# The characteristic polynomial of linear pencils of small size and the numerical range

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**Abstract** The numerical range of a linear pencil $(A,B)$ of matrices of size $n$ may be characterized in terms of a certain algebraic curve of class $n$, called the boundary generating curve, which is explicitly given by the characteristic polynomial of the pencil. We shall be specially concerned with the case of one of the matrices being Hermitian. For $n = 2$ and $n = 3$, each possible type of boundary generating curve can be completely described. For $n = 3$, the curve type is given by Newton's classification of cubic curves. Illustrative examples of the different possibilities are given.

## 1 Introduction

Let $A,B \in M_n$, the algebra of $n \times n$ complex matrices. The linear pencil $(A,B)$ is the set of matrices $A - \lambda B$, where $\lambda$ is a real or complex parameter. A pencil is said to be *regular* if the polynomial $\det(A - \lambda B)$ does not identically vanish, otherwise it is *singular*. If the matrix $B$ is nonsingular, the *spectrum* of the regular pencil denoted by $\sigma(A,B)$ consists of all the zeros $\lambda$ of the polynomial $\det(A - \lambda B)$. The spectral theory of pencils is an important issue in pure mathematics as well as in applications (e.g., see [3, 8, 12, 13] and their references). An informative containment region for the spectrum of $(A,B)$ is the numerical range.

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The *numerical range* (also called the *field of values*) of a linear pencil is defined and denoted as

$$W(A,B) = \{\lambda \in \mathbb{C} : x^*(A-\lambda B)x = 0, \ \text{for some} \ 0 \neq x \in \mathbb{C}^n\} \tag{1}$$

(cf. [10, 13]). If $B$ is singular, then the pencil $(A,B)$ may have an infinite eigenvalue $\lambda$, nevertheless (1) does not contain the point at infinity. So, from the above definition, $W(A,B)$ is not necessarily a spectral inclusion region for the generalized eigenvalue problem $Ax = \lambda Bx$. Indeed, we consider a slightly modified definition: if $A, B$ have a common null space, then

$$W(A,B) = \mathbb{C} \cup \{\infty\};$$

otherwise

$$W(A,B) = \left\{ \frac{x^*Ax}{x^*Bx} : 0 \neq x \in \mathbb{C}^n \right\}, \tag{2}$$

where $1/0$ is understood as the point at infinity. When $B$ is the identity matrix, (2) reduces to the (classical) field of values of the $n \times n$ matrix $A$,

$$W(A) = \{x^*Ax : \ x \in \mathbb{C}^n, \ \|x\| = 1\},$$

where $\|x\| = \langle x,x\rangle^{1/2} = (x^*x)^{1/2}$ is the usual Euclidean norm in $\mathbb{C}^n$. This concept has been extensively investigated; see, for instance, [7] and references therein.

Throughout, we shall be concerned with regular pencils $(A,B)$ of which either $A$ or $B$ is Hermitian. Let us assume that $B$ is Hermitian. The *characteristic polynomial* of the pencil $(A,B)$ is defined as

$$f(u,v,w) = \det(uH + vK + wB),$$

where $A = H + iK$, and

$$H = (A + A^*)/2, \quad K = (A - A^*)/(2i)$$

are Hermitian matrices.

The main goal of this article is to investigate connections between the characteristic polynomial $f(u,v,w)$ and the shape of $W(A,B)$. The paper is organized as follows. In Section 2 we recall some properties of algebraic curves used subsequently. In Section 3 we completely characterize the field of values of $2 \times 2$ linear pencils, distinguishing the cases of $B$ being definite, indefinite and singular. These results allow simple direct proofs of the convexity of $W(A,B)$ for $B$ definite or semi-definite Hermitian. In Section 4, each possible boundary type curve is described for $3 \times 3$ matrices of which one of them is Hermitian. In Section 5 illustrative examples are given.

## 2 The polynomial $f(u,v,w)$ and $W(A,B)$

As we shall see in the sequel, the characteristic polynomial of $(A,B)$ gives rise to the *boundary generating curve* of the numerical range $W(A,B)$. To investigate this relation and for the sake of completeness, we present some prerequisites concerning plane algebraic curves.

An ordered pair of complex numbers $(x,y)$ is a (complex) point in *nonhomogeneous point coordinates*. If $x$ and $y$ are real numbers, $(x,y)$ is called a *real* point. A point in *homogeneous point coordinates* is a triple of complex numbers $(x,y,z)$, not all zero. If $r$ is any non zero complex number, then $(x,y,z)$ and $(rx,ry,rz)$ represent the same point. We identify the point $(x,y,z)$ in homogeneous coordinates with the point $(x/z,y/z)$ in nonhomogeneous coordinates. On the other hand, the point $(x,y)$ becomes $(x,y,1)$ in homogeneous coordinates. Any point with $z=0$ is a *point at infinity*.

If $B$ is Hermitian positive definite (HPD), we clearly have

$$W(A,B) = W(B^{-1/2}AB^{-1/2}),$$

and so the numerical range of the pencil reduces to the classical numerical range. Toeplitz and Hausdorff have proven that the classical field of values is a convex set [7]. So, assuming that $B$ is positive definite, then $W(A,B)$ is convex.

