ON TUNNEL NUMBER DEGENERATION UNDER THE CONNECTED SUM OF PRIME KNOTS

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Abstract. We study 2-string free tangle decompositions of knots with tunnel number two. As an application, we construct infinitely many counter-examples to a conjecture in the literature stating that the tunnel number of the connected sum of prime knots doesn't degenerate by more than one: $t(K_1\#K_2)\geq t(K_1)+t(K_2)-1$, for K_1 and K_2 prime knots.

1. Introduction

Given a knot K in S^3, an *unknotting tunnel system* for K is a collection of arcs $t_1, t_2, \ldots, t_n$, properly embedded in the exterior of K, with the complement of a regular neighborhood of $K\cup t_1\cup\cdots\cup t_n$ being a handlebody[1]. The minimum cardinality of an unknotting tunnel system for a knot K is a knot invariant, referred to as the *tunnel number* of K and is denoted by $t(K)$.

A natural question of study on knot invariants is their behavior under the connected sum of knots. In the particular case of the tunnel number, it is known, by Norwood [21], that tunnel number one knots are prime. This result is now consequence of more general work. For instance, Scharlemann and Schultens prove in [26] that the tunnel number of the connected sum of knots is bigger than or equal to the number of summands:

$$t(K_1\#\cdots\#K_n)\geq n,$$

where $K_1\#\cdots\#K_n$ represents the connected sum of the knots $K_1,\ldots,K_n$. Also, in [4] Gordon and Reid prove that tunnel number one knots are, in fact, n-string prime[2] for any positive integer n.

On the tunnel number behavior under connected sum, it is a consequence from the definition of connected sum of knots that for two knots K_1 and K_2 in S^3 we have:

$$t(K_1\#K_2)\leq t(K_1)+t(K_2)+1.$$

For some time the only examples known had an additive behavior:

$$t(K_1\#K_2)=t(K_1)+t(K_2).$$

However, in the early nineties, Morimoto [13] constructed connected sum examples of prime knots K_1 with 2-bridge knots K_2 whose tunnel number degenerates by one[3]:

$$t(K_1\#K_2)=t(K_1)+t(K_2)-1.$$

[1] Note that every knot has an unknotting tunnel system obtained from the knot exterior triangulation.

[2] A knot is n-string prime if it has no n-string essential tangle decomposition. For definitions of n-string tangle decompositions of a knot we refer to section 4.1.3 of the survey paper [11] by Moriah, or section 3 of the paper [7] by Kobayashi.

[3] In [16], without mentioning it, Morimoto gives also the first examples of knots that when connected sum with themselves the tunnel number degenerates (by one): all tunnel number two 3-bridge knots with a 2-string free tangle decomposition (as the knot K_{149} from Rolfsen's list in [24]).

Shortly afterwards, Moriah and Rubinstein in [12], and also independently Morimoto, Sakuma and Yokota in [17], gave examples of knots with supper-additive behavior:

$$t(K_1\#K_2) = t(K_1) + t(K_2) + 1.$$

Furthermore, about the same time, Kobayashi in [7] constructed examples of knots that degenerate arbitrarily under connected sum: for any positive integer n, there are knots K_1 and K_2 where

$$t(K_1\#K_2) \leq t(K_1) + t(K_2) - n.$$

However, Kobayashi's examples to show arbitrarily hight degeneration of the tunnel number under connected sum are composite knots.

In this paper we study further the tunnel number degeneration under connected sum of prime knots. For this study we use the work of Morimoto in [16] that relates n-string free tangle decompositions of knots and high tunnel number degeneracy under the connected sum of prime knots. Within this setting, we study 2-string free tangle decompositions of knots with tunnel number two and we obtain Theorem 1, and its Corollary 1.1, for which statement we need the following definition.

Definition 1. Let s be a properly embedded arc in a ball B. Suppose the knot obtained by capping off s along ∂B has tunnel number one. We say that s is *μ-primitive* if there is a trivial arc t properly embedded in B, disjoint from s, such that the tangle $(B, s \cup t)$ is free[4].

Remark 1. Note that a string s is a μ-primitive if and only if the knot obtained by capping off s along ∂B is a μ-primitive knot[5].

Theorem 1. *Let K be a tunnel number two knot with a 2-string free tangle decomposition. Then both strings of some of the tangles are μ-primitive.*[6]

Corollary 1.1. *Let K be a knot with a 2-string free tangle decomposition where no tangle has both strings being μ-primitive. Then $t(K) = 3$.*

The only examples of prime knots whose tunnel number degenerates under connected sum are the ones given by Morimoto, and in this case the tunnel number only degenerates by one. Also, in [8] Kobayashi and Rieck, and also in [15] Morimoto, proved that the tunnel number of the connected sum of m-small[7] knots doesn't degenerate. With this and other results in perspective, Moriah conjectured in [11] that the tunnel number of the connected sum of prime knots doesn't degenerate by more than one: $t(K_1\#K_2) \geq t(K_1) + t(K_2) - 1$, for K_1 and K_2 prime knots.
In this paper, we construct infinitely many counter-examples to this conjecture as in Theorem 2 and its Corollary 2.1.

