On the Structure of Graded Leibniz Algebras

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Abstract

We study the structure of graded Leibniz algebras with arbitrary dimension and over an arbitrary base field $\mathbb{K}$. We show that any of such algebras $\mathfrak{L}$ with a symmetric G-support is of the form $\mathfrak{L} = U + \sum_j I_j$ with U a subspace of $\mathfrak{L}_1$, the homogeneous component associated to the unit element 1 in G, and any I_j a well described graded ideal of $\mathfrak{L}$, satisfying $[I_j, I_k] = 0$ if $j \neq k$. In the case of $\mathfrak{L}$ being of maximal length we characterize the gr-simplicity of the algebra in terms of connections in the support of the grading.

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1 Introduction and previous definitions

Throughout this paper, Leibniz algebras $\mathfrak{L}$ are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. It is worth to mention that, unless otherwise stated, there is not any restriction on $\dim \mathfrak{L}_g$ or the products $[\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, where $\mathfrak{L}_g$ denotes the homogeneous subspace associated to $g \in G$.

Leibniz algebras were introduced as a non-antisymmetric analogue of Lie algebras by Loday [28], being so the class of Leibniz algebras an extension of the one of Lie algebras. The structure of this kind of algebras has been considered in the frameworks of low dimensional algebras, nilpotence and related problems [4, 6, 7, 10, 16, 25, 26, 32]. The simple case was introduced in [1, 2] where some results concerning special cases of simple Leibniz algebras were also obtained.

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Recently, Liu and Hu have studied Leibniz algebras graded by finite root systems of types A_l, D_l and E_l, [27].

On the other hand, the interest on gradings on Lie algebras has been remarkable in the last years. The gradings of classical finite dimensional simple Lie algebras have been studied, among others works, in [8, 21, 23, 24, 30, 31]. The studies of gradings on exceptional Lie algebras are [9], [17] and [18], which describe the group gradings on $\mathfrak{g}_2$ and $\mathfrak{f}_4$. The study of the gradings of the real forms of complex Lie algebras begins in [22], where are considered the gradings on the real forms of classical simple complex Lie algebras. The description of the fine gradings of the real forms of the exceptional simple Lie algebras $\mathfrak{f}_4$ and $\mathfrak{g}_2$ are obtained in [14]. Respect to the group gradings on superalgebras, these have been considered, for the case of the Jordan superalgebra K_{10}, in [15].

In the present paper we begin the study of arbitrary graded Leibniz algebras, (not necessarily simple or finite-dimensional), introduced as the natural extension of graded Lie algebras, and over an arbitrary base field $\mathbb{K}$ by focussing on their structure. In §2 we extend the techniques of connections in the support of G developed for graded Lie algebras in [11] to the framework of graded Leibniz algebras $\mathfrak{L}$, so as to show that $\mathfrak{L}$ is of the form $\mathfrak{L} = U + \sum_j I_j$ with U a subspace of $\mathfrak{L}_1$, the homogeneous component associated to the unit element 1 in G, and any I_j a well described graded ideal of $\mathfrak{L}$, satisfying $[I_j, I_k] = 0$ if $j \neq k$. The gr-simple case is studied in §3 by characterizing the gr-simplicity of $\mathfrak{L}$ in terms of connections in the support of the grading.

Definition 1.1. *A* Leibniz algebra $\mathfrak{L}$ *is a vector space over a base field* $\mathbb{K}$ *endowed with a bilinear product* $[\cdot,\cdot]$ *satisfying the* Leibniz identity

$$[[y,z],x] = [[y,x],z] + [y,[z,x]],$$

for any $x, y, z \in \mathfrak{L}$.

Clearly Lie algebras are examples of Leibniz algebras.

For any $x \in \mathfrak{L}$, consider the *adjoint mapping* $\mathrm{ad}_x : \mathfrak{L} \longrightarrow \mathfrak{L}$ defined by $\mathrm{ad}_x(z) = [z,x]$. Observe that Leibniz identity is equivalent to assert that ad_x is a derivation for any $x \in \mathfrak{L}$. An *ideal* I of $\mathfrak{L}$ is a vector subspace such that $[I,\mathfrak{L}] + [\mathfrak{L}, I] \subset I$.

The term grading will always mean abelian group grading. That is:

Definition 1.2. *We say that a Leibniz algebra* $\mathfrak{L}$ *is a* graded *algebra, by means of the abelian group* G*, if* $\mathfrak{L}$ *decomposes as the direct sum of vector subspaces*

$$\mathfrak{L} = \bigoplus_{g \in G} \mathfrak{L}_g$$

where the homogeneous spaces satisfy $[\mathfrak{L}_g, \mathfrak{L}_h] \subset \mathfrak{L}_{gh}$ *(denoting by juxtaposition the product in* G*). We call the* G-support *of the grading to the set*

$$\Sigma_G := \{g \in G \setminus \{1\} : \mathfrak{L}_g \neq 0\}.$$

We will also say that the G-support of the grading is *symmetric* if $g \in \Sigma_G$ implies $g^{-1} \in \Sigma_G$. We finally note that graded Lie algebras and split Leibniz algebras are examples of graded Leibniz algebras and so the present paper extends the results in [11] and [13].

The usual regularity concepts will be understood in the graded sense. For instance, a *graded ideal* I of $\mathfrak{L}$ is an ideal which splits as $I = \bigoplus_{g \in G} I_g$ with any $I_g = I \cap \mathfrak{L}_g, g \in G$.

We note that the ideal $\mathfrak{I}$ generated by $\{[x, x] : x \in \mathfrak{L}\}$ plays an important role in the theory since determines the (possible) non-Lie character of $\mathfrak{L}$. It is straightforward to verify that if $\mathfrak{L}$ is a graded algebra then $\mathfrak{I}$ is also a graded ideal and so we can write $\mathfrak{I} = \bigoplus_{g \in G} \mathfrak{I}_g$, being any $\mathfrak{I}_g = \mathfrak{I} \cap \mathfrak{L}_g$. From the Leibniz identity, this ideal also satisfies

$$[\mathfrak{L}, \mathfrak{I}] = 0. \tag{1}$$

The usual definition of simple algebra lack of interest in the case of Leibniz algebras because would imply the ideal $\mathfrak{I} = \mathfrak{L}$ or $\mathfrak{I} = 0$, being so $\mathfrak{L}$ an abelian or a Lie algebra respectively. Abdykassymova and Dzhumadil'daev introduced in [1, 2] the following adequate definition.

Definition 1.3. *A Leibniz algebra $\mathfrak{L}$ is said to be* simple *if its product is nonzero and its only ideals are $\{0\}$, $\mathfrak{I}$ and $\mathfrak{L}$.*

It should be noted that the above definition agrees with the definition of simple Lie algebra, since $\mathfrak{I} = \{0\}$ in this case. Of course, a graded Leibniz algebra $\mathfrak{L}$ is called *gr-simple* if $[\mathfrak{L}, \mathfrak{L}] \neq 0$ and its only graded ideals are $\{0\}$, $\mathfrak{I}$ and $\mathfrak{L}$.

Example 1. Consider the complex (non-Lie) Leibniz algebra $\mathfrak{L}$ with the basis $\{e, h, f, p, q\}$ defined by the following multiplication, see [29]:

$$[e, h] = 2e, \quad [h, f] = 2f, \quad [e, f] = h,$$
$$[h, e] = -2e, \quad [f, h] = -2f, \quad [f, e] = -h,$$
$$[p, h] = p, \quad [p, f] = q,$$
$$[q, h] = -q, \quad [q, e] = -p,$$

where omitted products are equal to zero. The Leibniz algebra $\mathfrak{L}$ can be $\mathbb{Z}$-graded as

$$\mathfrak{L} = \bigoplus_{z \in \mathbb{Z}} \mathfrak{L}_z$$

where

$$\mathfrak{L}_0 = \langle h \rangle, \quad \mathfrak{L}_1 = \langle p \rangle, \quad \mathfrak{L}_{-1} = \langle q \rangle,$$
$$\mathfrak{L}_2 = \langle e \rangle, \quad \mathfrak{L}_{-2} = \langle f \rangle$$

and $\mathfrak{L}_z = 0$ for any $z \notin \{0, \pm 1, \pm 2\}$. The only graded ideals of $\mathfrak{L}$ are $\{0\}$, $\mathfrak{L}$ and $\mathfrak{I} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_1$. Hence, $\mathfrak{L}$ is gr-simple.