A *supporting line* of a convex set $S \subset \mathbb{C}$ is a line that intersects $S$ at least in one point and that defines two half-planes, such that one of them does not contain any point of $S$. Let $B$ be positive definite and let $A$ be arbitrary. It can be shown, using similar reasoning to the one in [9, Theorem 10] that

**Theorem 1.** *If $ux+vy+w=0$ is the equation of a supporting line of $W(A,B)$, then*

$$f(u,v,w) = \det(uH+vK+wB) = 0. \tag{3}$$

It can be easily proved that the above result holds for $B$ indefinite or semi-definite. Since $f(u,v,w)$ is a homogeneous polynomial of degree $n$, (3) may be viewed as the line equation of an algebraic curve in the complex projective plane $P\mathbb{C}^2$. The set of lines $(u:v:w)$ (with equation $ux+vy+wz=0$) such that $f(u,v,w)=0$, may be regarded as a set of lines in the plane whose envelope is a certain curve. Considering the *dual curve*, i.e., the curve in line coordinates,

$$\Gamma^* = \{(u:v:w) \in P\mathbb{C}^2 \colon f(u,v,w)=0\},$$

by dualization, we may easily determine:

$$\Gamma = \{(x:y:z) \in P\mathbb{C}^2 \colon xu+yv+zw=0 \text{ is a tangent of } \Gamma^*\}.$$

The real affine view of $\Gamma$, say

$$C(A,B) = \{(x,y) \in \mathbb{R}^2 \colon (x:y:1) \in \Gamma\},$$

is called the *associated curve* or *boundary generating curve* of $W(A,B)$.

For $(A,B) \in M_n$, with $B$ positive definite, it is a simple consequence of a result of Kippenhahn (see [9]) that the curve $C(A,B)$ generates $W(A,B)$ as its convex hull.

We recall that an usual procedure to find the point equation of the boundary generating curve $C(A,B)$ is to eliminate one of the indeterminates, say $u$, from (3) and $ux+vy+w=0$, dehomogenize the result by setting $w=1$, and to eliminate the remaining parameter $v$ from the equations

$$F(v,x,y) = f(-(1+vy)x^{-1}, v, 1) = 0 \quad \text{and} \quad \frac{\partial F(v,x,y)}{\partial v} = 0.$$

The curve $f(u,v,w) = 0$ has *class n* (because the defining polynomial has degree $n$), that is, through a general point in the plane there are $n$ lines (may be complex) tangent to the curve.

A point $P$, not equal to the *circular points at infinity* $(1:i:0)$ and $(1:-i:0)$, is called a *focus* of a curve $C$ if the line $l_1$ through $P$ and $(1:i:0)$ and the line $l_2$ through $P$ and $(1:-i:0)$ are tangent to $C$ at points other than the circular points at infinity. The coefficients of the polynomial $f(u,v,w)$ are real, as it can be easily seen. A curve of class $n$ with real coefficients has $n$ real foci, according to proper multiplicities, and $n^2 - n$ foci which are not real [14].

As a consequence of a result, independently obtained by Murnaghan [11] and Kippenhahn [9], the real foci of the algebraic curve defined by $\det(uH + vK + wB) = 0$, where $B$ is positive definite, are the eigenvalues of the matrix $B^{-1}A$, with $A = H + iK$. The corresponding result for $B$ indefinite is as follows [3].

**Theorem 2.** *Let $A,B \in M_n$, with $B$ indefinite. The $n$ real foci of the algebraic curve defined by the equation $f(u,v,w) = \det(uH + vK + wB) = 0$ are the eigenvalues of the pencil $(A,B)$, where $A = H + iK$ with $H$ and $K$ Hermitian.*

For details on plane algebraic curves, we refer the interested reader to [5].

## 3 Linear pencils generated by $2 \times 2$ matrices

For matrices $A$ and $B$ of dimension two, the boundary generating curve $C(A,B)$ is a curve of class two, more concretely, a conic. The three theorems that characterize the boundary of $W(A,B)$, for $B$ Hermitian, in terms of the invariants of the pencil $(A,B)$ are stated below. The case 2 by 2 is specially important, since the numerical range of an $n \times n$ pencil may be characterized by compression to the bidimensional setting [12].

**Theorem 3.** *(Elliptical Range Theorem) Let $A,B$ be $2 \times 2$ matrices with $B$ positive definite. Then $W(A,B)$ is a (possibly degenerate) closed elliptical disc, whose foci are the eigenvalues of $B^{-1}A$, $\lambda_1$ and $\lambda_2$. and the lengths of the major and minor axis are, respectively,*

$$M = \sqrt{\mathrm{Tr}\,(A^*B^{-1}AB^{-1}) - 2\mathrm{Re}\left(\overline{\lambda_1}\lambda_2\right)},$$

*and*

$$N = \sqrt{\mathrm{Tr}\,(A^*B^{-1}AB^{-1}) - |\lambda_1|^2 - |\lambda_2|^2}.$$

In the case of degeneracy, $W(A,B)$ may reduce to a line segment whose endpoints are $\lambda_1$ and $\lambda_2$, or to a singleton if and only if $\lambda_1 = \lambda_2$.

