A NOTE ON INVARIANT FACTORS

CRISTINA CALDEIRA AND JOÃO FILIPE QUEIRÓ

Dedicated to Manuela Sobral

ABSTRACT. This is a short survey about properties of invariant factors of matrices over principal ideal domains and the possibility of extending those properties to matrices over more general rings.

1. INTRODUCTION

Let R be a principal ideal domain (PID) and A an $n \times n$ nonsingular matrix over R. It is well known that A is equivalent to its *Smith normal form*, that is, there exist U and V invertible over R such that

$$UAV = \begin{bmatrix} a_n & 0 & \cdots & 0 \\ 0 & a_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 \end{bmatrix},$$

where $a_n \mid a_{n-1} \mid \cdots \mid a_1$ are the *invariant factors* of A.

The invariant factors are uniquely determined by A, as follows from the characterization

$$a_{n-k+1} = \frac{d_k(A)}{d_{k-1}(A)}\ , \quad k = 1, \ldots, n,$$

where, for each k, $d_k(A)$ is the gcd of all $k \times k$ minors of A, $d_0 \equiv 1$. By the Cauchy-Binet theorem for determinants, the d_k are invariant under equivalence. That $d_{k-1}(A)$ divides $d_k(A)$ follows from Laplace's theorem.

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There is a large body of literature concerning how invariant factors behave when adding or multiplying matrices, taking submatrices, etc. In the present note we are concerned with the problem of extending results — specially concerning products — to matrices over more general rings. For simplicity, we shall stay within the class of integral domains. Also, invariant factors may of course be defined for singular or rectangular matrices, and most of our remarks remain valid in those more general situations, but we'll work with nonsingular square matrices only.

2. Invariant factors of products

The most important problem about invariant factors concerns their behaviour under matrix multiplication. Specifically, one asks: What are the possible invariant factors $c_n \mid \cdots \mid c_1$ of a product AB, if A and B are $n \times n$ nonsingular matrices over R with invariant factors $a_n \mid \cdots \mid a_1$ and $b_n \mid \cdots \mid b_1$, respectively?

Examples of valid relations:

$$c_6 \,|\, a_2\, b_5 \,,$$

$$c_2\, c_4 \,|\, a_1\, a_4\, b_1\, b_3 \,,$$

$$c_3\, c_5\, c_9 \,|\, a_2\, a_3\, a_7\, b_2\, b_4\, b_5 \,.$$

This problem has been solved with a variety of approaches, starting with its p-module version in [6], where p is a prime in R. Indeed, all approaches start by "localizing" the problem at an arbitrary prime p, working in that context, and then recovering the global solution.

The solution in [6] immediately suggested a connection with the representation theory of $GL_n(\mathbb{C})$, namely the well-known question of finding which irreducible representations of $GL_n(\mathbb{C})$ occur in the tensor product of two given such irreducible representations.

To state and relate these problems we need some notation.

For each fixed prime $p \in R$, we restrict our attention to matrices over the local ring R_p, that is, we just work with powers of p:

$$a_i \to p^{\alpha_i} \;\;,\;\; b_i \to p^{\beta_i} \;\;,\;\; c_i \to p^{\gamma_i}$$

where $\alpha_1 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \cdots \geq \beta_n$, $\gamma_1 \geq \cdots \geq \gamma_n$ are nonnegative integers.

Denote by $I(\alpha, \beta)$ the set of possible γ in the invariant factor product problem. Then from the above we have the following examples of valid relations for the exponents $\gamma \in I(\alpha, \beta)$:

$$\gamma_6 \leq \alpha_2 + \beta_5 \,,$$

$$\gamma_2 + \gamma_4 \leq \alpha_1 + \alpha_4 + \beta_1 + \beta_3\,,$$
$$\gamma_3 + \gamma_5 + \gamma_9 \leq \alpha_2 + \alpha_3 + \alpha_7 + \beta_2 + \beta_4 + \beta_5\,.$$

Introduce the notation $\Lambda_n = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n : \alpha_1 \geq \cdots \geq \alpha_n \geq 0\}$. What T. Klein proved in [6] was that $I(\alpha, \beta) = LR(\alpha, \beta)$, where the latter is the set of $\gamma \in \Lambda_n$ which can be obtained from α and β using the combinatorial Littlewood-Richardson rule. This is what establishes the connection to representation theory. The irreducible representations of $GL_n(\mathbb{C})$ are the *Weyl modules* V_λ indexed by $\lambda \in \Lambda_n$. Then it is classical that in the decomposition of the tensor product of two irreducible representations, $V_\alpha \otimes V_\beta \cong \bigoplus_\gamma N_{\alpha\beta\gamma} V_\gamma$, the coefficient $N_{\alpha\beta\gamma}$ is at least 1 (*i.e.* V_γ occurs in the decomposition) if and only if $\gamma \in LR(\alpha, \beta)$, whence we have the chain of equivalences $\gamma \in I(\alpha, \beta) \Leftrightarrow \gamma \in LR(\alpha, \beta) \Leftrightarrow V_\gamma$ occurs in $V_\alpha \otimes V_\beta$.