2 Σ_G-Connections and Decompositions

From now on, $\mathfrak{L}$ denotes a graded Leibniz algebra with a symmetric G-support Σ_G, and

$$\mathfrak{L} = \bigoplus_{g\in G} \mathfrak{L}_g = \mathfrak{L}_1 \oplus (\bigoplus_{g\in\Sigma_G} \mathfrak{L}_g)$$

the corresponding grading. We begin by developing connection techniques in this framework.

Definition 2.1. *Let $g, g' \in \Sigma_G$. We shall say that g is* Σ_G-connected *to g' if there exist $\{g_1, g_2..., g_n\} \subset \Sigma_G$ such that*

1. $g_1 = g$,
2. $\{g_1g_2, g_1g_2g_3, ..., g_1g_2\cdots g_{n-1}\} \subset \Sigma_G$ *and*
3. $g_1g_2\cdots g_n \in \{g', (g')^{-1}\}$.

We shall also say that $\{g_1, ..., g_n\}$ is a Σ_G-connection *from g to g'.*

The next result shows the Σ_G-connection relation is of equivalence. Its proof is analogous to the one for graded Lie algebras given in [11, Proposition 2.1].

Proposition 2.1. *The relation $\sim$ in Σ_G, defined by $g \sim g'$ if and only if g is Σ_G-connected to g', is of equivalence.*

Given $g \in \Sigma_G$, we denote by

$$\mathfrak{C}_g := \{g' \in \Sigma_G : g' \text{ is } \Sigma_G\text{−connected to } g\}.$$

Clearly if $g' \in \mathfrak{C}_g$ then $(g')^{-1} \in \mathfrak{C}_g$ and, by Proposition 2.1, if $h \notin \mathfrak{C}_g$ then $\mathfrak{C}_g \cap \mathfrak{C}_h = \emptyset$.

Lemma 2.1. *If $g' \in \mathfrak{C}_g$ and $g'', g'g'' \in \Sigma_G$, then $g'', g'g'' \in \mathfrak{C}_g$.*

Proof. The Σ_G-connection $\{g', g''\}$ gives us $g' \sim g'g''$. Hence, by the transitivity of $\sim$, we finally get $g'g'' \in \mathfrak{C}_g$. To verify $g'' \in \mathfrak{C}_g$, observe that $\{g'g'', (g')^{-1}\}$ is a Σ_G-connection from $g'g''$ to g''. Now, taking into account $g'g'' \in \mathfrak{C}_g$, we conclude as above that $g'' \in \mathfrak{C}_g$. □

Our next goal is to associate an (adequate) graded ideal $I_{[g]}$ to any $\mathfrak{C}_g$. For $\mathfrak{C}_g$, $g \in \Sigma_G$, we define

$$\mathfrak{L}_{\mathfrak{C}_g,1} := span_{\mathbb{K}}\{[\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}] : g' \in \mathfrak{C}_g\}$$

and

$$V_{\mathfrak{C}_g} := \bigoplus_{g'\in\mathfrak{C}_g} \mathfrak{L}_{g'}.$$

We denote by $\mathfrak{L}_{\mathfrak{C}_g}$ the following subspace of $\mathfrak{L}$,

$$\mathfrak{L}_{\mathfrak{C}_g} := \mathfrak{L}_{\mathfrak{C}_g,1} \oplus V_{\mathfrak{C}_g}.$$

Proposition 2.2. *Let $g \in \Sigma_G$. Then the following assertions hold.*

1. $[\mathfrak{L}_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_g}] \subset \mathfrak{L}_{\mathfrak{C}_g}$.
2. *If $h \notin \mathfrak{C}_g$ then* $[\mathfrak{L}_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_h}] = 0$.

Proof. 1. We have

$$[\mathfrak{L}_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_g}] = [\mathfrak{L}_{\mathfrak{C}_g,1} \oplus V_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_g,1} \oplus V_{\mathfrak{C}_g}] \subset \tag{2}$$

$$[\mathfrak{L}_{\mathfrak{C}_g,1}, \mathfrak{L}_{\mathfrak{C}_g,1}] + [\mathfrak{L}_{\mathfrak{C}_g,1}, V_{\mathfrak{C}_g}] + [V_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_g,1}] + [V_{\mathfrak{C}_g}, V_{\mathfrak{C}_g}].$$

Consider the above second summand $[\mathfrak{L}_{\mathfrak{C}_g,1}, V_{\mathfrak{C}_g}]$. Taking into account $\mathfrak{L}_{\mathfrak{C}_g,1} \subset \mathfrak{L}_1$ and $[\mathfrak{L}_1, \mathfrak{L}_g] \subset \mathfrak{L}_g$ for any $g \in \Sigma_g$, we have $[\mathfrak{L}_{\mathfrak{C}_g,1}, V_{\mathfrak{C}_g}] \subset V_{\mathfrak{C}_g}$. In a similar way $[V_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_g,1}] \subset V_{\mathfrak{C}_g}$ and so

$$[\mathfrak{L}_{\mathfrak{C}_g,1}, V_{\mathfrak{C}_g}] + [V_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_g,1}] \subset V_{\mathfrak{C}_g}. \tag{3}$$

Consider now the fourth summand $[V_{\mathfrak{C}_g}, V_{\mathfrak{C}_g}]$ in equation (2) and suppose there exist $g', g'' \in \mathfrak{C}_g$ such that $[\mathfrak{L}_{g'}, \mathfrak{L}_{g''}] \neq 0$. If $g'' = (g')^{-1}$, clearly $[\mathfrak{L}_{g'}, \mathfrak{L}_{g''}] = [\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}] \subset \mathfrak{L}_{\mathfrak{C}_g,1}$. Otherwise, if $g'' \neq (g')^{-1}$, then $g'g'' \in \Sigma_G$ and Lemma 2.1 gives us $g'g'' \in \mathfrak{C}_g$. Hence, $[\mathfrak{L}_{g'}, \mathfrak{L}_{g''}] \subset \mathfrak{L}_{g'g''} \subset V_{\mathfrak{C}_g}$. In any case

$$[V_{\mathfrak{C}_g}, V_{\mathfrak{C}_g}] \subset \mathfrak{L}_{\mathfrak{C}_g}. \tag{4}$$

Finally, let us consider the first summand $[\mathfrak{L}_{\mathfrak{C}_g,1}, \mathfrak{L}_{\mathfrak{C}_g,1}]$ in equation (2). We have

$$[\mathfrak{L}_{\mathfrak{C}_g,1}, \mathfrak{L}_{\mathfrak{C}_g,1}] = [\sum_{g' \in \mathfrak{C}_g} [\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}], \sum_{g'' \in \mathfrak{C}_g} [\mathfrak{L}_{g''}, \mathfrak{L}_{(g'')^{-1}}]].$$

Taking now into account Leibniz identity we get

$$\sum_{g', g'' \in \mathfrak{C}_g} [[\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}], [\mathfrak{L}_{g''}, \mathfrak{L}_{(g'')^{-1}}]] \subset$$

$$\sum_{g', g'' \in \mathfrak{C}_g} ([[\mathfrak{L}_{g'}, [\mathfrak{L}_{g''}, \mathfrak{L}_{(g'')^{-1}}]], \mathfrak{L}_{(g')^{-1}}] + [\mathfrak{L}_{g'}, [\mathfrak{L}_{(g')^{-1}}, [\mathfrak{L}_{g''}, \mathfrak{L}_{(g'')^{-1}}]]]) \subset$$

$$\subset \sum_{g' \in \mathfrak{C}_g} [\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}] = \mathfrak{L}_{\mathfrak{C}_g,1}.$$

That is,

$$[\mathfrak{L}_{\mathfrak{C}_g,1}, \mathfrak{L}_{\mathfrak{C}_g,1}] \subset \mathfrak{L}_{\mathfrak{C}_g,1} \tag{5}$$

From equations (2)-(5) we conclude $[\mathfrak{L}_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_g}] \subset \mathfrak{L}_{\mathfrak{C}_g}$.