**Theorem 4.** *(Hyperbolical Range Theorem) Let $A,B$ be $2\times 2$ matrices with $B$ indefinite. Then $W(A,B)$ is bounded by a hyperbola with foci at $\lambda_1$ and $\lambda_2$, the eigenvalues of $B^{-1}A$, and transverse and non-transverse axis of length*

$$M = \sqrt{\mathrm{Tr}(B^{-1}A^*B^{-1}A) - 2\mathrm{Re}(\lambda_1\bar{\lambda_2})}$$

*and*

$$N = \sqrt{|\lambda_1|^2 + |\lambda_2|^2 - \mathrm{Tr}(B^{-1}A^*B^{-1}A)}.$$

If $\mathrm{Tr}(B^{-1}A^*B^{-1}A) - 2\mathrm{Re}(\lambda_1\bar{\lambda_2}) < 0$, the hyperbola degenerates and $W(A,B)$ is the whole complex plane. In the case of degeneracy of the hyperbola, $W(A,B)$ may also reduce to two half-lines of the line defined by $\lambda_1$ and $\lambda_2$, and with these endpoints.

Now, we consider $W(A,B)$ for $A,B \in M_2$, with $B$ positive (negative) semidefinite. Observing that

$$W(\mathrm{e}^{i\phi}(A+\zeta B), kB) = \frac{1}{k}\mathrm{e}^{i\phi}(W(A,B)+\zeta), \quad k,\phi \in \mathbb{R},\ \zeta \in \mathbb{C},$$

and using the invariance of $W(A,B)$ under unitary similarities, we may take

$$B = \mathrm{diag}(1,0), \quad A = \begin{bmatrix} a\mathrm{e}^{i\gamma} & c\mathrm{e}^{i\gamma} \\ d & b \end{bmatrix}, \quad c,d \geq 0,\ b > 0,\ a = \frac{cd}{b}. \tag{4}$$

Notice that $W(A,B) = \mathbb{C}$ if $b = 0$.

**Theorem 5.** *(Parabolical Range Theorem) Let $A,B$ be of the form (4). Then $W(A,B)$ is bounded by the (possibly degenerate) parabola with focus $\lambda_0 = 0$ and equation*

$$\frac{y^2}{4p^2} - \frac{x}{p} = 1,$$

*where*

$$p = \frac{a^2b^2 + c^4 - 2abc^2\cos\gamma}{4bc^2}.$$

In the case of degeneracy of the parabola, $W(A,B)$ may reduce to one half-line with $\lambda_0 = 0$ as endpoint.

We remark that for $A=(a_{ij})\in M_2$, with $a_{22}\neq 0$ and $B=\operatorname{diag}(1,0)$, the slope of the axis of the parabolic boundary, relatively to the positive semi real axis, is equal to $\theta_0=\operatorname{Arg}(a_{22})$, and the focus of the parabola is the (finite) eigenvalue of the pencil $(A,B)$. The vertex of the parabola is the point $u_0^*Au_0/u_0^*Bu_0$, where $u_0$ is an eigenvector of the Hermitian pencil

$$\left(\frac{1}{2}(A\mathrm{e}^{-i\theta_0}+A^*\mathrm{e}^{i\theta_0}),B\right)$$

associated with its single (finite) eigenvalue.

## 4 Characterization of $W(A,B)$ for $A,B\in M_3$ with $B$ Hermitian

### *4.1 $C(A,B)$ for $B$ positive definite and $A$ arbitrary*

Under the present assumptions, $W(A,B)$ is convex, bounded and closed, since it reduces to $W(B^{-1/2}AB^{-1/2})$, and so inherits the properties of the classical numerical range. Following the arguments in [9, Theorem 10], we can prove the following

**Theorem 6.** *The convex hull of $C(A,B)$ is $W(A,B)$ .*

Kippenhahn classified the associated curve in this context, considering the factorizability of the polynomial $f(u,v,w)$. Adopting this procedure, we easily conclude that the following possibilities may occur.

**1st Case:** The polynomial $f(u,v,w)$ factorizes into three linear factors. Each one of these factors corresponds to an eigenvalue of $B^{-1}A$.

**2nd Case:** Suppose that $B=\operatorname{diag}(b_1,b_2,b_3)$ and that $A\in M_3$ is a $B$-decomposable matrix, i.e., there exists a nonsingular matrix $V\in M_3$, such that

$$V^*BV=B,\quad V^*AV=\begin{bmatrix}cb_1 & 0\\ 0 & A_1\end{bmatrix}, \tag{5}$$

where $c\in\mathbb{C}$ and $A_1\in M_2$. Thus, $f(u,v,w)$ factorizes into a linear and a quadratic factor, and so $C(A,B)$ consists of the point $c$ and of the boundary of the elliptical disc $W(A_1,\operatorname{diag}(b_2,b_3))$.

**3rd Case:** The matrix $A$ is $B$-indecomposable, but the polynomial $f(u,v,w)$ factorizes into a linear and a quadratic factor. The linear factor corresponds to an eigenvalue of $B^{-1}A$. The quadratic factor corresponds to an ellipse. In fact, the conic can not be neither a parabola, because one of its real foci is a point at infinity and this contradicts Proposition 2, nor an hyperbola because this curve is unbounded. Therefore, $C(A,B)$ consists of an ellipse and a point.

