Gamma(\wedge A).$$

A direct computation gives

$$[\underline{N}, l_2]_{RN} = l_2^N \tag{21}$$

(see [4]).

The Nijenhuis torsion of N with respect to $\mu = [\cdot,\cdot]$, that we denote by $\mathcal{T}_\mu N$, is defined by

$$\mathcal{T}_\mu N(X, Y) := [NX, NY] - N[X, Y]_N,$$

for all sections $X, Y \in \Gamma(A)$, or, equivalently, by

$$\mathcal{T}_\mu N(X, Y) := \frac{1}{2}\left([X, Y]_{N,N} - [X, Y]_{N^2}\right), \tag{22}$$

where $[\cdot,\cdot]_{N,N} := \left([\cdot,\cdot]_N\right)_N$ and $N^2 = N \circ N$. If $\mathcal{T}_\mu N = 0$, then N is called *Nijenhuis* and $\left(A, \mu^N = [\cdot,\cdot]_N, \rho \circ N\right)$ is a Lie algebroid.

Lemma 2.6. [4] *For every Nijenhuis $\mathcal{C}^\infty$-linear bundle map N on a Lie algebroid $(A, [\cdot,\cdot], \rho)$, the extension by derivation of N, $\underline{N}$, is a Nijenhuis vector valued 1-form with respect to the L_∞-structure l_2 given by (16), on the Gerstenhaber algebra $(\Gamma(\wedge A)[2], [\cdot,\cdot]_{SN}, \wedge)$, with square $\underline{N^2}$, i.e.,*

$$\left[\underline{N}, [\underline{N}, l_2]_{RN}\right]_{RN} = \left[\underline{N^2}, l_2\right]_{RN} \quad \text{and} \quad \left[\underline{N}, \underline{N^2}\right]_{RN} = 0.$$

In the next theorem, we see how a Nijenhuis $\mathcal{C}^\infty$-linear bundle map on a Lie algebroid, appears as a Nijenhuis vector valued 1-form with respect to the associated L_∞-algebras, according to Theorem 2.3.

Theorem 2.7. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid and N a Nijenhuis $\mathcal{C}^\infty$-linear bundle map. Let $\mu := \sum_{i\geq 1} a_i l_i$ be the L_∞-algebra as in Theorem 2.3. Then $\underline{N}$ is a Nijenhuis vector valued 1-form with respect to μ, with square $\underline{N^2}$.*

Proof. Using the graded Jacobi identity of the Richardson-Nijenhuis bracket, Lemma 1.5 *(c)* and Equation (19), we have

$$\begin{aligned}
[\underline{N}, l_k]_{RN} &= \left[\underline{N}, [\mathcal{N}_{k-1}, l_2]_{RN}\right]_{RN} \\
&= \left[[\underline{N}, \mathcal{N}_{k-1}]_{RN}, l_2\right]_{RN} + \left[\mathcal{N}_{k-1}\, [\underline{N}, l_2]_{RN}\right]_{RN} \\
&= \left[\mathcal{N}_{k-1}\, [\underline{N}, l_2]_{RN}\right]_{RN}.
\end{aligned}$$

Hence, by (19) and Lemma 2.6, we get

$$\begin{aligned}
\left[\underline{N}, [\underline{N}, l_k]_{RN}\right]_{RN} &= \left[\underline{N}, \left[\mathcal{N}_{k-1}\, [\underline{N}, l_2]_{RN}\right]_{RN}\right]_{RN} \\
&= \left[[\underline{N}, \mathcal{N}_{k-1}]_{RN}, [\underline{N}, l_2]_{RN}\right]_{RN} + \left[\mathcal{N}_{k-1}, \left[\underline{N}, [\underline{N}, l_2]_{RN}\right]_{RN}\right]_{RN} \\
&= \left[\mathcal{N}_{k-1}, \left[\underline{N}, [\underline{N}, l_2]_{RN}\right]_{RN}\right]_{RN} \\
&= \left[\mathcal{N}_{k-1}, \left[\underline{N^2}, l_2\right]_{RN}\right]_{RN} \\
&= \left[\left[\mathcal{N}_{k-1}, \underline{N^2}\right]_{RN}, l_2\right]_{RN} + \left[\underline{N^2}, [\mathcal{N}_{k-1}, l_2]_{RN}\right]_{RN} \\
&= \left[\underline{N^2}, [\mathcal{N}_{k-1}, l_2]_{RN}\right]_{RN} \\
&= \left[\underline{N^2}, l_k\right]_{RN}.
\end{aligned}$$

Therefore,

$$\left[\underline{N}, [\underline{N}, \mu]_{RN}\right]_{RN} = \left[\underline{N^2}, \mu\right]_{RN}.$$

Since $\left[\underline{N}, \underline{N^2}\right]_{RN} = 0$ holds, we conclude that $\underline{N}$ is Nijenhuis with respect to μ with square $\underline{N^2}$. □

3. Exact Poisson quasi-Nijenhuis structures with background

In this section we give another example of co-boundary Nijenhuis on an L_∞-algebra, in relation with Poisson structures on Lie algebroids. Given a Lie algebroid A and a closed 3-form H, we may define an L_∞-structure μ on the graded vector space $\Gamma(\wedge A)[2]$ obtained out of the Lie algebroid structure and H. We study the conditions that a $\mathcal{C}^\infty$-linear bundle map $N : A \to A$, a bivector π and a 2-form ω on A should satisfy in order that the vector valued form $\mathcal{N} = \pi + \underline{N} + \underline{\omega}$ is a co-boundary Nijenhuis vector valued form with respect to μ, with a certain square. The conditions we shall obtain are similar to the conditions which define an exact Poisson quasi-Nijenhuis structure with background on A.

