Zero-inflated compound Poisson distributions in integer-valued GARCH models

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**Abstract.** In this paper we introduce a wide class of integer-valued stochastic processes that allows to take into consideration, simultaneously, relevant characteristics observed in count data namely zero inflation, overdispersion and conditional heteroscedasticity. This class includes, in particular, the compound Poisson, the zero-inflated Poisson and the zero-inflated negative binomial INGARCH models, recently proposed in literature.

The main probabilistic analysis of this class of processes is here developed. Precisely, first and second-order stationarity conditions are derived, the autocorrelation function is deduced and the strict stationarity is established in a large subclass. We also analyze in a particular model the existence of higher-order moments and deduce the explicit form for the first four cumulants, as well as its skewness and kurtosis.

**Keywords**: integer-valued GARCH model, zero inflation, compound distributions.

*Mathematical Subject Classification*: 62M10

# 1 Introduction

Count time series are quite common in various scientific fields like medicine, economics, finance, tourism and queuing systems. The modeling of these time series has been receiving increasing attention and several integer-valued stochastic models have been recently proposed and developed in order to best describe and analyze this kind of data.

The change of the series variability is often observed in count time series which lead to the proposal of conditionally heteroscedastic models. The integer-valued process proposed by Ferland *et al.* ([6]), denoted INGARCH$(p, q)$ and inspired in the GARCH models of Bollerslev ([3]), takes into account this characteristic. In fact, Ferland *et al.* ([6]) assume a Poissonian conditional distribution whose parameter evolves with the past of the process similarly to GARCH models. Following this idea, several models have been introduced in literature considering other deviates discrete distributions like the negative binomial ([18]), the generalized Poisson ([19]) or a general compound Poisson distribution proposed by Gonçalves *et al.* ([8]).

Excess of zeros is another fact often observed in count time series. The interest of this characteristic is clear because zero counts frequently have special status.

Neyman ([13]) and Feller ([5]) first introduced the concept of zero inflation to address the problem of excess of zeros. Since then, there have been extensive studies related to the development of zero-inflated probability processes, in particular Poisson models, essentially considered in econometric literature and in regression context. Application areas are diverse and have included situations that produce a low fraction of non-conforming units, road safety, species abundance and processes related to health where the monitoring of a rare disease is of interest. Ridout *et al.* ([14]) include several references and details.

A zero-inflated distribution can be viewed as a mixture of a degenerate distribution with mass at zero and a nondegenerate distribution. For example, the random variable $X$ is zero-inflated Poisson $(\lambda, \omega)$ distributed ([11]) if its probability mass function can be written in the form $P(X = k) = \omega\delta_{k,0} + (1 - \omega)\frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, 2, ...$, where $0 < \omega < 1$, $\delta_{k,0}$ is the Kronecker delta, i.e., $\delta_{k,0}$ is 1 when $k = 0$ and is zero when $k \neq 0$. This distribution is denoted by ZIP.

In time series context, Bakouch and Ristić ([2]), Jazi *et al.* ([10]) and Li *et al.* ([12]) are recent works dedicated to the proposal and study of zero-inflated models for integer-valued time series.

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In an heteroscedastic context, Zhu ([20]) introduces the zero-inflated Poisson and the Negative Binomial integer-valued GARCH models. For example, the zero-inflated Poisson integer-valued GARCH model, denoted by ZIP-INGARCH$(p, q)$, is defined as

$$\begin{cases} X_t|\underline{X}_{t-1} : ZIP(\lambda_t, \omega), \ \forall t \in \mathbb{Z}, \\ \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}, \end{cases}$$

where $0 < \omega < 1$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1, ..., p$, $k = 1, ..., q$, $p \geq 1$, $q \geq 0$ and $\underline{X}_{t-1}$ the $\sigma$-field generated by $\{X_{t-j}, j \geq 1\}$. The zero-inflated negative binomial INGARCH model, denoted by ZINB-INGARCH, is analogously defined by [20], giving weak stationarity conditions and the autocorrelation function and proposing an estimating procedure.

With the aim of enlarging and unifying the study of these several models we introduce in this paper a Zero-Inflated INteger-valued GARCH process with general Compound Poisson deviates. We include in this general class the CP-INGARCH models ([8]) corresponding to $\omega = 0$, and if $0 < \omega < 1$ we have, as particular cases, the ZIP-INGARCH and ZINB-INGARCH models.

Additionally to the zero-inflated characteristic, in most count data sets the conditional variance is greater in value than the conditional mean, often much greater. This characteristic is known as conditional overdispersion. For example, Xu *et al.* ([17]) present a study of weekly dengue cases observed in Singapure where the conditional overdispersion is highly significant. Our proposal have also the aim of modeling zero inflation, overdispersion and conditional heteroskedasticity in the same framework, by means of a general class of integer-valued conditional distributions.

The main probabilistic analysis of this model is organized as follows. In Section 2 we introduce the Zero-Inflated Compound Poisson INGARCH model by means of the conditional characteristic function, as it is a closed-form of characterizing this class of laws. The wide range of this proposal is stressed referring the most important models recently studied ([8],[20]) and also presenting a general procedure to obtain new models. A necessary and sufficient condition of first-order stationarity is given. Concerning the second-order stationarity a necessary and sufficient condition is stated in Section 3 based on a vectorial state space representation of the general ZICP-INGARCH process. Moreover, we present a solution for this general model and state its strict stationarity in a wide subclass. In Section 4 we focus on the ZICP-INGARCH(1, 1) model and establish a necessary and sufficient condition for the existence of all moments, directly expressed on the model coefficients, and give explicitly in some cases the first cumulants of $X_t$ from which skewness and kurtosis of the process are deduced. Section 5 presents some discussions and in Appendices 1 and 2 we summarize some auxiliary forms and calculations.

## 2 The model

Let $X = (X_t, t \in \mathbb{Z})$ be a stochastic process with values in $\mathbb{N}_0$.

**Definition 2.1 (ZICP-INGARCH*(p,q)* model)** *The process $X$ is said to satisfy a Zero-Inflated Compound Poisson INteger-valued GARCH model with orders $p$ and $q$, ($p$, $q \in \mathbb{N}$) if, $\forall t \in \mathbb{Z}$, the characteristic function of $X_t|\underline{X}_{t-1}$, $\Phi_{X_t|\underline{X}_{t-1}}$, is given by*

$$\begin{cases} \Phi_{X_t|\underline{X}_{t-1}}(u) = \omega + (1-\omega) \exp\left\{ i \frac{\lambda_t}{\varphi_t'(0)} \left[ \varphi_t(u) - 1 \right] \right\}, \quad u \in \mathbb{R}, \\ \\ \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}, \end{cases} \quad (1)$$

*for some constants $0 \leq \omega < 1$, $\alpha_0 > 0$, $\alpha_j \geq 0$ ($j = 1, ..., p$), $\beta_k \geq 0$ ($k = 1, ..., q$), and where $(\varphi_t, t \in \mathbb{Z})$ is a family of characteristic functions on $\mathbb{R}$, $\underline{X}_{t-1}$-measurable associated to a family of discrete laws with support $\mathbb{N}_0$ and finite mean. $i$ denotes the imaginary unit.*

We observe that the conditional distribution of $X_t$ is a mixture of the Dirac law at zero with a discrete compound Poisson law. The probability at zero is then inflated with value $\omega$.

The ZICP-INGARCH$(p,q)$ model with $q=1$ and $\beta_1=0$ is denoted by ZICP-INARCH$(p)$ and, as mentioned before, when $\omega=0$ we recover the CP-INGARCH$(p,q)$ model considered in Gonçalves *et al.* ([8]).

The conditional mean and variance of $X_t$ are, respectively, given by $E(X_t|\underline{X}_{t-1})=(1-\omega)\lambda_t$ and $V(X_t|\underline{X}_{t-1})=(1-\omega)\lambda_t\left(-i\frac{\varphi_t''(0)}{\varphi_t'(0)}+\omega\lambda_t\right)$, noting that, to assure the variance existence, we consider that the characteristic functions $(\varphi_t)$ are differentiable at zero up to order 2.

As can be seen, a wide class of processes is included in the ZICP-INGARCH$(p,q)$ model (1). In fact, let $X=(X_t,t\in\mathbb{Z})$ be a stochastic process defined by

$$X_t=\sum_{j=1}^{N_t}X_{t,j}$$

where $N_t$ is a random variable that, conditionally on $\underline{X}_{t-1}$, follows a zero-inflated Poisson law and $X_{t,1}$,..., $X_{t,N_t}$ are discrete random variables with support $\mathbb{N}_0$ that, conditionally on $\underline{X}_{t-1}$, are independent, independent of $N_t$ and with characteristic function $\varphi_t$ differentiable at zero. If the parameters of the probability mass function of $N_t$ are $(\lambda_t^*,\omega)$, with $\lambda_t^*=\frac{i\lambda_t}{\varphi_t'(0)}$ and $0\le\omega<1$, then the process $X$ satisfies the model (1) as we have

$$\Phi_{X_t|\underline{X}_{t-1}}(u)=\sum_{k=0}^{\infty}E[\exp\{iu\,(X_{t,1}+...+X_{t,N_t})\}|N_t=k]\cdot P(N_t=k)$$

$$=\omega\sum_{k=0}^{\infty}\varphi_t^k(u)\delta_{k,0}+(1-\omega)e^{-\lambda_t^*}\sum_{k=0}^{\infty}\varphi_t^k(u)\frac{(\lambda_t^*)^k}{k!}=\omega+(1-\omega)\exp\left\{\lambda_t^*\left[\varphi_t(u)-1\right]\right\}.$$

Based on this construction, many particular models can be deduced.

