On Some Generalizations of Malcev Algebras*†

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1 Introduction

We start recalling the notion of *Ω-algebra* over an associative commutative ring Φ with unity: this is a unital module over Φ, on which we define a system of multilinear algebraic operations $\Omega = \{\omega_i : |\omega_i| = n_i \in N,\ i \in I\}$, where $|\omega_i|$ denotes the arity of ω_i. An Ω-algebra is merely called an algebra.

Among these is the class of *n-Lie algebras*, recently rebaptized as *n-ary Filippov algebras* (see, for instance, [2], [13] and [10]), so-called in honor of Filippov's work [3], where this subject was first studied. An *n-ary Filippov algebra*, or simply *Filippov algebra* (if the arity of the operation can be omitted) is an Ω-algebra L with one n-ary operation $[.,\dots,.] : \times^n L \to L$, $n \geq 2$, satisfying the identities

$$[x_1,\dots,x_n] = sgn(\sigma)[x_{\sigma(1)},\dots,x_{\sigma(n)}], \tag{1}$$

$$[[x_1,\dots,x_n],y_2,\dots,y_n] = \sum_{i=1}^{n}[x_1,\dots,[x_i,y_2,\dots,y_n],\dots,x_n], \tag{2}$$

where σ is a permutation in the symmetric group S_n and $sgn(\sigma)$ stands for the sign of σ. The identity (1) is called *anticommutativity,* while (2) is a *generalized Jacobi identity,* also known as *Filippov identity.*

Recently, Pozhidaev [9] introduced another class of Ω-algebras, the *n-ary Malcev algebras,* which are a generalization of *Malcev algebras.* These were first introduced by A. I. Malcev in 1955 [6] and are defined by the identities

$$x^2 = 0, \tag{3}$$

$$J(x,y,xz) = J(x,y,z)x, \tag{4}$$

where $J(x,y,z) = (xy)z + (yz)x + (zx)y$ is the Jacobian of the elements x, y, z.

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We briefly recall the procedure under which this generalization was achieved. It is easy to see that (4) can be written as follows:

$$xyzx + yzxx + zxxy = (xy)(xz) \tag{5}$$

(in order to simplify notations, hereinafter the brackets in the left-normalized products shall be omitted, so that $xyzx = [(xy)z]x)$. Let $R_x : a \mapsto ax$ be the operator of right multiplication by x in a Malcev algebra A and let A^* be the multiplication algebra of A, *i.e.*, the associative algebra generated by the operators R_x. It follows from (5) that

$$R_x R_{xy} + R_{xy} R_x = R_x^2 R_y - R_y R_x^2 \tag{6}$$

holds in A^*. We can generalize (6) to the n-ary case:

$$R_x(\sum_{i=2}^{n} R_{x_2,\dots,x_i R_y,\dots,x_n}) + (\sum_{i=2}^{n} R_{x_2,\dots,x_i R_y,\dots,x_n})R_x = R_x^2 R_y - R_y R_x^2,$$

where $R_x = R_{x_2,\dots,x_n}$ and $R_y = R_{y_2,\dots,y_n}$ are right multiplication operators: $zR_x = [z, x_2, \dots, x_n]$. Being A an algebra with an n-ary anticommutative multiplication $[.,...,.]$, we can rewrite the last equality as follows:

$$\begin{aligned} &\sum_{i=2}^{n} [[z, x_2, \dots, x_n], x_2, \dots, [x_i, y_2, \dots, y_n], \dots, x_n] \\ &+ \sum_{i=2}^{n} [[z, x_2, \dots, [x_i, y_2, \dots, y_n], \dots, x_n], x_2, \dots, x_n] \\ = \; & [[[z, x_2, \dots, x_n], x_2, \dots, x_n], y_2, \dots, y_n] - [[[z, y_2, \dots, y_n], x_2, \dots, x_n], x_2, \dots, x_n], \end{aligned} \tag{7}$$

hereafter known as the *generalized Malcev identity* (GMI).

This way, as proposed by Pozhidaev [9], an *n-ary Malcev algebra* ($n \geq 2$) is a unitary Φ-module M equipped with an anticommutative n-ary operation $[.,\dots,.] : \times^n M \to M$ such that (7) holds.

Several notions have a natural translation to the theory of n-ary Malcev algebras. For the sake of completeness, we recall some of these. A subspace S of an n-ary Malcev algebra M is a *subalgebra* if $[S, \dots, S] \subseteq S$ and a subspace I is said to be an *ideal* of M if $[I, M, \dots, M] \subseteq I$. A particular ideal of M is the center, which is defined by

$$Z(M) = \{x \in M : [x, M, ..., M] = 0\}.$$

If $[M, ..., M] \neq 0$ and M has no nontrivial ideals we say that M is *simple*.

We recall that it has been proved that every n-ary Filippov algebra is an n-ary Malcev algebra [9]. Further, if M is an n-ary Malcev algebra, $n \geq 3$, with multiplication $[.,\dots,.]$, and $a \in M$ is an arbitrary element, it is possible to equip M with a new multiplication, $[.,\dots,.]_a$ defined by

$$[x_1, \dots, x_{n-1}]_a = [a, x_1, \dots, x_{n-1}].$$

Together with this multiplication, we define a *reduced algebra* of M, which is denoted by M_a. It has been proved [9] that the reduced algebras of an n-ary Malcev algebra are $(n-1)$-ary Malcev algebras.

In this paper we start presenting two examples of ternary Malcev algebras (the first of which introduced by Pozhidaev) defined by means of a composition algebra. This will be done in the second section and will be followed by a discussion of other possible generalizations of the Malcev identity (4). It turns out that none of the multiplications concerning the above mentioned examples satisfy the ternary versions of those generalizations.

The third section is devoted to the transference of the notions of solvability and nilpotence to the class of n-ary Malcev algebras, inspired by Kasimov's work [4] on n-ary Filippov algebras. Although the impossibility of all results being adopted, the invariance of the radical under derivations of an n-ary Malcev algebra (over a field of characteristic zero) is valid.

If A is a finite-dimensional n-ary Malcev algebra over a field of characteristic zero, then $Rad_k(A)$ is invariant under all derivations of A.

The description of the ternary Malcev algebras of low dimensions (≤ 4) is studied in the fourth section. The conclusion of this section is that all ternary Malcev algebras with dimension not greater than 4 are ternary Filippov algebras.

Finally, last section is devoted to the reduced Malcev algebras of the ternary Malcev algebras $M(A)$ defined on composition algebras. It turns out that all reduced Malcev algebras which arise by fixing the elements of an arbitrary orthonormal basis of A are simple.

2 Ternary Malcev algebras defined on composition algebras

In this section we present two examples of ternary Malcev algebras, the first of which already mentioned by Pozhidaev. We point out the possibility of different generalizations of Malcev algebras and present some counterexamples for these based on the two known examples of ternary Malcev algebras.

2.1 Two examples of ternary Malcev algebras

Let A be a composition algebra over a field Φ such that $char\Phi \neq 2$, with an involution $^- : a \mapsto \bar{a}$ and unity e. Admit that the symmetric bilinear form $\langle x, y\rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$ defined on A is nonsingular, being possible to define a norm n such that $n(a) = \langle a, a\rangle$ for every $a \in A$.

For the sake of completeness we present, without proof (which can be found, *e.g.*, in [9]), some results concerning composition algebras.