2. We have as in 1. that

$$[\mathfrak{L}_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_h}] \subset [\mathfrak{L}_{\mathfrak{C}_g,1}, \mathfrak{L}_{\mathfrak{C}_h,1}] + [\mathfrak{L}_{\mathfrak{C}_g,1}, V_{\mathfrak{C}_h}] + [V_{\mathfrak{C}_g}, \mathfrak{L}_{\mathfrak{C}_h,1}] + [V_{\mathfrak{C}_g}, V_{\mathfrak{C}_h}]. \tag{6}$$

Let us suppose that there exist $g' \in \mathfrak{C}_g$ and $h' \in \mathfrak{C}_h$ such that $[\mathfrak{L}_{g'}, \mathfrak{L}_{h'}] \neq 0$. Then $g'h' \in \Sigma_G$ and we have as consequence of Lemma 2.1 that g is connected to h, a contradiction. From here $[V_{\mathfrak{C}_g}, V_{\mathfrak{C}_h}] = 0$. Taking into account this equality and the fact $(g')^{-1} \in \mathfrak{C}_g$ for any $g' \in \mathfrak{C}_g$, we can argue with Leibniz identity in $[[\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}], \mathfrak{L}_{h'}]$ and in $[\mathfrak{L}_{g'}, [\mathfrak{L}_{h'}, \mathfrak{L}_{(h')^{-1}}]]$ to get $[[\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}], \mathfrak{L}_{h'}] = [\mathfrak{L}_{g'}, [\mathfrak{L}_{h'}, \mathfrak{L}_{(h')^{-1}}]] = 0$. Now a same argument can be applied to verify $[[\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}], [\mathfrak{L}_{h'}, \mathfrak{L}_{(h')^{-1}}]] = 0$. Taking into account equation (6) we have proved 2. □

Proposition 2.2 let us assert that for any $g \in \Sigma_G$, $\mathfrak{L}_{\mathfrak{C}_g}$ is a (graded) subalgebra of $\mathfrak{L}$ that we call the subalgebra of $\mathfrak{L}$ *associated* to $\mathfrak{C}_g$. Now, the following results can be proved as in [11, Theorem 2.1] and [11, Theorem 2.2] respectively.

Theorem 2.1. *The following assertions hold.*

1. *For any $g \in \Sigma_G$, the graded subalgebra $\mathfrak{L}_{\mathfrak{C}_g} = \mathfrak{L}_{\mathfrak{C}_g,1} \oplus V_{\mathfrak{C}_g}$ of $\mathfrak{L}$ associated to $\mathfrak{C}_g$ is a graded ideal of $\mathfrak{L}$.*

2. *If $\mathfrak{L}$ is gr-simple, then there exists a Σ_G-connection from g to g' for any $g, g' \in \Sigma_G$, and $\mathfrak{L}_1 = \sum_{g' \in \Sigma_G} [\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}]$.*

Theorem 2.2. *For the complementary vector space $\mathcal{U}$ of $span_{\mathbb{K}}\{[\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}] : g \in \Sigma_G\}$ in $\mathfrak{L}_1$, we have*

$$\mathfrak{L} = \mathcal{U} + \sum_{g \in \Sigma_G/\sim} I_{[g]},$$

where any $I_{[g]}$ is one of the graded ideals $\mathfrak{L}_{\mathfrak{C}_g}$ of $\mathfrak{L}$ described in Theorem 2.1, satisfying $[I_{[g]}, I_{[g']}] = 0$ if $[g] \neq [g']$.

Definition 2.2. *The* annihilator *of a Leibniz algebra $\mathfrak{L}$ is the set* $\mathrm{Z}(\mathfrak{L}) = \{x \in \mathfrak{L} : [x, \mathfrak{L}] + [\mathfrak{L}, x] = 0\}$.

The next corollary follows as in [11, Corollary 2.1].

Corollary 2.1. *If $\mathcal{Z}(\mathfrak{L}) = 0$ and $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, then $\mathfrak{L}$ is the direct sum of the graded ideals given in Theorem 2.1-1,*

$$\mathfrak{L} = \bigoplus_{[g] \in \Sigma_G/\sim} I_{[g]}.$$

3 The gr-simple components

In this section we focus on the gr-simplicity of graded Leibniz algebras by centering our attention in those of maximal length. This terminology is taking borrowed from the theory of gradations of Lie and Leibniz algebras, (see [3, 5, 19, 20]). See also [5, 11, 13, 33] for examples.

Definition 3.1. *We say that a graded Leibniz algebra $\mathfrak{L}$ is of* maximal length *if $\mathfrak{L}_1 \neq 0$ and $\dim \mathfrak{L}_g = 1$ for any $g \in \Sigma_G$.*

As an example of a graded Leibniz algebra of maximal length we can take the graded Leibniz algebra given in Example 1. Our target is to characterize the gr-simplicity of $\mathfrak{L}$ in terms of connectivity properties in Σ_G. We begin with a preliminary result which holds for non-necessarily of maximal length graded Leibniz algebras.

Lemma 3.1. *Let $\mathfrak{L}$ be a graded Leibniz algebra with $\mathrm{Z}(\mathfrak{L}) = 0$ and satisfying $\mathfrak{L}_1 = \sum\limits_{g\in\Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$. If I is a graded ideal of $\mathfrak{L}$ such that $I \subset \mathfrak{L}_1$ then $I = \{0\}$.*

Proof. Suppose there exists a nonzero graded ideal I of $\mathfrak{L}$ such that $I \subset \mathfrak{L}_1$. Then $[I, \bigoplus\limits_{g\in\Sigma_G} \mathfrak{L}_g] + [\bigoplus\limits_{g\in\Sigma_G} \mathfrak{L}_g, I] \subset (\bigoplus\limits_{g\in\Sigma_G} \mathfrak{L}_g) \cap \mathfrak{L}_1$ and so

$$[I, \bigoplus_{g\in\Sigma_G} \mathfrak{L}_g] + [\bigoplus_{g\in\Sigma_G} \mathfrak{L}_g, I] = 0.$$

From here, the fact $\mathcal{Z}(\mathfrak{L}) = 0$ gives us that necessarily $[I, \mathfrak{L}_1] + [\mathfrak{L}_1, I] \neq 0$. Taking into account $\mathfrak{L}_1 = \sum\limits_{g\in\Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, there exists $g_0 \in \Sigma_G$ such that either $[I, [\mathfrak{L}_{g_0}, \mathfrak{L}_{g_0^{-1}}]] \neq 0$ or $[[\mathfrak{L}_{g_0}, \mathfrak{L}_{g_0^{-1}}], I] \neq 0$. By Leibniz identity one of the following products is nonzero: $[I, \mathfrak{L}_{g_0}] \subset \mathfrak{L}_{g_0} \cap \mathfrak{L}_1$, $[I, \mathfrak{L}_{g_0^{-1}}] \subset \mathfrak{L}_{g_0^{-1}} \cap \mathfrak{L}_1$, $[\mathfrak{L}_{g_0}, I] \subset \mathfrak{L}_{g_0} \cap \mathfrak{L}_1$, $[\mathfrak{L}_{g_0^{-1}}, I] \subset \mathfrak{L}_{g_0^{-1}} \cap \mathfrak{L}_1$. In any case we have a contradiction since $\mathfrak{L}_{g_0} \cap \mathfrak{L}_1 = \mathfrak{L}_{g_0^{-1}} \cap \mathfrak{L}_1 = 0$. So $I = \{0\}$. $\square$

Let us return to a graded Leibniz algebra of maximal length $\mathfrak{L}$. In fact, from now on $\mathfrak{L} = \mathfrak{L}_1 \oplus (\bigoplus\limits_{g\in\Sigma_G} \mathfrak{L}_g)$ will denote such an algebra. Consider any nonzero graded ideal I of $\mathfrak{L}$, then the maximal length of $\mathfrak{L}$ gives us

$$I = (I \cap \mathfrak{L}_1) \oplus (\bigoplus_{g\in\Sigma_I} \mathfrak{L}_g) \tag{7}$$

where $\Sigma_I := \{g \in \Sigma_G : \mathfrak{L}_g \cap I \neq 0\} = \{g \in \Sigma_G : \mathfrak{L}_g \subset I\}$. In the particular, (an important) case $I = \mathfrak{I}$, we get

$$\mathfrak{I} = (\mathfrak{I} \cap \mathfrak{L}_1) \oplus (\bigoplus_{g\in\Sigma_\mathfrak{I}} \mathfrak{L}_g). \tag{8}$$