**Example 2.1** ***(a)*** *When $\omega=0$, as we recover the CP-INGARCH model we obtain, in particular, the INGARCH ([6]), negative binomial INGARCH, generalized Poisson INGARCH ([18],[19]) and negative binomial DINARCH ([17]) models. For $0<\omega<1$, we have the zero-inflated Poisson and the zero-inflated negative binomial INGARCH models ([20]) as particular cases.*

***(b)*** *Let us consider independent random variables $(X_{t,j},t\in\mathbb{Z})$ following a geometric law with parameter $p_t=\frac{r}{r+\lambda_t}$ and $r>0$ arbitrarily fixed, that is, $\varphi_t(u)=\frac{p_te^{iu}}{1-(1-p_t)e^{iu}}$, $u\in\mathbb{R}$, $t\in\mathbb{Z}$. If $N_t$ is a random variable independent of $X_{t,j}$ and following a zero-inflated Poisson law with parameters $(r,\omega)$, $0\le\omega<1$ then the process $X_t=\sum_{j=1}^{N_t}X_{t,j}$ satisfies, unless an additive parameter $r$, the model (1). In this case, the model will be denoted by ZIGEOMP-INGARCH$(p,q)$. For $\omega=0$, we obtain the GEOMP-INGARCH model studied in Gonçalves* et al. *([8]).*

***(c)*** *As in the previous example, let us consider a sequence of independent random variables $(X_{t,j},t\in\mathbb{Z})$ following a geometric law with parameter $p\in]0,1[$ and $N_t$ following a zero-inflated Poisson law with parameters $\lambda_t^*=p\lambda_t$ and $\omega$. Then, $X_t=\sum_{j=1}^{N_t}X_{t,j}$ also satisfies the model (1). In this case, the model will be denoted by ZIGEOMP2-INGARCH$(p,q)$.*

***(d)*** *If $(X_{t,j},t\in\mathbb{Z})$ are independent random variables following a Poisson distribution with parameter $\theta>0$ and $N_t$ is independent of $X_{t,j}$ and follows a zero-inflated Poisson law with parameters $\lambda_t^*=\frac{\lambda_t}{\theta}$ and $\omega$, the resulting process $X$ satisfies the model (1). When $\omega=0$, the $X_t|\underline{X}_{t-1}$ law is the Neyman type-A distribution with parameters $(\lambda_t^*,\theta)$ ([11]) and so we will denote this model by ZINTA-INGARCH$(p,q)$.*

We note that the characteristic function $\varphi_t$ may be a random function since the parameter involved in $\varphi_t$ may depend on the previous observations of the process, via $\lambda_t$, as in the ZIGEOMP-INGARCH model.

Figure 1 presents trajectories and the basic descriptives of the ZIGEOMP-INGARCH$(1,1)$ model with $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$, considering different values for $\omega$, namely $\omega = 0$, 0.2, 0.4, illustrating clearly the zero-inflated characteristic of these models.

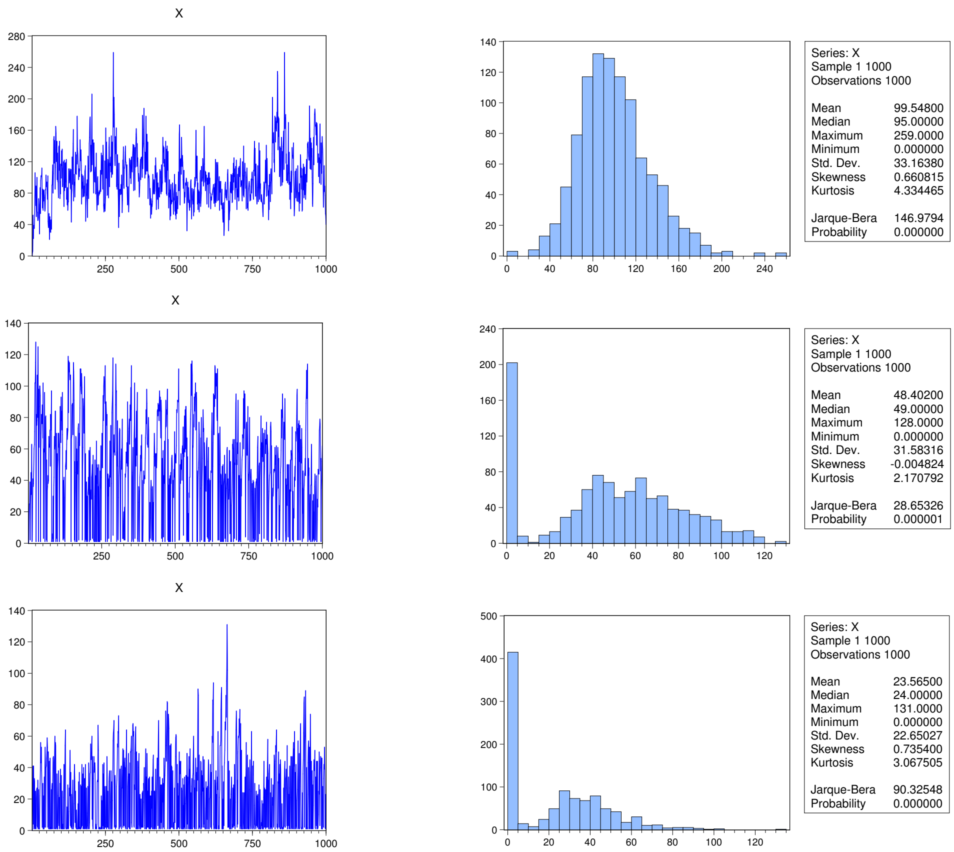


Figure 1: Trajectories and descriptives of ZIGEOMP-INGARCH $(1,1)$ model with $\omega = 0$ (on top), $\omega = 0.2$ (middle) and $\omega = 0.4$ (below): $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$.

We observe that

$$\frac{V\left(X_t|\underline{X}_{t-1}\right)}{E\left(X_t|\underline{X}_{t-1}\right)} = -i\frac{\varphi_t''(0)}{\varphi_t'(0)} + \omega\lambda_t = \frac{E\left(X_{t,1}^2\right)}{E\left(X_{t,1}\right)} + \omega\lambda_t = 1 + \frac{E\left(X_{t,1}\left(X_{t,1}-1\right)\right)}{E\left(X_{t,1}\right)} + \omega\lambda_t \geq 1 + \omega\lambda_t$$

whenever $\omega \geq 0$, with equality to 1 if and only if $X_{t,1} = 1$ almost surely (that is, the model is conditionally Poisson) and $\omega = 0$.

In consequence, this model is also overdispersed as it is easy to show that

$$\frac{V\left(X_t\right)}{E\left(X_t\right)} \geq \frac{E\left(V\left(X_t|\underline{X}_{t-1}\right)\right)}{E\left(X_t\right)},$$

from which the result is obtained supposing the conditional overdispersion, that is, $V\left(X_t|\underline{X}_{t-1}\right) > E\left(X_t|\underline{X}_{t-1}\right)$.

We also note that the definition of the model is still valuable when the parameter $\omega$ takes negative values, provided that $0 \leq \omega + (1-\omega)P(X_t = 0|\underline{X}_{t-1}) \leq 1$, which is equivalent to $\max\left\{-1, -\frac{P(X_t=0|\underline{X}_{t-1})}{1-P(X_t=0|\underline{X}_{t-1})}\right\} \leq \omega \leq 0$. For instance, if the conditional distribution is a mixture of a degenerate distribution with mass at zero and a Poisson law we obtain $\max\left\{-1, \frac{-e^{-\lambda_t}}{1-e^{-\lambda_t}}\right\} \leq \omega \leq 0$. To consider negative values for $\omega$ corresponds to a deflation at point zero. It introduces the possibility of considering underdispersed models as is the case of the generalization of the ZIP-INGARCH model when $\omega < 0$.

## 3 Stationarity properties

The study of first and second-order stationarity of these processes follows the approach developed for the compound Poisson INGARCH processes in Gonçalves *et al.* ([8]). In the following we summarize the main conclusions of this study. We note that the results obtained are not affected by the form of the conditional law but mainly by the evolution of $\lambda_t$.

In what concerns first-order stationarity we consider $\mu_t = E(X_t)$ and we deduce from the difference equation $\mu_t = (1-\omega)\alpha_0 + \sum_{j=1}^{p}(1-\omega)\alpha_j\mu_{t-j} + \sum_{k=1}^{q}\beta_k\mu_{t-k}$, that $X$ is first-order stationary if and only if $(1-\omega)\sum_{j=1}^{p}\alpha_j + \sum_{k=1}^{q}\beta_k < 1$. Moreover, under this condition, the processes $(X_t)$ and $(\lambda_t)$ are both first-order stationary and we have

$$\mu = E(X_t) = (1-\omega)E(\lambda_t) = \frac{(1-\omega)\alpha_0}{1-(1-\omega)\sum_{j=1}^{p}\alpha_j - \sum_{k=1}^{q}\beta_k}.$$

In order to obtain second-order stationarity conditions for the ZICP-INGARCH$(p,q)$ model (1) we assume that the family of characteristic functions $(\varphi_t, t\in\mathbb{Z})$ is differentiable at zero up to order 2 and restrict our study to the subclass of ZICP-INGARCH models with $\varphi_t$ satisfying the following condition:

**Hypothesis H1**: $-i\frac{\varphi_t''(0)}{\varphi_t'(0)} = v_0 + v_1\lambda_t$,

with $v_0 \geq 0$, $v_1 \geq 0$, not simultaneously zero. We stress that this particular case includes all the models presented in Example 2.1 **(a)** as well as a wide class of models not studied in literature like the models introduced in Example 2.1 **(b)** (for which $v_0 = 1$ and $v_1 = 2/r$), **(c)** (with $v_0 = (2-p)/p$ and $v_1 = 0$) and **(d)** (with $v_0 = 1+\theta$ and $v_1 = 0$).

A necessary and sufficient condition of second-order stationarity of $X$ is easily deduced from the vectorial state space representation presented below.