**4th Case:** Finally, suppose that the polynomial $f(u,v,w)$ is irreducible. The number of real cusps of an (irreducible) class three curve is 1 or 3, and the order of the boundary generating curve is 4 or 6. By Newton's classification of cubic curves

[1] and dual considerations, there are the following possibilities for the associated curve:

C1. $C(A,B)$ is a sextic, consisting of an oval and a closed tricuspid curve lying in its interior;
C2. $C(A,B)$ is a quartic, with one cusp and an ordinary double tangent at two of its points.

There are examples showing that all these types of curves appear as boundary generating curves of $W(A,B)$.

## 4.2 $C(A,B)$ for $B$ indefinite and $A$ arbitrary

**Theorem 7.** *Let $A$ be arbitrary and let $B$ be indefinite. Then $W(A,B)$ is pseudo-convex.*

*Proof.* Let us consider $\lambda_1 \neq \lambda_2 \in W(A,B)$. Then, there exist $0 \neq v_1,\ 0 \neq v_2 \in W(A,B)$ such that $v_i^* A v_i = \lambda_i v_i^* B v_i,\ i = 1,2$. Let $\tilde{v}_1,\ \tilde{v}_2$ be orthonormal vectors belonging to the subspace $\mathscr{H}_2$ spanned by $v_1,\ v_2$. Let $A_{\tilde{v}_1,\tilde{v}_2}$ and $B_{\tilde{v}_1,\tilde{v}_2}$ be the compressions of $A$ and $B$, respectively, to $\mathscr{H}_2$. Obviously, $W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2})$ is either an elliptical, parabolic or hyperbolical domain, depending on $B_{\tilde{v}_1,\tilde{v}_2}$ being definite, semidefinite or indefinite. If $W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2})$ is an elliptical or parabolical disc, it is convex. In this case, we have that

$$\{\lambda_1 + x(\lambda_2 - \lambda_1) : 0 \le x \le 1\} \subseteq W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2}) \subseteq W(A,B).$$

If $W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2})$ is hyperbolical, it is pseudo-convex. In this case, either

$$\{\lambda_1 + x(\lambda_2 - \lambda_1) : 0 \le x \le 1\} \subseteq W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2}) \subseteq W(A,B).$$

or

$$\{\lambda_1 + x(\lambda_2 - \lambda_1) : x \le 0 \ \text{ or } \ x \ge 1\} \subseteq W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2}) \subseteq W(A,B),$$

This completes the proof.

For $B$ indefinite, consider $\mathbb{C}^n$ endowed with the *$B$-inner product* $\langle Bx, y\rangle = y^* Bx$, and corresponding *$B$-norm* $\|x\|_B^2 = \langle Bx, x\rangle$ [6]. For arbitrary $A \in M_3$, $W(A,B)$ has been characterized in [3], following Kippenhahn's approach in the classical case.

Let us define

$$W(A,B) = \left\{ \frac{\langle Au, u\rangle}{\langle Bu, u\rangle} : u \in \mathbb{C}^n,\ \langle Bu, u\rangle \neq 0 \right\}.$$

For convenience, we also consider the sets

$$W_+(A,B) = \left\{ \frac{\langle Au, u\rangle}{\langle Bu, u\rangle} : u \in \mathbb{C}^n,\ \langle Bu, u\rangle > 0 \right\},$$

and

$$W_-(A,B) = \left\{ \frac{\langle Au,u\rangle}{\langle Bu,u\rangle} : u \in \mathbb{C}^n,\ \langle Bu,u\rangle < 0 \right\}.$$

Obviously,

$$W(A,B) = W_+(A,B) \cup W_-(A,B).$$

In our analysis, when $A$ and $B$ are both Hermitian, we shall consider the eigenvalues of *positive* and *negative* type, that is, the eigenvalues with associated eigenvectors with positive and negative $B$-norm, respectively. We shall denote by $\sigma_+(A,B)$ $(\sigma_-(A,B))$ the set of eigenvalues of positive (negative) type.

Let $X^+$ $(X^-)$ be a set of points in $W_+(A,B)$ $(W_-(A,B))$ and let $\Xi^+$ $(\Xi^-)$ be the convex hull of $X^+$ $(X^-)$. Consider the lines defined by points $z_+,\ z_-$ with $z_+ \in \Xi^+$ and $z_- \in \Xi^-$. The union of all half-lines with $z_+$ as endpoint not containing $z_-$ and the half-lines with $z_-$ as endpoint not containing $z_+$, is the so called *pseudo-convex hull* of $X^+$and $X^-$.

The curve $C(A,B)$ has branches of a well defined type, either *positive* or *negative*, say $C_+(A,B)$ and $C_-(A,B)$. The sign is determined by considering for the corresponding root $w$ of (3), an associated eigenvector $\xi$, such that

$$(uH + vK + wB)\xi = 0.$$

The type of each branch of $C(A,B)$ is characterized by the sign of the $B$-norm $\langle B\xi,\xi\rangle$.

For pencils of the class $\mathcal{ND}$ (see [4, Section 5]) the following holds. The proof follows analogous steps to those i [9, Theorem 10].

**Theorem 8.** *If the pencil $(A,B)$ is in $\mathcal{ND}$, then the pseudo-convex hull of $C_+(A,B)$ and $C_-(A,B)$ is $W(A,B)$.*

We classify the associated curve $C(A,B)$, considering the factorizability of the polynomial $f(u,v,w)$. Without loss of generality, we may assume that $B = \operatorname{diag}(b_1, b_2, -b_3),\ b_1,b_2,b_3 > 0$. The following possibilities may occur,

**1st Case:** The polynomial $f(u,v,w)$ factorizes into three linear factors. Each one of these factors corresponds to an eigenvalue of $B^{-1}A$.