From here, we can write

$$\Sigma_G = \Sigma_\mathfrak{I} \dot{\cup} \Sigma_{\neg\mathfrak{I}}, \tag{9}$$

where

$$\Sigma_\mathfrak{I} := \{g \in \Sigma_G : \mathfrak{L}_g \cap \mathfrak{I} \neq 0\} = \{g \in \Sigma_G : \mathfrak{L}_g \subset \mathfrak{I}\}$$

and

$$\Sigma_{\neg\mathfrak{I}} := \{g \in \Sigma_G : \mathfrak{L}_g \cap \mathfrak{I} = 0\}.$$

Consequently

$$\mathfrak{L} = \mathfrak{L}_1 \oplus (\bigoplus_{g\in\Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \oplus (\bigoplus_{h\in\Sigma_\mathfrak{I}} \mathfrak{L}_h). \tag{10}$$

Now, observe that the concept of Σ_G-connectivity among the elements of Σ_G given in Definition 2.1 is not strong enough to detect if a given $g \in \Sigma_G$ belongs to $\Sigma_{\mathfrak{I}}$ or to $\Sigma_{\neg\mathfrak{I}}$. Consequently we lose the information respect to whether a given $\mathfrak{L}_g$ is contained in $\mathfrak{I}$ or not, which is fundamental to study the gr-simplicity of $\mathfrak{L}$. So, we are going to refine the concept of Σ_G-connection in the setup of maximal length graded Leibniz algebras.

In the following, we suppose $\Sigma_{\mathfrak{I}}$ and $\Sigma_{\neg\mathfrak{I}}$ are *symmetric*, that is, satisfying that if $g \in \Sigma_\Upsilon$ then $g^{-1} \in \Sigma_\Upsilon$, for $\Upsilon \in \{\mathfrak{I}, \neg\mathfrak{I}\}$. Then we note that in case $\mathfrak{L}_1 = \sum_{g\in\Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, the decomposition given by equation (10) and equation (1) show

$$\mathfrak{L}_1 = \sum_{g\in\Sigma_{\neg\mathfrak{I}}} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]. \tag{11}$$

Definition 3.2. *Let $g, g' \in \Sigma_\Upsilon$ with $\Upsilon \in \{\mathfrak{I}, \neg\mathfrak{I}\}$. We say that g is* $\Sigma_{\neg\mathfrak{I}}$-connected *to g', denoted by $g \sim_{\neg\mathfrak{I}} g'$, if there exist*

$$g_2, ..., g_n \in \Sigma_{\neg\mathfrak{I}}$$

such that

1. $\{g_1, g_1g_2, g_1g_2g_3, ..., g_1g_2\cdots g_{n-1}g_n\} \subset \Sigma_\Upsilon$ *where* $g_1 := g$, *and*
2. $g_1g_2\cdots g_n \in \{g', (g')^{-1}\}$.

The set $\{g_1, ..., g_n\}$ is called a $\Sigma_{\neg\mathfrak{I}}$-connection *from g to g'.*

Proposition 3.1. *The following assertions hold.*

1. *The relation $\sim_{\neg\mathfrak{I}}$ is of equivalence in $\Sigma_{\neg\mathfrak{I}}$.*
2. *If $\mathrm{Z}(\mathfrak{L}) = 0$ and $\mathfrak{L}_1 = \sum_{g\in\Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, then the relation $\sim_{\neg\mathfrak{I}}$ is of equivalence in $\Sigma_{\mathfrak{I}}$.*

Proof. 1. Can be proved in a similar way to Proposition 2.1.

2. Let $h \in \Sigma_{\mathfrak{I}}$. Taking into account $\mathrm{Z}(\mathfrak{L}) = 0$ and $\mathfrak{L}_1 = \sum_{g\in\Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, Leibniz identity and equation (1) give us that there exists $g \in \Sigma_{\neg\mathfrak{I}}$ such that $[\mathfrak{L}_h, \mathfrak{L}_g] \neq 0$, being $g \neq h^{-1}$ by the symmetry of $\Sigma_{\mathfrak{I}}$ and equation (1). Hence, the symmetry of $\Sigma_{\neg\mathfrak{I}}$ and the character of ideal of $\mathfrak{I}$ let us assert that $\{h, g, g^{-1}\}$ is a $\Sigma_{\neg\mathfrak{I}}$-connection which gives us $h \sim_{\neg\mathfrak{I}} h$ and consequently $\sim_{\neg\mathfrak{I}}$ is reflexive in $\Sigma_{\mathfrak{I}}$. The symmetric and transitive character of $\sim_{\neg\mathfrak{I}}$ in $\Sigma_{\mathfrak{I}}$ follows as in Proposition 2.1. $\square$

Let us introduce the notion of Σ_G-multiplicativity in the framework of graded Leibniz algebras of maximal length, in a similar way to the ones for graded Lie algebras, split Lie triple systems and split Leibniz algebras (see [11, 12, 13] for these notions and examples).

Definition 3.3. *We say that a graded Leibniz algebra of maximal length* $\mathfrak{L}$ *is* Σ_G-multiplicative *if the below conditions hold.*

1. *Given* $g, g' \in \Sigma_{\neg\mathfrak{I}}$ *such that* $gg' \in \Sigma_G$ *then* $[\mathfrak{L}_g, \mathfrak{L}_{g'}] \neq 0$.

2. *Given* $g \in \Sigma_{\neg\mathfrak{I}}$ *and* $h \in \Sigma_{\mathfrak{I}}$ *such that* $gh \in \Sigma_{\mathfrak{I}}$ *then* $[\mathfrak{L}_h, \mathfrak{L}_g] \neq 0$.

Another interesting notion related to graded Leibniz algebras of maximal length $\mathfrak{L}$ is those of Lie-annihilator. Write $\mathfrak{L} = \mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \oplus (\bigoplus_{h \in \Sigma_{\mathfrak{I}}} \mathfrak{L}_h)$, (see equation (10)).

Definition 3.4. *The* Lie-annihilator *of a graded Leibniz algebra of maximal length* $\mathfrak{L}$ *is the set*

$$\mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = \{x \in \mathfrak{L} : [x, \mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g)] + [\mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g), x] = 0\}.$$

Clearly the above definition agrees with the definition of annihilator of a Lie algebra, since in this case $\Sigma_{\mathfrak{I}} = \emptyset$. We also have $\mathrm{Z}(\mathfrak{L}) \subset \mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L})$.

Consider the graded Leibniz algebra $\mathfrak{L} = \bigoplus_{z \in \mathbb{Z}} \mathfrak{L}_z$ given in Example 1, that we know is of maximal length. Since $\mathfrak{I} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_1$ we have $\Sigma_{\mathfrak{I}} = \{\pm 1\}$ and $\Sigma_{\neg\mathfrak{I}} = \{\pm 2\}$. From here, and taking in to account the multiplication in $\mathfrak{L}$, it is easy to verify that $\mathfrak{L}$ is Σ_G-multiplicative. We also have that in this example $\mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = \{0\}$.

Lemma 3.2. *Suppose* $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$ *and* $\mathfrak{L}$ *is* Σ_G*-multiplicative. If* $\Sigma_{\neg\mathfrak{I}}$ *has all of its elements* $\Sigma_{\neg\mathfrak{I}}$*-connected, then any nonzero graded ideal* I *of* $\mathfrak{L}$ *such that* $I \cap (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \neq \{0\}$ *satisfies that* $\mathfrak{L}_1 \subset I$ *and that for any* $g \in \Sigma_{\neg\mathfrak{I}}$*, either* $\mathfrak{L}_g \subset I$ *or* $\mathfrak{L}_{g^{-1}} \subset I$.