**Proposition 3.1** *Suppose that the process $X$ following the ZICP-INGARCH$(p,q)$ model is first-order stationary and satisfies the hypothesis* **H1**. *The vector of dimension $p+q-1$ given by*

$$W_t = \left[E(X_t^2) \;\; E(X_tX_{t-1}) \;\; \cdots \;\; E(X_tX_{t-(p-1)}) \;\; E(\lambda_t\lambda_{t-1}) \;\; \cdots \;\; E(\lambda_t\lambda_{t-(q-1)})\right]^T,$$

*$t\in\mathbb{Z}$, satisfies an autoregressive equation of order* $\max(p,q)$*:*

$$W_t = B_0 + \sum_{k=1}^{\max(p,q)} B_k W_{t-k}, \tag{2}$$

*where $B_0$ is a real vector of dimension $p+q-1$ and $B_k$ $(k=1,...,\max(p,q))$ are real squared matrices of order $p+q-1$.*

*Proof.* These equations may be deduced even if $E(X_t^2)$, $E(X_tX_{t-k})$ and $E(\lambda_t\lambda_{t-k})$ are not finite as they involve nonnegative measurable functions.

We focus on the case $p=q$, whereas the other cases can be obtained from this one setting additional parameters to 0. As $E(X_t^2) = E(E(X_t^2|\underline{X}_{t-1}))$ we obtain, following the same steps of Proposition 1 of Gonçalves *et al.* ([8]),

$$E(X_t^2) = C + (1+v_1)\left[\sum_{i=1}^{p}\left((1-\omega)\alpha_i^2 + \frac{2(1-\omega)\alpha_i\beta_i + \beta_i^2}{1+v_1}\right)E(X_{t-i}^2)\right.$$
$$\left. +2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\alpha_j((1-\omega)\alpha_i+\beta_i)E(X_{t-i}X_{t-j}) + 2(1-\omega)\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\beta_j((1-\omega)\alpha_i+\beta_i)E(\lambda_{t-i}\lambda_{t-j})\right] \tag{3}$$

$$E(X_tX_{t-k}) = \left[\alpha_0 - \frac{v_0\beta_k}{1+v_1}\right](1-\omega)\mu + (1-\omega)\left[\alpha_k + \frac{\beta_k}{1+v_1}\right]E(X_{t-k}^2) + \sum_{j=k+1}^{p}(1-\omega)^2\beta_jE(\lambda_{t-j}\lambda_{t-k})$$

$$+\sum_{j=1}^{k-1}((1-\omega)\alpha_j+\beta_j)E(X_{t-j}X_{t-k})+\sum_{j=k+1}^{p}(1-\omega)\alpha_j E(X_{t-j}X_{t-k}) \tag{4}$$

$$E(\lambda_t\lambda_{t-k})=\left[\alpha_0-\frac{v_0(\alpha_k+\beta_k)}{1+v_1}\right]\frac{\mu}{1-\omega}+\frac{\alpha_k+\beta_k}{(1-\omega)(1+v_1)}E(X_{t-k}^2)+\sum_{j=k+1}^{p}\beta_j E(\lambda_{t-j}\lambda_{t-k})$$

$$+\sum_{j=k+1}^{p}\frac{\alpha_j}{1-\omega}E(X_{t-j}X_{t-k})+\sum_{j=1}^{k-1}((1-\omega)\alpha_j+\beta_j)E(\lambda_{t-j}\lambda_{t-k}) \tag{5}$$

with $k\geq 1$ and $C=v_0\mu+(1+v_1)\left[2\alpha_0\mu-(1-\omega)\alpha_0^2\right]-v_0\mu\sum_{i=1}^{p}\left(2(1-\omega)\alpha_i\beta_i+\beta_i^2\right)>0$, independent of $t$. Using the above expressions it is clear that $W_t=B_0+\sum_{k=1}^{p}B_kW_{t-k}$, with $B_0$ the vector and $B_k$ $(k=1,...,p)$ the matrices presented in Appendix 1. □

In the following theorem we present the referred necessary and sufficient condition for weak stationarity of the process under study.

**Theorem 3.1** *Let $X$ be a first-order stationary process following an ZICP-INGARCH$(p,q)$ model such that* **H1** *is satisfied. This process is weakly stationary if and only if the polynomial matrix $P(z)=I_{p+q-1}-\sum_{k=1}^{\max(p,q)}B_kz^k$ is such that $\det P(z)$ has all its roots outside the unit circle, where $I_{p+q-1}$ is the identity matrix of order $p+q-1$ and $B_k$ $(k=1,...,\max(p,q))$ are the squared matrices of the autoregressive equation (2). Moreover, with $e_j$ denoting the $j-th$ row of the identity matrix,*

$$Cov(X_t,X_{t-j})=e_{j+1}[P(1)]^{-1}B_0-\mu^2,\qquad j=0,...,p-1,$$

$$Cov(\lambda_t,\lambda_{t-j})=e_{p+j}[P(1)]^{-1}B_0-\frac{\mu^2}{(1-\omega)^2},\quad j=1,...,q-1.$$

*Proof.* Let us consider $C_0=I_{p+q-1}$ and $C_k=B_k, k\geq 1$. Since $C_k=0$ when $k>\max(p,q)$, the autoregressive equation $W_t=B_0+\sum_{k=1}^{\max(p,q)}B_kW_{t-k}$ can be rewritten in the form

$$W_t=B_0+\sum_{k=1}^{\max(p,q)}C_kW_{t-k}\Longleftrightarrow W_t=B_0+\sum_{k=0}^{t}C_{t-k}W_t-W_t$$

when $t\geq\max(p,q)$. Introducing the $Z$-transform of $W_t$, $\widetilde{W}(z)=\sum_{k=0}^{+\infty}W_kz^{-k}$, and that of $C_t$, $\widetilde{C}(z)=C_0+\sum_{k=1}^{\max(p,q)}C_kz^{-k}$, and taking the $Z$-transform of both sides of last equation we get

$$\widetilde{W}(z)=B_0+\widetilde{C}(z)\widetilde{W}(z)-\widetilde{W}(z)\Leftrightarrow\left(I_{p+q-1}-\widetilde{C}(z)+I_{p+q-1}\right)\widetilde{W}(z)=B_0.$$

So, according to Elaydi [4] (p. 299), a necessary and sufficient condition for weak stationarity is

$$\det\left(I_{p+q-1}-\widetilde{C}(z)+I_{p+q-1}\right)\neq 0,\ \text{for all } z \text{ such that } |z|\geq 1$$

that is, $P\left(\frac{1}{z}\right)=\det\left(I_{p+q-1}-\sum_{k=1}^{\max(p,q)}B_kz^{-k}\right)$ has all its roots inside the unit circle. □

Let us consider a first-order stationary ZICP-INGARCH process with $p=q=1$. The previous study leads us to the following weak stationarity characterization which in this particular case can be easily proved from expression (3).

**Corollary 3.1** *Consider a first-order stationary ZICP-INGARCH$(1,1)$ model satisfying* **H1**. *A necessary and sufficient condition for weak stationarity is $(1-\omega)(1+v_1)\alpha_1^2+2(1-\omega)\alpha_1\beta_1+\beta_1^2<1$.*

We present now a result from which the autocorrelation function of the ZICP-INGARCH model can be obtained. The result extends those presented in Theorem 4 of Zhu ([20]) and is obtained using the same arguments. Moreover it states an ARMA$(p,q)$-like serial dependence structure for $X$.

**Theorem 3.2** *Suppose that $X$ follows a second-order stationary ZICP-INGARCH$(p,q)$ process. The autocovariances $\Gamma(k)=Cov(X_t,X_{t-k})$ and $\widetilde{\Gamma}(k)=Cov(\lambda_t,\lambda_{t-k})$ satisfy the linear equations*

$$\Gamma(k)=(1-\omega)\sum_{i=1}^{p}\alpha_i\cdot\Gamma(k-i)+\sum_{j=1}^{\min(k-1,q)}\beta_j\cdot\Gamma(k-j)+(1-\omega)^2\sum_{j=k}^{q}\beta_j\cdot\widetilde{\Gamma}(j-k),\quad k\geq 1,$$

$$\widetilde{\Gamma}(k)=(1-\omega)\sum_{i=1}^{\min(k,p)}\alpha_i\cdot\widetilde{\Gamma}(k-i)+\frac{1}{1-\omega}\sum_{i=k+1}^{p}\alpha_i\cdot\Gamma(i-k)+\sum_{j=1}^{q}\beta_j\cdot\widetilde{\Gamma}(k-j),\qquad k\geq 0,$$

*assuming that $\sum_{j=k}^{q}\beta_j\cdot\widetilde{\Gamma}(j-k)=0$ if $k>q$ and $\sum_{i=k+1}^{p}\alpha_i\cdot\Gamma(i-k)=0$ if $k+1>p$.*

**Example 3.1** *Supposing that $X$ follows an ZICP-INGARCH$(1,1)$ model, from Theorem 3.2, we have*

$$\Gamma(k)=[(1-\omega)\alpha_1+\beta_1]^{k-1}\cdot\frac{(1-\omega)\alpha_1[1-(1-\omega)\alpha_1\beta_1-\beta_1^2]}{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\cdot\Gamma(0),\quad k\geq 1,$$

$$\widetilde{\Gamma}(k)=[(1-\omega)\alpha_1+\beta_1]^{k}\cdot\widetilde{\Gamma}(0),\quad k\geq 1,$$

*from which the autocorrelation functions of $X$ and $\lambda$ can be obtained.*

*Under the hypothesis* **H1**, *the value of $\Gamma(0)$ can be deduced using the expression derived in Theorem 3.1. Indeed, $\Gamma(0)=V(X_t)=[P(1)]^{-1}B_0-\mu^2$, where*

$$P(1)=1-B_1=1-(1-\omega)(1+v_1)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2,$$

$$B_0=v_0\mu+(1+v_1)[2\alpha_0\mu-(1-\omega)\alpha_0^2]-v_0\mu[2(1-\omega)\alpha_1\beta_1+\beta_1^2],\quad \mu=\frac{(1-\omega)\alpha_0}{1-(1-\omega)\alpha_1-\beta_1}.$$

*Thus,* $\Gamma(0)=\frac{[1-2(1-\omega)\alpha_1\beta_1-\beta_1^2]}{1-(1-\omega)(1+v_1)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\left(v_0\mu+\frac{v_1+\omega}{1-\omega}\mu^2\right).$

*Using this result when $\omega=0$ we recover, in particular, the expressions stated in Weiß ([15]) for the Poisson distribution, in Zhu ([18], [19]) for the negative binomial and generalized Poisson laws. If we consider $v_0=(1-p)/p$ and $v_1=0$ we obtain expressions for the case where the conditional distribution of the process is the geometric Poisson law. For $\omega\neq 0$, we recover the results stated in Zhu ([20]) for the zero-inflated Poisson and the zero-inflated negative binomial distributions.*

We conclude this Section with a brief reference to the strict stationarity and, in particular, we construct a solution of the model with this property.