Let $(A, \mu = [\cdot,\cdot], \rho)$ be a Lie algebroid and $\pi \in \Gamma\left(\wedge^2 A\right)$ a Poisson bivector, that is, $[\pi,\pi]_{SN} = 0$. The *Koszul bracket* $\{\cdot,\cdot\}^\pi_\mu$ on the space $\Gamma(A^*)$ is defined by:

(23) $$\{\alpha,\beta\}^{\pi}_{\mu} := \mathcal{L}_{\pi^{\#}(\alpha)}\beta - \mathcal{L}_{\pi^{\#}(\beta)}\alpha - \mathsf{d}^{A}\left(\pi\left(\alpha,\beta\right)\right),$$

where $\pi^{\#}(\alpha)$ is the section defined by $\left\langle \beta, \pi^{\#}(\alpha)\right\rangle = \pi(\alpha,\beta)$ and $\mathcal{L}_{\pi^{\#}(\alpha)}\beta$ is the Lie derivative of the form β in direction of the section $\pi^{\#}(\alpha)$. It is well known [12] that $\left(A^{*}, \{\cdot,\cdot\}^{\pi}_{\mu}, \rho\circ\pi^{\#}\right)$ is a Lie algebroid and its differential is given by $\mathsf{d}^{A^{*}} = [\pi,\cdot]_{SN}$.

Let $\{\alpha,\beta\}^{\pi}_{\mu^{N}}$ be the Koszul bracket (23) on $\Gamma(A^{*})$, associated with π and $\mu^{N} = [\cdot,\cdot]_{N}$ (see(20)). The Magri-Morosi concomitant $C(\pi,N)$ is defined, for all $\alpha,\beta \in \Gamma(A^{*})$, by

$$C(\pi,N)(\alpha,\beta) := \left(\{\alpha,\beta\}^{\pi}_{\mu}\right)_{N^{*}} - \{\alpha,\beta\}^{\pi}_{\mu^{N}},$$

where $N^{*} : A^{*} \to A^{*}$ is defined by $\langle N^{*}\alpha, X\rangle = \langle \alpha, NX\rangle$, for all $\alpha \in \Gamma(A^{*})$ and $X \in \Gamma(A)$, and $\left(\{\cdot,\cdot\}^{\pi}_{\mu}\right)_{N^{*}}$ is the deformed bracket of $\{\cdot,\cdot\}^{\pi}_{\mu}$, according to (20).

Any 2-form $\omega \in \Gamma(\wedge^{2}A^{*})$ determines a morphism $\omega^{\flat} : A \to A^{*}$ given by $\langle Y, \omega^{\flat}(X)\rangle = \omega(X,Y)$, for all $X,Y \in A$. Given $\omega \in \Gamma(\wedge^{2}A^{*})$ and a $\mathcal{C}^{\infty}$-linear bundle map $N : A \to A$ such that $\omega^{\flat}\circ N = N^{*}\circ\omega^{\flat}$, we denote by ω_{N} the 2-form defined by

$$\omega_{N}(X,Y) = \omega(NX,Y) = \omega(X,NY),$$

for all $X,Y \in \Gamma(A)$.

Let us recall the notion of an exact Poisson quasi-Nijenhuis structure with background on a Lie algebroid.

Definition 3.1. [3] An exact Poisson quasi-Nijenhuis structure with background on a Lie algebroid $(A, [\cdot,\cdot], \rho)$ is a quadruple (π, N, ω, H), where π is a bivector, $N : A \to A$ is a $\mathcal{C}^{\infty}$-linear bundle map, ω is a 2-form and H is a closed 3-form such that $N\circ\pi^{\#} = \pi^{\#}\circ N^{*}$, $\omega^{\flat}\circ N = N^{*}\circ\omega^{\flat}$ and the following conditions are satisfied:

(a) π is Poisson;
(b) $C(\pi,N)(\alpha,\beta) = 2H(\pi^{\#}\alpha, \pi^{\#}\beta, \cdot)$, for all $\alpha,\beta \in \Gamma(A^{*})$;
(c) $\mathcal{T}_{\mu}N(X,Y) = \pi^{\#}\left(-H(NX,Y,\cdot) - H(X,NY,\cdot) + \mathsf{d}\omega(X,Y,\cdot)\right)$, for all $X,Y \in \Gamma(A)$;
(d) $\iota_{N}\mathsf{d}\omega - \mathsf{d}\omega_{N} - \mathcal{H} = \lambda H$, for some $\lambda \in \mathbb{R}$, where $\mathcal{H}(X,Y,Z) = H(NX,NY,Z) + \circlearrowleft_{X,Y,Z}$ and $\circlearrowleft_{X,Y,Z}$ stands for the circular permutation on X, Y and Z, and $\iota_{N}\mathsf{d}\omega(X,Y,Z) = \mathsf{d}\omega(NX,Y,Z) + \mathsf{d}\omega(X,NY,Z) + \mathsf{d}\omega(X,Y,NZ)$, for all $X,Y,Z \in \Gamma(A)$.

Similar to the case of a $\mathcal{C}^{\infty}$-linear bundle map, for a k-form on a Lie algebroid we also consider its extension by derivation [4]. More precisely, if $\kappa \in \Gamma(\wedge^{k}A^{*})$, the extension of κ by derivation is a k-linear map, denoted by $\underline{\kappa}$, given by

$$\underline{\kappa}(P_{1},\cdots,P_{k}) := \sum_{i_{1},\cdots,i_{k}=1}^{p_{1},\cdots,p_{k}} (-1)^{\spadesuit}\kappa(P_{1,i_{1}},\cdots,P_{k,i_{k}})\widehat{P_{1,i_{1}}}\wedge\cdots\wedge\widehat{P_{k,i_{k}}},$$

for all homogeneous multi-sections $P_{i} = P_{i,1}\wedge\cdots\wedge P_{i,p_{i}} \in \Gamma(\wedge^{p_{i}}A)$, with $i = 1,\cdots,k$, where $1 \le i_{j} \le p_{j}$ for all $1 \le j \le k$,

$$\widehat{P_{j,i_{j}}} = P_{j,1}\wedge\cdots\wedge P_{j,i_{j}-1}\wedge P_{j,i_{j}+1}\wedge\cdots\wedge P_{j,p_{j}} \in \Gamma(\wedge^{p_{j}-1}A)$$

and

$$\spadesuit = 2p_{1} + 3p_{2} + \cdots + (k+1)p_{k}.$$

It follows from its definition that $\underline{\kappa}$ is a derivation on the graded vector space $\Gamma(\wedge A)[2]$ and that it is a symmetric vector valued k-form of degree $k-2$ on $\Gamma(\wedge A)[2]$.

