Determinantal Inequalities for J-Accretive Dissipative Matrices

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Abstract. In this note we determine bounds for the determinant of the sum of two J-accretive dissipative matrices with prescribed spectra.

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1 Results

Consider the complex n-dimensional space $\mathbf{C}^n$ endowed with the indefinite inner product

$$[x,y]_J = y^* J x, \quad x,y \in \mathbf{C}^n,$$

where $J = I_r \oplus -I_{n-r}$, and corresponding J-norm $[x,x]_J = |x_1|^2 + \ldots + |x_r|^2 - |x_{r+1}|^2 - \ldots - |x_n|^2$. In the sequel we shall assume that $0 < r < n$, except where otherwise stated. The *J-adjoint* of $A \in \mathbf{C}^{n\times n}$ is defined and denoted as

$$[A^{\#}x, x] = [x, Ax]$$

or, equivalently, $A^{\#} := JA^*J$. The matrix A is said to be *J-Hermitian* if $A^{\#} = A$, and is *J-positive definite* (*semi-definite*) if JA is positive definite (semi-definite). This kind of matrices appears on Quantum Physics and in Symplectic Geometry [10]. An arbitrary matrix $A \in \mathbf{C}^{n\times n}$ may be uniquely written in the form

$$A = \mathrm{Re}^J A + i\mathrm{Im}^J A,$$

where

$$\mathrm{Re}^J A = (A + A^{\#})/2, \ \mathrm{Im}^J A = (A - A^{\#})/(2i)$$

are J-Hermitian. This is the so-called J-Cartesian decomposition of A. J-Hermitian matrices share properties with Hermitian matrices, but they also have important differences. For instance, they have real and complex eigenvalues, these occurring in conjugate pairs. Nevertheless, the eigenvalues of a J-positive matrix are all real, being r positive and $n-r$ negative, according to the J-norm of the associated eigenvectors being positive or negative. A matrix A is said to be *J-accretive* (resp. *J-dissipative*) if $J\mathrm{Re}^J A$ (resp. $J\mathrm{Im}^J A$) is positive definite. If both matrices $J\mathrm{Re}^J A$ and $J\mathrm{Im}^J A$ are positive definite the matrix is said to be *J-accretive dissipative.* We are interested in obtaining determinantal inequalities for J-accretive dissipative matrices.

Throughout, we shall be concerned with the set

$$D^J(A,C) = \{\det(A + VCV^{\#}) : V \in \mathcal{U}(r, n-r)\},$$

where $A, C \in \mathbf{C}^{n\times n}$ are J-unitarily diagonalizable with prescribed eigenvalues and $\mathcal{U}(r,n-r)$ is the group of J-unitary transformations in $\mathbf{C}^n$ (V is *J-unitary* if $VV^{\#} = I$). The so-called J-unitary group is connected, nevertheless it is not compact. As a consequence, $D^J(A,C)$ is connected. This set is invariant under the transformation $C \to UCU^{\#}$ for every J-unitary matrix U, and, for short, $D^J(A,C)$ is said to be J-unitarily invariant.

In the sequel we use the following notation. By S_n we denote the symmetric group of degree n, and we shall also consider

$$S_n^r = \{\sigma \in S_n : \sigma(j) = j, \ j = r+1, \ldots, n\}, \tag{1}$$

$$\hat{S}_n^r = \{\sigma \in S_n : \sigma(j) = j,\ j = 1, \dots, r\}. \tag{2}$$

Let $\alpha_j,\ \gamma_j \in \mathbf{C}$, $j = 1, \dots, n$ denote the eigenvalues of A and C, respectively. The $r!(n-r)!$ points

$$z_\sigma = z_{\xi\tau} = \prod_{j=1}^{r}(\alpha_j + \gamma_{\xi(j)}) \prod_{j=r+1}^{n} (\alpha_j + \gamma_{\tau(j)}),\ \xi \in S_n^r,\ \tau \in \hat{S}_n^r. \tag{3}$$

belong to $D^J(A, C)$.

The purpose of this note, which is in the continuation of [1], is to establish the following results.