Theorem 2. *There are infinitely many tunnel number three prime knots K_1 such that, for any 3-bridge knot K_2, $t(K_1\#K_2) \leq 3$.*

Corollary 2.1. *There are infinitely many prime knots K_1 and K_2 where*

$$t(K_1\#K_2) = t(K_1) + t(K_2) - 2.$$

[4]A tangle is *free* if the complement of a regular neighborhood of the strings is a handlebody.

[5]For a definition of *μ-primitive* knot see Definition 5.13 of the survey paper [11] by Moriah.

[6]The correspondent result to Theorem 1 for links is proved by the author in [20].

[7]A knot is said m-small if there is no incompressible surface with meridional boundary components in its complement.

In [27], Scharlemann and Schultens introduced the concept of degeneration ratio for the connected sum of two prime knots, K_1 and K_2:

$$d(K_1, K_2) = \frac{t(K_1) + t(K_2) - t(K_1\#K_2)}{t(K_1) + t(K_2)}.$$

If the knots K_1 and K_2 behave additively we have $d(K_1, K_2) = 0$.
In case the knots K_1 and K_2 have supper-additive behavior then $-\frac{1}{2} \leq d(K_1, K_2) < 0$. The minimum is achieved by the examples of Morimoto, Sakuma and Yokota in [17]. From the examples of Moriah and Rubinstein in [12] we can choose a sequence of pairs of prime knots (K_1, K_2), with super-additive behavior, where $d(K_1, K_2)$ converges to zero.
For the sub-additive behavior of the tunnel number, the degeneration ratio is not so well understood. Naturally $d(K_1, K_2) > 0$, and from Corollary 9.2 in [27], $d(K_1, K_2) \leq \frac{3}{5}$. The examples of Morimoto in [13] have degeneration ratio $\frac{1}{3}$. The examples from Corollary 2.1 have degeneration ratio $\frac{2}{5}$. If K_1 is a knot as in the statement of the Theorem 2 and K_2 is any 3-bridge knot with tunnel number one, from the main theorem of Morimoto in [14], $t(K_1\#K_2) = 3$. Hence, the degeneration ratio for these knots is $\frac{1}{4}$. So, for sub-additive behavior, from the results in this paper we have the lowest known degeneration ratio for the connected sum of prime knots[8], $\frac{1}{4}$, and also the highest, $\frac{2}{5}$.

The proof of Theorem 2 is a consequence of Morimoto's work in [16] and Theorem 1, and is explained in Section 8. For the proof of Theorem 1, we present the setting in Section 2. In Sections 3 and 4 we prove some auxiliary technical lemmas that are used along the paper. In Sections 5 and 6 we present the main lemmas that together give an outline of the proof. And finally in Section 7 we organize all the information to prove Theorem 1. For this proof, new and deeper arguments of innermost-arc type are developed to study the 2-string free tangle decomposition of K with respect to a minimal unknotting tunnel system of K.

2. Preliminaries

Let K be a tunnel number two knot in S^3 with a 2-string essential[9] free tangle decomposition defined by the 2-sphere S. We represent this tangle decomposition by $(S^3, K) = (B_1, \mathcal{T}_1) \cup_S (B_2, \mathcal{T}_2)$. As the tangles are free, their strings have no local knots[10]. This property and the next lemma will be frequently used along this paper.

Lemma 2.1. *The two strings of a 2-string essential free tangle are not parallell*[11]*.*

Proof. Let $(B, s_1 \cup s_2)$ be a 2-string essential free tangle. Suppose that s_1 and s_2 are parallel, and let D be a disk in B with boundary the strings $s_1 \cup s_2$ and two arcs in ∂B connecting the ends of these strings. As s_1 and s_2 are parallel, from Theorem 1' of [2], the strings are knotted in B.
Let N be a regular neighborhood of D in B. Hence, N is a regular neighborhood of s_1 and of

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[8]From work of Morimoto in [16], there are pairs of prime knots with tunnel number two that also have degeneration ratio of $\frac{1}{4}$. In this paper, the same degeneration ratio is obtained with a tunnel number three knot and a tunnel number one knot.

[9]Along the following sections we are assuming the tangle decomposition is essential. The inessential case is observed in the proof of Theorem 1.

[10]We say that a tangle $(B, \mathcal{T})$ contains a *local knot*, if there is a ball in B intersecting a single string of $\mathcal{T}$ at a knotted arc.

[11]We say that the strings of a tangle $(B; s_1, s_2)$ are *parallel* if there is an embedded disk D in B with boundary the strings $s_1 \cup s_2$ and two arcs in ∂B connecting the ends of these strings.

s_2. We have that $B - int\, N$ is embedded in $B - s_1 \cup s_2$ and ∂N is a proper essential surface in $B - s_1 \cup s_2$. So, $\pi_1(B - intN)$, that is not a free group, injects into $\pi_1(B - (s_1 \cup s_2))$, that is free, which is a contradiction. So, the strings $s_1 \cup s_2$ are not parallel. $\square$

As in the statement of Theorem 1, we want to prove that the two strings of $(B_1, \mathcal{T}_1)$ or $(B_2, \mathcal{T}_2)$ are μ-primitive. With this purpose, it is useful to consider the following characterization of μ-primitive string.

