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A FIEDLER–TYPE THEOREM FOR THE DETERMINANT OF J–POSITIVE MATRICES

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Abstract. In this note we characterize the set of all possible values attained by the determinant of the sum of two J-positive matrices with prescribed spectra, under a natural compatibility condition.

1. Introduction

Let A and C be Hermitian $n\times n$ matrices with prescribed eigenvalues, $a_1 \geq \ldots \geq a_n$ and $c_1 \geq \ldots \geq c_n$, respectively. Fiedler [7] proved that $\det(A+C)$ lies between the minimum and the maximum of $\prod_{i=1}^{n}(a_i + c_{\sigma(i)})$, where S_n denotes the symmetric group of degree n.

This result has been generalized in several ways (cf. [3, 5, 8] and the references therein), and is in the origin of the longstanding conjecture of Marcus-de Oliveira [9, 12] on the determinant of the sum of two normal matrices.

Marcus-de Oliveira Conjecture *Let A and C be $n\times n$ matrices with prescribed complex eigenvalues $a_1,\ldots,a_n$ and $c_1,\ldots,c_n$, respectively. Let Δ be the subset of $\mathbf{C}$ given by*

$$\Delta = co\left\{\prod_{j=1}^{n}(a_j + c_{\sigma(j)}) : \sigma \in S_n\right\}.$$

Then,

$$\det(A+C) \in \Delta.$$

For details see [3, 1, 6]. The goal of the present note is to obtain bounds for the determinant of the sum of J-Hermitian matrices. Matrices of this type appear in relativistic quantum mechanics and in quantum physics, and inequalities involving them deserve the attention of researchers (cf. [2] and therein references).

Next, we recall some useful facts.

Given a selfadjoint involution $J \in \mathbf{C}^{n\times n}$, that is, $J = J^*, J^2 = I$, let us consider $\mathbf{C}^n$ endowed with the indefinite inner product $[.,.]$ defined by

$$[x,y] := \langle Jx, y\rangle = y^*Jx, \quad x,y \in \mathbf{C}^n.$$

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Assume that $(r, n-r)$, $0 \leq r \leq n$, is the inertia of J. The J-*adjoint* of a complex matrix A, is defined and denoted as

$$[A^{\#}x, y] := [x, Ay] \text{ for all } x, y \in \mathbf{C}^n.$$

A matrix A is said to be J-*selfadjoint* or J-*Hermitian* if $A = A^{\#}$ or equivalently $A = JA^*J$. If, in addition, $[Ax, x] > 0$ for any $x \in \mathbf{C}^n$, equivalently, $A = JP$, for some positive definite matrix P, then A is called J-*positive definite*. The eigenvalues of a J-selfadjoint matrix may not be real, nevertheless its spectrum must be closed under complex conjugation. Note that the eigenvalues of a J-positive matrix are all real, nevertheless, some of them are positive and others are negative, according to the J-norm of the associated eigenvectors. A matrix U is J-*unitary* if $UJU^* = J$. The J-unitary matrices form a connected but non-compact group, usually denoted by $U(r, n-r)$ [11].

Throughout this note, we assume that A and C are J-Hermitian matrices with real eigenvalues a_j and c_j, $j = 1, \cdots, n$, respectively.

We define $|\sigma_+^J(A)|$ and $|\sigma_-^J(A)|$ as the positive and negative indices of A, respectively. In the sequel, we shall assume that the eigenvalues of A, C are arranged according to the J-order,

$$a_1 \geq \ldots \geq a_r > 0 > a_{r+1} \geq \ldots \geq a_n, \; c_1 \geq \ldots \geq c_r > 0 > c_{r+1} \geq \ldots, \geq c_n \tag{1}$$

We say that A, C are *compatible* when their indeces are the same. Compatible J-Hermitian matrices are J-unitarily diagonalizable, i.e., there exist $U, V \in U(r, n-r)$ such that $UAU^{\#} = \mathrm{diag}(a_1, \ldots, a_n)$ and $VCV^{\#} = \mathrm{diag}(c_1, \ldots, c_n)$.

Our main aim is to prove the following

THEOREM 1. *Let A, C be two compatible J-positive matrices with negative index p. If p is even, then*

$$\det(A + C) \geq \prod_{j=1}^{n} (a_j + c_j).$$

If p is odd, then the inequality reverses.