**2nd Case:** Suppose that $A \in M_3$ is $B$-decomposable, i.e., there exists a nonsingular matrix $V$, such that $V^*BV = B = \operatorname{diag}(b_1,b_2,-b_3)$ and

$$V^*AV = \begin{bmatrix} cb_1 & 0 \\ 0 & A_1 \end{bmatrix}, \tag{6}$$

or

$$V^*AV = \begin{bmatrix} A_1 & 0 \\ 0 & -cb_3 \end{bmatrix}, \tag{7}$$

where $c \in \mathbb{C}$ and $A_1 \in M_2$.

If $A$ is of the form (6), then $C(A_1, \mathrm{diag}(b_2, -b_3))$ is an hyperbola with one branch in $W_+(A,B)$ and the other one in $W_-(A,B)$. We may write

$$C(A_1, \mathrm{diag}(b_2, -b_3)) = C_+(A_1, \mathrm{diag}(b_2, -b_3)) \cup C_-(A_1, \mathrm{diag}(b_2, -b_3)),$$

where $C_\pm(A_1, \mathrm{diag}(b_2, -b_3)) \subset W_\pm(A,B)$. Clearly, $c \in W_+(A,B)$. Let $X_+ = \mathrm{conv}(c, C_+(A_1, \mathrm{diag}(b_2, -b_3)))$. The *pseudo-convex hull* of $X_+$ and $C_-(A_1, \mathrm{diag}(b_2, -b_3))$ coincides with $W(A,B)$.

Suppose, now, that $A$ is of the form (7). Notice that $c \in W_-(A,B)$ and $C(A_1, \mathrm{diag}(b_1, b_2)) \subset W_+(A,B)$. Then $W(A,B)$ is the pseudo-convex hull of $c$ and an ellipse (possibly degenerate): $C(A_1, \mathrm{diag}(b_1, b_2))$.

**3rd Case:** The matrix $A$ is $B$-indecomposable, but the polynomial $f(u,v,w)$ factorizes into a linear and a quadratic factor. The quadratic factor corresponds to an hyperbola or to an ellipse. The conic can not be a parabola, because one of its real foci is a point at infinity and this contradicts Proposition 2.

Therefore, $C(A,B)$ consists of: 1) one point, produced by vectors with a negative $B$-norm, and an ellipse produced by vectors with a positive $B$-norm, 2) one point, produced by vectors with a positive $B$-norm, and an hyperbola, with one branch produced by vectors with a negative $B$-norm and the other branch produced by vectors with a positive $B$-norm.

In case 1), $W(A,B) = \mathbb{C}$. In case 2), $W(A,B) = \mathbb{C}$, whenever the point lies inside the hyperbolic disc of negative type, otherwise $W(A,B)$ is a hyperbolical disc.

**4th Case:** Finally, suppose that the polynomial $f(u,v,w)$ is irreducible. The number of real cusps of an (irreducible) class three curve is 1 or 3, and the order of the boundary generating curve is 4 or 6. By Newton's classification of cubic curves and dual considerations, there are the following possibilities for the associated curve:

- C1. $C(A,B)$ is a sextic, with three real cusps and at least one oval component;
- C2. $C(A,B)$ is a quartic, with three real cusps and a real double tangent (at two complex points of the curve);
- C3. $C(A,B)$ is a quartic with one real cusp and a real double tangent (at two real points of the curve);
- C4. $C(A,B)$ is a cubic with a real cusp and a real flex;
- C5. $C(A,B)$ is a sextic, with three real cusps and not containing neither oval components nor double tangents.

There are examples showing that all the above curves may occur as boundary generating curves [3]. The characterization of $W(A,B)$ requires the determination of the signs of each branch of $C(A,B)$, in order to obtain the pseudo-convex hull of the boundary generating curve.

### *4.3 $C(A,B)$ for $B$ positive semi-definite and $A$ arbitrary*

**Theorem 9.** *Let $A$ be arbitrary and let $B$ be positive semi-definite. Then $W(A,B)$ is convex.*

*Proof.* Let us consider $\lambda_1 \neq \lambda_2 \in W(A,B)$. Then, there exist $0 \neq v_1,\ 0 \neq v_2 \in W(A,B)$ such that $v_i^* A v_i = \lambda_i v_i^* B v_i,\ i = 1,2$. Let $\tilde{v}_1,\ \tilde{v}_2$ be orthonormal vectors belonging to the subspace $\mathscr{H}_2$ spanned by $v_1,\ v_2$. Let $A_{\tilde{v}_1,\tilde{v}_2}$ and $B_{\tilde{v}_1,\tilde{v}_2}$ be the compressions of $A$ and $B$, respectively, to $\mathscr{H}_2$. Obviously, $W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2})$ is either a parabolic or elliptical disc, so it is convex. Thus, $[\lambda_1,\lambda_2] \in W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2}) \subseteq W(A,B)$, which completes the proof.

We next characterize $W(A,B)$, for $B$ positive semi-definite and an arbitrary $A \in M_3$, using again Kippenhahn's approach. We classify the associated curve $C(A,B)$, considering the factorizability of the polynomial $f(u,v,w)$.