Lemma 2.2. *Let s be a string properly embedded in a ball B. Then s is μ-primitive if and only if s is trivial in a solid torus T in B intersecting ∂B in a single disk and whose complement in B is also a solid torus.*

Proof. Assume s is μ-primitive in B. Then there is a trivial string t in B, disjoint from s and where $(B, s \cup t)$ is a free tangle. Let $T' = B - intN(t)$[12]. As t is trivial in B we have that T' is a solid torus and, from Theorem 1' in [2], s is trivial in T'. Consider the annulus $A = \partial B \cap \partial T'$. Let D' be a disk in A that is a regular neighborhood of an arc in A connecting the two boundary components of A. We have that $A - intD'$ is also a disk D. Consider a regular neighborhood of D' in T' and isotope $\partial T'$, along the neighborhood of D', away from D'. We are left with a solid torus T in B, intersecting ∂B at the disk D. Furthermore, the complement of T in B is also a solid torus, it is a 1-handle attached to a ball, and s is trivial in T.
Assume now that s is a trivial string in a solid torus T in B intersecting ∂B in a single disk and whose complement in B is also a solid torus. Take a meridian disk L of the complement of T in B not intersecting S. Add the 2-handle with core L to T. We have that $R = N(L) \cup T$ is a ball intersecting ∂B in a single disk. So, the complement of R in B is a ball. We isotope ∂R to ∂B along this ball, and from T we obtain the solid torus T', and from the disk L we obtain the disk L'. We have that $\partial T' \cap \partial B$ is an annulus and the complement of T' in B is the cylinder $N(L')$, where $N(L')$ intersects ∂B in two disks. Let t be the co-core arc of $N(L')$. Hence, as T' is a solid torus, t is a trivial string in B. Also, $N(t) = N(L')$ and s is trivial in the complement of $N(t)$. Therefore, $(B, t \cup s)$ is a free tangle, and s is μ-primitive. $\square$

Consider an unknotting tunnel system of K, $\{t_1, t_2\}$, and the respective union of regular neighborhoods to be $V = N(K) \cup (N(t_1) \cup N(t_2))$. So, $W = S^3 - intV$ is a handlebody and $S^3 = V \cup W$ is a genus three Heegaard decomposition of S^3. Taking $K \cup t_1 \cup t_2$ in general position with respect to S, we can assume that $S \cap V$ is a collection of essential disks: $S \cap V = D_1^* \cup \cdots \cup D_{n_1}^* \cup D_1 \cup \cdots \cup D_{n_2}$, where D_i^* , $i = 1, \ldots, n_1$, are the disks of $S \cap V$ intersecting K. Let $\mathcal{D}^* = D_1^* \cup \cdots \cup D_{n_1}^*$ and $\mathcal{D} = D_1 \cup \cdots \cup D_{n_2}$.

Lemma 2.3.

(a) *There is no 2-sphere in V defining a tangle decomposition of K isotopic to the one defined by S.*

(b) *Let C be a component of $V - V \cap S$ that intersects K. Then $C \cap K$ is parallel to the boundary of C.*

Proof. As $V = N(K) \cup N(t_1) \cup N(t_2)$ there is an annulus A in V with $\partial A = K \cup b$, where b is a simple closed curve in ∂V in general position with $S \cap V$. As $K \cap \mathcal{D}^*$ is non-empty, $A \cap S$ is also non-empty. Assume that $|A \cap S|$ is minimal.
First assume that some arc γ of $A \cap S$ has both ends in a string s from the tangle decomposition, and also that γ co-bounds a disk in A together with the string s, that intersects $S \cap V$ only at γ. As γ is in S, we have that s is trivial in the respective tangle decomposition, which contradicts

[12]For a manifold X smoothly embedded in the manifold Y, we denote by $N(X)$ the regular neighborhood of X in Y.

the tangle decomposition being essential.
Suppose $A \cap S$ contains a simple closed curve c essential in A. Then K is isotopic to c. As c is a simple closed curve in S it bounds a disk in S. Therefore, in this case, K would be unknotted, which is a contradiction. Therefore, if c is a simple closed curve of $A \cap S$ then c bounds a disk in A.

(a) Suppose there is a 2-sphere in V defining a tangle decomposition isotopic to the one defined by S, and, abusing notation, denote it also by S. Hence, $S \subset V$. So, there cannot be arcs of $A \cap S$ between K and b. Let B be the ball bounded by S in V. Suppose $A \cap S$ contains some simple closed curve c. As observed before, c bounds a disk D in A. Suppose that c is an innermost simple closed curve of $A \cap S$ in A. Then, D intersects S only at c. As $S \subset V$, if c bounds a disk S disjoint from K we can reduce $|A \cap S|$, which is a contradiction to the minimality of $|A \cap S|$. Otherwise, both disks bounded by c in S intersect K, which contradicts the surface $S - intN(K)$ being essential in $S^3 - N(K)$. Then, $A \cap S$ contains no simple closed curves. Then, from the previous observations, the components of $A \cap B$ are two disks co-bounded by the strings of the tangle in B and two arcs of $A \cap S$. As each disk of $A \cap B$ intersects S only at a single arc in its boundary, we have that both strings of the tangle $(B, B \cap K)$ are trivial, which is a contradiction to the tangle decomposition defined by S being essential.