2. Proofs

Before proving our main result, some considerations are in order. We are interested in the characterization of the set

$$D^J(A, C) = \{\det(A + UCU^{\#}) : U \in \mathscr{U}(r, n-r)\}. \tag{2}$$

As $U \in \mathscr{U}(r, n-r)$ is connected and $D^J(A, C)$ is the range of the continuous map from $\mathscr{U}(r, n-r)$ to $\mathbf{C}$ defined by $U \to \det(A + UCU^{\#})$, $D^J(A, C)$ is a connected set in the complex plane. Since the determinant is J-unitarily invariant, without loss of generality we may consider $A = \mathrm{diag}(a_1, \ldots, a_n)$ and $C = \mathrm{diag}(c_1, \ldots, c_n)$. Obviously, if either A or C is scalar, then $D^J(A, C)$ reduces to a singleton. If J is the identity, then A, C are Hermitian matrices, and the theorem of Fiedler [7] applies.

As usual, the permutation matrix associated to $\sigma \in S_n$, is defined by $(P_\sigma)_{ij} = \delta_{\sigma(i),j}$ (the Kroenecker symbol which equals one if $\sigma(i) = j$ and zero otherwise). In the sequel we use the following notation

$$S_n^r = \{\sigma \in S_n : \sigma(j) = j,\ j = r+1,\dots,n\}, \tag{3}$$

$$\hat{S}_n^r = \{\sigma \in S_n : \sigma(j) = j,\ j = 1,\dots,r\}. \tag{4}$$

PROPOSITION 2.1. *Let A and C be compatible J-Hermitian matrices. The following occurs:*

(i) The set $D^J(A,C)$ is the half-line

$$D^J(A,C) = \{(a_1+c_1)(a_2+c_2) - s(a_1-a_2)(c_1-c_2) : s \geq 0\},$$

for 2×2 matrices.

(ii) The $r!(n-r)!$ points

$$z_\sigma = z_{\xi\tau} = \prod_{j=1}^{r}(a_j + c_{\xi(j)}) \prod_{j=r+1}^{n}(a_j + c_{\tau(j)}),\ \xi \in S_n^r,\ \tau \in \hat{S}_n^r. \tag{5}$$

belong to $D^J(A,C)$,

(iii) The line segments defined by two σ-points generated by permutations that differ by a transposition are contained in $D^J(A,C)$. On the other hand, the $r!(n-r)!r(n-r)$ half-lines

$$\begin{aligned} L_{i,j,\sigma,\tau} &= [(a_i + c_{\sigma(i)})(a_{r+j} + c_{\tau(r+j)}) - s(a_i - a_{r+j})(c_{\sigma(i)} - c_{\tau(r+j)})] \\ &\times \prod_{k\neq i}\prod_{l\neq j}(a_k + c_{\sigma(k)})(a_{r+l} + c_{\tau(r+l)}) : s \geq 0,\ \sigma \in S_n^r, \tau \in \hat{S}_n^r,\ 1 \leq i \leq r < j \leq n \end{aligned} \tag{6}$$

are also contained in $D^J(A,C)$.

Proof. (i) Considering in (2) matrices A and B of order 2 and

$$U = \begin{pmatrix} \mathrm{ch}\,u\,\mathrm{e}^{i\psi} & \mathrm{sh}\,u\,\mathrm{e}^{i\phi} \\ \mathrm{sh}\,u\,\mathrm{e}^{-i\phi} & \mathrm{ch}\,u\,\mathrm{e}^{-i\psi} \end{pmatrix},\ u \in \mathbf{R},\ \phi, \psi \in [0, 2\pi[\,,$$

which belongs to $U(1,1)$, by direct computation, we easily find

$$D^J(A,C) = \ ]-\infty, (a_1+c_1)(a_2+c_2)].$$

(ii) The points are produced taking in (2), $U = P_\sigma P_\tau$.

(iii) The line segments are described considering in (2) the matrix $U = VP_\sigma P_\tau$, where $\sigma \in S_n^r, \tau \in \hat{S}_n^r$ and V is the matrix obtained from the identity replacing the entries $(i,i), (i,j), (j,i)$ and (j,j) by $\cos\theta, \sin\theta, \cos\theta$ and $-\sin\theta$, respectively, for $1 \leq i < j \leq r$ or for $r+1 \leq i < j \leq n$. The half-line $L_{i,j,\sigma,\tau}$ is described considering in

(2) the matrix $U = VP_\sigma P_\tau$, where $\sigma \in S_n^r, \tau \in \hat{S}_n^r$ and V is the matrix obtained from the identity replacing the entries $(i,i),(i,j+r),(j+r,i)$ and $(j+r,j+r)$ by $\operatorname{ch}u, \operatorname{sh}u, \operatorname{ch}u$ and $\operatorname{sh}u$, respectively.