Proof. By equations (7) and (9) we can write

$$I = (I \cap \mathfrak{L}_1) \oplus (\bigoplus_{g_i \in \Sigma_{\neg\mathfrak{I}, I}} \mathfrak{L}_{g_i}) \oplus (\bigoplus_{h_j \in \Sigma_{\mathfrak{I}, I}} \mathfrak{L}_{h_j})$$

where $\Sigma_{\neg\mathfrak{I}, I} = \Sigma_{\neg\mathfrak{I}} \cap \Sigma_I$ and $\Sigma_{\mathfrak{I}, I} = \Sigma_{\mathfrak{I}} \cap \Sigma_I$. Since $I \cap (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \neq \{0\}$ we have $\Sigma_{\neg\mathfrak{I}, I} \neq \emptyset$ and so we can fix some $g_0 \in \Sigma_{\neg\mathfrak{I}, I}$ being then

$$\mathfrak{L}_{g_0} \subset I. \tag{12}$$

For any $g' \in \Sigma_{\neg\mathfrak{I}} \setminus \{g_0, g_0^{-1}\}$, the fact that g_0 and g' are $\Sigma_{\neg\mathfrak{I}}$-connected gives us a $\Sigma_{\neg\mathfrak{I}}$-connection $\{g_1, g_2, ..., g_n\} \subset \Sigma_{\neg\mathfrak{I}}$ from g_0 to g' such that $g_1 = g_0$,

$$g_1 g_2, g_1 g_2 g_3, ..., g_1 g_2 g_3 \cdots g_{n-1} \in \Sigma_{\neg\mathfrak{I}}$$

and

$$g_1 g_2 g_3 \cdots g_n \in \{g', (g')^{-1}\}.$$

Consider g_1, g_2 and g_1g_2. Since $g_1, g_2 \in \Sigma_{\neg\mathfrak{I}}$, the Σ_G-multiplicativity and maximal length of $\mathfrak{L}$ show $0 \neq [\mathfrak{L}_{g_1}, \mathfrak{L}_{g_2}] = \mathfrak{L}_{g_1g_2}$, and by equation (12)

$$0 \neq \mathfrak{L}_{g_1g_2} \subset I.$$

We can argue in a similar way from g_1g_2, g_3 and $g_1g_2g_3$ to get

$$0 \neq \mathfrak{L}_{g_1g_2g_3} \subset I.$$

Following this process with the $\Sigma_{\neg\mathfrak{I}}$-connection $\{g_1, ..., g_n\}$ we obtain that

$$0 \neq \mathfrak{L}_{g_1g_2g_3\cdots g_n} \subset I$$

and so either $\mathfrak{L}_{g'} \subset I$ or $\mathfrak{L}_{(g')^{-1}} \subset I$. That is,

$$\mathfrak{L}_\mu \subset I \text{ for any } g' \in \Sigma_{\neg\mathfrak{I}} \text{ and some } \mu \in \{g', (g')^{-1}\}.$$

Since $\mathfrak{L}_1 = \sum_{g\in\Sigma_{\neg\mathfrak{I}}} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, (see equation (11)), we get $\mathfrak{L}_1 \subset I$. □

Lemma 3.3. *Suppose $\mathfrak{L}_1 = \sum_{g\in\Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, $\mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$ and $\mathfrak{L}$ is Σ_G-multiplicative. If $\Sigma_{\neg\mathfrak{I}}$ has all of its elements $\Sigma_{\neg\mathfrak{I}}$-connected, then any nonzero graded ideal I of $\mathfrak{L}$ such that $I \cap (\bigoplus_{g\in\Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \neq \{0\}$ satisfies $\mathfrak{L}_1 \oplus (\bigoplus_{g\in\Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \subset I$.*

Proof. By Lemma 3.2, for any $g \in \Sigma_{\neg\mathfrak{I}}$, either $\mathfrak{L}_g \subset I$ or $\mathfrak{L}_{g^{-1}} \subset I$. Let us show that there exists $g_0 \in \Sigma_{\neg\mathfrak{I}}$ such that $\mathfrak{L}_{g_0} \subset I$ and $\mathfrak{L}_{g_0^{-1}} \subset I$. To do that, suppose there is not any $g_0 \in \Sigma_{\neg\mathfrak{I}}$ such that $\mathfrak{L}_{g_0} \subset I$ and $\mathfrak{L}_{g_0^{-1}} \subset I$. Since $\mathfrak{L}_1 \neq 0$ by the maximal length of $\mathfrak{L}$, and $\mathfrak{L}_1 = \sum_{g\in\Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, we can take some $g_0 \in \Sigma_{\neg\mathfrak{I}}$ such that

$$[\mathfrak{L}_{g_0}, \mathfrak{L}_{g_0^{-1}}] \neq 0 \text{ or } [\mathfrak{L}_{g_0^{-1}}, \mathfrak{L}_{g_0}] \neq 0 \tag{13}$$

with $\mathfrak{L}_{g_0} \subset I$ and $\mathfrak{L}_{g_0^{-1}} \cap I = \{0\}$. Let us suppose $[\mathfrak{L}_{g_0}, \mathfrak{L}_{g_0^{-1}}] \neq 0$. Taking into account $\mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$, $\mathfrak{L}_1 = \sum_{g\in\Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}] = \sum_{h\in\Sigma_{\neg\mathfrak{I}}} [\mathfrak{L}_h, \mathfrak{L}_{h^{-1}}]$ and Leibniz identity, there exists $h \in \Sigma_{\neg\mathfrak{I}}$ such that $[\mathfrak{L}_h, [\mathfrak{L}_{g_0}, \mathfrak{L}_{g_0^{-1}}]] + [[\mathfrak{L}_{g_0}, \mathfrak{L}_{g_0^{-1}}], \mathfrak{L}_h] \neq 0$, being so $\mathfrak{L}_h \subset I$. That is, $h \in \Sigma_{\neg\mathfrak{I},I}$. By Leibniz identity, either

$$[[\mathfrak{L}_h, \mathfrak{L}_{g_0}], \mathfrak{L}_{g_0^{-1}}] + [[\mathfrak{L}_{g_0}, \mathfrak{L}_h], \mathfrak{L}_{g_0^{-1}}] \neq 0 \tag{14}$$

or

$$[[\mathfrak{L}_h, \mathfrak{L}_{g_0^{-1}}], \mathfrak{L}_{g_0}] + [\mathfrak{L}_{g_0}, [\mathfrak{L}_{g_0^{-1}}, \mathfrak{L}_h]] \neq 0. \tag{15}$$

If equation (14) holds, then $hg_0 \in \Sigma_{\neg\mathfrak{I},I}$ and we get by Σ_G-multiplicativity $[\mathfrak{L}_h, \mathfrak{L}_{h^{-1}g_0^{-1}}] = \mathfrak{L}_{g_0^{-1}} \subset I$ which is a contradiction. By the other hand, if equation (15) holds, then $hg_0^{-1} \in \Sigma_{\neg\mathfrak{I},I}$ and we also get by Σ_G-multiplicativity $[\mathfrak{L}_{hg_0^{-1}}, \mathfrak{L}_{h^{-1}}] = \mathfrak{L}_{g_0^{-1}} \subset I$, a contradiction. Finally, note that if $[\mathfrak{L}_{g_0}, \mathfrak{L}_{g_0^{-1}}] = 0$ and by equation (13) we have the case $[\mathfrak{L}_{g_0^{-1}}, \mathfrak{L}_{g_0}] \neq 0$, we obtain as above a contradiction. Consequently, there exists $g_0 \in \Sigma_{\neg\mathfrak{I}}$ such that $\mathfrak{L}_{g_0} \subset I$ and $\mathfrak{L}_{g_0^{-1}} \subset I$. From here, for any $g' \in \Sigma_{\neg\mathfrak{I}} \setminus \{g_0, g_0^{-1}\}$, the fact that g_0 and g' are $\Sigma_{\neg\mathfrak{I}}$-connected, the Σ_G-multiplicativity and the maximal length of $\mathfrak{L}$ give us as in Lemma 3.2 a $\Sigma_{\neg\mathfrak{I}}$-connection $\{g_0, g_2, ..., g_n\} \subset \Sigma_{\neg\mathfrak{I}}$ from g_0 to g' such that $\mathfrak{L}_{g_0 g_2 g_3 \cdots g_n} = \mathfrak{L}_\zeta \subset I$ for some $\zeta \in \{g', (g')^{-1}\}$. Now we also have that $\{g_0^{-1}, g_2^{-1}, ..., g_n^{-1}\}$ is a $\Sigma_{\neg\mathfrak{I}}$-connection from g_0^{-1} to g' but satisfying now $[[\cdots[\mathfrak{L}_{g_0^{-1}}, \mathfrak{L}_{g_2^{-1}}], \cdots], \mathfrak{L}_{g_n^{-1}}] = \mathfrak{L}_{\zeta^{-1}} \subset I$ and so $\mathfrak{L}_{g'} + \mathfrak{L}_{(g')^{-1}} \subset I$ for any $g' \in \Sigma_{\neg\mathfrak{I}}$. Since by Lemma 3.2 we also have $\mathfrak{L}_1 \subset I$ we get $\mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \subset I$. □

Proposition 3.2. *Suppose* $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, $\mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$ *and* $\mathfrak{L}$ *is* Σ_G-*multiplicative. If* $\Sigma_{\neg\mathfrak{I}}$ *and* $\Sigma_{\mathfrak{I}}$ *have all of their elements* $\Sigma_{\neg\mathfrak{I}}$-*connected, then any nonzero graded ideal* I *of* $\mathfrak{L}$ *such that* $I \cap (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \neq \{0\}$ *satisfies* $I = \mathfrak{L}$.