Lemma 2.1. For every elements a, b, c of a composition algebra A, the following equalities hold:

1. $a\bar{a}b = a(\bar{a}b) = n(a)b = b\bar{a}a = b(\bar{a}a)$;
2. $\bar{a}b\bar{a} = -n(a)\bar{b} + 2\langle a, b\rangle \bar{a}$;
3. $a\bar{b}c + a\bar{c}b = 2\langle b, c\rangle a$;
4. $a(\bar{b}c) + b(\bar{a}c) = 2\langle a, b\rangle c$;
5. $\langle ab, c\rangle = \langle b, \bar{a}c\rangle = \langle a, c\bar{b}\rangle$;
6. $\langle \bar{a}, \bar{b}\rangle = \langle a, b\rangle$, $\langle \bar{a}, b\rangle = \langle a, \bar{b}\rangle$.

Some of these results have a short version if we are dealing with orthonormal vectors.

Corollary 2.2. If a, b, c are elements on an orthonormal basis of A, then

1. $\bar{a}b\bar{a} = -\bar{b}$;

2. $a\bar{b}c = -a\bar{c}b$;
3. $a(\bar{b}c) = -b(\bar{a}c)$.

Finally, since every composition algebra is an alternative algebra, the *Moufang identities* hold in A (see, *e.g.*, [12]):

$$yzyx = y(z(yx)) \quad \text{(the left Moufang identity)}, \tag{8}$$

$$(xy)(zx) = x(yz)x \quad \text{(the middle Moufang identity)}, \tag{9}$$

$$xyzy = x(yzy) \quad \text{(the right Moufang identity)}. \tag{10}$$

As proved by Brown and Gray [1] on the classification of algebras with vector cross product, there is a class of 8-dimensional algebras with ternary vector cross product, being these defined either by

$$[x, y, z] = x\bar{y}z - \langle y, z\rangle x + \langle x, z\rangle y - \langle x, y\rangle z \tag{11}$$

or

$$[x, y, z] = x(\bar{y}z) - \langle y, z\rangle x + \langle x, z\rangle y - \langle x, y\rangle z. \tag{12}$$

Examples of ternary Malcev algebras defined on composition algebras can be given using (11) and (12).

Theorem 2.3. *Let A be a composition algebra with unity e and with an involution $^- : a \mapsto \bar{a}$. Equip A with a ternary multiplication $[.,.,.]$ by one of the rules, (11) or (12). Then A is a simple ternary Malcev algebra with respect to $[.,.,.]$, which will be denoted by $M(A)$.*

Proof The proof concerning the algebra with multiplication defined by (11) has already been given in [9].

Admit that $[.,.,.]$ is defined by (12). It is easy to use properties from lemma 2.1 to deduce the anticommutativity of $[.,.,.]$.

Recall that a ternary version of (7) is given by:

$$[[x, y, z], [y, u, v], z] + [[x, y, z], y, [z, u, v]] + [[x, [y, u, v], z], y, z] + [[x, y, [z, u, v]], y, z]$$

$$= [[[x, y, z], y, z], u, v] - [[[x, u, v], y, z], y, z]. \tag{13}$$

In order to prove that (13) is satisfied when the multiplication is given by (12), we denote it by $[.,.,.]_2$, while the first will be denoted by $[.,.,.]_1$. The mapping $\phi : a \to \bar{a}$ from $M_1 = (A, [.,.,.]_1)$ to $M_2 = (A, [.,.,.]_2)$ is an anti-isomorphism of ternary Malcev algebras. In other words, ϕ is an automorphism of the linear space A which satisfies

$$\phi([a, b, c]_1) = -[\phi(a), \phi(b), \phi(c)]_2, \text{ for all } a, b, c \text{ in } A, \tag{14}$$

as it is simple to verify. Applying ϕ to both sides of the identity (13) with respect to $[.,.,.]_1$ and using (14), it is possible to observe (13) written with respect to $[.,.,.]_2$ in terms of $\overline{x}, \overline{y}, \overline{z}, \overline{u}, \overline{v}$. Thus, the same identity will hold when expressed in terms of x, y, z, u, v. Henceforth, M_2 is also a ternary Malcev algebra.

To prove the simplicity of M_2, admit that I is a non-trivial ideal of M_2 and consider φ the inverse mapping of ϕ, which is also an anti-isomorphism of A, but from M_2 to M_1. By definition of ideal, we have $[I, A, A]_2 \subseteq I$ and therefore,

$$\varphi([I, A, A]_2) \subseteq \varphi(I).$$

Observing that $\varphi([I, A, A]_2) = [\varphi(I), A, A]_1$ and recalling that φ is an automorphism of A, the above inclusion implies that $\varphi(I)$ is a non-trivial ideal of M_1. But this can't happen, because M_1 is simple. The result is proved.

2.2 Different generalizations

It is possible to obtain different generalizations of Malcev algebras. Indeed, if A is a Malcev algebra, the identity (4), which is equivalent to the Malcev identity, also leads to the following identity in A^* :

$$R_y R_x^2 - R_x^2 R_y = R_x R_{yx} + R_{yx} R_x. \tag{15}$$

Proceeding analogously to what's described in the introduction, if A is now an algebra with an n-ary anticommutative operation $[., ..., .]$ and A^* its multiplication algebra, we can generalize (15) writing

$$R_y R_x^2 - R_x^2 R_y = R_x \left(\sum_{i=2}^{n} R_{y_2,\dots,y_i R_x,\dots,y_n} \right) + \left(\sum_{i=2}^{n} R_{y_2,\dots,y_i R_x,\dots,y_n} \right) R_x$$

where $R_x, R_y \in A^*$ are defined as before. Finally, the correspondent identity in A can be derived from the last one, leading to:

$$\begin{aligned} & \sum_{i=2}^{n} [[z, x_2, \dots, x_n], y_2, \dots, [y_i, x_2, \dots, x_n], \dots, y_n] \\ & + \sum_{i=2}^{n} [[z, y_2, \dots, [y_i, x_2, \dots, x_n], \dots, y_n], x_2, \dots, x_n] \\ = \; & [[[z, y_2, \dots, y_n], x_2, \dots, x_n], x_2, \dots, x_n] - [[[z, x_2, \dots, x_n], x_2, \dots, x_n], y_2, \dots, y_n], \end{aligned} \tag{16}$$

which will be denoted by the *second generalized Malcev identity (2GMI)*.

Unfortunately, the notion of n-ary Malcev algebra which arise replacing (7) by (16) doesn't seem to be as interesting as the one we have first introduced. Indeed, we claim that none of the products defined by (11) or (12) satisfies

$$\begin{aligned} & [[x, y, z], [u, y, z], v] + [[x, y, z], u, [v, y, z]] + [[x, [u, y, z], v], y, z] + [[x, u, [v, y, z]], y, z] \\ & = [[[x, u, v], y, z], y, z] - [[[x, y, z], y, z], u, v], \end{aligned} \tag{17}$$

which is the ternary case of (16). The proof will only include a case where it fails, restricted to the product defined by (11). [1]

Putting $x = y$, (17) reduces to

$$[[x,[u,x,z],v],x,z]+[[x,u,[v,x,z]],x,z]=[[[x,u,v],x,z],x,z] \tag{18}$$

Let us take x, z, u, v pairwise different belonging to an orthonormal basis. Developing the term on the right side of (18), using the fact that $[a, b, c]$ is orthogonal to a, b, c, together with properties of lemma 2.1 and its corollary, we have:

$$\begin{aligned}
[[[x,u,v],x,z],x,z] &= [[x,u,v],x,z]\overline{x}z \\
&= x\overline{u}v\overline{x}z\overline{x}z + \langle z, x\overline{u}v\rangle x\overline{x}z \\
&= -x\overline{u}v + \langle z, x\overline{u}v\rangle z.
\end{aligned}$$