The eigenvalues of A are said *to interlace* if

$$a_1 \geq \ldots \geq a_r, \quad a_{r+1} \geq \ldots \geq a_n, \quad a_1 \neq a_r, \quad a_{r+1} \neq a_n, \quad a_r \ngeq a_{r+1}, \quad a_n \ngeq a_1.$$

PROPOSITION 2.2. *Let A, C be J-Hermitian compatible matrices. If either the eigenvalues of A interlace and $a_r \neq a_{r+1}$, $a_r \neq a_n$ or the eigenvalues of C interlace and $c_r \neq c_{r+1}$, $c_r \neq c_n$, then $D^J(A,C)$ is the whole real line.*

Proof. Under the assumptions, there will be half-lines $L_{i,j,\sigma,\tau}$ with the same end point, described considering in (2) the matrix $U = VP_\sigma P_\tau$ (where $\sigma \in S_n^r, \tau \in \hat{S}_n^r$ and V is the matrix obtained from the identity replacing the entries $(i,i),(i,j+r),(j+r,i)$ and $(j+r,j+r)$ by $\operatorname{ch}u, \operatorname{sh}u, \operatorname{ch}u$ and $\operatorname{sh}u$, respectively), some of which are directed to the right, and some to the left.

For brevity, the points z_σ will be called σ*-points*.

We remark that the converse of last proposition is not valid, as the following example shows.

EXAMPLE 1. Let

$$A_0 = C_0 = \operatorname{diag}(3,-1,-2), \ J_3 = \operatorname{diag}(1,1,-1).$$

We investigate the set

$$D^J(A_0,C_0) = \{\det(A_0 + UC_0J_3U^*) : UJ_3U^* = J_3\}.$$

We easily find for $J_2 = \operatorname{diag}(1,-1)$,

$$\left\{(-2)\det\left(\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix} + V\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}J_2V^*\right) : VJ_2V^* = J_2\right\} = \{20s^2 : s \in \mathbf{R}\} = [0,+\infty[,$$

and

$$\left\{6\det\left(\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix} + V\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}J_2V^*\right) : VJ_2V^* = J_2\right\} = \{-108s^2 : s \in \mathbf{R}\} =]-\infty,0].$$

Thus,

$$D^J(A_0,C_0) = \mathbf{R}.$$

LEMMA 1. *Let A, C be J-Hermitian matrices under the assumptions of Theorem 1. If all the σ-points do not have the same sign, then $D^J(A,C)$ is the whole real line.*

Proof. Assume that some σ-points are positive and some are negative. Accordingly, some of the half-lines $L_{i,j,\sigma,\tau}$ which are described considering in (2) the matrix $U = VP_\sigma P_\tau$ (where $\sigma \in S_n^r, \tau \in \hat{S}_n^r$ and V is the matrix obtained from the identity replacing the entries $(i,i),(i,j+r),(j+r,i)$ and $(j+r,j+r)$ by $\mathrm{ch}\,u, \mathrm{sh}\,u, \mathrm{ch}\,u$ and $\mathrm{sh}\,u$, respectively) are directed to the right, and some to the left, depending on the end-points being positive or negative. Having in mind that $D^J(A,C)$ is connected, the result follows.

The following question arises. Let A,C be compatible J-Hermitian matrices. If all the σ-points have the same sign, is $D^J(A,C)$ the whole real line, or a half-line? A partial answer is given in the proof of Theorem 1, namely, if p is even (odd), then $D^J(A,C)$ is half-line to the left (right).

Proof of Theorem 1

Since A,C are J-positive, so is $A+C$. Thus, $\det(J(A+C)) = (-1)^{n-r}\det(A+C) > 0$, so that $\det(A+C) > 0$ if p is even and $\det(A+C) < 0$ if p is odd. Moreover, $a_r > 0 > a_{r+1}$ and $c_r > 0 > c_{r+1}$. According to the hypothesis, the matrix A is J-unitarily similar to $A_0 = \mathrm{diag}(a_1,\dots,a_n)$ and C is J-unitarily similar to $C_0 = \mathrm{diag}(c_1,\dots,c_n)$.

Clearly, $\det(A+C) = \det(A_0 + VC_0V^\#)$ for some J-unitary matrix V. Assume p is even, so that $\det(A+C) > 0$. Then, the closure of $D^J(A;C)$ is a half-ray with its end-point. There exists a J-unitary matrix V_0 such that

$$\det(A_0 + V_0C_0V_0^\#) = \inf_{V\in U(r,n-r)} \det(A_0 + VC_0V^\#).$$

We claim that

$$[A_0, VC_0V^\#] = 0,$$

where $[X,Y] = XY - YX$. Let $C_0' = VC_0V^\#$ so that $\det(A_0 + C_0')$ is an extremal point of $D^J(A,C)$. For simplicity, assume that $a_1 > \cdots > a_n$. Let $H \in \mathbf{C}^{n\times n}$ be J-Hermitian. For any real t, the matrix $\mathrm{e}^{itH} = I + itH - 1/2t^2H^2 + \cdots$ is J-unitary. Consider the one-parameter curve in $D(A,C)$

$$t \to \det(A_0 + \mathrm{e}^{-itH}C_0'\mathrm{e}^{itH}) = \det(A_0 + C_0')\left[1 + it\mathrm{Tr}\,(A_0 + C_0')^{-1}[H, C_0']\right] + \Theta(t^2).$$