The next two lemmas, proved in [4], give some results about the Richardson-Nijenhuis bracket of these $\underline{\kappa}$'s.

Lemma 3.2. [4] *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, $\alpha \in \Gamma(\wedge^k A^*)$ a k-form, $\beta \in \Gamma(\wedge^l A^*)$ an l-form, $\omega \in \Gamma(\wedge^2 A^*)$ a 2-form and $N : A \to A$ is a $\mathcal{C}^\infty$-linear bundle map. Then,*

(a) $[\underline{\alpha}, \underline{\beta}]_{RN} = 0$;

(b) $[\underline{N}, \underline{\omega}]_{RN} = 2\underline{\omega_N}$.

Lemma 3.3. [4] *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, with differential d^A and with associated L_∞-structure l_2, given by (16), on $\Gamma(\wedge A)[2]$. Then,*

$$[\underline{\alpha}, l_2]_{RN} = \underline{\mathsf{d}^A \alpha},$$

for all $\alpha \in \Gamma(\wedge^k A^*)$.

The next theorem contains the main result of this section: we give necessary and sufficient conditions for a vector valued form $\mathcal{N} := \pi + \underline{N} + \underline{\omega}$ to be co-boundary Nijenhuis with respect to $\mu = l_2 + \underline{H}$, where π is a bivector, $N : A \to A$ is a $\mathcal{C}^\infty$-linear map, ω is a 2-form and H is a closed 3-form.

Theorem 3.4. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, $H \in \Gamma\left(\wedge^3 A^*\right)$ a closed 3-form, $\pi \in \Gamma\left(\wedge^2 A\right)$, $\omega \in \Gamma\left(\wedge^2 A^*\right)$, $N : A \to A$ a $\mathcal{C}^\infty$-linear bundle map such that $N \circ \pi^\# = \pi^\# \circ N^*$ and $\omega^\flat \circ N = N^* \circ \omega^\flat$. Then, $\mathcal{N} = \pi + \underline{N} + \underline{\omega}$ is a co-boundary Nijenhuis vector valued form with respect to the L_∞-algebra $(\Gamma(\wedge A)[2], \mu = l_2 + \underline{H})$ with square $\underline{N^2} + [\underline{\omega}, \pi]_{RN}$, if and only if the quadruple $(\pi, N, -\omega, H)$ is an exact Poisson quasi-Nijenhuis structure with background, in the sense of Definition 3.1, with coefficient $\lambda = 0$.*

The lemmas and propositions that follow are needed to prove Theorem 3.4. We start with a lemma that establishes some formulas involving the Schouten-Nijenhuis and the Richardson-Nijenhuis brackets.

Lemma 3.5. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, H a 3-form, π a bivector and $N : A \to A$ a $\mathcal{C}^\infty$-linear map such that $N \circ \pi^\# = \pi^\# \circ N^*$. Consider the associated L_∞-structure l_2 on $\Gamma(\wedge A)[2]$, given by (16). Then, for all $\alpha, \beta \in \Gamma(A^*)$ and $X \in \Gamma(A)$,*

(a) $\underline{N}[\pi, X]_{SN}(\alpha, \beta) = [\pi, X]_{SN}(N^*\alpha, \beta) + [\pi, X]_{SN}(\alpha, N^*\beta)$,

(b) $\left([\pi, [\underline{N}, l_2]_{RN}]_{RN} + [\underline{N}, [\pi, l_2]_{RN}]_{RN}\right)(X)(\alpha, \beta) = -C(\pi, N)(\alpha, \beta)(X)$,

(c) $[\pi, [\pi, \underline{H}]_{RN}]_{RN}(X)(\alpha, \beta) = 2H\left(\pi^\#\alpha, \pi^\#\beta, X\right)$.

Proof. For the sake of simplicity and without loss of generality, assume that $\pi = \pi_1 \wedge \pi_2$, with $\pi_1, \pi_2 \in \Gamma(A)$. Notice that

$$\begin{aligned}\underline{N}[\pi, X]_{SN} &= \underline{N}([\pi_1, X] \wedge \pi_2 - [\pi_2, X] \wedge \pi_1) \\ &= N[\pi_1, X] \wedge \pi_2 + [\pi_1, X] \wedge N\pi_2 - N[\pi_2, X] \wedge \pi_1 - [\pi_2, X] \wedge N\pi_1.\end{aligned}$$

Hence, using the fact that $\langle \alpha, NX \rangle = \langle N^*\alpha, X \rangle$, for all $\alpha \in \Gamma(A^*), X \in \Gamma(A)$ and after considering suitable terms together, we have

$$\begin{aligned}\underline{N}[\pi, X]_{SN}(\alpha, \beta) &= ([\pi_1, X] \wedge \pi_2 - [\pi_2, X] \wedge N\pi_1)(N^*\alpha, \beta) \\ &\quad + ([\pi_1, X] \wedge \pi_2 - [\pi_2, X] \wedge N\pi_1)(\alpha, N^*\beta). \\ &= [\pi, X]_{SN}(N^*\alpha, \beta) + [\pi, X]_{SN}(\alpha, N^*\beta).\end{aligned}$$

This proves (a). Let $[\cdot,\cdot]^N_{SN}$ be the Schouten-Nijenhuis bracket associated to the deformed bracket μ^N (see (20)). Then, using the definition of the Richardson-Nijenhuis bracket and Equations (1) and (21), for every section $X \in \Gamma(A)$ we have

(24)
$$\left([\pi, [\underline{N}, l_2]_{RN}]_{RN} + [\underline{N}, [\pi, l_2]_{RN}]_{RN}\right)(X) = -[\pi, X]^N_{SN} - [\pi, NX]_{SN} + \underline{N}[\pi, X]_{SN}.$$