Concerning the first summand on the left side, we have:

$$[[x,[u,x,z],v],x,z] = x\overline{u}v - \langle z, x\overline{u}v\rangle z.$$

The second summand on the left side of (18) can be obtained from the first one, interchanging u and v. In fact, we have

$$\begin{aligned}
[[x,u,[v,x,z]],x,z] &= -[[x,[v,x,z],u],x,z] \\
&= -x\overline{v}u + \langle z, x\overline{v}u\rangle z \\
&= x\overline{u}v - \langle z, x\overline{u}v\rangle z.
\end{aligned}$$

Therefore, the left side of (18) is equal to $2x\overline{u}v - 2\langle z, x\overline{u}v\rangle z$ while the right side is equal to $-x\overline{u}v + \langle z, x\overline{u}v\rangle z$ (note that if $char\Phi = 3$ we obtain a trivial identity, since the difference between the left and the right side equals to $3x\overline{u}v + 3\langle x\overline{z}, u\overline{v}\rangle z$, which is zero in this case).

Both identities, (7) and (16), being though generalizations of the Malcev identity to $n = 3$, are nonlinear on all variables, but just on two sets of variables (although linearizable). We will now analyze two generalizations which result from (4) and are already linear on all variables.

If A is a Malcev algebra over an associative and commutative ring Φ with unity and $char\Phi \neq 2$, it has been proved [11] that (4) is equivalent to

$$(xy)(zw) = x(wyz) + w(yzx) + y(zxw) + z(xwy),$$

which in turn can be written in the operator form, as follows:

$$R_xR_yR_z - R_yR_zR_x = R_zR_{xR_y} - R_{zR_x}R_y - R_{(yR_z)R_x},$$

for every right multiplication operator $R_x, R_y, R_z \in A^*$. Generalizing this to the n-ary case, we have

$$\begin{aligned}
R_xR_yR_z - R_yR_zR_x &= \sum_{i=2}^{n}\left(R_zR_{x_2,\ldots,x_iR_y,\ldots,x_n} - R_{z_2,\ldots,z_iR_x,\ldots,z_n}R_y\right) \\
&\quad - \sum_{\substack{i,j=2\\ i\neq j}}^{n} R_{y_2,\ldots,y_iR_z,\ldots,y_jR_x,\ldots,y_n} - \sum_{i=2}^{n} R_{y_2,\ldots,(y_iR_z)R_x,\ldots,y_n},
\end{aligned}$$

[1]In this and forthcoming counterexamples, we could deal with the multiplication defined by (12). By analogous procedures, we would arrive to the same conclusions.

where R_x is such that $bR_x = [b, x_2, ..., x_n]$. A ternary version of this identity can be expressed in A as follows:

$$\begin{aligned}
&[[[x,y,z],u,v],s,t] - [[[x,u,v],s,t],y,z] \qquad (19)\\
&= [[x,s,t],[y,u,v],z] + [[x,s,t],y,[z,u,v]] - [[x,[s,y,z],t],u,v] - [[x,s,[t,y,z]],u,v]\\
&- [x,[u,s,t],[v,y,z]] - [x,[u,y,z],[v,s,t]] - [x,[[u,s,t],y,z],v] - [x,u,[[v,s,t],y,z]].
\end{aligned}$$

Let A be a composition algebra under the same conditions as those described in the previous subsection and consider the multiplication $[.,.,.]$ defined by (11). Consider $f(x,y,z,u,v,s,t)$ as the difference between the left and the right sides of (19), which is a linear function of all variables. Admit that x,y,z,u,v,s,t belong to an orthonormal basis and put $x=y=u$, being x,z,v,s,t pairwise different. Then

$$\begin{aligned}
f(x,x,z,x,v,s,t) &= -[[x,s,t],x,[z,x,v]] + [[x,[s,x,z],t],x,v] \qquad (20)\\
&\quad + [[x,s,[t,x,z]],x,v] + [x,[x,s,t],[v,x,z]] + [x,[[x,s,t],x,z],v]\\
&= [[[x,s,z],x,t] - [[x,t,z],x,s] - [[x,s,t],x,z],x,v].
\end{aligned}$$

Now, it is an easy task to apply properties form lemma 2.1 and its corollary to conclude that

$$\begin{aligned}
[[x,s,z],x,t] &= -z\overline{s}t + \langle x, z\overline{s}t\rangle x,\\
[[x,t,z],x,s] &= z\overline{s}t - \langle x, z\overline{s}t\rangle x,\\
[[x,s,t],x,z] &= z\overline{s}t - \langle x, z\overline{s}t\rangle x.
\end{aligned}$$

This way, replacing in (20), we obtain

$$f(x,x,z,x,v,s,t) = -3\,[z\overline{s}t, x, v],$$

which is nonzero in general (except if $char\Phi = 3$). Thus, the proposed ternary Malcev identity (19) is not satisfied by the multiplication defined by (11).

It is well known [11] that being A a Malcev algebra, for every $x,y \in A$ the operator defined by

$$D(x,y) = R_{xy} + [R_x, R_y], \qquad (21)$$

where $[.,.]$ stands for the commutator, is a derivation of A, that is,

$$(zw)D(x,y) = (zD(x,y))\,w + z\,(wD(x,y)) \qquad (22)$$

is satisfied. If A is now an algebra with an n-linear anticommutative multiplication $[.,\ldots,.]$, a generalization of (21) is the operator defined by

$$D(x,y) = \sum_{i=2}^{n} R_{x_2,\ldots,x_iR_y,\ldots,x_n} + [R_x, R_y], \qquad (23)$$

where $x = (x_2,...,x_n)$, $y = (y_2,...,y_n)$, and $R_x, R_y \in A^*$. One may ask if $D(x,y)$ is a derivation of A, that is, if

$$[z_1,...,z_n]\,D(x,y) = \sum_{i=1}^{n} [z_1,...,z_iD(x,y),...,z_n] \qquad (24)$$

holds. Written in terms of right multiplication operators, (24) is equivalent to:

$$[R_z, D(x,y)] = \sum_{i=2}^{n} R_{z_2,\ldots,z_i D(x,y),\ldots,z_n}.$$

Using (23), from the above identity we may conclude that (24) is also equivalent to:

$$\begin{aligned} &\sum_{i=2}^{n} \left(R_z R_{x_2,\ldots,x_i R_y,\ldots,x_n} - R_{x_2,\ldots,x_i R_y,\ldots,x_n} R_z\right) + R_z (R_x, R_y) - (R_x, R_y) R_z \\ = &\sum_{i,j=2}^{n} R_{z_2,\ldots,z_i R_{\left(x_2,\ldots,x_j R_y,\ldots,x_n\right)},\ldots,z_n} + \sum_{i=2}^{n} R_{z_2,\ldots,z_i (R_x,R_y),\ldots,z_n}. \end{aligned} \tag{25}$$