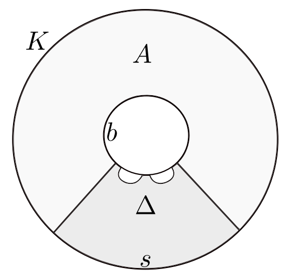

FIGURE 1: An annulus A in V with boundary being K and a curve b in the boundary of V. The disk Δ is a disk in the component C of $V - V \cap S$, with boundary being the string s and a curve in the boundary of C.

(b) To prove part (b) of this lemma we just need to prove that $A \cap S$ contains no simple closed curves, and that no arc of $A \cap S$ has both ends in ends of strings from the tangle decomposition. Assume now $A \cap S$ contains a simple closed curve, c. As observed before, c bounds a disk D in A; suppose that it is an innermost curve with this property. Let L be the disk bounded by c in $S \cap V$. If L doesn't intersect K then we can reduce $|A \cap S|$, which contradicts the minimality of $|A \cap S|$. If L intersects K in less than four points then D contradicts the tangle decomposition defined by S being essential. If L intersects K in four points then the tangles decomposition defined by $D \cup L \subset V$ and S are isotopic, which is a contradiction to (a). Then $A \cap (S \cap V)$ contains no simple closed curve.
From the previous arguments all arcs of $A \cap S$ either have both ends in b or one end in b and the other at an end of a string in C. Also, as A is in general position with $S \cap V$, each string end is attached to a single arc of $A \cap S$. Let C be a component of $V - V \cap S$ that intersects K. Then each string s of C belongs to the boundary of a properly embedded disk component of $A - A \cap S$ in C, disjoint from the other string components in C, as in Figure 1. Therefore, all components of $C \cap K$ are independently parallel to the boundary of C, which gives us the statement (b) of the lemma. □

Considering the previous lemma and that all 2-spheres in S^3 intersect K an even number of times, no disk of $\mathcal{D}$ is parallel to a disk of $\mathcal{D}^*$ in V.

From the work of Ozawa [23], we know that if a knot has an essential 2-string free tangle decomposition then this decomposition is unique up to *isotopy*[13], and, furthermore, K is n-string prime for $n \neq 2$. (In particular, K is prime.) This is a result frequently throghout this paper, and we refer to it as Ozawa's unicity theorem.
We assume the unknotting tunnel system and the tangle decomposition defined by S up to isotopy are such that $S \cap V$ is a collection of disks with minimum cardinality $|S \cap V|$[14]. From Lemma 2.3 and the minimality of $|S \cap V|$, we can assume that all disks $S \cap V$ are essential in V. As S decomposes K in two 2-string tangles we have $n_1 \leq 4$. If $n_1 \geq 3$, we denote the string with one end in D_i^* and the other end in D_j^* by s_{ij}.

Let P denote the planar surface $S \cap W$. By the minimality of $|S \cap V|$ and the incompressibility of $S - int(N(K))$ in $S^3 - int(N(K))$, we have that P is essential in W.
For a complete system of meridian disks[15] of W, $\{E_1, E_2, E_3\}$, we write $E = E_1 \cup E_2 \cup E_3$. Considering E and P in general position, we choose E such that $|P \cap E|$ is minimal between the complete systems of meridian disks of W.
By the incompressibility of P and the minimality of $|P \cap E|$, no component of $P \cap E$ is a simple closed curve. Also, if an arc component of $P \cap E$ is a loop co-bounding a disk in P disjoint from $P \cap E$, using this disk, we can change the complete system of meridian disks of W to E' with $|P \cap E| > |P \cap E'|$. This is a contradiction, and therefore $P \cap E$ is a collection of essential arcs[16] in P.
With the arcs of $P \cap E$ we define a graph in S that we denote by G_P: the vertices are the disks from $S \cap V$, each of which corresponds to a boundary component of P, and the edges are the arcs $P \cap E$. The graph G_P is connected: in fact, if the graph G_P is not connected then by cutting along a complete system of meridian disks of W we can find a compressing disk for P in W, which is a contradiction as P is essential. As G_P is a connected graph in a 2-sphere, from the arcs $P \cap E$ in E we can create a sequence of isotopies of type A[17] over a sequence of arcs $\alpha_1, \alpha_2, \dots, \alpha_m$ of $P \cap E$ such that the closure of the components of $P - \alpha_1 \cup \dots \cup \alpha_m$ is a collection of disks.

The vertices of G_P associated to $\mathcal{D}$, resp. $\mathcal{D}^*$, are referred to as *d-vertices*, resp. *d*-vertices*, and are illustrated as white disks, resp. dark disks. Between the edges of G_P it is useful to define some types of arcs as follows. (See Figure 2.)