Assume that $A \in M_3$ and $B$ is positive semi-definite. The following possibilities for $C(A,B)$ may occur.

**1st Case:** Suppose that $B = \mathrm{diag}(b_1,b_2,0)$, $b_1,b_2 > 0$, and $A \in M_3$ is a $B$-decomposable matrix, i.e., there exists a nonsingular matrix $V$ such that $V^*BV = B$ and $V^*AV$ is as in (6). Then, $W(A,B)$ is the convex hull of $c$ and $C(A_1, \mathrm{diag}(b_2,0))$.

**2nd Case:** Suppose that $B = \mathrm{diag}(b_1,b_2,0)$, $b_1,b_2 > 0$, and $A$ is a $3\times 3$ $B$-decomposable matrix, i.e., there exists a non-singular matrix $V$, such that $V^*BV = B$ and

$$V^*AV = \begin{bmatrix} A_1 & 0 \\ 0 & c \end{bmatrix}, \tag{8}$$

where $c \in \mathbb{C}$ and $A_1$ is a $2\times 2$ matrix. Thus, $W(A,B)$ is the convex hull of a certain point at infinity and $C(A_1, \mathrm{diag}(b_1,b_2))$ (cf. Example 5.4).

**3rd Case:** Suppose that $B = \mathrm{diag}(b_1,b_2,0)$, $b_1,b_2 > 0$, and the matrix $A$ is $B$-indecomposable, but the polynomial $f(u,v,w)$ factorizes into a linear and a quadratic factor. The linear factor corresponds to an eigenvalue of the pencil, and the quadratic factor corresponds to a parabola. Therefore, $C(A,B)$ consists of one real point and a parabola (cf. Example 5.3), being $W(A,B)$ its convex hull.

**4th Case:** Suppose that $B = \mathrm{diag}(b_1,b_2,0)$, $b_1,b_2 > 0$, and the polynomial $f(u,v,w)$ is irreducible. By Newton's classification of cubic curves and dual considerations, there are the following possibilities for the associated curve:

C1. $C(A,B)$ is a sextic, with three real cusps and at least one oval component (cf. Example 5.1);
C2. $C(A,B)$ is a quartic, with one cusp and an ordinary double tangent at two of its real points (cf. Example 5.2).

**5th Case:** Suppose that $B = \mathrm{diag}(b_1,0,0)$, $b_1 > 0$. There exists a non-singular matrix $V$, such that $V^*BV = B$ and

$$V^*AV = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

In order to avoid the existence of vectors $\xi \neq 0$ such that $\xi^* A\xi = \xi^* B\xi = 0$, we assume that $a_{22}a_{33} \neq 0$. We also assume that $\{a_{12}, a_{13}\} \neq \{0, 0\}$, so that $A$ is not $B$-decomposable. Notice that

$$W_1 = W\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

is a parabola with focus $a_{11}/b_1$ and axis with slope equal to $\mathrm{Arg}(a_{22})$, while

$$W_2 = W\left(\begin{bmatrix} a_{11} & a_{13} \\ 0 & a_{33} \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

is a parabola with focus $a_{11}/b_1$ and axis with slope equal to $\mathrm{Arg}(a_{33})$. Considering $\mathrm{Co}(W_1, W_2)$, we conclude the following.

If $|\mathrm{Arg}(a_{22}/a_{33})| \geq \pi/2$, then $W(A,B)$ is the whole complex plane (cf. Example 5.6). If $|\mathrm{Arg}(a_{22}/a_{33})| < \pi/2$ , then $W(A,B)$ is a proper subset of the complex plane bounded by a certain algebraic curve, which is a quartic, if the characteristic polynomial is irreducible (cf. Example 5.5), and a conic if the characteristic polynomial is factorizable (cf. Example 5.6).

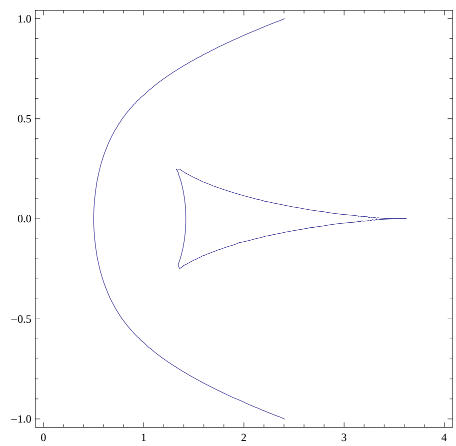


**Fig. 1** Boundary generating curve of $W(A,B)$ (Example 5.1).

### *4.4* $C(A,B)$ *for* $B$ *indefinite singular and* $A$ *arbitrary*

Let $A$ be arbitrary, $B = \mathrm{diag}(b_1, -b_2, 0)$, with $b_1, b_2 > 0$. We say that $\theta \in [0, 2\pi[$ is an *admissible* direction if the Hermitian pencil $(H(\mathrm{e}^{-i\theta}A), B)$ has real eigenvalues with associated non-isotropic eigenvectors, and for $\sigma_+(H(\mathrm{e}^{-i\theta}A), B) = \{\alpha_\theta\}$, $\sigma_-(H(\mathrm{e}^{-i\theta}A), B) = \{\beta_\theta\}$, we have $(\alpha_\theta - \beta_\theta)\ u^*Au > 0$, where $u = (0,0,1)^T$. The condition $(\alpha_\theta - \beta_\theta)\ u^*Au > 0$, ensures that $W(H(\mathrm{e}^{-i\theta}A), B) \neq \mathbb{R}$. If admissible directions do not exist, $W(A,B) = \mathbb{C}$.