Note that π is not assumed to be Poisson, so that $(A^*, \{\cdot,\cdot\}_\mu^\pi, \rho\circ\pi^\#)$ is a pre-Lie algebroid with a pre-differential d^{A^*} which is a derivation of $\Gamma(\wedge(A^*)^*)\simeq\Gamma(\wedge A)$ given for all $X\in\Gamma(A)$ by:

$$\mathsf{d}^{A^*}(X)(\alpha,\beta)=\rho\circ\pi^\#\alpha\left\langle\beta,X\right\rangle-\rho\circ\pi^\#\beta\left\langle\alpha,X\right\rangle-\left\langle\{\alpha,\beta\}_\mu^\pi,X\right\rangle. \tag{25}$$

Above, we have implicitely identified $(A^*)^*$ and A. A simple computation yields that for all $X\in\Gamma(A)$ and $\alpha,\beta\in\Gamma(A^*)$:

$$\mathsf{d}^{A^*}(X)(\alpha,\beta)=[\pi,X]_{SN}(\alpha,\beta). \tag{26}$$

Equations (26) and (25) imply that

$$\left\langle\{\alpha,\beta\}_\mu^\pi,X\right\rangle=-[\pi,X]_{SN}(\alpha,\beta)+\rho\circ\pi^\#\alpha\left\langle\beta,X\right\rangle-\rho\circ\pi^\#\beta\left\langle\alpha,X\right\rangle. \tag{27}$$

Applying, successively, (27) for μ, $N^*\alpha$, β and X, then for μ, α, $N^*\beta$ and X, and finally for μ, α, β and NX, we have

$$\left\langle\{N^*\alpha,\beta\}_\mu^\pi,X\right\rangle=-[\pi,X]_{SN}(N^*\alpha,\beta)+\rho\circ\pi^\# N^*\alpha\left\langle\beta,X\right\rangle-\rho\circ\pi^\#\beta\left\langle N^*\alpha,X\right\rangle, \tag{28}$$

$$\left\langle\{\alpha,N^*\beta\}_\mu^\pi,X\right\rangle=-[\pi,X]_{SN}(\alpha,N^*\beta)+\rho\circ\pi^\#\alpha\left\langle N^*\beta,X\right\rangle-\rho\circ\pi^\# N^*\beta\left\langle\alpha,X\right\rangle \tag{29}$$

and

$$\begin{aligned}\left\langle N^*\{\alpha,\beta\}_\mu^\pi,X\right\rangle &= \left\langle\{\alpha,\beta\}_\mu^\pi,NX\right\rangle\\ &= -[\pi,NX]_{SN}(\alpha,\beta)+\rho\circ\pi^\#\alpha\left\langle\beta,NX\right\rangle-\rho\circ\pi^\#\beta\left\langle\alpha,NX\right\rangle,\end{aligned} \tag{30}$$

respectively. Now, applying (27) for μ^N, α, β and X we have

$$\left\langle-\{\alpha,\beta\}_{\mu^N}^\pi,X\right\rangle=[\pi,X]_{SN}^N(\alpha,\beta)-\rho\circ N\circ\pi^\#\alpha\left\langle\beta,X\right\rangle+\rho\circ N\circ\pi^\#\beta\left\langle\alpha,X\right\rangle. \tag{31}$$

Summing up the equations (28), (29), (30) and (31) and using item (a) we get

$$\left\langle C(\pi,N)(\alpha,\beta),X\right\rangle=\left([\pi,X]_{SN}^N+[\pi,NX]_{SN}-\underline{N}[\pi,X]_{SN}\right)(\alpha,\beta).$$

This together with (24) prove (b). Let $\alpha,\beta\in\Gamma(\wedge A^*)$. Then,

$$\pi^\#(\alpha)=\iota_\alpha(\pi_1\wedge\pi_2)=\pi_1(\alpha)\pi_2-\pi_2(\alpha)\pi_1 \tag{32}$$

and, by definition of $\underline{H}$ and Equation (1), we get

$$\left[\pi,[\pi,\underline{H}]_{RN}\right]_{RN}(X)=\underline{H}(\pi,\pi,X)=2H(\pi_1,\pi_2,X)\,\pi_1\wedge\pi_2.$$

Hence,

$$\left[\pi,[\pi,\underline{H}]_{RN}\right]_{RN}(X)(\alpha,\beta)=2H(\pi_1,\pi_2,X)\,\pi(\alpha,\beta). \tag{33}$$

On the other hand,

$$\begin{aligned}2H\left(\pi^\#(\alpha),\pi^\#(\beta),X\right) &= 2H\left(\pi_1(\alpha)\pi_2-\pi_2(\alpha)\pi_1,\pi_1(\beta)\pi_2-\pi_2(\beta)\pi_1,X\right)\\ &= 2H(\pi_1,\pi_2,X)\,\pi(\alpha,\beta).\end{aligned} \tag{34}$$

Equations (33) and (34) prove (c). □

Proposition 3.6. *Let $(A,[\cdot,\cdot],\rho)$ be a Lie algebroid, H a 3-form, π a bivector and $N:A\to A$ a $\mathcal{C}^\infty$-linear map. Consider the associated L_∞-structure l_2 on $\Gamma(\wedge A)[2]$, given by (16). Then, for all $\alpha,\beta\in\Gamma(A^*)$ and $X\in\Gamma(A)$,*

$$\left[\pi,[\pi,\underline{H}]_{RN}\right]_{RN}+\left[\pi,[\underline{N},l_2]_{RN}\right]_{RN}+\left[\underline{N},[\pi,l_2]_{RN}\right]_{RN}=0 \tag{35}$$

if and only if

$$C\left(\pi,N\right)\left(\alpha,\beta\right)=2H\left(\pi^{\#}\alpha,\pi^{\#}\beta,\cdot\right). \tag{36}$$

Proof. Assume that (35) holds. Then, items (b) and (c) in Lemma 3.5 imply (36).