Type I: is an edge connected to a single vertex.
Type II: is an edge connected to two distinct vertices.
d-arc: is an edge with at least one end attached to a d-vertex.
d-arc*: is an edge with at least one end attached to a d*-vertex.

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[13] We say that two tangle decompositions of a knot K defined by the 2-spheres S and S' are *isotopic*, if there is an ambient isotopy of $S \cup K$ to $S' \cup K$.

[14] For a topological space X, $|X|$ denotes the number of connected components of X.

[15] A *complete system of meridian disks* of a handlebody H is a collection of disks in H whose complement is a ball.

[16] An arc α in P is *essential* if the components closure of $P - \alpha$ doesn't contain any disk component.

[17] See chapter 2 of [1] by Jaco for a definition of an *isotopy of type A*, and section 2 of [22] by Ochiai for a definition of the latter and also of an *inverse isotopy of Type A*.

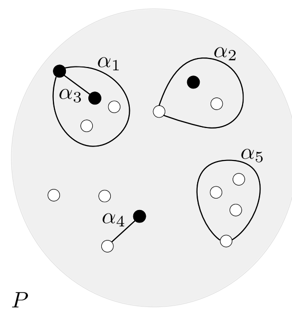


FIGURE 2: An illustration of some arc components of $E \cap P$ in P. The arc α_1 (resp., α_2) is a sk-arc (resp., st-arc). The arc α_3 is a d*-arc that is also a k-arc, and α_4 is both a d-arc and a d*-arc. Note also that the arc α_5 is an example of a type I d-arc that is not a st-arc.

t-arc: is an edge of type II, α_i, in a sequence of isotopies of type A as above, connected to some d-vertex D and where α_j, $j < i$, is disjoint from D. (See Remark 2 and also Lemma 1 of [22] by Ochiai.)

k-arc: is a type II arc connecting two d*-vertices.

st-arc: is a type I d-arc separating P into two planar surfaces, each with some boundary component of $\mathcal{D}^*$.

sk-arc: is a type I d*-arc separating P into two planar surfaces, each with some boundary component of $\mathcal{D}^*$.

Remark 2. Suppose α is a type II d-arc with one end in the d-vertex associated with D. If one of the disks separated by α from E intersects the disk D only at the end of α in D, then all arcs of $E \cap P$ in this disk have no end in D. This implies that α is a t-arc. (See Figure 3.) In Lemma 2.4 we prove that such arcs cannot exist.

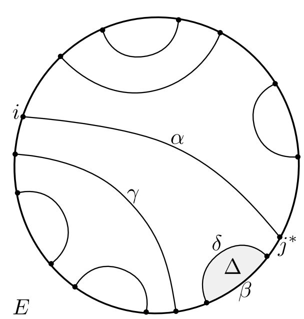


FIGURE 3: An illustration of arc components of $E \cap P$ in some component of E. If an arc of $E \cap P$ in E has one end in the disk D_i, resp. D^*_j, then we label the end of the arc in E by i, resp. by j^*. The ends of the arc α in the figure exemplify this notation. If all arcs in one of the disks separated by α from E have no ends being i, then α is a t-arc.

We say that an arc δ of $P \cap E$ is an *outermost arc*, if δ separates a disk component Δ of $E - E \cap S$ from E. We have $\Delta \cap P = \delta$ and $\partial\Delta = \delta \cup \beta$ with $\beta \subset \partial E$. The disk Δ is said to be an *outermost disk*. (See Figure 3.) An outermost disk is said to be *over* a component of

$V - V \cap S$ if the correspondent arc β is in the (boundary) of the component.

In the next lemma, we study the arcs $P \cap E$ in P and in E and obtain properties that give the base setting for these arcs along the work in this paper.

Lemma 2.4.

(a) *All outermost arcs are of type I.*

(b) *If n is the number of vertices of G_P then either $n = 1$ and the graph G_P has no edges, or $n \geq 3$ and at least two vertices of G_P are not adjacent to edges of type I.*

(c) *No arc of $E \cap P$ is a t-arc.*

(d) *The outermost d-arcs of $E \cap P$ in E are of type I.*

(e) *Each type I arc of $E \cap P$ is a st-arc or sk-arc.*

(f) *All d-vertices are adjacent to a st-arc.*

(g) *Each outermost arc of $E \cap P$ in E is a st-arc or a sk-arc.*

Proof.

(a) Suppose some outermost arc is of type II. Then, proceeding with an isotopy of type A along the respective outermost disk we can reduce $|S \cap V|$, which is a contradiction to the minimality of $|S \cap V|$.

(b) If $n = 1, 2$ and there is some loop in G_P, the outermost loop co-bounds a disk in P. Furthermore, if G_P has no loops and $n = 2$ then the outermost arcs of $E \cap P$ in E are of type II, which is a contradiction to (a). If $n \geq 3$ and at most one vertex is not adjacent to a loop, then one outermost loop co-bounds a disk in P. In both cases we contradict the fact that all edges of G_P are essential in P.