In A, this identity is expressed the following way:

$$\begin{aligned} &\sum_{i=2}^{n} \left([[w,z_2,\ldots,z_n], x_2,\ldots,[x_i,y_2,\ldots,y_n],\ldots,x_n] - [[w,x_2,\ldots,[x_i,y_2,\ldots,y_n],\ldots,x_n], z_2,\ldots,\ldots,z_n]\right) \\ &+ [[[w,z_2,\ldots,z_n], x_2,\ldots,x_n], y_2,\ldots,y_n] - [[[w,z_2,\ldots,z_n], y_2,\ldots,y_n], x_2,\ldots,x_n] \\ &- [[[w,x_2,\ldots,x_n], y_2,\ldots,y_n], z_2,\ldots,z_n] + [[[w,y_2,\ldots,y_n], x_2,\ldots,x_n], z_2,\ldots,z_n] \\ = &\sum_{i,j=2}^{n} [w,z_2,\ldots,[z_i,x_2,\ldots,[x_j,y_2,\ldots,y_n],\ldots,x_n],\ldots,z_n] \\ &+ \sum_{i=2}^{n} \left([w,z_2,\ldots,[[z_i,x_2,\ldots,x_n],y_2,\ldots,y_n],\ldots,z_n] - [w,z_2,\ldots,[[z_i,y_2,\ldots,y_n],x_2,\ldots,x_n],\ldots,z_n]\right). \end{aligned} \tag{26}$$

Therefore, $D(x,y)$ is a derivation of the n-ary algebra A if (26) is satisfied.

Let A be an algebra with multilinear and anticommutative multiplication $[.,\ldots,.]$ which satisfies (26). We may investigate if, in the ternary case, any of the multiplications given in a composition algebra A by (11) or by (12) satisfy the ternary version of (26). That is, we are asking if any of those multiplications verify

$$\begin{aligned} &[[x,s,t],[y,u,v],z] - [[x,[y,u,v],z],s,t] + [[x,s,t],y,[z,u,v]] - [[x,y,[z,u,v]],s,t] \\ &+ [[[x,s,t],y,z],u,v] - [[[x,s,t],u,v],y,z] - [[[x,y,z],u,v],s,t] + [[[x,u,v],y,z],s,t] \\ = &[x,[s,[y,u,v],z],t] + [x,[s,y,[z,u,v]],t] + [x,s,[t,[y,u,v],z]] + [x,s,[t,y,[z,u,v]]] \\ &+ [x,[[s,y,z],u,v],t] - [x,[[s,u,v],y,z],t] + [x,s,[[t,y,z],u,v]] - [x,s,[[t,u,v],y,z]]. \end{aligned} \tag{27}$$

As usual, defining $f(x,y,z,u,v,s,t)$ as the difference between the left and the right hand sides of (27), it is easy to observe that f is linear on all variables and skew-symmetric on the pairs of variables $(y,z),(u,v),(s,t)$. We may consider x,y,z,u,v,s,t belonging to an orthonormal basis of A. Taking $x=y$, and considering the multiplication defined by (11), we have:

$$\begin{aligned} f(x,x,z,u,v,s,t) = &[[x,s,t],[x,u,v],z] - [[x,[x,u,v],z],s,t] + [[x,s,t],x,[z,u,v]] \\ &+ [[[x,s,t],x,z],u,v] - [[[x,s,t],u,v],x,z] + [[[x,u,v],x,z],s,t] \\ &- [x,[s,[x,u,v],z],t] - [x,[s,x,[z,u,v]],t] - [x,s,[t,[x,u,v],z]] \\ &- [x,s,[t,x,[z,u,v]]] - [x,[[s,x,z],u,v],t] + [x,[[s,u,v],x,z],t] \\ &- [x,s,[[t,x,z],u,v]] + [x,s,[[t,u,v],x,z]], \end{aligned}$$

which may not be zero. Indeed, being e,a,b,ab,c,ac,bc,abc an orthonormal basis of A, where e is the unity, we have $f(e,e,a,b,c,ab,ac) = 4a$.

3 k-solvability and k-nilpotence of n-ary Malcev algebras

In this section A stands for an n-ary Malcev algebra (over an associative and commutative ring Φ with unit) with multiplication denoted by $[.,\dots,.]$. Without further properties for the multiplication, we can just say that the sum of two ideals of A is again an ideal of A, but if $I_1,\dots,I_n$ are ideals of A $[I_1,\dots,I_n]$ may not be an ideal of A, oppositely to what happens concerning n-ary Filippov algebras (a counterexample can be found in [11], when $n=2$).

We are going to discuss different generalizations of the notions of solvability and nilpotence concerning the n-ary Malcev algebra A. Those have been analyzed by Kasymov but with respect to n-ary Filippov algebras.

The following definitions are adopted from those proposed by E. N. Kuz'min (about n-ary Filippov algebras). Being I an ideal of A, consider two descending sequences of subalgebras of A, $I^{(s)}, s\geq 0$, and $I^s, s\geq 1$, recursively defined by

$$\begin{cases} I^{(0)} = I \\ I^{(s+1)} = \left[I^{(s)}, I^{(s)}, A,\dots,A\right], \quad s\geq 0 \end{cases}$$

and

$$\begin{cases} I^1 = I \\ I^{s+1} = \left[I^s, I, A,\dots,A\right], \quad s\geq 1, \end{cases}$$

respectively.

Definition 3.1. We say that I is a *solvable* ideal of A if $I^{(r)}=0$ for some $r\geq 0$.

Definition 3.2. We say that I is a *nilpotent* ideal of A if $I^r=0$ for some $r\geq 1$. In particular, if $I^2=0$ we say that the ideal I is *abelian*.

It is easy to observe that both $I^{(s)}$ and I^s are descending series of subalgebras of A, but not necessarily of ideals of A.

Filippov's version of these notions can be adopted, too. Given an ideal I of an n-ary Malcev algebra A, consider the following descending sequences of subalgebras of A, denoted respectively by $I^{(s)}$, $s\geq 0$, and $I^{\langle s\rangle}, s\geq 1$, such that:

$$\begin{cases} I^{(0)} = I \\ I^{(s+1)} = \left[I^{(s)},\dots,I^{(s)}\right], \quad s\geq 0 \end{cases}$$

and

$$\begin{cases} I^{\langle 0\rangle} = I \\ I^{\langle s+1\rangle} = \left[I^{\langle s\rangle}, I,\dots,I\right], \quad s\geq 1. \end{cases}$$

Definition 3.1.' An ideal I of A is said to be *solvable* if $I^{(r)}=0$ for some $r\geq 0$.

Definition 3.2.' An ideal I of A is said to be *nilpotent* if $I^{\langle r\rangle}=0$ for some $r\geq 0$.

Observe that an ideal I of A is solvable (nilpotent) in the sense of Filippov if it is a solvable (nilpotent) subalgebra in the sense of Filippov.

Though both pairs of definitions agree when $n=2$, it is possible a certain gradation of the concepts of solvability and nilpotence for the ideals of an n-ary Malcev algebra A, as Kasymov did concerning n-ary Filippov algebras. Let k be a fixed integer in $\{2,\dots,n\}$, I an ideal of A and consider the following sequences of subalgebras of A, $I^{(s,k)}$, $s\geq 0$, and $I^{s,k}$, $s\geq 1$, such that

$$\begin{cases} I^{(0,k)} = I \\ I^{(s+1,k)} = [\underbrace{I^{(s,k)}, \ldots, I^{(s,k)}}_{k}, A, \ldots, A], s \geq 0 \end{cases}$$

and

$$\begin{cases} I^{1,k} = I \\ I^{s+1,k} = [I^{s,k}, \underbrace{I, \ldots, I}_{k-1}, A, \ldots, A], s \geq 1. \end{cases}$$

Definition 3.3. For a fixed integer $k \in \{2, ..., n\}$, an ideal I of A is said to be *k-solvable* if $I^{(r,k)} = 0$ for some $r \geq 0$.