Theorem 1.1 *Let $J = I_r \oplus -I_{n-r}$, and A and C be J- positive matrices with prescribed real eigenvalues*

$$\alpha_1 \geq \dots \geq \alpha_r > 0 > \alpha_{r+1} \geq \dots \geq \alpha_n \tag{4}$$

and

$$\gamma_1 \geq \dots \geq \gamma_r > 0 > \gamma_{r+1} \geq \dots \geq \gamma_n, \tag{5}$$

respectively. Then

$$|\det(A + iC)| \geq \left((\alpha_1^2 + \gamma_1^2) \dots (\alpha_n^2 + \gamma_n^2)\right)^{1/2}.$$

Corollary 1.1 *Let $J = I_r \oplus -I_{n-r}$, and B be a J-accretive dissipative matrix. Assume that the eigenvalues of $\mathrm{Re}^J B$ and $\mathrm{Im}^J B$ satisfy (4) and (5), respectively. Then,*

$$|\det(B)| \geq \left((\alpha_1^2 + \gamma_1^2) \dots (\alpha_n^2 + \gamma_n^2)\right)^{1/2}.$$

Example 1.1 *In order to illustrate the necessity of A and C to be J-positive matrices in Theorem 1.1, let $A = \mathrm{diag}(\alpha_1, \alpha_2)$, $C = \mathrm{diag}(\gamma_1, \gamma_2)$, with $\alpha_1 = \gamma_1 = 1,\ \alpha_2 = 3/2,\ \gamma_2 = -2$, and $J = \mathrm{diag}(1, -1)$. We find $(\alpha_1^2 + \gamma_1^2)(\alpha_2^2 + \gamma_2^2) = 27/2$. However, the minimum of $|\det(A + iVBV^{\#}|^2$, for V ranging over the J-unitary group , is $49/4$.*

Theorem 1.2 *Let $J = I_r \oplus -I_{n-r}$, and A and C be J-unitary matrices with prescribed eigenvalues*

$$\alpha_1, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_n$$

and

$$\gamma_1 \dots, \gamma_r, \gamma_{r+1}, \dots . \gamma_n,$$

respectively. Assume moreover that

$$\frac{\Im\alpha_1}{2(1 + \Re\alpha_1)} \leq \dots \leq \frac{\Im\alpha_r}{2(1 + \Re\alpha_r)} < 0 < \frac{\Im\alpha_{r+1}}{2(1 + \Re\alpha_{r+1})} \leq \dots \leq \frac{\Im\alpha_n}{2(1 + \Re\alpha_n)} \tag{6}$$

and

$$\frac{\Im\gamma_1}{2(1 - \Re\gamma_1)} \leq \dots \leq \frac{\Im\gamma_r}{2(1 - \Re\gamma_r)} < 0 < \frac{\Im\gamma_{r+1}}{2(1 - \Re\gamma_{r+1})} \leq \dots \leq \frac{\Im\gamma_n}{2(1 - \Re\gamma_n)}. \tag{7}$$

Then

$$D^J(A, C) = (\alpha_1 + \gamma_1) \dots (\alpha_n + \gamma_n)[1, +\infty[\ .$$

We shall present the proofs of the above results in the next section.

2 Proofs

Lemma 2.1 *Let $g : \mathcal{U}(r, n-r) \to \mathbf{R}$ be the real valued function defined by*

$$g(U) = \det(I + A_0^{-1}UC_0JU^*JA_0^{-1}UC_0JU^*J),$$

where $A_0 = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$, $C_0 = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$ and α_i, γ_j satisfy (4) and (5). Then the set

$$\{U \in \mathcal{U}(r, n-r) : g(U) \leq a\},$$

where

$$a > \prod_{j=1}^{n} \left(1 + \frac{\gamma_j^2}{\alpha_j^2}\right),$$

is compact.