(c) If there is a t-arc, then by a sequence of isotopies of type A followed by a sequence of inverse isotopies of type A, as in Lemma 1 of [22] by Ochiai, we can ambient isotope S, in the exterior of K, to some 2-sphere S' where $|S \cap V| > |S' \cap V|$. This is a contradiction to the minimality of $|S \cap V|$.

(d) If an outermost d-arc of $E \cap P$ in E is of type II then it is a t-arc, which is a contradiction to (c). Therefore, the outermost d-arcs of $E \cap P$ in E are of type I.

(e) Let α be a type I arc of $E \cap P$. As α is essential in P it separates P into two components that are not disks. If one of these components, say F, only contains boundary components corresponding to d-vertices there is some arc of $E \cap P$ in F that is a t-arc, which contradicts (c).

(f) Assume there is a d-vertex D that is only adjacent to edges of type II. Then there is a t-arc with respect to D (choose an outermost arc, in E, between the edges of type II attached to D), which is a contradiction to (c). Hence, there is at least one edge of type I with ends in D, and from (e) it is a st-arc.

(g) From (a) the outermost arcs are of type I, and from (e) the type I arcs are st or sk-arcs. □

3. Outermost disks over ball components of $V - V \cap S$

In this section we study the case when there is an outermost disk over some ball component of $V - V \cap S$, as in Lemma 3.2. We also have presented other crucial lemmas relating ball components of $V - V \cap S$ and certain disks of $E - E \cap P$, together with the next lemma where we show several properties of tangles obtained from balls in B_1 or B_2.

Lemma 3.1. *Suppose there is a ball Q in one of the tangles defined by S that intersects each string of the tangle in a single arc.*

(a) *Let Q^c denote the complement of Q in S^3. The tangle $(Q^c, Q^c \cap K)$ is essential.*

(b) *If one of the strings of $Q \cap K$ is unknotted in Q then either the tangle $(Q, Q \cap K)$ is trivial or some string of some tangle defined by S is unknotted.*

(c) *Suppose both strings of the tangle are in Q and have ends in one or two disk components of $Q \cap S$. Then the tangle $(Q, Q \cap K)$ is essential.*
(d) *If a ball component of $V - S \cap V$ contains a string with both ends in the same component of $\mathcal{D}^*$, then some string of some tangle is unknotted.*

Proof.
Assume that the tangle containing Q is $(B_1, \mathcal{T}_1)$.

(a) Suppose that $(Q^c, Q^c \cap K)$ isn't essential. As this tangle contains only two strings, both strings are trivial in it. Let s' be an arc component of $Q^c \cap K$, and D' be a disk in Q^c with interior disjoint from K and with boundary being the union of s' and an arc in ∂Q^c. Let s be the string from the tangle decomposition defined by S that is a subset of s'. So, s is a string of the tangle $(B_2, \mathcal{T}_2)$. As s' contains only the string s of $K - S \cap K$, we have that $\partial D'$ intersects S only at two points, which are the end points of s. Considering a minimal collection $D' \cap S$ and following an innermost curve or arc type of argument, we can prove that $D' \cap S$ is a single arc a with ends being the ends of $s \subset \partial D'$. Let D be the disk in D' with boundary defined by the arcs $a \subset S$ and s. Then D is in the tangle $(B_2, \mathcal{T}_2)$ and the interior of D doesn't intersect S. Therefore, the string s is trivial in $(B_2, \mathcal{T}_2)$, which is a contradiction to the tangle decomposition defined by S being essential.

(b) Assume that one of the strings of $Q \cap K$ is unknotted in B. If the tangle $(Q, Q \cap K)$ is essential then, from (a), the 2-sphere ∂Q defines a 2-string essential tangle decomposition of K. By Ozawa's unicity theorem, the tangle decompositions given by S and ∂Q are isotopic. Hence, as one string of $(Q, Q \cap K)$ is unknotted, some string of some tangle defined by S is also unknotted. Otherwise, the tangle $(Q, Q \cap K)$ is trivial.

(c) Suppose the tangle $(Q, Q \cap K)$ is trivial. Let Q' be obtained from Q after we isotope away from S in B_1 the components of $Q \cap S$ that don't contain any string ends. If Q' intersects S in a disk with all string ends in it, as the strings are trivial in Q' they are both unknotted in $(B_1, \mathcal{T}_1)$. From Theorem 1' in [2], this is a contradiction to the tangle $(B_1, \mathcal{T}_1)$ being essential. Otherwise, if Q' intersects S in two disks that also contain the strings ends in them. As the tangle $(B_1, \mathcal{T}_1)$ is free, following an argument as in Lemma 2.1, the complement of Q' in B_1 is a solid torus. Then $\partial Q'$ is ambient isotopic to S in $S^3 - K$, which is also a contradiction to the tangle $(B_1, \mathcal{T}_1)$ being essential. So, the tangle $(Q, Q \cap K)$ is essential.