Definition 3.4. For a fixed integer $k \in \{2, ..., n\}$, an ideal I of A is said to be *k-nilpotent* if $I^{r,k} = 0$ for some $r \geq 1$.

At the light of these, Kuz'min's and Filippov's definitions are particular cases, with $k = 2$ and $k = n$, respectively. Clearly, if $k < r \leq n$ the k-solvability (k-nilpotence) of an ideal implies it r-solvability (r-nilpotence).

The example described by Kasymov concerning n-ary Filippov algebras can also be adopted to the present study to show that for different values of k we obtain distinct notions of k-solvability (k-nilpotence), even when the ideal coincides with the n-ary Malcev algebra.

Example Consider an $(n+1)$-dimensional space A with basis $e_1, \ldots, e_{n+1}$ and anticommutative, n-linear multiplication defined by

$$[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}] = \alpha_i e_i, \quad \alpha_i \in F, \ i = 1, ..., n+1.$$

Then A is an n-ary Malcev algebra (further, it is an n-ary Filippov algebra) for any choice of the constants. Considering $\alpha_1 ... \alpha_k \neq 0$ and $\alpha_{k+1} = ... = \alpha_{n+1} = 0$, $2 \leq k \leq n$, it is easy to observe that the ideal $A^{(1,l)} = A^2$ has basis $e_1, ..., e_k$ for all $l \in \{2, ..., n\}$ and that

$$A^{(2,k)} = [\underbrace{A^2, \ldots, A^2}_{k}, A, \ldots, A] = 0.$$

Thus, A is k-solvable. However, since

$$A^{(2,k-1)} = A^2 = A^{(1,k-1)}, \quad \text{for } k \geq 3,$$

we conclude that A is not $(k-1)$-solvable.

Observe that in the above example the ideal $I = A^2$ is an abelian subalgebra, since $I^2 = \left[A^2, ..., A^2\right] = 0$. However, when $k \geq 3$, it is not a $(k-1)$-nilpotent ideal and not even a $(k-1)$-solvable ideal, since

$$I^{(1,k-1)} = I = I^{(0,k-1)}.$$

Thus, if $k < n$, an ideal of A that is a k-solvable (k-nilpotent) algebra doesn't need to be a k-solvable (k-nilpotent) ideal.

Returning to the context of definitions 3 and 4, we have the following basic results:

Theorem 3.1. If I and J are ideals of an n-ary Malcev algebra A such that $I \supseteq J$, J is a k-solvable ideal of A and $\overline{I} = I/J$ is a k-solvable ideal of $\overline{A} = A/J$, then I is also a k-solvable ideal of A.

Proof Considering the canonical epimorphism $\varphi : A \to \overline{A}$, we see that $\varphi(I) = \overline{I}$ is, by hypothesis, a k-solvable ideal of $\overline{A}$. Therefore, $\overline{I}^{(r,k)} = \overline{0}$ for some $r \geq 0$. Since $\varphi\left(I^{(r,k)}\right) = \overline{I}^{(r,k)}$ we conclude that $I^{(r,k)} \subseteq J$. But, being J k-solvable, we have $J^{(s,k)} = 0$ for some $s \geq 0$. Thus, $I^{(r+s,k)} = 0$, and the result is proved.

Corollary 3.2. If J_1 and J_2 are k-solvable ideals of A, then $J_1 + J_2$ is also a k-solvable ideal.

Proof Considering the canonical epimorphism $\varphi : A \to A/J_2$, we have

$$\varphi(J_1 + J_2) = \varphi(J_1).$$

Since $\varphi(J_1)$ is a k-solvable ideal of A/J_2 (for it is the image of a k-solvable ideal by an epimorphism), we obtain the desired conclusion.

Therefore, if A is a finite dimensional n-ary Malcev algebra, for each fixed $k \in \{2, ..., n\}$ there is a maximal k-solvable ideal of A, which is called the *k-radical* of A and is denoted by $Rad_k(A)$. If $Rad_k(A) = 0$, A is said to be a *k-semisimple* n-ary Malcev algebra. Of course, by theorem 3.1, we can conclude that $A/Rad_k(A)$ is k-semisimple.

Remark 3.1. Kasymov proposed yet a different definition of k-solvability (k-nilpotence) of an ideal, which asserts that an ideal of an algebra A is said to be k-solvable (k-nilpotent) if it is a k-solvable (k-nilpotent) subalgebra. However, he pointed some deficiencies in this definition. First, if $k < n-1$ we are not allowed to define the k-radical of an algebra, since the sum of k-solvable ideals of a finite dimensional algebra in the sense of this definition is not necessarily a k-solvable ideal. Further, if $k = n$ the above example can be used to show that the quotient $A/Rad_{n-1}(A)$, where $Rad_{n-1}(A)$ is the $(n-1)$-radical in the sense of this definition, is not semisimple. Thus, one of the basic properties of the radicals doesn't hold. By these reasons, we will not adopt that definition to the present algebras we are studying.

Let $Der(A)$ be the algebra of derivations of A.

Theorem 3.3. If A is a finite-dimensional n-ary Malcev algebra over a field of characteristic zero, then $Rad_k(A)$ is invariant under all derivations of A.

Proof Let $\mathcal{D} \in Der(A)$ and consider $I = Rad_k(A)$. It is easy to observe that

$$J = I + I\mathcal{D}$$

is an ideal of A. If we prove that J is k-solvable, then $J \subseteq I$, which implies that $I\mathcal{D} \subseteq I$, as we wish to prove.

Since I is k-solvable, we have $I^{(r,k)} = 0$ for some $r \geq 0$. The elements of $I^{(r,k)}$ can be represented by $f_r(x_1, ..., x_m, y_1, ..., y_l)$ where $m = k^r$, $x_1, ..., x_m \in I$, $y_1, ..., y_l \in A$ and f_r involves multiplications of these elements by means of $[., ..., .]$. Therefore, we have $f_r = 0$ and thus $f_r\mathcal{D}^m = 0$. Whence,

$$m! f_r(x_1\mathcal{D}, ..., x_m\mathcal{D}, y_1, ..., y_l) \in I. \tag{28}$$

Observe that

$$J^{(r,k)} \subseteq I + (I\mathcal{D})^{(r,k)} \tag{29}$$

Since all elements in $(I\mathcal{D})^{(r,k)}$ have the shape $f_r(x_1\mathcal{D},...,x_m\mathcal{D},y_1,...,y_l)$, it follows from (28) and (29) that $J^{(r,k)} \subseteq I$. Thus, $J^{(2r,k)} \subseteq I^{(r,k)} = 0$ and J is k-solvable. This ends the proof.

4 Description of ternary Malcev algebras of low dimensions

The description of ternary Malcev algebras of dimension ≤ 4 can be summarized and stated as follows:

Theorem 4.1. Let M be a ternary Malcev algebra with dimension not greater than 4 over a field Φ of arbitrary characteristic. Then M is a ternary Filippov algebra.

In order to prove this assertion, we have to consider several cases.