Assume that (36) holds. Set

$$I:=\left[\pi,\left[\pi,\underline{H}\right]_{RN}\right]_{RN}+\left[\pi,\left[\underline{N},l_2\right]_{RN}\right]_{RN}+\left[\underline{N},\left[\pi,l_2\right]_{RN}\right]_{RN}.$$

Note that I is a vector valued 1-form of degree 1. Hence, if $P=X_1\wedge\cdots\wedge X_n\in\Gamma\left(\wedge^nA\right)$ is a n-section, then $I(P)\in\Gamma\left(\wedge^{n+1}A\right)$ is a $(n+1)$-section. Since $\underline{H}$ and $\underline{N}$ are derivations, I is a derivation, and so we have

$$\begin{array}{rcl}\iota_{\alpha_1\wedge\cdots\wedge\alpha_{n+1}}I\left(P\right) & = & \sum_{i=1}^{n}\iota_{\alpha_1\wedge\cdots\wedge\alpha_{n+1}}I\left(X_i\right)\wedge\widehat{P}_i\\ & = & \sum_{i=1}^{n}\sum_{\sigma\in Sh(2,n-1)}I\left(X_i\right)\left(\alpha_{\sigma(1)},\alpha_{\sigma(2)}\right)\iota_{\alpha_{\sigma(3)}\wedge\cdots\wedge\alpha_{\sigma(n+1)}}\widehat{P}_i,\end{array}$$

for all $\alpha_1,\cdots,\alpha_{n+1}\in\Gamma\left(\wedge A^*\right)$, where $\widehat{P}_i=X_1\wedge\cdots\wedge X_{i-1}\wedge X_{i+1}\wedge\cdots\wedge X_n$. But items (b) and (c) in Lemma 3.5 imply that

$$I\left(X_i\right)\left(\alpha_{\sigma(1)},\alpha_{\sigma(2)}\right)=-C\left(\pi,N\right)\left(\alpha_{\sigma(1)},\alpha_{\sigma(2)}\right)\left(X_i\right)+2H\left(\pi^{\#}\alpha_{\sigma(1)},\pi^{\#}\alpha_{\sigma(2)},X_i\right)=0,$$

and so $I(P)=0$, which implies that $I=0$. □

Lemma 3.7. *Let $(A,[\cdot,\cdot],\rho)$ be a Lie algebroid, H a 3-form, π a bivector and $N:A\to A$ a $\mathcal{C}^\infty$-linear map such that $N\circ\pi^{\#}=\pi^{\#}\circ N^*$. Then, for all $X,Y\in\Gamma\left(A\right)$,*

(a) $\pi^{\#}\left(H\left(X,Y,\cdot\right)\right)=\underline{H}\left(\pi,X,Y\right)$,

(b) $\left(\left[\pi,\left[\underline{N},\underline{H}\right]_{RN}\right]_{RN}+\left[\underline{N},\left[\pi,\underline{H}\right]_{RN}\right]_{RN}\right)\left(X,Y\right)=2\underline{H}\left(\pi,NX,Y\right)+2\underline{H}\left(\pi,X,NY\right)$.

Proof. Without loss of generality, we assume that $\pi=\pi_1\wedge\pi_2$, with $\pi_1,\pi_2\in\Gamma(A)$. Then, for all $X,Y\in\Gamma\left(A\right)$, using (32), we have

$$\pi^{\#}\left(H\left(X,Y,\cdot\right)\right)=H\left(X,Y,\pi_1\right)\pi_2-H\left(X,Y,\pi_2\right)\pi_1.$$

So, by definition of $\underline{H}$, we have

$$\pi^{\#}\left(H\left(X,Y,\cdot\right)\right)=\underline{H}\left(X,Y,\pi\right)$$

and, since $\underline{H}$ is graded symmetric, we get (a). A direct computation, using the definition of Richardson-Nijenhuis bracket, shows that for all $X,Y\in\Gamma\left(A\right)$,

$$\begin{array}{c}\left(\left[\pi,\left[\underline{N},\underline{H}\right]_{RN}\right]_{RN}+\left[\underline{N},\left[\pi,\underline{H}\right]_{RN}\right]_{RN}\right)\left(X,Y\right)\quad=\\ 2\underline{H}\left(\pi,NX,Y\right)+2\underline{H}\left(\pi,X,NY\right)+\underline{H}\left(\underline{N}\pi,X,Y\right)-2\underline{N}\circ\underline{H}\left(\pi,X,Y\right)\end{array} \tag{37}$$

holds. With a similar argument as in the proof of (a) in Lemma 3.5, it can be shown that for all $\alpha,\beta\in\Gamma\left(A^*\right)$,

$$\left(\underline{N}\pi\right)\left(\alpha,\beta\right)=\pi\left(N^*\alpha,\beta\right)+\pi\left(\alpha,N^*\beta\right),$$

which is equivalent to

$$\left\langle\left(\underline{N}\pi\right)^{\#}\alpha,\beta\right\rangle=\left\langle\pi^{\#}(N^*\alpha),\beta\right\rangle+\left\langle\pi^{\#}\alpha,N^*\beta\right\rangle.$$

Now, $N\circ\pi^{\#}=\pi^{\#}\circ N^*$ implies that for all $\alpha,\beta\in\Gamma\left(A^*\right)$,

$$\left\langle\left(\underline{N}\pi\right)^{\#}\alpha,\beta\right\rangle=2\left\langle N(\pi^{\#}\alpha),\beta\right\rangle,$$

which proves that

$$\left(\underline{N}\pi\right)^{\#}=2N\circ\pi^{\#}.$$

This implies that $\left(\underline{N}\pi\right)^{\#}\left(H\left(X,Y,\cdot\right)\right)=2N\circ\pi^{\#}\left(H\left(X,Y,\cdot\right)\right)$. Then, using (a), we get

$$\underline{H}\left(\underline{N}\pi,X,Y\right)-2\underline{N}\circ\underline{H}\left(\pi,X,Y\right)=0.$$