**Proposition 4.1** *Let $(A,B)$ be a $3\times 3$ self-adjoint pencil with $B = \mathrm{diag}(b_1, -b_2, 0)$, $b_1, b_2 > 0$, such that $W(A,B) \neq \mathbb{C}$. Let $u = (0,0,1)^T$, $\sigma_+(A,B) = \{\alpha\}$, $\sigma_-(A,B) = \{\beta\}$.*
*i) If $(\alpha - \beta)\ u^*Au > 0$, then $W(A,B) = ]-\infty, \min(\alpha,\beta)] \cup [\max(\alpha,\beta), +\infty[$ .*
*ii) If $(\alpha - \beta)\ u^*Au < 0$, then $W(A,B) = \mathbb{R}$.*

For $A \in M_3$ and $B$ semi-indefinite, the different possibilities that may occur for $C(A,B)$ can be identified according with the procedures in the previous sections (cf.Example 5.8).

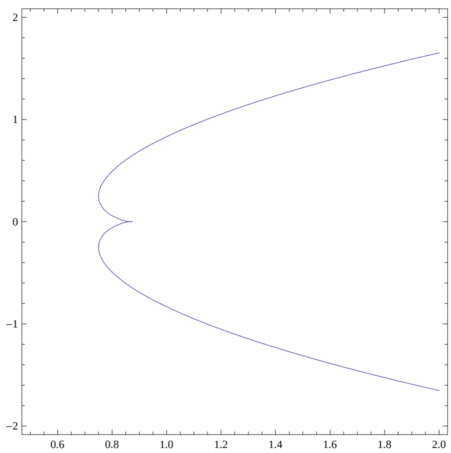


**Fig. 2** Boundary generating curve of $W(A_4, B)$ (Example 5.2).

## 5 Examples

The figures presented in this section have been produced with *Mathematica 5.1*, also used to determine the point equation of $C(A,B)$. The associated curve is represented in the figures. The boundaries of $W(A,B)$ are represented by thick lines.

**Example 5.1** *Let*

$$A = \begin{bmatrix} 1 & 1 & 4/5 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \mathrm{diag}(1,1,0).$$

*The characteristic polynomial of the pencil is*

$$f(u,v,w) = \frac{1}{100}(71u^3 - 29uv^2 + 192u^2w - 8v^2w + 100uw^2).$$

*The Cartesian equation of the boundary generating curve of* $W(A,B)$ *is*

$$\begin{aligned}&-1731619 + 6115752x - 6709556x^2 + 3123808x^3 - 655104x^4 + 51200x^5\\ &-1891452y^2 + 7557408xy^2 - 17370208x^2y^2 + 9142400x^3y^2 - 160000x^4y^2\\ &-15865104y^4 + 51091200xy^4 - 21320000x^2y^4 - 21160000y^6 = 0.\end{aligned}$$

*The boundary of* $W(A,B)$ *is represented in Fig. 1 by the outer curve.*

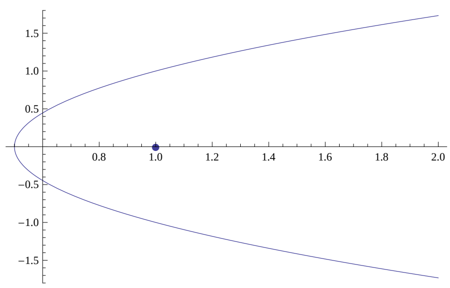


**Fig. 3** Boundary of $W(A_1,B)$ (Example 5.3).

**Example 5.2** *Let*

$$A_4 = \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \mathrm{diag}(1,1,0)$$

*The characteristic polynomial of the pencil* $(A_4,B)$ *is*

$$f(u,v,w) = \frac{1}{16}(9u^3 - 7uv^2 + 24u^2w - 8v^2w + 16uw^2).$$

*The Cartesian equation of the boundary generating curve of* $W(A_4,B)$ *is*

$$-343 + 1176x - 1344x^2 + 512x^3 - 592y^2 + 1024xy^2 - 256x^2y^2 - 256y^4 = 0.$$

$W(A_4,B)$ *is the convex hull of* $C(A_4,B)$*, represented in Fig. 2, and has a flat portion on the boundary parallel to the imaginary axis.*