(d) Suppose there is a ball component C of $V - V \cap S$ containing a string s with both ends in the same component of $\mathcal{D}^*$. From Lemma 2.3, the tangle $(C, C \cap K)$ is trivial. Consequently, the string s is trivial in C. As the ends of s are in the same disk of $C \cap S$, it is also unknotted in the respective tangle defined by S. □

Lemma 3.2. *If there is an outermost disk over a ball component of $V - (S \cap V)$ then some string of some tangle is unknotted.*

Proof. Suppose there is an outermost disk Δ over a ball component C of $V - (S \cap V)$, and let δ be the respective outermost arc. Without loss of generality assume that $C \subset B_1$. Let A be the annulus in the intersection of $C \subset V$ with the 2-sphere obtained after an isotopy, along a regular neighborhood of Δ, of a regular neighborhood of δ in S into V. The component C is either disjoint from K or contains one or both strings of $\mathcal{T}_1$. Then, a core of A bounds a disk D in ∂C that is either disjoint or intersects K in one or two points. We isotope $int\, D$ into C, slightly, such that $D \cap S = \partial D$.

Assume D is disjoint from K. The arc δ union an arc component of $\partial P - \partial \delta$ is a simple closed curve parallel, in $S - K$, to a core of A. Also, the arc δ separates P into two planar surfaces containing boundary components of $\mathcal{D}^*$. Therefore, D separates the strings of the tangle $(B_1, \mathcal{T}_1)$

and intersects S only at ∂D, which is a contradiction to the tangle decomposition defined by S being essential.

Assume that $|D \cap K| = 1$. Let D' be the disk in S with $\partial D' = \partial D$ and $|D' \cap K| = 1$, and Q be the ball in B_1 bounded by $D \cup D'$. Then $Q \cap \mathcal{T}_1$ is a single trivial arc in Q. So, considering the 2-sphere $S' = (S - D') \cup D$, the tangle decompositions defined by S and S' are isotopic with $|S' \cap V| < |S \cap V|$, which is a contradiction to the minimality of $|S \cap V|$.

Assume now that $|D \cap K| = 2$. Then D splits the tangle $(B_1, \mathcal{T}_1)$ in two 2-string tangles: $(B_1', \mathcal{T}_1')$ and $(B_1'', \mathcal{T}_1'')$. If D intersects K in the same string of $\mathcal{T}_1$ then one string of this tangle, say s_1, is either in $(B_1', \mathcal{T}_1')$ or in $(B_1'', \mathcal{T}_1'')$. Assume, without loss of generality, that s_1 is in $(B_1', \mathcal{T}_1')$. From Lemma 3.1 (a), if the tangle $(B_1', \mathcal{T}_1')$ is essential then $\partial B_1'$ defines an essential 2-string tangle decomposition of K with $|\partial B_1' \cap V| < |S \cap V|$, which contradicts the minimality of $|S \cap V|$. Hence, the tangle $(B_1', \mathcal{T}_1')$ is trivial. So, s_1 is trivial in $(B_1', \mathcal{T}_1')$ and therefore unknotted in $(B_1, \mathcal{T}_1)$. Otherwise, assume that D intersects K in different strings of $\mathcal{T}_1$. By a similar argument as when D intersects K in the same string we can prove that the tangles $(B_1', \mathcal{T}_1')$ and $(B_1'', \mathcal{T}_1'')$ are trivial. Then the string $s_1 \cap B_1'$ is trivial in B_1' and the string $s_1 \cap B_1''$ is trivial in B_1'', which implies that s_1 is unknotted in $(B_1, \mathcal{T}_1)$. □

Remark 3. From Lemma 3.2, if some outermost disk is over a ball component of $V - S \cap V$ then we have Theorem 1. So, we can assume that all outermost disks are over components of $V - S \cap V$ other than balls.

We say that two arcs of $E \cap P$ are *parallel* in E if the union of these arcs cuts a disk component of $E - E \cap P$ from E. An arc outermost in E between the arcs of $E \cap P$ not in a sequence of parallel arcs to a outermost arc is said to be a *second-outermost arc.* A disk of $E - E \cap S$ in the outermost side of a second-outermost arc is called a *second-outermost disk.* The arcs α and γ in Figure 3 are examples of second-outermost arcs.

Let γ and γ' be two type I arcs of $E \cap P$ parallel in E attached to disks D and D' of $S \cap V$, resp., parallel in V. Denote by Γ the disk cut by $\gamma \cup \gamma'$ from E, and by C the ball component of $S - S \cap V$ cut by $D \cup D'$ from V. Suppose that C and Γ are in the same ball component bounded by S, say B_1. Then Γ is a proper surface in the complement of the solid torus $B_2 \cup_{D \cup D'} C$ in S^3, which is $B_1 - intC$. The curve $\partial\Gamma$ is inessential in the boundary of $B_2 \cup_{D \cup D'} C$, and as we are in S^3, it bounds a disk in $\partial(B_2 \cup_{D \cup D'} C)$. Let L be the disk bounded by $\partial\Gamma$ in $\partial(B_1 - intC)$. Note that L intersects S in two disks and C in a disk band from D to D'. Let R be the ball bounded by $\Gamma \cup L$ in $B_1 - intC$. For the next lemma, denote by q a core arc of C in B_1, this is a proper arc in B_1 with regular neighborhood C. This construction and the following lemma will be frequently used throughout this paper.