4.1 Case: $\dim M \leq 3$

It is easy to see [3] that if A is an Ω-algebra with one anticommutative n-ary multiplication (simply anticommutative Ω-algebra) such that $\dim A < n$, then A is abelian. Thus, every ternary Malcev algebra with dimension lower than 3 is abelian. It has also been proved that the only anticommutative Ω-algebras A, up to an isomorphism, with $\dim A = n$, are those with multiplication given by:

$$[e_1, \dots, e_n] = 0$$

and

$$[e_1, \dots, e_n] = e_1,$$

where $e_1, \dots, e_n$ is a basis of A. Any of such algebras is an n-ary Filippov algebra and thus an n-ary Malcev algebra. Henceforth, the only nonabelian 3-dimensional ternary Malcev algebra M has multiplication isomorphic to

$$[e_1, e_2, e_3] = e_1,$$

where e_1, e_2, e_3 is a basis of M.

4.2 Case: $\dim M = 4$

In order to recognize two isomorphic 4-dimensional ternary Malcev algebras, we start recalling a necessary and sufficient condition for two anticommutative $(n+1)$-dimensional Ω-algebras being isomorphic [3].

Theorem 4.2. Let A and $\overline{A}$ be two anticommutative $(n+1)$-dimensional Ω-algebras, with multiplications matricially defined by

$$\left(e^1, \dots, e^{n+1}\right) = (e_1, \dots, e_{n+1})\, B \tag{30}$$

and

$$\left(\overline{e^1}, \dots, \overline{e^{n+1}}\right) = (\overline{e_1}, \dots, \overline{e_{n+1}})\, \overline{B},$$

where $e_1, \dots, e_{n+1}$ and $\overline{e_1}, \dots, \overline{e_{n+1}}$ are basis of A and $\overline{A}$, respectively, and

$$e^i = (-1)^{n+i+1} [e_1, \dots, \widehat{e_i}, \dots, e_{n+1}], \quad i = 1, ..., n+1.$$

Then A and $\overline{A}$ are isomorphic if and only if there is a nonsingular matrix T such that

$$\overline{B} = |T|^{-1} TBT'.$$

Recall also that $rank\ B = \dim A^1$, being $A^1 = [A, ..., A]$ the square of A.

In what follows, M is a 4-dimensional ternary Malcev algebra, with multiplication defined by

$$\left(e^1, \ldots, e^4\right) = \left(e_1, \ldots, e_4\right) B, \tag{31}$$

that is,

$$e^i = b_{1i}e_1 + \ldots + b_{4i}e_4, \quad i = 1, ..., 4,$$

where $e_1, \ldots, e_4$ is a basis of M and

$$e^i = (-1)^i \left[e_1, \ldots, \widehat{e_i}, \ldots, e_4\right]. \tag{32}$$

Therefore, M must satisfy (13).

4.2.1 Case: $\dim M = 4$ and $\dim M^1 = 1$

Admit that M is nonabelian (the abelian case means that M has trivial multiplication) and suppose that $\dim M^1 = 1$. It is easy to see that Filippov's work on the description of n-ary Filippov algebras can be adopted to conclude the following. The only possible multiplications are isomorphic to those described by:

$$\left\{ \begin{array}{c} [e_2, e_3, e_4] = e_1 \\ {[e_1, e_2, e_3] = [e_1, e_2, e_4] = [e_1, e_3, e_4] = 0} \end{array} \right. \quad \text{if } M^1 \subseteq Z(M);$$

and

$$\left\{ \begin{array}{c} [e_1, e_2, e_3] = e_1 \\ {[e_1, e_2, e_4] = [e_1, e_3, e_4] = [e_2, e_3, e_4] = 0}\ , \end{array} \right. \quad \text{if } M^1 \nsubseteq Z(M).$$

4.2.2 Case: $\dim M = 4$ and $\dim M^1 = 2$

Let M be a 4-dimensional ternary Malcev algebra such that $\dim M^1 = 2$. It is clear that we may choose the basis $e_1, ..., e_4$ of M such that $M^1 = \left\langle e^1, e^2 \right\rangle$. Thus,

$$e^3 = \alpha e^1 + \beta e^2 \quad \text{and} \quad e^4 = \gamma e^1 + \delta e^2,$$

for some $\alpha, \beta, \gamma, \delta \in F$. Further, without loss of generality, we may admit that these scalars are all zero, since M is isomorphic to $\overline{M}$ with basis $\overline{e_1}, ..., \overline{e_4}$ such that $\overline{M}^1 = \left\langle \overline{e^1}, \overline{e^2} \right\rangle$ and $\overline{e^3} = \overline{e^4} = 0$. Indeed, take

$$T = T_{31}(-\alpha)\, T_{32}(-\beta)\, T_{41}(-\gamma)\, T_{42}(-\delta),$$

where

$$T_{ij}(\alpha) = E + \alpha e_{ij}, \quad i, j = 1, ..., 4,$$

and apply theorem 4.2. Then, the ternary Malcev algebra $\overline{M}$ with multiplication matricially defined by $\overline{B}$ such that

$$\overline{B} = |T|^{-1} TBT'$$

satisfies the mentioned assertions.

In what follows, e^1, e^2 is a basis of M such that

$$e^i = (-1)^i [e_1, ..., \widehat{e_i}, ..., e_4] = b_{1i}e_1 + b_{2i}e_2 + b_{3i}e_3 + b_{4i}e_4, \quad i = 1, 2,$$

and $e^3 = e^4 = 0$. Thus $B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & 0 \end{bmatrix}$ where $rankB = 2$. Being M a 4-dimensional anticommutative algebra, it is known [3] that a necessary and sufficient condition for M to be ternary Filippov is

$$(b_{ij} - b_{ji})\, e^k + (b_{ki} - b_{ik})\, e^j + (b_{jk} - b_{kj})\, e^i = 0, \quad i, j, k = 1, ..., 4.$$

Due to the present circumstances, this is equivalent to

$$\begin{cases} b_{32}e^1 - b_{31}e^2 = 0 \\ b_{42}e^1 - b_{41}e^2 = 0 \end{cases}$$

Since e^1, e^2 is a basis of M, we must conclude that M is ternary Filippov if and only if

$$B_{21} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} = 0.$$

Now, putting

$$x = e_3; \quad y = e_4; \quad z = e_1; \quad u = e_4; \quad v = e_2 \tag{33}$$

in (13), we have:

$$b_{32}^2 e^1 - b_{31}b_{32}e^2 = 0,$$

which implies $b_{32} = 0$. Further, by suitable changes of indexes in (33), we conclude that $b_{42} = b_{31} = b_{41} = 0$. Therefore, $B_{21} = 0$ and M is a ternary Filippov algebra.

By the initial considerations, we may conclude that every 4-dimensional ternary Malcev algebra M such that $\dim M^1 = 2$ is a ternary Filippov algebra.

The following result, due to Filippov, will be useful to analyze the next cases.

Theorem 4.3. Let A be an anticommutative $(n+1)$-dimensional Ω-algebra defined by (30) and such that $\dim A^1 > 2$. Then A is an n-ary Filippov algebra if and only if B is symmetric.