In order to construct such a solution, let us consider a sequence $M=(M_t,t\in\mathbb{Z})$ of i.i.d. Bernoulli random variables with parameter $(1-\omega)$, $\omega\in\,]0,1[$, and let us define a process $X^*=(X_t^*,t\in\mathbb{Z})$ such that

$$X_t^*=\begin{cases}0, & M_t=0\\ Y_t, & M_t=1\end{cases}$$

where $Y=(Y_t,t\in\mathbb{Z})$ is a CP-INGARCH process, independent of $M$, for which the conditional distribution of $Y_t$ given $\underline{Y}_{t-1}$ satisfies

$$\Phi_{Y_t|\underline{Y}_{t-1}}(u)=\exp\left\{i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\}$$
$$E\left(Y_t|\underline{Y}_{t-1}\right)=\lambda_t=\alpha_0+\sum_{j=1}^{p}\alpha_jY_{t-j}+\sum_{k=1}^{q}\beta_k\lambda_{t-k}.$$

We remember that in order to have this process it is sufficient to consider $Y_t = \sum_{j=1}^{N_t} Y_{t,j}$ where, conditionally on the past, $N_t$ follows a Poisson law with parameter $i\frac{\lambda_t}{\varphi_t'(0)}$ and $Y_{t,1}$,..., $Y_{t,N_t}$ are discrete independent random variables, independent of $N_t$ and with characteristic function $\varphi_t$.

If $X = (X_t, t \in \mathbb{Z})$ is the process following the model ZICP-INGARCH defined in (1), we show in the following that

$$\Phi_{X_t^*|(\underline{Y}_{t-1},\underline{M}_{t-1})}(u) = \Phi_{X_t|\underline{X}_{t-1}}(u).$$

In fact,

$$\begin{aligned}
\Phi_{X_t^*|(\underline{Y}_{t-1},\underline{M}_{t-1})}(u) &= E\left(\exp\left(iu\mathbb{I}_{\{M_t=1\}}Y_t\right) \mid \left(\underline{Y}_{t-1}, \underline{M}_{t-1}\right)\right) \\
&= E\left(\mathbb{I}_{\{M_t=0\}} + \exp\left(iuY_t\right)\mathbb{I}_{\{M_t=1\}} \mid \left(\underline{Y}_{t-1}, \underline{M}_{t-1}\right)\right) \\
&= \omega + E\left(\exp\left(iuY_t\right)\mathbb{I}_{\{M_t=1\}} \mid \left(\underline{Y}_{t-1}, \underline{M}_{t-1}\right)\right) \\
&= \omega + (1-\omega)\, E\left(\exp\left(iuY_t\right) \mid \left(\underline{Y}_{t-1}, \underline{M}_{t-1}\right)\right)
\end{aligned}$$

as $(Y_t)$ is independent of $(M_t)$ and $(M_t)$ are independent variables. So, from the independence between the processes $Y$ and $M$, we have

$$\begin{aligned}
\Phi_{X_t^*|(\underline{Y}_{t-1},\underline{M}_{t-1})}(u) &= \omega + (1-\omega)\, E\left(\exp\left(iuY_t\right) |\underline{Y}_{t-1}\right) \\
&= \omega + (1-\omega)\exp\left\{i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(u) - 1\right]\right\} \\
&= \Phi_{X_t|\underline{X}_{t-1}}(u).
\end{aligned}$$

So, a solution of our ZICP-INGARCH model defined in (1) may be obtained by this way.

We are now in conditions to state the strict stationarity of this kind of solution of model (1).

**Proposition 3.2** *Let us consider the model ZICP-INGARCH defined in (1) with $\varphi_t$ deterministic and independent of $t$. There is a strictly stationary process in $L^1$ that is a solution of this model if and only if $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$. Moreover, the first two moments of this process are finite.*

*Proof.* Let us consider the model (1) associated to a given family of characteristic functions deterministic and independent of $t$, that is,

$$\forall t \in \mathbb{Z}, \quad \varphi_t = \varphi \text{ and } \varphi \text{ deterministic.}$$

Following the study developed in Gonçalves *et al.* ([8]), Section 3.3, there is a strictly stationary solution of a CP-INGARCH model associated to the referred family of characteristic functions if and only if $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$. We denote this solution by $Y^* = (Y_t^*, t \in \mathbb{Z})$ and remember that $E\left(Y_t^*|\underline{Y^*}_{t-1}\right) = \lambda_t$ and $\Phi_{Y_t^*|\underline{Y^*}_{t-1}}(u) = \exp\left\{i\frac{\lambda_t}{\varphi'(0)}\left[\varphi(u) - 1\right]\right\}$. Defining the process $X^* = (X_t^*, t \in \mathbb{Z})$ as

$$X_t^* = Y_t^*\mathbb{I}_{\{M_t=1\}}$$

with $M = (M_t, t \in \mathbb{Z})$ a sequence of i.i.d. Bernoulli random variables with parameter $(1-\omega)$, $\omega \in ]0,1[$ and independent of $Y$, we obtain a solution of model (1) taking into account that

$$\Phi_{X_t^*|(\underline{Y^*}_{t-1},\underline{M}_{t-1})}(u) = \omega + (1-\omega)\exp\left\{i\frac{\lambda_t}{\varphi'(0)}\left[\varphi(u) - 1\right]\right\}.$$

The process $X^*$ is strictly stationary as it is a measurable function of the process $((Y_t^*, M_t), t \in \mathbb{Z})$, which is strictly stationary as $Y^*$ and $M$ are independent and strictly stationary processes.
As $Y^*$ is a second order process, the same happens to $X^*$, and so $X^*$ is also a weakly stationary process if $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$. □

## 4 High order moments of ZICP-INGARCH(1,1) models

In this section, we focus on model (1) with $p = q = 1$, *i.e.*,

$$\Phi_{X_t|\underline{X}_{t-1}}(u) = \omega + (1-\omega)\exp\left\{ i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\}, u \in \mathbb{R}, \qquad \lambda_t = \alpha_0 + \alpha_1 X_{t-1} + \beta_1 \lambda_{t-1}.$$

We also consider that the characteristic function $\varphi_t$ satisfies the condition:

**Hypothesis H2**: $\varphi_t$ is a deterministic function.

In this particular case we are able to enlarge the probabilistic study of the model. Namely, we obtain conditions assuring the existence of all orders moments and closed-forms expressions for the cumulants up to order 4. The skewness and kurtosis are deduced in consequence. Contrarily to the analysis in Section 3, this study strongly involves the conditional law of the process.

To study the moments of $X_t$ let us assume in the following that $\varphi_t$ is differentiable as many times as necessary. We start by stating a necessary and sufficient condition for the existence of all the moments of the process. This result includes Proposition 6 of Ferland *et al.* ([6]) in which $\varphi_t(u)$ is equal to $e^{iu}$ and, in accordance with these authors, we point out that it is an unexpected result taking into consideration what is known on the complexity analysis and on the conditions of moments existence for conditional heteroscedastic models in general.

**Theorem 4.1** *The moments of a ZICP-INGARCH*$(1,1)$ *model satisfying the hypothesis* ***H2*** *are all finite if and only if* $(1-\omega)\alpha_1 + \beta_1 < 1$.

*Proof.* According to Grubbström and Tang ([9]), since $X_t|\underline{X}_{t-1}$ is a compound random variable where the counting distribution is the zero-inflated Poisson law, its $m$th moment is given by

$$E[X_t^m|\underline{X}_{t-1}] = (1-\omega)\sum_{r=0}^{m}\frac{1}{r!}\frac{\lambda_t^r}{(\varphi_t'(0))^r}\sum_{j=0}^{r}\binom{r}{j}\frac{(-1)^{r-j}}{i^{m-r}}\left(\varphi_t^j\right)^{(m)}(0), \qquad m \geq 1,$$

with $\varphi_t^j = \prod_{k=1}^{j}\varphi_t$ and $(\varphi_t^j)^{(m)}$ the $m$th derivative of $\varphi_t^j$, namely,

$$\left(\varphi_t^j\right)^{(m)}(u) = \sum_{n=m-j}^{m-1}\left\{\frac{j!}{(j-m+n)!}\,\varphi_t^{j-m+n}(u) \times \right.$$
$$\left.\sum_{\substack{k_1+\ldots+k_m=m-n\\ k_1+2k_2+\ldots+mk_m=m\\ k_r\in\mathbb{N}_0}}(m;k_1,...,k_m)\,[\varphi_t'(u)]^{k_1}...[\varphi_t^{(m)}(u)]^{k_m}\right\}, \quad m \geq j, \tag{6}$$

where $(m;k_1,...,k_m) = \frac{m!}{k_1!k_2!...k_m!(1!)^{k_1}(2!)^{k_2}...(m!)^{k_m}}$ ([1], p. 823). For a proof of expression (6), see Appendix 2. So,

$$E[X_t^m] = (1-\omega)\sum_{r=0}^{m}\sum_{j=0}^{r}\frac{1}{r!}\binom{r}{j}\frac{(-1)^{r-j}\left(\varphi_t^j\right)^{(m)}(0)}{i^{m-r}(\varphi_t'(0))^r}\,E[\lambda_t^r], \tag{7}$$

and

$$\lambda_t^r = (\alpha_0 + \alpha_1 X_{t-1} + \beta_1\lambda_{t-1})^r = \sum_{n=0}^{r}\binom{r}{n}\alpha_0^{r-n}\sum_{l=0}^{n}\binom{n}{l}\alpha_1^l\beta_1^{n-l}X_{t-1}^l\lambda_{t-1}^{n-l}$$
$$= \alpha_0^r + \sum_{n=1}^{r}\binom{r}{n}\alpha_0^{r-n}\left[\beta_1^n\lambda_{t-1}^n + \sum_{l=1}^{n}\binom{n}{l}\alpha_1^l\beta_1^{n-l}X_{t-1}^l\lambda_{t-1}^{n-l}\right].$$