Proof. Notice that $JA_0 > 0, \; JC_0 > 0$, so we may write

$$g(U) = \det(I + WW^*WW^*),$$

where

$$W = (JA_0)^{-1/2}U(JC_0)^{1/2}.$$

The condition $g(U) \leq a$ implies that W is bounded, and is satisfied if we require that $WW^* \leq \kappa I$, for $\kappa > 0$ such that $(1+\kappa^2)^n \leq a$. Thus, also U is bounded. The result follows by Heine-Borel Theorem. ■

Proof of Theorem 1.1

Under the hypothesis, A is nonsingular. Since the determinant is J-unitarily invariant and C is J-unitarily diagonalizable, we may consider $C = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$. We observe that

$$|\det(A + iC)|^2 = \det\left((A + iC)(A - iC)\right) = \left(\prod_{i=1}^{n} \alpha_i\right)^2 \det\left((I + iA^{-1}C)(I - iA^{-1}C)\right)$$

Clearly,

$$\det\left((I + iA^{-1}C)(I - iA^{-1}C)\right) = \det(I + A^{-1}CA^{-1}C).$$

The set of values attained by $|\det(A+iC)|^2$ is an unbounded connected subset of the positive real line. In order to prove the unboundedness, let us consider the J-unitary matrix V obtained from the identity matrix I through the replacement of the entries $(r, r), \; (r+1, r+1)$ by $\cosh\; u$, and the replacement of the entries $(r, r+1), \; (r+1, r)$ by $\sinh\; u$, $u \in R$. We may assume that $A_0 = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$. A simple computation shows that

$$|\det(A_0 + iVCV^{\#})|^2 = \prod_{j=1}^{n} (\alpha_j^2 + \gamma_j^2)$$
$$-2(\alpha_r - \alpha_{r+1})(\gamma_r - \gamma_{r+1})(\alpha_{r+1}\gamma_r + \alpha_r\gamma_{r+1})(\sinh\; u)^2 + (\alpha_r - \alpha_{r+1})^2(\gamma_r - \gamma_{r+1})^2(\sinh\; u)^4.$$

Thus, the set of values attained by $|\det(A_0 + iVCV^{\#})|$ is given by

$$[(\alpha_1^2 + \gamma_1^2)^{1/2} \dots (\alpha_n^2 + \gamma_n^2)^{1/2}, +\infty[\ .$$

As a consequence of Lemma 2.1, the set of values attained by $|\det(A + iC)|^2$ is closed and a half-ray in the positive real line. So, there exist matrices A, C such that the endpoint of the half-ray is given by $|\det(A+iC)|^2$. Let us assume that the endpoint of this half-ray is attained at $|\det(A+iC)|^2$. We prove that A commutes with C. Indeed, for $\epsilon \in \mathbf{R}$ and an arbitrary J-Hermitian X, let us consider the J-unitary matrix given as

$$\mathrm{e}^{iX} = i + i\epsilon X - \frac{\epsilon^2}{2}X^2 + \dots .$$

We obtain by some computations

$$\begin{aligned} &f(\epsilon) := \det(I + A^{-1}\mathrm{e}^{-i\epsilon X}C\mathrm{e}^{i\epsilon X}A^{-1}\mathrm{e}^{-i\epsilon X}C\mathrm{e}^{i\epsilon X}) \\ &= \det(I + A^{-1}CA^{-1}C - i\epsilon(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C]) + \mathcal{O}(\epsilon^2) \\ &= \det(I + A^{-1}CA^{-1}C) \\ &\times \det\left(I - i\epsilon(I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C])\right) + \mathcal{O}(\epsilon^2) \\ &= \det(I + A^{-1}CA^{-1}C) \\ &\times \exp\left(-i\epsilon \mathrm{tr}((I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C]))\right) + \mathcal{O}(\epsilon^2), \end{aligned}$$

where $[X,Y] = XY - YX$ denotes the commutator of the matrices X and Y. The function $f(\epsilon)$ attains its minimum at $\det(I + A^{-1}CA^{-1}C)$, if