4.2.3 Case: $\dim M = 4$ and $\dim M^1 = 3$

Since $\dim M^1 = 3$, we may, without loss of generality, assume that e^1, e^2, e^3 is a basis of M^1. Observe also that M is isomorphic to a ternary Malcev algebra $\overline{M}$ with basis $\overline{e^1}, \overline{e^2}, \overline{e^3}$ and such that $\overline{e^4} = 0$. Indeed, consider the multiplication in M matricially defined by (31), being e_1, e_2, e_3 and e^1, e^2, e^3 basis of M and M^1 respectively. Let $\alpha, \beta, \gamma \in F$ such that $e^4 = \alpha e^1 + \beta e^2 + \gamma e^3$. Then take

$$T = T_{41}(-\alpha) T_{42}(-\beta) T_{43}(-\gamma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha & -\beta & -\gamma & 1 \end{bmatrix}$$

and apply theorem 4.2. Computing $\overline{B} = |T|^{-1} TBT'$, we have

$$\overline{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 \\ b_{21} & b_{22} & b_{23} & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ -\alpha b_{11} - \beta b_{21} - \gamma b_{31} & -\alpha b_{12} - \beta b_{22} - \gamma b_{32} & -\alpha b_{13} - \beta b_{23} - \gamma b_{33} & 0 \end{bmatrix}$$

i.e., $\overline{e^4} = 0$. Thus, we can consider M a ternary Malcev algebra such that $\dim M = 4$, e^1, e^2, e^3 is a basis of M^1 and $e^4 = 0$.

Put

$$x = e_4; \quad y = e_1; \quad z = e_2; \quad u = e_1; \quad v = e_3,$$

and replace in (13). Then

$$\begin{cases} b_{43}^2 = 0 \\ b_{42} b_{43} = 0 \end{cases},$$

and thus $b_{43} = 0$. Analogously, considering the cases

$$x = e_4; \quad y = e_1; \quad z = e_3; \quad u = e_1; \quad v = e_2,$$

and

$$x = e_4; \quad y = e_3; \quad z = e_2; \quad u = e_3; \quad v = e_1,$$

we may conclude that $b_{42} = 0$ and $b_{41} = 0$, respectively.

Consider now

$$x = e_1; \quad y = e_4; \quad z = e_3; \quad u = e_4; \quad v = e_2,$$

and replace in (13). We obtain

$$\begin{cases} b_{22}(b_{31} - b_{13}) = b_{32}(b_{21} - b_{12}) \\ b_{11}(b_{32} - b_{23}) = -b_{13}(b_{21} + b_{12}) + 2b_{12}b_{31} \\ b_{12}(b_{21} - b_{12}) = 0 \end{cases} \tag{34}$$

It's not difficult to see that, interchanging e_1 with e_2 in the previous case, we arrive to the following identities:

$$\begin{cases} b_{11}(b_{32} - b_{23}) = b_{31}(b_{12} - b_{21}) \\ b_{22}(b_{31} - b_{13}) = -b_{23}(b_{12} + b_{21}) + 2b_{21}b_{32} \\ b_{21}(b_{12} - b_{21}) = 0 \end{cases} \tag{35}$$

If $b_{21} \neq b_{12}$, from the third identities in (34) and in (35) it would be $b_{21} = b_{12} = 0$, which is a contradiction. Therefore, $b_{21} = b_{12}$. Analogously reasoning, the consideration of the two pair of cases:

$$x = e_1; \quad y = e_4; \quad z = e_2; \quad u = e_4; \quad v = e_3,$$

$$x = e_3; \quad y = e_4; \quad z = e_2; \quad u = e_4; \quad v = e_1,$$

and

$$x = e_3; \quad y = e_4; \quad z = e_1; \quad u = e_4; \quad v = e_2,$$

$$x = e_2; \quad y = e_4; \quad z = e_1; \quad u = e_4; \quad v = e_3,$$

allows us to conclude that $b_{31} = b_{13}$ and $b_{32} = b_{23}$, respectively (no long computations are needed: just a convenient change of indexes will produce equalities which are analogous to (34) and (35)). Therefore, $B = B'$.

This way, if M is a 4-dimensional ternary Malcev algebra with $\dim M^1 = 3$, then M is a ternary Filippov algebra.

4.2.4 Case: $\dim M = 4$ and $\dim M^1 = 4$

Since $\dim M^1 = 4$, we may admit, without loss of generality, that in the matricial definition of the multiplication we have $b_{44} \neq 0$. Further, it is possible to take $b_{i4} = 0$, $i = 1, 2, 3$. This happens because M is always isomorphic to a ternary Malcev algebra, $\overline{M}$, with basis $\overline{e^1}, \ldots, \overline{e^4}$ such that $\overline{b_{i4}} = 0$, $i = 1, 2, 3$ and $\overline{b_{44}} \neq 0$, as it is easy to show by a proper choice of T in Theorem 4.2. Indeed, the ternary Malcev algebra $\overline{M}$ with multiplication matricially defined by $\overline{B}$ such that

$$\overline{B} = T_{14}(\alpha_3)T_{24}(\alpha_2)T_{34}(\alpha_1)\, B\, (T_{14}(\alpha_3)T_{24}(\alpha_2)T_{34}(\alpha_1))'$$

where

$$\alpha_1 = -b_{34}b_{44}^{-1}, \quad \alpha_2 = -b_{24}b_{44}^{-1}, \quad \alpha_1 = -b_{14}b_{44}^{-1},$$

is isomorphic to M. It is easy to see that $\overline{b_{i4}} = 0$, $i = 1, 2, 3$ and $\overline{b_{44}} \neq 0$. Finally, being $b_{i4} = 0$, $i = 1, 2, 3$, we can also admit that $b_{44} = 1$. In fact, M is isomorphic to $\overline{M}$, with $\overline{b_{44}} = 1$, by considering

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b_{44}^{-1} \end{bmatrix}$$

and applying theorem 4.2.

Thus, in what follows, the basis of M is such that

$$e^4 = [e_1, e_2, e_3] = e_4.$$

Putting

$$x = e_1; \quad y = e_2; \quad z = e_3; \quad u = e_1; \quad v = e_4, \tag{36}$$

in (13), since $e^1, ..., e^4$ are linearly independent, we obtain:

$$\begin{cases} 2(b_{32} - b_{23}) = 0 \\ b_{13} = b_{31} \\ b_{12} = b_{21} \\ b_{12}b_{43} = b_{13}b_{42} \end{cases}$$

Considering now

$$x = e_4; \quad y = e_1; \quad z = e_2; \quad u = e_1; \quad v = e_3, \tag{37}$$

we get

$$\begin{cases} -b_{33} = b_{43}^2 - b_{33} \\ b_{23} = b_{32} - b_{42}b_{43} \\ 0 = b_{33}b_{42} - b_{23}b_{43} \end{cases},$$

which is equivalent to

$$\begin{cases} b_{43} = 0 \\ b_{23} = b_{32} \\ b_{33}b_{42} = 0 \end{cases}$$

A suitable change of indexes in (36) allows us to obtain:

$$\begin{cases} b_{42} = 0 \\ b_{32} = b_{23} \\ b_{22}b_{43} = 0 \end{cases} \quad \text{and} \quad \begin{cases} b_{41} = 0 \\ b_{21} = b_{12} \\ b_{11}b_{42} = 0 \end{cases}$$

Therefore, B is symmetric and, consequently, M is a ternary Filippov algebra.

Thus, every 4-dimensional ternary Malcev algebra M with $\dim M^1 = 4$ is a ternary Filippov algebra.

5 Reduced Malcev algebras of the ternary Malcev algebras $M(A)$

Consider an 8-dimensional composition algebra A over a field Φ such that $char\Phi \neq 2$, with an involution $^- : a \mapsto \bar{a}$ and unity e, and the related ternary Malcev algebra $M(A)$ which arise defining the following multiplication (11). Fix an orthonormal basis $e_1, ..., e_8$ and, for each $i = 1, ..., 8$, consider the reduced Malcev algebra $M_{e_i} = (A, [\cdot, \cdot]_{e_i})$, where

$$[x, y]_{e_i} = [e_i, x, y],$$

for all $x, y \in A$. In order to simplify notations, we will write M_i and $[\cdot, \cdot]_i$ instead of M_{e_i} and $[\cdot, \cdot]_{e_i}$, respectively.