Proof. By Lemma 3.3, $\mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \subset I$. If $I \subset \mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g)$ then $I = \mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g)$ and, by equation (10), we have

$$\mathfrak{L} = I \oplus \mathfrak{I}.$$

Since $[I, \mathfrak{I}] \subset [\mathfrak{L}, \mathfrak{I}] = \{0\}$ and $[\mathfrak{I}, I] \subset \mathfrak{I} \cap I = \{0\}$ we get $\mathfrak{I} \subset \mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$ which implies $\mathfrak{L} = I$.

Consider then the case in which $I \not\subseteq \mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g)$. Then there exists $h_0 \in \Sigma_{\mathfrak{I}}$ such that $\mathfrak{L}_{h_0} \subset I$, that is, $h_0 \in \Sigma_{\mathfrak{I},I}$. Taking into account that $\Sigma_{\mathfrak{I}}$ have all of their elements $\Sigma_{\neg\mathfrak{I}}$-connected, we can argue with the Σ_G-multiplicativity and the maximal length of $\mathfrak{L}$ as in Lemma 3.2 to conclude that given any $h \in \Sigma_{\mathfrak{I}}$, there exists a $\Sigma_{\neg\mathfrak{I}}$-connection $\{h_0, g_2, ..., g_n\}$ from h_0 to h such that

$$[[\cdots[\mathfrak{L}_{h_0}, \mathfrak{L}_{g_2}], \cdots], \mathfrak{L}_{g_n}] \subset \mathfrak{L}_\zeta$$

and so $\mathfrak{L}_\zeta \subset I$ for some $\zeta \in \{h, h^{-1}\}$. That is,

$$\zeta \in \Sigma_{\mathfrak{I},I} \text{ for any } h \in \Sigma_{\mathfrak{I}} \text{ and some } \zeta \in \{h, h^{-1}\}.$$

Let us show that there exists some $h_1 \in \Sigma_{\mathfrak{I},I}$ such that $h_1^{-1} \in \Sigma_{\mathfrak{I},I}$. Indeed, in the opposite case. That is, there is not any $h \in \Sigma_{\mathfrak{I},I}$ such that $h^{-1} \in \Sigma_{\mathfrak{I},I}$, we can write

$$\Sigma_{\mathfrak{I}} = \Sigma_{\mathfrak{I},I} \dot{\cup} (\Sigma_{\mathfrak{I},I})^{-1} \tag{16}$$

where $(\Sigma_{\mathfrak{I},I})^{-1} := \{h^{-1} : h \in \Sigma_{\mathfrak{I},I}\}$ and denote by $K := \bigoplus_{h \in \Sigma_{\mathfrak{I},I}} \mathfrak{L}_{h^{-1}}$, being then

$$\mathfrak{L} = I \oplus K \text{ with } K \neq 0$$

as consequence of equations (10) and (16). Observe that $[\mathfrak{L}_1 \oplus (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g), K] \subset [\mathfrak{L}, \mathfrak{I}] = 0$. We also have $[K, \bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g] = 0$. In fact, if there were $k_0 \in (\Sigma_{\mathfrak{I},I})^{-1}$

and $g_0 \in \Sigma_{\neg\mathfrak{I}}$ such that

$$[\mathfrak{L}_{k_0}, \mathfrak{L}_{g_0}] \neq 0, \tag{17}$$

since $k_0 g_0 \in \Sigma_{\mathfrak{I}}$ we would get by the Σ_G-multiplicativity of $\mathfrak{L}$, the symmetries of $\Sigma_{\neg\mathfrak{I}}$ and $\Sigma_{\mathfrak{I}}$, and the fact $\mathfrak{L}_{k_0^{-1}} \subset I$ that $0 \neq [\mathfrak{L}_{k_0^{-1}}, \mathfrak{L}_{g_0^{-1}}] = \mathfrak{L}_{k_0^{-1} g_0^{-1}} \subset I$. That is, $k_0^{-1} g_0^{-1} \in \Sigma_{\mathfrak{I},I}$. Hence, $k_0 g_0 \in (\Sigma_{\mathfrak{I},I})^{-1}$ and so $[\mathfrak{L}_{k_0}, \mathfrak{L}_{g_0}] \subset K$. But the fact $\mathfrak{L}_{g_0} \subset I$ also gives us $[\mathfrak{L}_{k_0}, \mathfrak{L}_{g_0}] \subset I$ and so $[\mathfrak{L}_{k_0}, \mathfrak{L}_{g_0}] \subset I \cap K = \{0\}$, which contradicts equation (17). Therefore, $[K, \bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g] = 0$. By Leibniz identity and the fact $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}] = \sum_{h \in \Sigma_{\neg\mathfrak{I}}} [\mathfrak{L}_h, \mathfrak{L}_{h^{-1}}]$ we also get $[K, \mathfrak{L}_1] = 0$. That is, $K \subset \mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$, which contradicts the fact $K \neq 0$. So there exists some $h_1 \in \Sigma_{\mathfrak{I},I}$ such that $h_1^{-1} \in \Sigma_{\mathfrak{I},I}$.

We can argue with the above $h_1 \in \Sigma_{\mathfrak{I},I}$ as we did as the beginning of the proof with h_0 to get that for any $h \in \Sigma_{\mathfrak{I}}$ there exists a $\Sigma_{\neg\mathfrak{I}}$-connection $\{h_1, g_2, ..., g_n\}$ from h_1 to h such that $[[\cdots[\mathfrak{L}_{h_1}, \mathfrak{L}_{g_2}], \cdots], \mathfrak{L}_{g_n}] \subset \mathfrak{L}_\zeta \subset I$ for some $\zeta \in \{h, h^{-1}\}$. Taking into account $h_1^{-1} \in \Sigma_{\mathfrak{I},I}$, we also have that

$$\{h_1^{-1}, g_2^{-1}, ..., g_n^{-1}\}$$

is a $\Sigma_{\neg\mathfrak{I}}$-connection from h_1^{-1} to h satisfying

$$[[\cdots[\mathfrak{L}_{h_1^{-1}}, \mathfrak{L}_{g_2^{-1}}], \cdots], \mathfrak{L}_{g_n^{-1}}] = \mathfrak{L}_\zeta^{-1} \subset I$$

and so $\mathfrak{L}_h + \mathfrak{L}_{h^{-1}} \subset I$ for any $h \in \mathfrak{I}$. From here $\mathfrak{I} \subset I$ and so $I = \mathfrak{L}$. □

Proposition 3.3. *Suppose* $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}], \mathrm{Z}(\mathfrak{L}) = 0$ *and* $\mathfrak{L}$ *is* Σ_G*-multiplicative. If* $\Sigma_{\mathfrak{I}}$ *has all of its elements* $\Sigma_{\neg\mathfrak{I}}$*-connected, then any nonzero graded ideal* I *of* $\mathfrak{L}$ *such that* $I \subset \mathfrak{I}$ *satisfies either* $I = \mathfrak{I}$ *or* $\mathfrak{I} = I \oplus P$ *with* P *a graded ideal of* $\mathfrak{L}$.