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0.$$

Then we must have

$$\mathrm{tr}\left((I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C])\right) = 0,$$

for every J-Hermitian X. That is

$$[C, (A^{-1}C(I + A^{-1}CA^{-1}C)^{-1}A^{-1} + (I + A^{-1}CA^{-1}C)^{-1}A^{-1}CA^{-1})] = 0,$$

and so, performing some computations, we find

$$\begin{aligned} &[C, (A^{-1}C(I + A^{-1}CA^{-1}C)^{-1}A^{-1}C + (I + A^{-1}CA^{-1}C)^{-1}A^{-1}CA^{-1}C)] \\ &= 2\left[C,\ \frac{A^{-1}CA^{-1}C}{I + A^{-1}CA^{-1}C}\right] = 2\left[C,\ I - \frac{I}{I + A^{-1}CA^{-1}C}\right] \\ &= -2\left[C,\ \frac{I}{I + A^{-1}CA^{-1}C}\right] = \frac{2I}{I + (A^{-1}C)^2}\left[C,\ (A^{-1}C)^2\right]\frac{I}{I + (A^{-1}C)^2} = 0. \end{aligned}$$

Thus

$$[C, (A^{-1}C)^2] = 0.$$

Assume that C, which is in diagonal form, has distinct eigenvalues. Then $(A^{-1}C)^2$ is a diagonal matrix as well as $((JA)^{-1}JC)^2$. Furthermore, $((JC)^{1/2}(JA)^{-1}(JC)^{1/2})^2$ is diagonal. Since $(JC)^{1/2}(JA)^{-1}(JC)^{1/2}$ is positive definite, it is also diagonal, and so are $(JA)^{-1}JC$ and $A^{-1}C$. Henceforth, A is also a diagonal matrix and commutes with C. (If C has multiple eigenvalues we can apply a perturbative technique and use a continuity argument).

For $\sigma \in S_n$, such that $\sigma(1), \dots, \sigma(r) \leq r$, we have

$$(\alpha_1^2 + \gamma_{\sigma(1)}^2) \dots (\alpha_n^2 + \gamma_{\sigma(n)}^2) \geq (\alpha_1^2 + \gamma_1^2) \dots (\alpha_n^2 + \gamma_n^2).$$

Thus, the result follows. ■

In the proof of Theorem 1.2, the following lemma is used (cf. [1, Theorem 1.1]).

Lemma 2.2 *Let B, D be J-positive matrices with eigenvalues satisfying*

$$\beta_1 \geq \ldots \geq \beta_r > 0 > \beta_{r+1} \geq \ldots > \beta_n,$$

and

$$\delta_1 \geq \ldots \geq \delta_r > 0 > \delta_{r+1} \geq \ldots > \delta_n.$$

Then

$$D^J(B, D) = \{(\beta_1 + \delta_1) \ldots (\beta_n + \delta_n)\ t : t \geq 1\}.$$

Proof of Theorem 1.2

Since, by hypothesis, A, C, are J-unitary matrices, considering convenient Möbius transformations, it follows that

$$B = \frac{i}{2}\frac{A - I}{A + I}, \quad D = -\frac{i}{2}\frac{C + I}{C - I} \tag{8}$$

are J-Hermitian matrices. Since

$$B + D = -i(A + I)^{-1}(C + A)(C - I)^{-1},$$

we obtain

$$\det(B + D) = i^n \frac{\det(A + C)}{\prod_{j=1}^n (1 + \alpha_j)(1 - \gamma_j)}.$$

Assume that the eigenvalues of B and D are

$$\sigma(B) = \{\beta_1, \ldots, \beta_n\},\ \sigma(D) = \{\delta_1, \ldots, \delta_n\},$$

respectively. From (8) we get,

$$\beta_j = -\frac{\Im \alpha_j}{2(1 + \Re \alpha_j)}, \quad \delta_j = -\frac{\Im \gamma_j}{2(1 - \Re \gamma_j)}.$$

From (6) and (7) we conclude that

$$\beta_1 \geq \ldots \geq \beta_r > 0 > \beta_{r+1} \geq \ldots > \beta_n,$$

and

$$\delta_1 \geq \ldots \geq \delta_r > 0 > \delta_{r+1} \geq \ldots > \delta_n,$$

so that the matrices B and D are J-positive. From Lemma 2.2 it follows that

$$D^J(B, D) = (\beta_1 + \delta_1) \ldots (\beta_n + \delta_n)[1, +\infty[\ .$$

Thus, $D^J(A, C)$ is a half-line with endpoint at

$$(\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n),$$

or, more precisely,

$$D^J(A, C) = \{(\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n)\ t : t \geq 1\}.$$

■

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