This, together with (37), proves (b). □

Proposition 3.8. *Let* $(A, [\cdot,\cdot], \rho)$ *be a Lie algebroid,* H *a 3-form,* π *a bivector and* $N : A \to A$ *a* $\mathcal{C}^\infty$*-linear map such that* $N \circ \pi^\# = \pi^\# \circ N^*$*. Consider the associated* L_∞*-structure* l_2 *on* $\Gamma(\wedge A)[2]$*, given by (16). Then,*

(38)
$$\left[\pi, [\underline{N}, \underline{H}]_{RN}\right]_{RN} + \left[\underline{N}, [\pi, \underline{H}]_{RN}\right]_{RN} + \left[\underline{N}, [\underline{N}, l_2]_{RN}\right]_{RN} + [\pi, \underline{\mathsf{d}\omega}]_{RN} + \left[\underline{\omega}, [\pi, l_2]_{RN}\right]_{RN} = \left[\underline{N^2}, l_2\right]_{RN} + \left[[\underline{\omega}, \pi]_{RN}, l_2\right]_{RN},$$

if and only if

$$\mathcal{T}_\mu N(X,Y) = -\pi^\#\left(H(NX,Y,\cdot) + H(X,NY,\cdot) + \mathsf{d}\omega(X,Y,\cdot)\right), \tag{39}$$

for all $X, Y \in \Gamma(A)$.

Proof. Assume that (38) holds. Using the graded Jacobi identity of the Richardson-Nijenhuis bracket, we get

$$\left[\pi, [\underline{\omega}, l_2]_{RN}\right]_{RN} + \left[[\underline{\omega}, \pi]_{RN}, l_2\right]_{RN} = \left[\underline{\omega}, [\pi, l_2]_{RN}\right]_{RN}$$

or, by Lemma 3.3,

$$[\pi, \underline{\mathsf{d}\omega}]_{RN} + \left[[\underline{\omega}, \pi]_{RN}, l_2\right]_{RN} = \left[\underline{\omega}, [\pi, l_2]_{RN}\right]_{RN}.$$

Hence, (38) is equivalent to

$$\left[\pi, [\underline{N}, \underline{H}]_{RN}\right]_{RN} + \left[\underline{N}, [\pi, \underline{H}]_{RN}\right]_{RN} + 2[\pi, \underline{\mathsf{d}\omega}]_{RN} = \left[\underline{N^2}, l_2\right]_{RN} - \left[\underline{N}, [\underline{N}, l_2]_{RN}\right]_{RN}.$$

In particular, for all sections $X, Y \in \Gamma(A)$ we have,

$$\begin{aligned}&\left(\left[\pi, [\underline{N}, \underline{H}]_{RN}\right]_{RN} + \left[\underline{N}, [\pi, \underline{H}]_{RN}\right]_{RN} + 2[\pi, \underline{\mathsf{d}\omega}]_{RN}\right)(X,Y)\\ &= \left(\left[\underline{N^2}, l_2\right]_{RN} - \left[\underline{N}, [\underline{N}, l_2]_{RN}\right]_{RN}\right)(X,Y)\end{aligned} \tag{40}$$

which, from item (b) of Lemma 3.7 and Equation (22), is equivalent to

$$\underline{H}(\pi, NX, Y) + \underline{H}(\pi, X, NY) + \underline{\mathsf{d}\omega}(\pi, X, Y) = -\mathcal{T}_\mu N(X,Y). \tag{41}$$

By item (a) of Lemma 3.7, (41) is equivalent to (39). Let $P, Q \in \Gamma(\wedge A)$ be any multi-sections on A. Since $\underline{N}$, $\underline{N^2}$, $\underline{H}$, $\underline{\omega}$ and $\underline{\mathsf{d}\omega}$ are extensions by derivation, both sides of Equation (38) are derivations in each variable and $\mathcal{C}^\infty$-linear on functions. Then, from the equivalence of (39) and (40), we get that

$$\begin{aligned}&\left(\left[\pi, [\underline{N}, \underline{H}]_{RN}\right]_{RN} + \left[\underline{N}, [\pi, \underline{H}]_{RN}\right]_{RN} + 2[\pi, \underline{\mathsf{d}\omega}]_{RN}\right)(P,Q)\\ &= \left(\left[\underline{N^2}, l_2\right]_{RN} - \left[\underline{N}, [\underline{N}, l_2]_{RN}\right]_{RN}\right)(P,Q)\end{aligned}$$

if and only if Equation (39) holds. This proves the proposition. □

Lemma 3.9. *Let* $(A, [\cdot,\cdot], \rho)$ *be a Lie algebroid,* H *a 3-form,* π *a bivector,* $N : A \to A$ *a* $\mathcal{C}^\infty$*-linear map and* ω *a 2-form. Then,*

$$\left[\underline{N}, [\underline{N}, \underline{H}]_{RN}\right]_{RN} = \left[\underline{N^2}, \underline{H}\right]_{RN} + 2\mathcal{H} - 2\underline{N} \circ [\underline{N}, \underline{H}]_{RN},$$

where $\mathcal{H}(P,Q,R) = \underline{H}(\underline{N}P, \underline{N}Q, R) + \circlearrowleft_{P,Q,R}$*, for all* $P, Q, R \in \Gamma(\wedge A)$.