Using the fact that $\lambda_{t-1}^{n-l}$ is $\underline{X}_{t-2}$-measurable we deduce that

$$E[\lambda_t^r|\underline{X}_{t-2}] = \alpha_0^r + \sum_{n=1}^{r}\binom{r}{n}\alpha_0^{r-n}\left[\beta_1^n\lambda_{t-1}^n + \sum_{l=1}^{n}\binom{n}{l}\alpha_1^l\beta_1^{n-l}\lambda_{t-1}^{n-l}\,E[X_{t-1}^l|\underline{X}_{t-2}]\right]$$

$$= \alpha_0^r + \sum_{n=1}^{r} \left( \begin{array}{c} r \\ n \end{array} \right) \alpha_0^{r-n} \beta_1^n \lambda_{t-1}^n + (1-\omega) \sum_{n=1}^{r} \left( \begin{array}{c} r \\ n \end{array} \right) \alpha_0^{r-n} \sum_{l=1}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \times$$

$$\times \sum_{v=0}^{l} \frac{\alpha_1^l \beta_1^{n-l}}{v!(\varphi'_{t-1}(0))^v} \sum_{x=0}^{v} \left( \begin{array}{c} v \\ x \end{array} \right) \frac{(-1)^{v-x}}{i^{l-v}} \left(\varphi_{t-1}^x\right)^{(l)}(0) \lambda_{t-1}^{v+n-l}. \qquad (8)$$

Let $\Lambda_t = (\lambda_t^m, ..., \lambda_t)^T$. In the algebraic expression of $E[\lambda_t^r | \underline{X}_{t-2}]$, for $r = 1, ..., m$, all the powers of $\lambda_{t-1}$ are $\leq r$. Therefore, the following equation is satisfied:

$$E[\Lambda_t | \underline{X}_{t-2}] = \mathbf{d} + \mathbf{D}\Lambda_{t-1},$$

with $\mathbf{d} = (\alpha_0^m, \cdots, \alpha_0^2, \alpha_0)^T$ and $\mathbf{D} = (d_{ij})$, $i, j = 1, ..., m$, the upper triangular matrix given by

$$\mathbf{D} = \left[ \begin{array}{cccc} (1-\omega)(\alpha_1+\beta_1)^m + \omega\beta_1^m & \cdots & d_{1,m-1} & d_{1m} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & (1-\omega)(\alpha_1+\beta_1)^2 + \omega\beta_1^2 & d_{m-1,m} \\ 0 & \cdots & 0 & (1-\omega)\alpha_1 + \beta_1 \end{array} \right].$$

Indeed, let us prove that the diagonal entries of the matrix $\mathbf{D}$ are those given above. The $k$th diagonal entry of the matrix $\mathbf{D}$ corresponds to the case where in equation (8), we consider $r = m - k + 1$. Thus, to obtain the coefficient of $\lambda_{t-1}^{m-k+1}$, we look at the terms corresponding to $n = m - k + 1$ and $l = v$. Then,

$$d_{kk} = \beta_1^{m-k+1} + (1-\omega) \sum_{l=1}^{m-k+1} \left( \begin{array}{c} m-k+1 \\ l \end{array} \right) \frac{\alpha_1^l \beta_1^{m-k+1-l}}{l!(\varphi'_{t-1}(0))^l} \sum_{x=0}^{l} \left( \begin{array}{c} l \\ x \end{array} \right) (-1)^{l-x} \left(\varphi_{t-1}^x\right)^{(l)}(0)$$

$$= \omega\beta_1^{m-k+1} + (1-\omega) \sum_{l=0}^{m-k+1} \left( \begin{array}{c} m-k+1 \\ l \end{array} \right) \frac{\alpha_1^l \beta_1^{m-k+1-l}}{l!(\varphi'_{t-1}(0))^l} \sum_{x=0}^{l} \left( \begin{array}{c} l \\ x \end{array} \right) (-1)^{l-x} \left(\varphi_{t-1}^x\right)^{(l)}(0).$$

Using the expression (6) we obtain

$$\sum_{x=0}^{l} \left( \begin{array}{c} l \\ x \end{array} \right) (-1)^{l-x} \left(\varphi_{t-1}^x\right)^{(l)}(0) = \sum_{x=0}^{l} \left( \begin{array}{c} l \\ x \end{array} \right) (-1)^{l-x} \sum_{j=l-x}^{l-1} \frac{x!}{(x-l+j)!} \varphi_{t-1}^{x-l+j}(0)$$

$$\times \sum_{\substack{k_1+...+k_l=l-j \\ k_1+2k_2+...+lk_l=l}} (l; k_1, ..., k_l) \left[\varphi'_{t-1}(0)\right]^{k_1} ... [\varphi_{t-1}^{(l)}(0)]^{k_l},$$

and therefore, for any arbitrarily fixed $k_1, ..., k_l \in \mathbb{N}_0$ such that $k_1 + ... + k_l = l - j$ and $k_1 + 2k_2 + ... + lk_l = l$, the coefficient of $\left[\varphi'_{t-1}(0)\right]^{k_1} \ldots [\varphi_{t-1}^{(l)}(0)]^{k_l}$ is given by

$$\left[ \sum_{x=0}^{l} \left( \begin{array}{c} l \\ x \end{array} \right) (-1)^{l-x} \frac{x!}{(x-(k_1+...+k_l))!} \right] (l; k_1, ..., k_l)$$

$$= \left[ \sum_{x=k_1+...+k_l}^{l} \frac{(-1)^{l-x}}{(l-x)!(x-(k_1+...+k_l))!} \right] l!(l; k_1, ..., k_l)$$

$$= \frac{(-1)^{l-(k_1+...+k_l)}}{(l-(k_1+...+k_l))!} \left[ \sum_{m=0}^{l-(k_1+...+k_l)} \left( \begin{array}{c} l-(k_1+...+k_l) \\ m \end{array} \right) (-1)^{-m} \right] l!(l; k_1, ..., k_l).$$

When $k_1 = l$, $k_2 = \ldots = k_l = 0$, we obtain the coefficient $l!(l; l, 0, ..., 0) = l!$. Otherwise, the coefficient is zero. Therefore, we finally conclude that

$$\sum_{x=0}^{l} \left( \begin{array}{c} l \\ x \end{array} \right) (-1)^{l-x} \left(\varphi_{t-1}^x\right)^{(l)}(0) = l! \; [\varphi'_{t-1}(0)]^l, \qquad (9)$$

and then the $k$th diagonal entry of the matrix $\mathbf{D}$ is

$$d_{kk} = \omega\beta_1^{m-k+1} + (1-\omega)\sum_{l=0}^{m-k+1}\binom{m-k+1}{l}\alpha_1^l\beta_1^{m-k+1-l}$$
$$= (1-\omega)(\alpha_1+\beta_1)^{m-k+1} + \omega\beta_1^{m-k+1}.$$

So, we conclude that the eigenvalues of $\mathbf{D}$ (which coincide with its diagonal entries because it is a triangular matrix) are inside the unit circle if and only if $(1-\omega)\alpha_1 + \beta_1 < 1$. Using this fact and proceeding as in Proposition 6 of Ferland *et al.* ([6]), we can write $E[\Lambda_t|\underline{X}_{t-k}] = (I_m - \mathbf{D})^{-1}(I_m - \mathbf{D}^{k-1})\mathbf{d} + \mathbf{D}^{k-1}\Lambda_{t-(k-1)}$, where $I_m$ is the identity matrix of order $m$. Since $\mathbf{D}^{k-1} \to 0$ when $k \to \infty$, we have $E[\Lambda_t] = \lim_{k\to\infty} E[\Lambda_t|\underline{X}_{t-k}] = (I_m - \mathbf{D})^{-1}\mathbf{d}$, and then from (7) all the moments of $X_t$ of order $\leq m$ are finite. □

Now let us consider that the characteristic function $\varphi_t$ satisfies the condition:

**Hypothesis H3**: $\varphi_t$ is deterministic and independent of $t$.

This is equivalent to say that $v_1 = 0$ in hypothesis **H1**. Henceforward we simply denote $\varphi_t$ as $\varphi$. We stress that this particular case still includes a wide class of new models not studied in literature as those introduced in Example 2.1 **(c)** and **(d)**.

Let us consider a first-order stationary ZICP-INARCH(1) model. In the following we illustrate the derivation of its first three order cumulants. To do this, let us start by recalling that if $\Phi_{X_t}$ denotes the characteristic function of $X_t$, its cumulant generating function is given by $\kappa_{X_t}(z) = \ln(\Phi_{X_t}(z))$, and the coefficient $\kappa_j(X_t)$ of the series expansion $\kappa_{X_t}(z) = \sum_{j=1}^{\infty}\kappa_j(X_t)\cdot(iz)^j/j!$ is referred to as the $j$-cumulant with $\kappa_j(X_t) = (-i)^j\kappa_{X_t}^{(j)}(0)$. Using the characteristic function of the conditional distribution and $\lambda_t = \alpha_0 + \alpha_1 X_{t-1}$ we obtain, for $z \in \mathbb{R}$,

$$\Phi_{X_t}(z) = E\left(e^{izX_t}\right) = E\left[E\left(e^{izX_t}|\underline{X}_{t-1}\right)\right] = E\left[\omega + (1-\omega)\exp\left(i\frac{\lambda_t}{\varphi'(0)}[\varphi(z)-1]\right)\right]$$
$$= \omega + (1-\omega)\exp\left(\frac{i\alpha_0}{\varphi'(0)}[\varphi(z)-1]\right)\cdot\Phi_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}[\varphi(z)-1]\right) = \omega + (1-\omega)A(z), \tag{10}$$

hence, the cumulant generating function of $X_t$ is given by $\kappa_{X_t}(z) = \ln(\omega + (1-\omega)A(z))$.