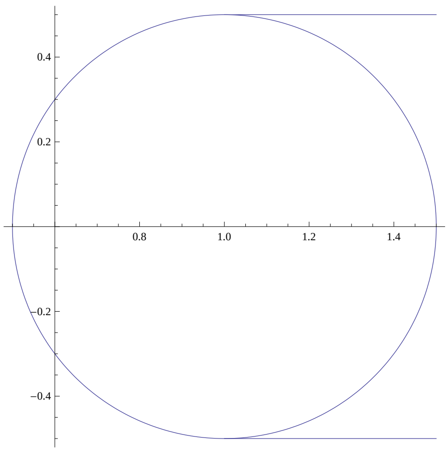


**Fig. 4** Boundary of $W(A,B)$ (Example 5.4).

**Example 5.3** *Let*

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \mathrm{diag}(1,1,0).$$

*The characteristic polynomial of the pencil* $(A_1,B)$ *is*

$$f(u,v,w) = \frac{1}{2}(u+w)(u^2 - v^2 + 2uw).$$

*The Cartesian equation of the boundary generating curve of* $W(A_1,B)$ *is*

$$(y^2 - 2x + 1)((x-1)^2 + y^2) = 0.$$

*Cf. Fig. 3.*

**Example 5.4** *Let*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \mathrm{diag}(1,1,0)$$

*The characteristic polynomial of* $(A,B)$ *is*

$$f(u,v,w) = \frac{1}{4}u(3u^2 - v^2 + 8uw + 4w^2).$$

*The Cartesian equation of* $C(A,B)$ *is* $(x-1)^2 + y^2 = \frac{1}{4}$ *and is represented in Fig. 4.*

**Example 5.5** *Let* $B = \mathrm{diag}(1,0,0)$ *and* $A = A_1$ *in Example 5.3. The characteristic polynomial of the pencil is* $f(u,v,w) = (2u^3 - 2uv^2 + 3u^2w - v^2w)/4$. *The Cartesian equation of the boundary of* $W(A,B)$ *is*

$$16 - 48x + 48x^2 - 20x^3 + 3x^4 + 36y^2 - 36xy^2 - 18x^2y^2 + 27y^4 = 0$$

*and is represented in Fig. 5.*

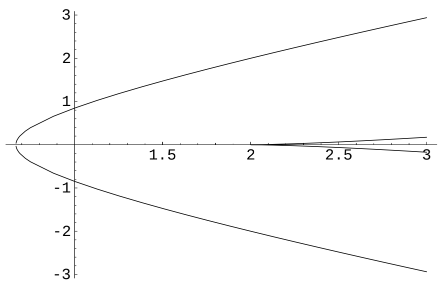


**Fig. 5** Boundary of $W(A,B)$ (Example 5.5).

**Example 5.6** *Let* $B = \mathrm{diag}(1,0,0)$ *and*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

*The characteristic polynomial of* $(A,B)$ *is* $f(u,v,w) = 1/4(-4u^3 - 5u^2w - v^2w)$. *The Cartesian equation of* $C(A,B)$ *is the deltoid*

$$-4x^3 + 5x^4 + 108y^2 - 180xy^2 + 50x^2y^2 + 125y^4 = 0.$$

*Since $f(u,v,w)$ has complex roots in $w$ for all $u,v$, it follows that $W(A,B) = C$.*

**Example 5.7** *Let $B = \mathrm{diag}(1,0,0)$, and*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*The characteristic polynomial of $(A,B)$ is given by $f(u,v,w) = 1/2u(u^2 - v^2 + 2uw)$. The boundary of $W(A,B)$ is parabolic and its Cartesian equation is*

$$y^2 - 2x + 1 = 0.$$

**Example 5.8** *Let $B = \mathrm{diag}(1,-1,0)$, and*

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

*The characteristic polynomial of $(A,B)$ is given by*

$$f(u,v,w) = \frac{1}{2}3u^3 - \frac{1}{2}5uv^2 - \frac{1}{4}3u^2w - \frac{1}{4}3v^2w - uw^2.$$

*The boundary generating curve $C(A,B)$ is represented in Fig. 6, it has Cartesian*

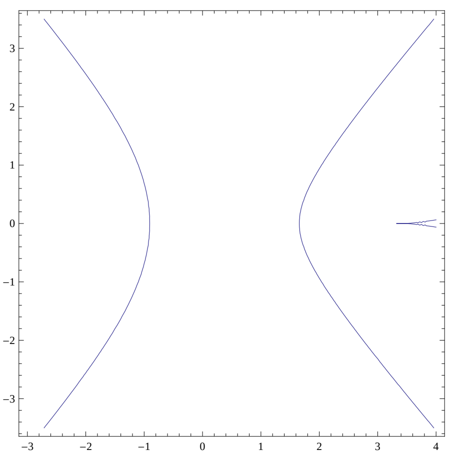


**Fig. 6** $C(A,B)$ for Example 5.8.

*equation*

$$6000-2400x-5080x^2+4248x^3-1161x^4+108x^5+2808y^2+1752xy^2+1678x^2y^2$$
$$-2184x^3y^2+36x^4y^2+2007y^4+2316xy^4-568x^2y^4+420y^6=0$$

*and is constituted of 2 branches, $C_+(A,B)$ for $x \leq (3-\sqrt{105})/8$ and $C_-(A,B)$ for $x \geq (3+\sqrt{105})/8$. The pseudoconvex hull of $C_+(A,B)$ and $C_-(A,B)$ is $W(A,B)$.*

## 6 Final Remarks

We presented the classification of the boundary generating curves of $W(A,B)$ for $2 \times 2$ and $3 \times 3$ matrices $A,B$, following Kipenhann's approach for the classical numerical range of a matrix. We have considered linear pencils generated by a pair $(A,B)$ of which at least one of the matrices is Hermitian. It would be challenging to drop this constraint. The systematic investigation of the existence of flat portions on the boundary, as well as its implications on the matrix structure, are open problems deserving the attention of researchers.

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