Proof. Let $P, Q, R \in \Gamma(\wedge A)$. Then, by definition of the Richardson-Nijenhuis bracket, we have

(42)
$$\begin{aligned}\left[\underline{N}, [\underline{N}, \underline{H}]_{RN}\right]_{RN}(P,Q,R) \;=\;& [\underline{N}, \underline{H}]_{RN}(\underline{N}P, Q, R) + [\underline{N}, \underline{H}]_{RN}(P, \underline{N}Q, R)\\ &+ [\underline{N}, \underline{H}]_{RN}(P, Q, \underline{N}R) - \underline{N} \circ [\underline{N}, \underline{H}]_{RN}(P,Q,R).\end{aligned}$$

Adding and subtracting $\underline{N^2} \circ \underline{H}(P,Q,R)$ to the right hand side of (42) and collecting suitable terms together we get the desired result. □

Proposition 3.10. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, H a 3-form, π a bivector, $N : A \to A$ a $\mathcal{C}^\infty$-linear map and ω a 2-form. Consider the associated L_∞-structure l_2 on $\Gamma(\wedge A)[2]$, given by (16). Then,*

$$\left[\underline{N}, [\underline{N}, \underline{H}]_{RN}\right]_{RN} + \left[\underline{N}, [\underline{\omega}, l_2]_{RN}\right]_{RN} + \left[\underline{\omega}, [\pi, \underline{H}]_{RN}\right]_{RN} + \left[\underline{\omega}, [\underline{N}, l_2]_{RN}\right]_{RN} = \left[\underline{N^2} + [\underline{\omega}, \pi]_{RN}, \underline{H}\right]_{RN}$$

if and only if

$$\mathcal{H} - \underline{N} \circ [\underline{N}, \underline{H}]_{RN} + [\underline{N}, \mathsf{d}\omega]_{RN} - \underline{\mathsf{d}\omega_N} = 0, \tag{43}$$

where $\mathcal{H}(P,Q,R) = \underline{H}(\underline{N}P, \underline{N}Q, R) + \circlearrowleft_{P,Q,R}$, for all $P,Q,R \in \Gamma(\wedge A)$.

Proof. Using the graded Jacobi identity of the Richardson-Nijenhuis bracket and Lemma 3.3, we get

$$\left[\underline{\omega}, [\underline{N}, l_2]_{RN}\right]_{RN} = [\underline{N}, \underline{\mathsf{d}\omega}]_{RN} + \left[[\underline{\omega}, \underline{N}]_{RN}, l_2\right]_{RN}. \tag{44}$$

From Lemma 3.2 (a) and using again the graded Jacobi identity of the Richardson-Nijenhuis bracket, we obtain

$$\left[\underline{\omega}, [\pi, \underline{H}]_{RN}\right]_{RN} = \left[[\underline{\omega}, \pi]_{RN}\, \underline{H}\right]_{RN}. \tag{45}$$

Equations (44) and (45) together with Lemmas 3.9 and 3.2 (b) imply that

$$\left[\underline{N}, [\underline{N}, \underline{H}]_{RN}\right]_{RN} + \left[\underline{N}, [\underline{\omega}, l_2]_{RN}\right]_{RN} + \left[\underline{\omega}, [\pi, \underline{H}]_{RN}\right]_{RN} + \left[\underline{\omega}, [\underline{N}, l_2]_{RN}\right]_{RN} = \left[\underline{N^2} + [\underline{\omega}, \pi]_{RN}, \underline{H}\right]_{RN},$$

if and only if

$$\mathcal{H} - \underline{N} \circ [\underline{N}, \underline{H}]_{RN} + [\underline{N}, \underline{\mathsf{d}\omega}]_{RN} - \underline{\mathsf{d}\omega_N} = 0.$$

□

Recall that $\underline{N}(f) = 0$, for all $f \in \mathcal{C}^\infty(M)$. The next corollary is a consequence of the fact that

$$\underline{N} \circ [\underline{N}, \underline{H}]_{RN}(X,Y,Z) = 0 \tag{46}$$

and $[\underline{N}, \underline{\mathsf{d}\omega}]_{RN}(X,Y,Z) = \iota_N \mathsf{d}\omega(X,Y,Z)$, for all $X,Y,Z \in \Gamma(A)$.

Corollary 3.11. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, H a 3-form, π a bivector, $N : A \to A$ a $\mathcal{C}^\infty$-linear map and ω a 2-form. Then,*

$$\left[\underline{N}, [\underline{N}, \underline{H}]_{RN}\right]_{RN} + \left[\underline{N}, [\underline{\omega}, l_2]_{RN}\right]_{RN} + \left[\underline{\omega}, [\pi, \underline{H}]_{RN}\right]_{RN} + \left[\underline{\omega}, [\underline{N}, l_2]_{RN}\right]_{RN} = \left[\underline{N^2} + [\underline{\omega}, \pi]_{RN}, \underline{H}\right]_{RN} \tag{47}$$

if and only if

$$(\mathcal{H} + \iota_N \mathsf{d}\omega - \mathsf{d}\omega_N)(X,Y,Z) = 0, \tag{48}$$

for all $X,Y,Z \in \Gamma(A)$, where $\mathcal{H}(X,Y,Z) = H(NX, NY, Z) + \circlearrowleft_{X,Y,Z}$.

Proof. Taking into account Equation (46), condition (48) implies that Equation (43) holds, when restricted to sections of A. So, by Proposition 3.10, (48) is equivalent to Equation (47), when restricted to sections of A. Since both sides of (47) are derivations in each variable and C^∞-linear on functions, the result follows. □

Proposition 3.12. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid and $H \in \Gamma(\wedge^3 A^*)$ a closed 3-form. Then $\mu := l_2 + \underline{H}$ is an L_∞-algebra structure on $\Gamma(\wedge A)[2]$.*

Proof. Observe that l_2 and $\underline{H}$ are both graded symmetric vector valued forms of degree 1. Using Lemmas 3.2 and 3.3, we have

$$[l_2 + \underline{H}, l_2 + \underline{H}]_{RN} = [l_2, l_2]_{RN} + 2\,[l_2, \underline{H}]_{RN} + [\underline{H}, \underline{H}]_{RN} = 0.$$

Therefore, by Theorem 1.3, μ is an L_∞-algebra structure on $\Gamma(\wedge A)[2]$. □

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. Using Lemmas 3.3 and 3.2, we have

$$[\mathcal{N},\mu]_{RN} = [\pi,l_2]_{RN}+[\pi,\underline{H}]_{RN}+[\underline{N},l_2]_{RN}+[\underline{N},\underline{H}]_{RN}+\underline{\mathsf{d}\omega}.$$