Taking derivatives on both sides, it follows that

$$\kappa'_{X_t}(z) = \frac{\Phi'_{X_t}(z)}{\Phi_{X_t}(z)} = \frac{(1-\omega)A'(z)}{\omega + (1-\omega)A(z)} = \frac{(1-\omega)\frac{A'(z)}{A(z)}}{(1-\omega) + \frac{\omega}{A(z)}},$$

$$\kappa''_{X_t}(z) = \frac{\Phi''_{X_t(z)}}{\Phi_{X_t(z)}} - \left[\frac{\Phi'_{X_t(z)}}{\Phi_{X_t(z)}}\right]^2 = \frac{(1-\omega)\frac{A''(z)}{A(z)}}{(1-\omega) + \frac{\omega}{A(z)}} - \left(\kappa'_{X_t}(z)\right)^2,$$

$$\kappa'''_{X_t}(z) = \frac{\Phi'''_{X_t(z)}}{\Phi_{X_t(z)}} - 3\frac{\Phi'_{X_t(z)}\Phi''_{X_t(z)}}{\left(\Phi_{X_t(z)}\right)^2} + 2\left[\frac{\Phi'_{X_t(z)}}{\Phi_{X_t(z)}}\right]^3 = \frac{(1-\omega)\frac{A'''(z)}{A(z)}}{(1-\omega) + \frac{\omega}{A(z)}} - 3\kappa'_{X_t}(z)\kappa''_{X_t}(z) - \left(\kappa'_{X_t}(z)\right)^3,$$

where, taking $h(z) = \frac{\varphi(z)-1}{\varphi'(0)}$,

$$A'(z) = \frac{i\alpha_0\varphi'(z)}{\varphi'(0)}\cdot\exp(i\alpha_0\cdot h(z))\cdot\Phi_{X_{t-1}}(\alpha_1\cdot h(z)) + \exp(i\alpha_0\cdot h(z))\cdot\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\cdot\Phi'_{X_{t-1}}(\alpha_1\cdot h(z)),$$

$$A''(z) = \left[\frac{i\alpha_0\varphi''(z)}{\varphi'(0)} + \left(\frac{i\alpha_0\varphi'(z)}{\varphi'(0)}\right)^2\right]\cdot\exp(i\alpha_0\cdot h(z))\cdot\Phi_{X_{t-1}}(\alpha_1\cdot h(z))$$
$$+\left[\frac{\alpha_1\varphi''(z)}{\varphi'(0)} + 2i\alpha_0\alpha_1\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^2\right]\cdot\exp(i\alpha_0\cdot h(z))\cdot\Phi'_{X_{t-1}}(\alpha_1\cdot h(z))$$

$$+\left(\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right)^2 \cdot \exp\left(i\alpha_0 \cdot h(z)\right) \cdot \Phi''_{X_{t-1}}\left(\alpha_1 \cdot h(z)\right),$$

$$A'''(z) = \left[\frac{i\alpha_0\varphi'''(z)}{\varphi'(0)} - 3\alpha_0^2\frac{\varphi'(z)\varphi''(z)}{(\varphi'(0))^2} - i\alpha_0^3\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3\right] \cdot \exp\left(i\alpha_0 \cdot h(z)\right) \cdot \Phi_{X_{t-1}}\left(\alpha_1 \cdot h(z)\right)$$

$$+\left[6i\alpha_0\alpha_1\frac{\varphi'(z)\varphi''(z)}{(\varphi'(0))^2} - 3\alpha_0^2\alpha_1\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3 + \alpha_1\frac{\varphi'''(z)}{\varphi'(0)}\right] \cdot \exp\left(i\alpha_0 \cdot h(z)\right) \cdot \Phi'_{X_{t-1}}\left(\alpha_1 \cdot h(z)\right)$$

$$+\left[3\alpha_1^2\frac{\varphi'(z)\varphi''(z)}{(\varphi'(0))^2} + 3i\alpha_0\alpha_1^2\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3\right] \cdot \exp\left(i\alpha_0 \cdot h(z)\right) \cdot \Phi''_{X_{t-1}}\left(\alpha_1 \cdot h(z)\right)$$

$$+\left(\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right)^3 \cdot \exp\left(i\alpha_0 \cdot h(z)\right) \cdot \Phi'''_{X_{t-1}}\left(\alpha_1 \cdot h(z)\right).$$

Inserting $z = 0$ into the previous equations and noting that $A(0) = 1$ and $h(0) = 0$, we obtain

$$\kappa'_{X_t}(0) = (1-\omega)\left[i\alpha_0 + \alpha_1 \cdot \kappa'_{X_{t-1}}(0)\right]$$

and

$$\kappa''_{X_t}(0) = (1-\omega)\frac{\varphi''(0)}{\varphi'(0)}\left[i\alpha_0 + \alpha_1 \cdot \kappa'_{X_{t-1}}(0)\right] + (1-\omega)(i\alpha_0)^2$$
$$+ 2i\alpha_0\left[\kappa'_{X_t}(0) - i(1-\omega)\alpha_0\right] + (1-\omega)\alpha_1^2 \cdot \kappa''_{X_{t-1}}(0) - (1-\omega)\alpha_1^2\mu^2 + \mu^2$$
$$\Leftrightarrow \left[1-(1-\omega)\alpha_1^2\right] \cdot \kappa''_{X_t}(0) = i\frac{\varphi''(0)}{\varphi'(0)}\mu + 2\alpha_0\mu + (1-\omega)\alpha_0^2 + (1-\omega)\left(\frac{i\mu}{1-\omega} - i\alpha_0\right)^2 + \mu^2.$$

So,

$$\kappa_1(X_t) = \frac{(1-\omega)\alpha_0}{1-(1-\omega)\alpha_1} = \mu,$$

$$\kappa_2(X_t) = \frac{v_0\mu + \frac{\omega\mu^2}{1-\omega}}{1-(1-\omega)\alpha_1^2} = V(X_t),$$

as stated in Example 3.1. Moreover,

$$\kappa'''_{X_t}(0) = i\frac{\varphi'''(0)}{\varphi'(0)}\mu + (1-\omega)\frac{\varphi''(0)}{\varphi'(0)}\left(-3\alpha_0^2 - 6\alpha_0\alpha_1\mu + 3\alpha_1^2[\kappa''_{X_{t-1}}(0) - \mu^2]\right)$$
$$+(1-\omega)\left(-i\alpha_0^3 - 3i\alpha_0^2\alpha_1\mu + 3i\alpha_0\alpha_1^2[\kappa''_{X_{t-1}}(0) - \mu^2] + 3i\alpha_1^3\mu\kappa''_{X_{t-1}}(0) - i\alpha_1^3\mu\right)$$
$$+(1-\omega)\alpha_1^3\kappa'''_{X_{t-1}}(0) - 3i\mu\kappa''_{X_t}(0) + i\mu^3.$$

So, the third-order cumulant of $X_t$ is

$$\kappa_3(X_t) = \frac{d_0\mu + 3\alpha_1^2\kappa_2(X_t)[(1-\omega)v_0 - \omega\mu] + \frac{\omega\mu^2}{1-\omega}[3v_0 + \alpha_0 + \mu(\alpha_1 - 2)]}{1-(1-\omega)\alpha_1^3},$$

with $d_0 = -\varphi'''(0)/\varphi'(0)$.

When $\omega = 0$, the first four cumulants may be explicited using the technique given in Weiß ([15]). So we derive these expressions for a CP-INARCH(1) process, under the hypothesis **H3**. The skewness and the flatness of the distribution of the process are consequently available. Let us fix some notation:

$$v_0 = -i\,\frac{\varphi''(0)}{\varphi'(0)}, \quad d_0 = -\frac{\varphi'''(0)}{\varphi'(0)}, \quad c_0 = i\frac{\varphi^{(iv)}(0)}{\varphi'(0)}, \quad f_k = \alpha_0 / \prod_{j=1}^{k}(1-\alpha_1^j), \;\; k \in \mathbb{N}.$$

**Theorem 4.2** *Let $X$ be a first order stationary CP-INARCH(1) process admitting fourth order moment and such that the hypothesis* **H3** *is satisfied with $\varphi$ differentiable up to order 4. Then, the first four cumulants of $X_t$ are given by*

$$\kappa_1(X_t) = f_1, \quad \kappa_2(X_t) = v_0 f_2, \quad \kappa_3(X_t) = f_3[d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2],$$

$$\kappa_4(X_t) = f_4[c_0(1-\alpha_1^2)(1-\alpha_1^3) + v_0^3(3\alpha_1^2 + 15\alpha_1^5) + v_0 d_0(4\alpha_1^2 + 6\alpha_1^3 - 10\alpha_1^5)].$$

*Proof.* Using the expression (10) with $\omega = 0$, we deduce the cumulant generating function of $X_t$

$$\kappa_{X_t}(z) = \frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right] + \kappa_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right).$$

Taking derivatives on both sides, it follows that

$$\kappa'_{X_t}(z) = \frac{i\alpha_0}{\varphi'(0)}\varphi'(z) + \frac{\alpha_1}{\varphi'(0)}\varphi'(z)\cdot\kappa'_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right),$$

$$\kappa^{(n)}_{X_t}(z) = \frac{i\alpha_0}{\varphi'(0)}\varphi^{(n)}(z) + \sum_{j=1}^{n-1} a_{n-1,j}\cdot\kappa^{(j)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$
$$+ \left[\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right]^n\cdot\kappa^{(n)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right), \quad n = 2,3,4, \tag{11}$$

where the second formula is proved by induction, with the coefficients $a_{n-1,j}$ given by $a_{n-1,1} = \frac{\alpha_1}{\varphi'(0)}\varphi^{(n)}(z)$, $a_{n-1,j} = \left[\frac{\alpha_1}{\varphi'(0)}\right]^j \sum_{\substack{k_1+\ldots+k_n=j \\ k_1+2k_2+\ldots+nk_n=n \\ k_r\in\mathbb{N}_0}} (n;k_1,\ldots,k_n)\,[\varphi'(z)]^{k_1}\ldots[\varphi^{(n)}(z)]^{k_n}$, $j\geq 2$.