Thus the invariant factor product problem, in its local “primary” version, has a complete and interesting solution, although not a clearly explicit one, via the Littlewood-Richardson rule. In particular, this solution is *not* given as a family of divisibility relations of the kind exemplified above.

3. Inequalities and divisibility relations

At the end of the 1990s, a spectacular connection to another, completely different, matrix problem, which had long been suspected, was established. Starting from a 1912 paper by H. Weyl [12], a lot of attention had been given to finding relations between the eigenvalues of two Hermitian matrices and those of their sum. Let $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_n$ be arbitrary real numbers and put $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$. The question is: What are the possible spectra $\gamma_1 \geq \cdots \geq \gamma_n$ of a sum $A + B$, where A and B are $n \times n$ Hermitian matrices with spectra α and β, respectively?

Denote by $E(\alpha, \beta)$ the set of possible $\gamma = (\gamma_1, \ldots, \gamma_n)$. We wish to describe $E(\alpha, \beta)$. Since it is the set of spectra of matrices of the form $D_\alpha + UD_\beta U^*$, where $D_\alpha = \operatorname{diag}(\alpha)$, $D_\beta = \operatorname{diag}(\beta)$ and U runs over the unitary group, we know that $E(\alpha, \beta)$ is compact and connected, as the map sending a Hermitian matrix to its spectrum is continuous.

The Hermitian sum problem attracted the attention of many mathematicians throughout the 20th century. The most important contribution came from A. Horn in [3], where a deep analysis was carried out and an important conjecture was proposed, according to which the full solution should come as a complicated collection, recursively defined, of linear inequalities relating the three n-tuples.

This is not the place to tell the full story of what happened (we refer the interested reader to [1], [2] and [11]; for later developments see [4]). The main protagonist was A. Klyachko, who in [7] proved, among other important results on $E(\alpha, \beta)$, that the set is indeed described by a set of linear inequalities of the type mentioned by Horn. These inequalities are precisely those obtained by looking at sums of scattered eigenvalues of a Hermitian matrix as extrema of Rayleigh traces of the matrix on Schubert varieties, and then considering conditions for the intersection of those varieties. These conditions are, in turn, describable in terms of the Littlewood-Richardson rule (the so-called *Schubert calculus*). Additional combinatorial work by Knutson and Tao [8] finally yielded Horn's conjecture.

A fascinating by-product of this work is that, for $\alpha, \beta \in \Lambda_n$, the $\gamma \in \Lambda_n$ occurring in $E(\alpha, \beta)$ are exactly the n-tuples in $LR(\alpha, \beta)$. In other words, we have $I(\alpha, \beta) = E(\alpha, \beta) \cap \mathbb{Z}^n$. From this we get a description of $I(\alpha, \beta)$ by a system of linear inequalities, recursively defined from lower dimensions due to the form of Horn's conjecture. Therefore, the original invariant factor problem for products of matrices has a solution in terms of (a long family of) divisibility relations.

4. Elementary divisor domains

The above results on invariant factors and their known proofs all depend crucially on the p-localization argument.

Now, invariant factors may be defined for matrices over more general rings. The more natural rings in this context are the *elementary divisor domains* (EDDs) introduced by Kaplansky in [5]. These are the integral domains R where every matrix over R is equivalent to a Smith normal form exactly as above. This is a strictly larger class of rings than PIDs, and arguments using reduction to the primary case do not work here, as EDDs are not in general unique factorization domains.

So a question naturally arises: what can we say about invariant factors of matrices over EDDs? Of course, results established using only the Smith normal form, without reduction to the primary case, immediately carry over to EDDs. An example is the general relation

$$a_i b_j \mid c_{i+j-n} ,$$

valid for invariant factors of products with notations as above. This was proved in [10] using just the Smith normal form, although stated for matrices over PIDs.

Another example concerns invariant factors of *sums*, rather than products. If $c_n \mid \cdots \mid c_1$ are now the invariant factors of $A + B$, one has that

$$\gcd\{a_i, b_j\} \mid c_{i+j-n} .$$

This was also proved in [10] in a form valid for EDDs.

But what about the huge family of divisibility relations, mentioned in the previous section, valid for invariant factors of products of matrices over PIDs (and which actually give the complete answer to the product problem in that setting)? Extending those results to EDDs presents an interesting challenge, necessitating a change in the proofs.

And it would bring an added bonus, since, by a localization argument due to Krull [9], any divisibility relation generally valid in EDDs actually generalizes to GCD domains, rings where every finite collection of elements has a gcd inside the ring. This technique was essentially already used by Kaplansky in [5].

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(C. Caldeira and J. F. Queiró) CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal

E-mail address: caldeira@mat.uc.pt; jfqueiro@mat.uc.pt