Let

$$\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^0_{RN}=l_2(\pi,\pi),$$

$$\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^1_{RN}=\left[\pi,[\pi,\underline{H}]_{RN}\right]_{RN}+\left[\pi,[\underline{N},l_2]_{RN}\right]_{RN}+\left[\underline{N},[\pi,l_2]_{RN}\right]_{RN},$$

$$\begin{aligned}\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^2_{RN} &= \left[\pi,[\underline{N},\underline{H}]_{RN}\right]_{RN}+\left[\underline{N},[\pi,\underline{H}]_{RN}\right]_{RN}+\left[\underline{N},[\underline{N},l_2]_{RN}\right]_{RN}\\ &\quad+[\pi,\underline{\mathsf{d}\omega}]_{RN}+\left[\underline{\omega},[\pi,l_2]_{RN}\right]_{RN},\\ \left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^3_{RN} &= \left[\underline{N},[\underline{N},\underline{H}]_{RN}\right]_{RN}+\left[\underline{N},[\underline{\omega},l_2]_{RN}\right]_{RN}+\left[\underline{\omega},[\pi,\underline{H}]_{RN}\right]_{RN}\\ &\quad+\left[\underline{\omega},[\underline{N},l_2]_{RN}\right]_{RN}\end{aligned}$$

and

$$\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^4_{RN}=\left[\underline{\omega},[\underline{N},\underline{H}]_{RN}\right]_{RN}=0.$$

Then,

$$\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]_{RN}=\sum_{i=0}^{4}\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^i_{RN}.$$

By construction, each $\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^i_{RN}$ is a vector valued i-form and

$$\left[\underline{N^2}+[\underline{\omega},\pi]_{RN},\mu\right]_{RN}$$

is the sum of a vector valued 2-form and a vector valued 3-form. Therefore,

$$\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]_{RN}=\left[\underline{N^2}+[\underline{\omega},\pi]_{RN},\mu\right]_{RN},$$

holds, if and only if

$$\begin{aligned}\left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^0_{RN}&=0,\\ \left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^1_{RN}&=0,\\ \left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^2_{RN}&=\left[\underline{N^2}+[\underline{\omega},\pi]_{RN},l_2\right]_{RN},\\ \left[\mathcal{N},[\mathcal{N},\mu]_{RN}\right]^3_{RN}&=\left[\underline{N^2}+[\underline{\omega},\pi]_{RN},\underline{H}\right]_{RN}.\end{aligned}$$

Hence, using Propositions 3.6, 3.8 and Corollary 3.11, $\mathcal{N}$ is co-boundary Nijenhuis with respect to $\mu=l_2+\underline{H}$, with a certain square, if and only if

(a) $l_2(\pi,\pi)=0$, i.e. π is Poisson,
(b) $C(\pi,N)(\alpha,\beta)=2H(\pi^{\#}\alpha,\pi^{\#}\beta,\cdot)$, for all $\alpha,\beta\in\Gamma(A^*)$,
(c) $\mathcal{T}_\mu N(X,Y)=-\pi^{\#}(H(NX,Y,\cdot)+H(X,NY,\cdot)+\mathsf{d}\omega(X,Y,\cdot))$, for all $X,Y\in\Gamma(A)$,
(d) $(\mathcal{H}+\iota_N\mathsf{d}\omega-\mathsf{d}\omega_N)(X,Y,Z)=0$, for all $X,Y,Z\in\Gamma(A)$.

According to Definition 3.1, items $(a)-(d)$ mean that the quadruple $(\pi,N,-\omega,H)$ is an exact Poisson quasi-Nijenhuis structure with background, with $\lambda=0$. This proves the theorem. □

The next corollary establishes a relation between co-boundary Nijenhuis vector valued forms and Poisson quasi-Nijenhuis structures on manifolds in the sense of Stiénon and Xu [17].

Corollary 3.13. *Let M be a manifold, π a bivector field, $N : TM \to TM$ a $(1,1)$-tensor and ω a 2-form such that $N \circ \pi^{\#} = \pi^{\#} \circ N^*$ and $\omega^{\flat} \circ N = N^* \circ \omega^{\flat}$. Then there exist, around each point of M, a 2-form α such that $\mathcal{N} = \pi + \underline{N} + \underline{\omega}$ is a co-boundary Nijenhuis vector valued form with respect to the L_∞-algebra $(\Gamma(\wedge TM)[2], \mu = l_2)$ with square $\underline{N^2} + [\underline{\omega}, \pi]_{RN} + \underline{\alpha}$, if and only if, the triple $(\pi, N, -\mathsf{d}\omega)$ is a Poisson quasi-Nijenhuis structure on M in the sense of Stiénon and Xu.*

Proof. Recall that Stiénon and Xu in [17] defined Poisson quasi-Nijenhuis structures on manifolds as triples $(\pi, N, \phi = -\mathsf{d}\omega)$ such that

(a) π is Poisson,
(b) $C(\pi, N)(\alpha, \beta) = 0$,
(c) $T_\mu N(X, Y) = -\pi^{\#}(\mathsf{d}\omega(X, Y, \cdot))$, for all $X, Y \in \Gamma(A)$,
(d) $\mathsf{d}(\iota_N \mathsf{d}\omega) = 0$.

A direct computation, done as in the proof of Theorem 3.4 for $H = 0$, gives that $\mathcal{N} = \pi + \underline{N} + \underline{\omega}$ is a co-boundary Nijenhuis vector valued form with respect to the L_∞-algebra $(\Gamma(\wedge TM)[2], \mu = l_2)$ with square $\underline{N^2} + [\underline{\omega}, \pi]_{RN} + \underline{\alpha}$, if and only if (a), (b) and (c) hold together with the condition $(\iota_N \mathsf{d}\omega - \mathsf{d}\omega_N - \mathsf{d}\alpha)(X, Y, Z) = 0$, for all vector fields X, Y and Z on M. Then, by Poincaré Lemma, such an α exists around each point of M, if and only if $\mathsf{d}(\iota_N \mathsf{d}\omega) = 0$. □

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