Inserting $z = 0$ into the previous equations, one obtains

$$\kappa'_{X_t}(0) = i\alpha_0 + \alpha_1\cdot\kappa'_{X_{t-1}}(0) \Rightarrow \kappa_1(X_t) = \frac{\alpha_0}{1-\alpha_1},$$

$$\kappa_n(X_t) = \sum_{j=1}^{n-1} b_{n-1,j}\cdot\kappa_j(X_{t-1}) + \alpha_1^n\cdot\kappa_n(X_{t-1}), \quad n = 2,3,4,$$

where the coefficients $b_{n-1,j}$ are given by

$$b_{n-1,1} = (-i)^{n-1}\frac{\varphi^{(n)}(0)}{\varphi'(0)}, \qquad b_{n-1,n-1} = -i\,\frac{n(n-1)}{2}\frac{\varphi''(0)}{\varphi'(0)}\alpha_1^{n-1},$$

$$b_{n-1,j} = (-i)^{n-j}\left[\frac{\alpha_1}{\varphi'(0)}\right]^j \sum_{\substack{k_1+\ldots+k_n=j \\ k_1+2k_2+\ldots+nk_n=n \\ k_r\in\mathbb{N}_0}} (n;k_1,\ldots,k_n)\,[\varphi'(0)]^{k_1}\ldots[\varphi^{(n)}(0)]^{k_n},$$

for $1<j<n-1$. From here, it follows

$$(1-\alpha_1^2)\cdot\kappa_2(X_t) = b_{1,1}\cdot\kappa_1(X_t) \Rightarrow \kappa_2(X_t) = -i\frac{\varphi''(0)}{\varphi'(0)}\frac{\alpha_0}{(1-\alpha_1)(1-\alpha_1^2)},$$

and, analogously,

$$\kappa_3(X_t) = -\alpha_0\,\frac{(1-\alpha_1^2)\frac{\varphi'''(0)}{\varphi'(0)} + 3\alpha_1^2\left[\frac{\varphi''(0)}{\varphi'(0)}\right]^2}{(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$

$$\kappa_4(X_t) = i\alpha_0\,\frac{(1-\alpha_1^2)(1-\alpha_1^3)\frac{\varphi^{(iv)}(0)}{\varphi'(0)} + (3\alpha_1^2+15\alpha_1^5)\left[\frac{\varphi''(0)}{\varphi'(0)}\right]^3 + (4\alpha_1^2+6\alpha_1^3-10\alpha_1^5)\frac{\varphi''(0)\varphi'''(0)}{(\varphi'(0))^2}}{(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)(1-\alpha_1^4)},$$

that ends the proof, using the notation indicated above. □

**Observation 4.1** *As a consequence of Theorem 4.2, X is an asymmetric process around the mean and is leptokurtic since its skewness and kurtosis are, respectively, given by*

$$S_{X_t} = \frac{(1-\alpha_1^2)d_0 + 3\alpha_1^2 v_0^2}{v_0(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{1+\alpha_1}{v_0\alpha_0}},$$

$$K_{X_t} = 3 + \frac{(1-\alpha_1^2)(1-\alpha_1^3)c_0 + (3\alpha_1^2+15\alpha_1^5)v_0^3 + (4\alpha_1^2+6\alpha_1^3-10\alpha_1^5)v_0 d_0}{\alpha_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)v_0^2}.$$

In the following we illustrate the expressions displayed for the skewness and kurtosis of a CP-INARCH(1) process considering some particular compound Poisson distributions.

**Example 4.1** ***(1) Poisson law:***
*We have $\varphi(u) = e^{iu}$ and so $\varphi^{(n)}(0) = i^n$, $v_0 = d_0 = c_0 = 1$. The skewness and the kurtosis of $X_t$, respectively, are given by*

$$S_{X_t} = \frac{1+2\alpha_1^2}{1+\alpha_1+\alpha_1^2}\sqrt{\frac{1+\alpha_1}{\alpha_0}}, \quad K_{X_t} = 3 + \frac{1+6\alpha_1^2+5\alpha_1^3+6\alpha_1^5}{\alpha_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}.$$

*Denoting by $\left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\}$ the Stirling number of the second kind ([1]), an alternative form to that used by Weiß ([15]) to determine recursively the cumulants of the INARCH$(1)$ model is*

$$\kappa_1(X_t) = \frac{\alpha_0}{1-\alpha_1}, \qquad \kappa_n(X_t) = (1-\alpha_1^n)^{-1}\cdot\sum_{j=1}^{n-1} b_{n-1,j}\cdot\kappa_j(X_t), \quad n \geq 2,$$

*where* $b_{n-1,1} = 1, \quad b_{n-1,n-1} = \frac{n(n-1)}{2}\,\alpha_1^{n-1}, \quad b_{n-1,j} = \left\{ \begin{matrix} n \\ j \end{matrix} \right\}\,\alpha_1^j, \quad 1 < j < n-1.$

***(2) generalized Poisson law:***
*In this case $\varphi$ is the characteristic function of the variables $X_{t,j}$, $j = 1, ..., N_t$ having the Borel law with parameter $\kappa$. For $0 < \kappa < 1$, all the moments of the Borel distribution exist and $\varphi^{(k)}(0) = i^k E(X_{t,1}^k)$. As $E(X_{t,1}) = (1-\kappa)^{-1}, E(X_{t,1}^2) = (1-\kappa)^{-3}, E(X_{t,1}^3) = (2\kappa+1)(1-\kappa)^{-5}$, $E(X_{t,1}^4) = (6\kappa^2+8\kappa+1)(1-\kappa)^{-7}$, we have $\varphi'(0) = i(1-\kappa)^{-1}, \varphi''(0) = -(1-\kappa)^{-3}, \varphi'''(0) = -i(2\kappa+1)(1-\kappa)^{-5}$, $\varphi^{(iv)}(0) = (6\kappa^2+8\kappa+1)(1-\kappa)^{-7}$, $v_0 = \frac{1}{(1-\kappa)^2}$, $d_0 = \frac{2\kappa+1}{(1-\kappa)^4}$, $c_0 = \frac{6\kappa^2+8\kappa+1}{(1-\kappa)^6}$.*
*Thus we obtain the cumulants*

$$\kappa_2(X_t) = \frac{\alpha_0}{(1-\kappa)^2(1-\alpha_1)(1-\alpha_1^2)}, \quad \kappa_3(X_t) = \frac{\alpha_0(1-\alpha_1^2)(2\kappa+1)+3\alpha_0\alpha_1^2}{(1-\kappa)^4(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$

$$\kappa_4(X_t) = \alpha_0\,\frac{6\kappa^2+8\kappa+1-6\alpha_1^2(\kappa^2+1)-\alpha_1^3(6\kappa^2-4\kappa-5)+6\alpha_1^5(\kappa^2-2\kappa+1)}{(1-\kappa)^6(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)(1-\alpha_1^4)},$$

*and the skewness and the kurtosis of $X_t$*

$$S_{X_t} = \frac{(1-\alpha_1^2)(2\kappa+1)+3\alpha_1^2}{(1-\kappa)(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{1+\alpha_1}{\alpha_0}},$$

$$K_{X_t} = 3 + \frac{6\kappa^2+8\kappa+1-6\alpha_1^2(\kappa^2+1)-\alpha_1^3(6\kappa^2-4\kappa-5)+6\alpha_1^5(\kappa^2-2\kappa+1)}{\alpha_0(1-\kappa)^2(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}.$$

*We stress that using the fact that the generalized Poisson distribution is a compound Poisson instead of the procedure adopted by Zhu ([19]) made much prompt the deduction of the first four cumulants and completes the results of [19]. In Figure 2 the trajectory and descriptives of* 1000 *observations of a GP-INARCH$(1)$ process are presented from which is evident the closeness of the theoretical values ($S_{X_t} \simeq 1.0362$ and $K_{X_t} = 4.2527$, according to the above formulas) and the empirical ones.*

***(3) geometric Poisson law*** *(like in Example 2.1 (c)):*

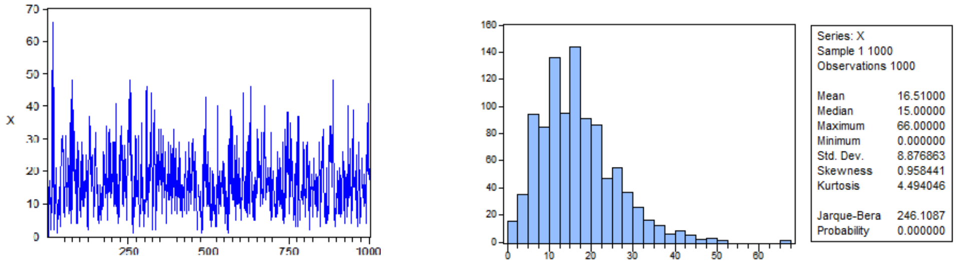


Figure 2: Trajectory and descriptives of GP-INARCH(1) model: $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\kappa = 0.5$.