Lemma 5.1. For each $i = 1, ..., 8$, we have $Z(M_i) = \langle e_i \rangle_\Phi$.

Proof Let us fix $i \in \{1, ..., 8\}$. It is clear that $Z(M_i) \supseteq \langle e_i \rangle_\Phi$. Now, admit that $a = \sum_{j=1}^{8} \alpha_j e_j \in Z(M_i)$. We claim that $a = \alpha_i e_i$. Indeed, by definition of ideal, we have:

$$u_k = [a, e_k]_i = \sum_{\substack{j=1 \\ j \neq i,k}}^{8} \alpha_j e_i \overline{e_j} e_k \in Z(M_i),$$

for all $k \in \left\{1, ..., \widehat{i}, ..., 8\right\}$. On the other hand, as a consequence of the above inclusion, we have:

$$v_k = [e_k, u_k]_i = 0. \tag{39}$$

Computing v_k by means of the properties of lemma 2.1 and its corollary, we have:

$$\begin{aligned} v_k &= \sum_{\substack{j=1 \\ j \neq i,k}}^{8} \alpha_j \left[e_i, e_k, e_i \overline{e_j} e_k\right] = \\ &= \sum_{\substack{j=1 \\ j \neq i,k}}^{8} \alpha_j \left((e_i \overline{e_k})(e_i \overline{e_j} e_k) - \langle e_i, e_k \rangle e_i \overline{e_j} e_k - \langle e_k, e_i \overline{e_j} e_k \rangle e_i + \langle e_i \overline{e_j} e_k, e_i \rangle e_k\right) \\ &= \sum_{\substack{j=1 \\ j \neq i,k}}^{8} \alpha_j e_j. \end{aligned}$$

Therefore, from (39) and from the linear independence of the vectors e_j we conclude that

$$\alpha_j = 0,$$

for every $j \in \{1, ..., 8\} \setminus \{i, k\}$. Observing that this conclusion can be achieved for all $k \in \left\{1, ..., \widehat{i}, ..., 8\right\}$ we obtain $\alpha_j = 0$ for all $j \in \left\{1, ..., \widehat{i}, ..., 8\right\}$. Thus, $a = \alpha_i e_i$ and the result is proved.

In order to simplify notations, being $a \in M$, we often call reduced algebra of M not to M_a but to $M_a / Z(M_a)$. Under this language simplification we have:

Theorem 5.2. The reduced algebras of the ternary Malcev algebras $M(A)$ which arise by fixing the elements of an orthonormal basis of A are 7-dimensional simple Malcev algebras.

Proof Let $e_1, ..., e_8$ be an arbitrary orthonormal basis of A. We want to prove that, for each $i \in \{1, ..., 8\}$, $M_i / Z(M_i)$ is a 7-dimensional simple Malcev algebra. Fixing $i \in \{1, ..., 8\}$, let M_i be written as a direct sum of linear spaces, as follows:

$$M_i = Z(M_i) \oplus N,$$

where $N = \langle e_1, ..., \widehat{e_i}, .., e_8 \rangle_\Phi$. Being $Z(M_i) = \langle e_i \rangle_\Phi$, if we conclude that N is a simple Malcev subalgebra of M_i the result will be proved, since $\overline{M_i} \cong N$. First, it is clear that $N^2 = [N, N]_i \neq 0$ and further, N is an ideal of M_i. Indeed, since $\langle e_i, [x, y]_i \rangle = \langle e_i, [e_i, x, y] \rangle = 0$ for all $x, y \in A$, if we take arbitrary different basis elements $e_j, e_k \in N$ and $a = \sum_{r=1}^{8} \alpha_r e_r$ we must have $\alpha_i = 0$ (remember that we are dealing with an orthonormal basis). Thus, $a \in N$ and $[N, N]_i \subseteq N$. By the above decomposition, we also have $[N, M_i]_i \subseteq N$.

Now, let I be a nonzero ideal of N and consider

$$u = \sum_{\substack{r=1 \\ r \neq i}}^{m} \alpha_r e_r \in I \backslash \{0\}$$

with minimal length and this length is greater than 1, *i.e.*,

$$length(u) = \begin{cases} m, \; m \geq 2, & \text{if } i > m \\ m-1, \; m \geq 3, & \text{if } i \leq m \end{cases}$$

By definition of ideal, $v = [u, e_k]_i \in I$, for all $k \in \left\{1, ..., \widehat{i}, ..., 8\right\}$. Computing v, we have:

$$v = -[u, e_i, e_k] = \begin{cases} -\sum\limits_{r=1}^{m} \alpha_r [e_r, e_i, e_k], & \text{if } i > m \\ -\sum\limits_{\substack{r=1 \\ r \neq i}}^{m} \alpha_r [e_r, e_i, e_k], & \text{if } i \leq m \end{cases}$$

In the case $i > m$, consider $k \in \{1, ..., m\}$. Then we have $v = -\sum\limits_{\substack{r=1 \\ r \neq k}}^{m} \alpha_r e_r \overline{e_i} e_k \in I$. On the other hand,

$$w = [v, e_k]_i = -\sum_{\substack{r=1 \\ r \neq k}}^{m} \alpha_r e_r \in I \backslash \{0\}.$$

If $i \leq m$, considering again $k \in \left\{1, ..., \widehat{i}, ..., m\right\}$, we now have $v = -\sum\limits_{\substack{r=1 \\ r \neq k,i}}^{m} \alpha_r e_r \overline{e_i} e_k \in I$. Further, it is a simple task to conclude that

$$w = [v, e_k]_i = -\sum_{\substack{r=1 \\ r \neq i,k}}^{m} \alpha_r e_r \in I \backslash \{0\}.$$

In both cases, we have obtained an element $w \in I \backslash \{0\}$ such that $length(w) < length(u)$, which is absurd by the choice of u. Therefore, we may consider that there exists a basis element e_j of N belonging to I. We claim that, this way, I coincides with N. Indeed, by definition of ideal, we have

$$u_k = [e_j, e_k]_i = e_i \overline{e_j} e_k \in I,$$

for all $k \in \{1, ..., 8\} \backslash \{i, j\}$. Further, $v_k = [u_k, e_j]_i \in I$. Note that

$$\begin{aligned} v_k &= [e_i, e_i \overline{e_j} e_k, e_j] \\ &= -e_i \overline{e_j} e_k \overline{e_i} e_j + \langle e_i \overline{e_j} e_k, e_i \rangle e_j + \langle e_i, e_j \rangle e_i \overline{e_j} e_k - \langle e_j, e_i \overline{e_j} e_k \rangle e_i \\ &= e_k. \end{aligned}$$

Therefore, $e_k \in I$ for all $k \in \{1, ..., 8\} \setminus \{i, j\}$ and thus $I = N$. The result is proved.

Remark 5.1. Observe that not all reduced Malcev algebras of the ternary Malcev algebra $M(A)$ are simple. In fact, consider an orthonormal basis $e, a, b, ab, c, ac, bc, abc$, of A,where e is the unity. Admit that there exists $\alpha \in \Phi$ such that $\alpha^2 = -1$ and take $u = e + \alpha a$. Considering the pair $(M(A), [\cdot, \cdot]_u)$ it is possible to prove that the reduced algebra M_u is solvable.

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