Proof. By equations (7), and (9) we can write

$$I = (I \cap \mathfrak{L}_1) \oplus (\bigoplus_{h_j \in \Sigma_{\mathfrak{I},I}} \mathfrak{L}_{h_j})$$

with $\Sigma_{\mathfrak{I},I} = \Sigma_{\mathfrak{I}} \cap \Sigma_I$. Observe that the fact $\mathrm{Z}(\mathfrak{L}) = 0$ implies

$$\mathfrak{I} \cap \mathfrak{L}_1 = \{0\}. \tag{18}$$

Indeed, $[\mathfrak{L}_1, \mathfrak{I} \cap \mathfrak{L}_1] + [\mathfrak{L}_g, \mathfrak{I} \cap \mathfrak{L}_1] + [\mathfrak{I} \cap \mathfrak{L}_1, \mathfrak{I}] \subset [\mathfrak{L}, \mathfrak{I}] = 0$ for any $g \in \Sigma_G$, and if $[\mathfrak{I} \cap \mathfrak{L}_1, \mathfrak{L}_g] = 0$ for some $g \in \Sigma_{\neg\mathfrak{I}}$ then $\mathfrak{L}_g \subset \mathfrak{I}$, being then $g \in \Sigma_{\mathfrak{I}}$, a contradiction. So $[\mathfrak{I} \cap \mathfrak{L}_1, \mathfrak{L}_g] = 0$ for any $g \in \Sigma_{\neg\mathfrak{I}}$. From here, we also have $[\mathfrak{I} \cap \mathfrak{L}_1, \mathfrak{L}_1] = 0$ as consequence of Leibniz identity and the fact $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}] = \sum_{h \in \Sigma_{\neg\mathfrak{I}}} [\mathfrak{L}_h, \mathfrak{L}_{h^{-1}}]$. We have showed $\mathfrak{I} \cap \mathfrak{L}_1 \subset \mathrm{Z}(\mathfrak{L}) = 0$. Hence, we can write

$$I = \bigoplus_{h_j \in \Sigma_{\mathfrak{I},I}} \mathfrak{L}_{h_j},$$

with $\Sigma_{\mathfrak{I},I} \neq \emptyset$, and so we can take some $h_0 \in \Sigma_{\mathfrak{I},I}$ such that $\mathfrak{L}_{h_0} \subset I$. We can argue with the Σ_G-multiplicativity and the maximal length of $\mathfrak{L}$ as in Proposition 3.2 to conclude that given any $h \in \Sigma_{\mathfrak{I}}$, there exists a $\Sigma_{\neg\mathfrak{I}}$-connection $\{h_0, g_2, ..., g_n\}$ from h_0 to h such that

$$0 \neq [[\cdots[\mathfrak{L}_{h_0}, \mathfrak{L}_{g_2}], \cdots], \mathfrak{L}_{g_n}] = \mathfrak{L}_\zeta \subset I$$

for some $\zeta \in \{h, h^{-1}\}$. That is

$$\zeta \in \Sigma_{\mathfrak{I},I} \text{ for any } h \in \Sigma_{\mathfrak{I}} \text{ and some } \zeta \in \{h, h^{-1}\}.$$

Suppose $h_0^{-1} \in \Sigma_{\mathfrak{I},I}$. Then we also have that $\{h_0^{-1}, g_2^{-1}, ..., g_n^{-1}\}$ is a $\Sigma_{\neg\mathfrak{I}}$-connection from h_0^{-1} to h satisfying

$$0 \neq [[\cdots[\mathfrak{L}_{h_0^{-1}}, \mathfrak{L}_{g_2^{-1}}], \cdots], \mathfrak{L}_{g_n^{-1}}] = \mathfrak{L}_{\zeta^{-1}} \subset I$$

and so $\mathfrak{L}_h + \mathfrak{L}_{h^{-1}} \subset I$ for any $h \in \Sigma_{\mathfrak{I}}$. Equations (8) and (18) let us now conclude $I = \mathfrak{I}$.

Now suppose there is not any $h_0 \in \Sigma_{\mathfrak{I},I}$ such that $h_0^{-1} \in \Sigma_{\mathfrak{I},I}$. Then we can write $\Sigma_{\mathfrak{I}} = \Sigma_{\mathfrak{I},I} \dot\cup (\Sigma_{\mathfrak{I},I})^{-1}$ where $(\Sigma_{\mathfrak{I},I})^{-1} = \{h^{-1} \in \Sigma_{\mathfrak{I}} : h \in \Sigma_{\mathfrak{I},I}\}$ and, (joint with equations (8) and (18)), assert that by denoting $P := \bigoplus_{h \in \Sigma_{\mathfrak{I},I}} \mathfrak{L}_{h^{-1}}$ we have

$$\mathfrak{I} = I \oplus P.$$

Let us finally show that P is an ideal of $\mathfrak{L}$. We have $[\mathfrak{L}, P] \subset [\mathfrak{L}, \mathfrak{I}] = 0$ and

$$[P, \mathfrak{L}] \subset [P, \mathfrak{L}_1] + [P, \bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g] + [P, \bigoplus_{h \in \Sigma_{\mathfrak{I}}} \mathfrak{L}_h] \subset P + [P, \bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g].$$

Let us consider the last summand $[P, \bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g]$ and suppose there exist $h_0 \in \Sigma_{\mathfrak{I},I}$ and $g_0 \in \Sigma_{\neg\mathfrak{I}}$ such that $[\mathfrak{L}_{h_0^{-1}}, \mathfrak{L}_{g_0}] \neq 0$. Since $\mathfrak{L}_{h_0^{-1}} \subset P \subset \mathfrak{I}$, we get $h_0^{-1} g_0 \in \Sigma_{\mathfrak{I}}$. By the Σ_G-multiplicativity of $\mathfrak{L}$, the symmetries of $\Sigma_{\neg\mathfrak{I}}$ and $\Sigma_{\mathfrak{I}}$, and the fact $\mathfrak{L}_{h_0} \subset I$ we obtain $0 \neq [\mathfrak{L}_{h_0}, \mathfrak{L}_{g_0^{-1}}] = \mathfrak{L}_{h_0 g_0^{-1}} \subset I$, that is $h_0 g_0^{-1} \in \Sigma_{\mathfrak{I},I}$. Hence, $h_0^{-1} g_0 \in (\Sigma_{\mathfrak{I},I})^{-1}$ and so $[\mathfrak{L}_{h_0^{-1}}, \mathfrak{L}_{g_0}] \subset P$. Consequently $[P, \bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g] \subset P$ and P is a graded ideal of $\mathfrak{L}$. □

We introduce the definition of gr-primeness in the framework of graded Leibniz algebras following the same motivation that in the case of gr-simplicity (see Definition 1.3).

Definition 3.5. *A graded Leibniz algebra $\mathfrak{L}$ is said to be* gr-prime *if given two graded ideals I, P of $\mathfrak{L}$ satisfying $[I, P] + [P, I] = 0$, then either $I \in \{0, \mathfrak{I}\}$ or $P \in \{0, \mathfrak{I}\}$.*

As an example of a gr-prime graded Leibniz algebra we have the graded Leibniz algebra given in Example 1. We also note that the above definition agrees with the definition of gr-prime Lie algebra, since $\mathfrak{I} = \{0\}$ in this case.

Under the hypotheses of Proposition 3.3 we have:

Corollary 3.1. *If furthermore $\mathfrak{L}$ is gr-prime, then any nonzero graded ideal I of $\mathfrak{L}$ such that $I \subset \mathfrak{I}$ satisfies $I = \mathfrak{I}$.*

Proof. Observe that, by Proposition 3.3, we could have $\mathfrak{I} = I \oplus P$ with I, P graded ideals of $\mathfrak{L}$, being $[I, P] + [P, I] = 0$ as consequence of $I, P \subset \mathfrak{I}$. The gr-primeness of $\mathfrak{L}$ completes the proof. $\square$

Consider $\mathfrak{L}$ with $\mathrm{Z}(\mathfrak{L}) = 0$ and $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, (see Proposition 3.1). Given any $g \in \Sigma_\Upsilon, \Upsilon \in \{\mathfrak{I}, \neg\mathfrak{I}\}$ we denote by

$$\mathfrak{C}_g^\Upsilon := \{g' \in \Sigma_\Upsilon : g' \sim_{\neg\mathfrak{I}} g\}.$$

If $g \in \Sigma_\Upsilon$, let us write $\mathfrak{L}_{1,\mathfrak{C}_g^\Upsilon} := span_{\mathbb{K}}\{[\mathfrak{L}_{g'}, \mathfrak{L}_{(g')^{-1}}] : g' \in \mathfrak{C}_g^\Upsilon\} \subset \mathfrak{L}_1$, and $V_{\mathfrak{C}_g^\Upsilon} := \bigoplus_{g' \in \mathfrak{C}_g^\Upsilon} \mathfrak{L}_{g'}$. We also denote by $\mathfrak{L}_{\mathfrak{C}_g^\Upsilon} := \mathfrak{L}_{1,\mathfrak{C}_g^\Upsilon} \oplus V_{\mathfrak{C}_g^\Upsilon}$.