*For this distribution we have* $\varphi'(0) = \frac{i}{p}$, $\varphi''(0) = \frac{p-2}{p^2}$, $\varphi'''(0) = -i\,\frac{6-6p+p^2}{p^3}$, $\varphi^{(iv)}(0) = \frac{16-16p+2p^2-p^3}{p^4}$, $v_0 = \frac{2-p}{p}$, $d_0 = \frac{6-6p+p^2}{p^2}$, $c_0 = \frac{16-16p+2p^2-p^3}{p^3}$, *from where we deduce, for instance, the cumulants*

$$\kappa_2(X_t) = \frac{\alpha_0(2-p)}{p(1-\alpha_1)(1-\alpha_1^2)}, \quad \kappa_3(X_t) = \alpha_0\,\frac{6(1+\alpha_1^2) - 6(1-\alpha_1^2)p + (1+2\alpha_1^2)p^2}{p^2(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$

*and the skewness and the kurtosis of* $X_t$*, respectively,*

$$S_{X_t} = \frac{6-6p+6p^2+2p^2\alpha_1^2}{(2p-p^2)(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{p(1+\alpha_1)}{\alpha_0(2-p)}},$$

$$K_{X_t} = 3 + \frac{(1-\alpha_1^2)(1-\alpha_1)(16-16p+12p^2-p^3)}{\alpha_0 p(2-p)^2(1+\alpha_1^2)} + \frac{\alpha_1^2(3+15\alpha_1^3)(2-p)}{\alpha_0 p(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)} - \frac{2\alpha_1^2(1-\alpha_1)(5\alpha_1^2+5\alpha_1+2)(p^2-6p+6)}{\alpha_0 p(2-p)(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}.$$

# 5 Conclusion

A new class of models which includes the main INGARCH processes present in literature is proposed and developed in this paper enlarging and unifying the analysis of those processes, and accomplishing the practical goal of modeling simultaneously different stylized facts that have been recorded in real count data. In fact, considering a mixture of a Dirac at zero with a general discrete compound Poisson as conditional distribution of INGARCH processes, we define the Zero-Inflated Compound Poisson INGARCH model, denoted ZICP-INGARCH, that may capture in the same framework characteristics of zero inflation and, in a general distributional context, overdispersion and conditional heteroscedasticity. A general procedure to obtain new models is developed showing the main nature of the processes that are solution of the model equations, namely the fact that they may be expressed as a random sum of random variables. Conditions for stationarity of these models are established and also illustrated for particular important cases. Furthermore, for ZICP-INGARCH$(1,1)$ processes, a simple condition on the model coefficients assuring the existence of all moments and closed-form expressions for the cumulants up to order 4 are deduced, from which the skewness and kurtosis of the processes are derived. Analogously to Ferland *et al.* ([6]), we point out that the existence of all moments is a surprising feature.

These results are useful in some probabilistic developments of these models as, in particular, the study of the Taylor property (Gonçalves *et al.*, [7]) or other type of applications ([16]). As illustrated in the ZIP-INGARCH process, we point out that this proposal may also include underdispersed models, analyzing in each case the possibility of negative values for the additional weight, $\omega$, on zero, that is, models with deflation in zero. Finally, we stress that, using the same methodology and slightly heavier calculations, this study is valid when the inflation takes place in a nonzero point.

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**Appendix 1. Autoregressive equation of $W_t$**

From (3), (4) and (5) it follows that the vector $W_t$ satisfies the autoregressive equation of order $p$, $W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k}$ where $B_0 = (b_j)$ is such that

$$b_j = \begin{cases} C, & j = 1 \\ (1-\omega)\mu\left[\alpha_0 - \frac{v_0\beta_{j-1}}{1+v_1}\right], & j = 2,...,p \\ \frac{\mu}{1-\omega}\left[\alpha_0 - \frac{v_0(\alpha_{j-p}+\beta_{j-p})}{1+v_1}\right], & j = p+1,...,2p-1 \end{cases}$$

and $B_k$ $(k = 1,...,p)$ are the squared matrices having generic element $b_{ij}^{(k)}$ given by:

- row $i = 1$:

$$b_{1j}^{(k)} = \begin{cases} (1-\omega)(1+v_1)\alpha_k^2 + 2(1-\omega)\alpha_k\beta_k + \beta_k^2, & j = 1 \\ 2(1+v_1)[(1-\omega)\alpha_k + \beta_k]\alpha_{j+k-1}, & j = 2,...,p \\ 2(1-\omega)(1+v_1)[(1-\omega)\alpha_k + \beta_k]\beta_{j+k-p}, & j = p+1,...,2p-1 \end{cases}$$

- row $i = k+1$, $(k \neq p)$:

$$b_{k+1,j}^{(k)} = \begin{cases} (1-\omega)\left[\alpha_k + \frac{\beta_k}{1+v_1}\right], & j = 1 \\ (1-\omega)\alpha_{j+k-1}, & j = 2,...,p \\ (1-\omega)^2\beta_{j+k-p}, & j = p+1,...,2p-1 \end{cases}$$

- row $i = k+p$:

$$b_{k+p,j}^{(k)} = \begin{cases} \frac{\alpha_k+\beta_k}{(1-\omega)(1+v_1)}, & j = 1 \\ \frac{\alpha_{j+k-1}}{1-\omega}, & j = 2,...,p \\ \beta_{j+k-p}, & j = p+1,...,2p-1 \end{cases}$$

- row $i = k+j$:

$$b_{k+j,j}^{(k)} = \begin{cases} (1-\omega)\alpha_k + \beta_k, & j = 2,...,p-k,p+1,...,2p-1-k \\ 0 & j = p-k+1,...,p \end{cases}$$

and for any other case $b_{ij}^{(k)} = 0$, where we consider $\alpha_j = \beta_j = 0$, for $j > p$.

**Appendix 2. Proof of expression (6) in Theorem 4.1**

*Proof.* Without loss of generality, let us consider $m \geq j$. For $m = 1$ the result is valid since $\left(\varphi_t^j\right)'(u) = j\varphi_t^{j-1}(u)\varphi_t'(u)$. Now, let us assume that the formula has been shown for an arbitrarily fixed value of $m$ and let us prove that it holds for $m+1$. We have

$$\left(\varphi_t^j\right)^{(m+1)}(u) = \frac{d}{du}\left(\sum_{n=m-j}^{m-1} \frac{j!\ \varphi_t^{j-m+n}(u)}{(j-m+n)!} \sum_{\substack{k_1+\ldots+k_m=m-n \\ k_1+\ldots+mk_m=m}} (m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1}...[\varphi_t^{(m)}(u)]^{k_m}\right)$$

$$= \sum_{n=m-j+1}^{m-1} \frac{j!\ \varphi_t^{j-m+n-1}(u)}{(j-m+n-1)!}\left[\sum_{\substack{k_1+\ldots+k_m=m-n \\ k_1+\ldots+mk_m=m}} (m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1+1}\left[\varphi_t''(u)\right]^{k_2}\ldots[\varphi_t^{(m)}(u)]^{k_m}\right.$$

$$+ \sum_{\substack{k_1+\ldots+k_m=m-n+1 \\ k_1+\ldots+mk_m=m}} k_1(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1-1}\left[\varphi_t''(u)\right]^{k_2+1}\left[\varphi_t'''(u)\right]^{k_3}\ldots[\varphi_t^{(m)}(u)]^{k_m}$$

$$+ \sum_{\substack{k_1+\ldots+k_m=m-n+1 \\ k_1+\ldots+lk_m=m}} k_2(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1}\left[\varphi_t''(u)\right]^{k_2-1}\left[\varphi_t'''(u)\right]^{k_3+1}[\varphi_t^{(iv)}(u)]^{k_4}\ldots[\varphi_t^{(m)}(u)]^{k_m}$$

$$\left.+\ldots+ \sum_{\substack{k_1+\ldots+k_m=m-n+1 \\ k_1+\ldots+mk_m=m}} k_m(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1}\ldots[\varphi_t^{(m)}(u)]^{k_m-1}\varphi_t^{(m+1)}(u)\right] + j\ \varphi_t^{j-1}(u)\varphi_t^{(m+1)}(u)$$

where the last term results from the second sum when $n = m-1$, since in this case one obtains $(m;0,\ldots,0,1)\times 0 + 0 + \ldots + 0 + (m;0,\ldots,0,1)\varphi_t^{(m+1)}(u)$. Thus,

$$\left(\varphi_t^j\right)^{(m+1)}(u) = \sum_{n=m-j+1}^{m-1} \frac{j!\ \varphi_t^{j-m+n-1}(u)}{(j-m+n-1)!}\left[\sum_{\substack{c_1+\ldots+c_m=m+1-n \\ c_1+\ldots+mc_m=m+1}} (m+1;c_1,...,c_m,0)\left[\varphi_t'(u)\right]^{c_1}\ldots[\varphi_t^{(m)}(u)]^{c_m}\right.$$

$$\left.+ \sum_{\substack{c_1+\ldots+c_m=m+1-n \\ c_1+\ldots+mc_m=m+1}} (m+1;c_1,...,c_m,1)\left[\varphi_t'(u)\right]^{c_1}\ldots[\varphi_t^{(m)}(u)]^{c_m}\varphi_t^{(m+1)}(u)\right]$$

$$+ j\ \varphi_t^{j-1}(u)\ (m+1;0,...,0,1)\varphi_t^{(m+1)}(u),$$

using the fact that

$$(c_i+1)\ (m;c_1,...,c_{i-1},c_i+1,c_{i+1}-1,...,c_m) = \frac{(i+1)c_{i+1}}{m+1}(m+1;c_1,...,c_m,0), \quad i=1,...,m-1,$$
$$(m;c_1-1,c_2,...,c_m) = \frac{c_1}{m+1}(m+1;c_1,...,c_m,0),$$
$$(c_m+1)\ (m;c_1,...,c_{m-1},c_m+1) = (m+1;c_1,...,c_m,1), \tag{12}$$

and hence

$$(m;c_1-1,c_2,...,c_m) + \sum_{i=1}^{m-1}(c_i+1)\ (m;c_1,...,c_{i-1},c_i+1,c_{i+1}-1,...,c_m)$$
$$= (m+1;c_1,...,c_m,0)\left[\frac{c_1+2c_2+...+mc_m}{m+1}\right] = (m+1;c_1,...,c_m,0).$$

The result is then obtained. □