Since we are assuming that $\det(A_0 + C_0')$ is an extremal point,

$$0 = \mathrm{Tr}\,(A_0 + C_0')^{-1}[H, C_0'] = \mathrm{Tr}\,H[(A_0 + C_0')^{-1}, C_0']$$

for all J-Hermitian H. Henceforth,

$$[C_0', (A_0 + C_0')^{-1}] = C_0'(A_0 + C_0')^{-1} - (A_0 + C_0')^{-1}C_0' = 0.$$

Consequently, $(A_0 + C_0')C_0' - C_0'(A_0 + C_0') = 0$ and the claim follows. Since A_0 is in diagonal form, $VC_0V^\#$ is also in diagonal form. Thus, $\det(A_0 + C_0')$ is a σ-point, and so, the minimum is attained and belongs to $D^J(A,C)$. We drop the assumption that the eigenvalues of A are distinct by a continuity argument.

Let p be even. It is clear that the origin does not belong to $D^J(A,C)$. According to the hypothesis, the points in (5) and the half-lines in (6), are in the positive real line.

Having in mind that $D^J(A,C)$ is a connected set, it follows that it is a half-line whose endpoint is a σ-point by the first part of the proof. Since for $i<j$ and $i'<j'$

$$(a_i+c_{i'})(a_j+c_{j'})-(a_i+c_{j'})(a_j+c_{i'})=-(a_i-a_j)(c_{i'}-c_{j'})<0$$

and recalling that every permutation can be expressed as a product of transpositions, it is clear that

$$\min_{\sigma\in S_n^r,\,\tau\in\hat{S}_n^r}\prod_{j=1}^{r}(a_j+c_{\sigma(j)})\prod_{j=r+1}^{n}(a_j+c_{\tau(j)})=\prod_{j=1}^{n}(a_j+c_j).$$

Let p be odd. By similar arguments, it is easy to conclude that

$$\max_{\sigma\in S_n^r,\,\tau\in\hat{S}_n^r}\prod_{j=1}^{r}(a_j+c_{\sigma(j)})\prod_{j=r+1}^{n}(a_j+c_{\tau(j)})=\prod_{j=1}^{n}(a_j+c_j).$$

■

3. Other determinantal inequalities

For A and C are $n\times n$ J-positive matrices, as JA and JC are positive definite matrices, the following inequalities hold [10]

$$\begin{aligned}&\det(J(A+C))\geq\det(JA)+\det(JC),\\&(\det(J(A+C)))^{1/n}\geq(\det(JA))^{1/n}+(\det(JC))^{1/n},\\&\det(\lambda JA+(1-\lambda)JC)\geq(\det(JA))^{\lambda}+(\det(JC))^{1-\lambda},\ \ 0\leq\lambda\leq 1,\end{aligned}$$

As consequence the following inequalities are valid.

PROPOSITION 3.1. *If A,C are J-positive matrices for $J=I_r\oplus -I_{n-r}$, then*

$$\det(A+C)\geq\det(A)+\det(C),$$

if $n-r$ is even, and

$$\det(A+C)\leq\det(A)+\det(C),$$

if $n-r$ is odd.

PROPOSITION 3.2. *If A,C are $n\times n$ J-positive matrices for $J=I_r\oplus -I_{n-r}$, then*

$$(\det(A+C))^{1/n}\geq(\det(A))^{1/n}+(\det(C))^{1/n},$$

if $n-r$ is even, and

$$(\det(A+C))^{1/n}\leq-|\det(A)|^{1/n}-|\det(C)|^{1/n},$$

if $n-r$ is odd.

PROPOSITION 3.3. *For $0 \leq \lambda \leq 1$, A, C J-positive matrices and $J = I_r \oplus -I_{n-r}$, then*

$$\det(\lambda A + (1-\lambda)C) \geq (\det(A))^{\lambda} + (\det(C))^{1-\lambda},$$

if $n-r$ is even, and

$$\det(\lambda A + (1-\lambda)C) \leq -|\det(A)|^{\lambda} - |\det(C)|^{1-\lambda},$$

if $n-r$ is odd.

We remark that the estimates for the determinant of the sum of J-positive matrices A, C in Theorem 1 are the best possible in terms of the eigenvalues of A and C.

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