Lemma 3.4. *If $\mathrm{Z}(\mathfrak{L}) = 0$ and $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}]$, then $\mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}$ is a graded ideal of $\mathfrak{L}$ for any $h \in \Sigma_\mathfrak{I}$.*

Proof. From $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}] = \sum_{h \in \Sigma_{\neg\mathfrak{I}}} [\mathfrak{L}_h, \mathfrak{L}_{h^{-1}}]$ we get $\mathfrak{L}_{1,\mathfrak{C}_h^\mathfrak{I}} = 0$ and so

$$\mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}} = V_{\mathfrak{C}_h^\mathfrak{I}} = \bigoplus_{h' \in \mathfrak{C}_h^\mathfrak{I}} \mathfrak{L}_{h'}.$$

We have $[\mathfrak{L}, \mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}] + [\mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}, \mathfrak{I}] \subset [\mathfrak{L}, \mathfrak{I}] = 0$ and $[\mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}, \mathfrak{L}_1] \subset \mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}$. Finally $[\mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}, \mathfrak{L}_{g'}] \subset \mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}$ for any $g' \in \mathfrak{C}_g^{\neg\mathfrak{I}}$. Indeed, given any $h' \in \mathfrak{C}_h^\mathfrak{I}$ such that $[\mathfrak{L}_{h'}, \mathfrak{L}_{g'}] \neq 0$ we have $h'g' \in \Sigma_\mathfrak{I}$ and so $\{h', g'\}$ is a $\Sigma_{\neg\mathfrak{I}}$-connection from h' to $h'g'$. By the symmetry and transitivity of $\sim_{\neg\mathfrak{I}}$ in $\Sigma_\mathfrak{I}$ we get $h'g' \in \mathfrak{C}_h^\mathfrak{I}$. Hence $[\mathfrak{L}_{h'}, \mathfrak{L}_{g'}] \subset \mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}$. Taking into account equation (10) we conclude $\mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}$ is a graded ideal of $\mathfrak{L}$. $\square$

Theorem 3.1. *Suppose $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}], \mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$ and $\mathfrak{L}$ is Σ_G-multiplicative. Then, $\mathfrak{L}$ is gr-simple if and only if it is gr-prime and $\Sigma_\mathfrak{I}$, $\Sigma_{\neg\mathfrak{I}}$ have all of their elements $\Sigma_{\neg\mathfrak{I}}$-connected.*

Proof. Suppose $\mathfrak{L}$ gr-simple. If $\Sigma_\mathfrak{I} \neq \emptyset$ and we take $h \in \Sigma_\mathfrak{I}$, Lemma 3.4 gives us $\mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}}$ is a nonzero graded ideal of $\mathfrak{L}$ and so, (by gr-simplicity), $\mathfrak{L}_{\mathfrak{C}_h^\mathfrak{I}} = \mathfrak{I} = \bigoplus_{h_j \in \Sigma_\mathfrak{I}} \mathfrak{L}_{h_j}$ (see equations (8) and (18)). Hence, $\mathfrak{C}_h^\mathfrak{I} = \Sigma_\mathfrak{I}$ and consequently

$$\Sigma_\mathfrak{I} \text{ has all of its elements } \Sigma_{\neg\mathfrak{I}}\text{-connected.}$$

Consider now any $g \in \Sigma_{\neg\mathfrak{I}}$ and the subspace $\mathfrak{L}_{\mathfrak{C}_g^{\neg\mathfrak{I}}}$. Let us denote by $I(\mathfrak{L}_{\mathfrak{C}_g^{\neg\mathfrak{I}}})$ the (graded) ideal of $\mathfrak{L}$ generated by $\mathfrak{L}_{\mathfrak{C}_g^{\neg\mathfrak{I}}}$. By gr-simplicity $I(\mathfrak{L}_{\mathfrak{C}_g^{\neg\mathfrak{I}}}) = \mathfrak{L}$.

Observe that the fact that $\mathfrak{I}$ is an ideal of $\mathfrak{L}$ let us assert that $I(\mathfrak{L}_{\mathfrak{C}_g^{\neg\mathfrak{I}}}) \cap (\bigoplus_{g' \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_{g'})$ is contained in the linear span of the set

$$\{[[\cdots[v_h, v_{g_1}], \cdots], v_{g_n}];\ [v_{g_n}, [\cdots[v_{g_1}, v_h], \cdots]];$$

$$[[\cdots[v_{g_1}, v_h], \cdots], v_{g_n}];\ [v_{g_n}, [\cdots[v_h, v_{g_1}], \cdots]] \text{ with } 0 \neq v_h \in \mathfrak{L}_{\mathfrak{C}_g^{\neg\mathfrak{I}}},$$

$$0 \neq v_{g_i} \in \mathfrak{L}_{g_i}, g_i \in \mathfrak{C}_g^{\neg\mathfrak{I}} \text{ and } n \in \mathbb{N}\ \}.$$

From here, given any $\tilde{g} \in \Sigma_{\neg\mathfrak{I}}$, the above observation and Leibniz identity give us we can write $\tilde{g} = hg_1 \cdots g_n$ with $h \in \mathfrak{C}_g^{\neg\mathfrak{I}}$, any $g_i \in \Sigma_{\neg\mathfrak{I}}$ and being the partial products nonzero. Hence $\{h, g_1, ..., g_n\}$ is a $\Sigma_{\neg\mathfrak{I}}$-connection from h to $\tilde{g}$. By the transitivity of $\sim_{\neg\mathfrak{I}}$ in $\Sigma_{\neg\mathfrak{I}}$ we deduce that g is $\Sigma_{\neg\mathfrak{I}}$-connected to any $\tilde{g} \in \Sigma_{\neg\mathfrak{I}}$. Consequently $\mathfrak{C}_g^{\neg\mathfrak{I}} = \Sigma_{\neg\mathfrak{I}}$ and we can assert that

$$\Sigma_{\neg\mathfrak{I}} \text{ has all of its elements } \Sigma_{\neg\mathfrak{I}}\text{-connected.}$$

Finally, since $\mathfrak{L}$ is gr-simple then is gr-prime.

Let us see the converse. Consider I a nonzero graded ideal of $\mathfrak{L}$ and let us show that necessarily either $I = \mathfrak{I}$ or $I = \mathfrak{L}$. If $I \subset \mathfrak{I}$, Corollary 3.1 gives us $I = \mathfrak{I}$. If $I \cap (\bigoplus_{g \in \Sigma_{\neg\mathfrak{I}}} \mathfrak{L}_g) \neq \{0\}$, Proposition 3.2 implies now $I = \mathfrak{L}$. Hence, we have just to study the case in which $I = (I \cap \mathfrak{L}_1) \oplus (\bigoplus_{h_j \in \Sigma_{\mathfrak{I},I}} \mathfrak{L}_{h_j})$, with $I \cap \mathfrak{L}_1 \neq 0$. But this possibility never happens. Indeed, if there was such an ideal, we would have $[I \cap \mathfrak{L}_1, \mathfrak{L}_g] + [\mathfrak{L}_g, I \cap \mathfrak{L}_1] \subset I \cap \mathfrak{L}_g = 0$ for any $g \in \Sigma_{\neg\mathfrak{I}}$. From here, taking also into account $\mathfrak{L}_1 = \sum_{g \in \Sigma_G} [\mathfrak{L}_g, \mathfrak{L}_{g^{-1}}] = \sum_{h \in \Sigma_{\neg\mathfrak{I}}} [\mathfrak{L}_h, \mathfrak{L}_{h^{-1}}]$, Leibniz identity would imply $[\mathfrak{L}_1, I \cap \mathfrak{L}_1] + [I \cap \mathfrak{L}_1, \mathfrak{L}_1] = 0$ and so $I \cap \mathfrak{L}_1 \subset \mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$, a contradiction. We conclude either $I = \mathfrak{I}$ or $I = \mathfrak{L}$ and $\mathfrak{L}$ is gr-simple. □

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