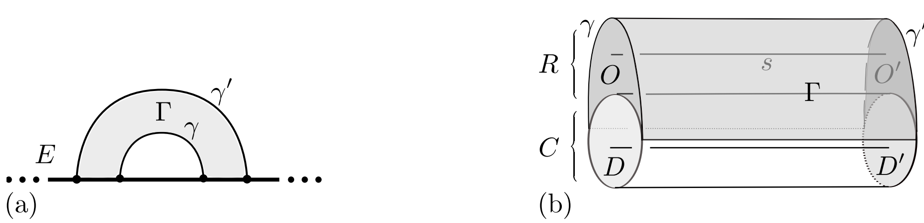


FIGURE 4

Lemma 3.3. *The ball R contains a single string of $\mathcal{T}_1$, and this string is parallel to q in B_1.*

Proof. Denote by O and O' the disks of $L \cap S$, which are the disks cut by γ and γ', resp., in $S - int(D \cup D')$. As γ, and γ', is a st or sk-arc we have that O and O' contain some component of $\mathcal{D}^*$. This means that R contains some string(s) of $\mathcal{T}_1$.
Suppose that $\mathcal{T}_1$ is in R. From Lemma 3.1(a) and (c), we have that ∂R defines a 2-string essential tangle decomposition of K. As the 2-string essential tangle decomposition of K is unique, we have that the tangle decompositions defined by S and ∂R are isotopic. But $|\partial R \cap V| < |S \cap V|$, which is a contradiction to the minimality of $|S \cap V|$.
Then R contains a single string, s, of $\mathcal{T}_1$. As there are no local knots, s is trivial in R. The intersection of R with C is the disk $L - O \cup O'$, that intersects each D and D' at an arc. Let a be an arc in $L - O \cup O'$ with one end in $D \cup O$ and other in $D' \cup O'$. Then, s (resp., q) is parallel to a in R (resp., C) through a disk with boundaries being $s \cup a$ (resp., $a \cup q$) and two arcs in $O \cup O'$ (resp., $D \cup D'$). As R intersects C in $L - O \cup O'$, we have that s and q are parallel through a disk with boundaries being $s \cup q$ and two arcs in S. Consequently, s and q are parallel in B_1. □

Corollary 2.2. *The disks D and D' cannot be disks of $\mathcal{D}^*$.*

Proof. As no disk of $\mathcal{D}$ is parallel to a disk of $\mathcal{D}^*$ in V, suppose both D and D' are disks of $\mathcal{D}^*$. Then C contains some string(s) of $\mathcal{T}_1$. As R contains a string of $\mathcal{T}_1$ we have that C contains a single string of $\mathcal{T}_1$, and from Lemma 2.3(b) this string is also a core of C. Therefore, from Lemma 3.3, the strings of $\mathcal{T}_1$ in R and in C are parallel in B_1, which is a contradiction to Lemma 2.1. □

Lemma 3.4. *Let D_k, D_i^* and D_j^* be disks of $S \cap V$ where $D_k \cup D_i^* \cup D_j^*$ cuts a ball component C of $V - V \cap S$ from V; assume that C intersects K at a single string, with one end at D_i^* and the other at D_j^*. Suppose there is a disk component of $E - E \cap P$, in the same tangle component as C, that intersects S in arcs where all but one of these arcs have both ends in D_k, and the remaining arc has either at least one end in D_k, or one end in D_i^* and the other in D_j^*. Then some string of some tangle is unknotted.*

Proof. Denote by Γ the disk component of $E - E \cap P$ referred to in the statement, and by γ the arc of $\Gamma \cap S$ that doesn't have by assumption both ends in D_k, as in Figure 5(a). Without loss of generality, suppose C is in B_1. Let s and s' be the strings in this tangle with s in C, and C_i denote the cylinder obtained from C after an isotopy pushing D_j^* away from S in B_1. Consider also the solid torus T_i defined by $B_2 \cup_{D_i^* \cup D_k} C_i$.
Assume that γ also has both ends in D_k.

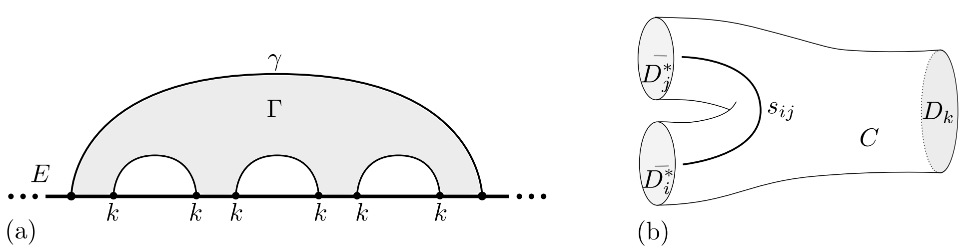


FIGURE 5: (a) Arc γ after the outermost arcs attached to D_k. The label k at an end of an arc means the end is at the disk D_k. (b) The ball C cut by $D_i^* \cup D_j^* \cup D_k$ from V, and the string s_{ij} in it.

The curve $\partial\Gamma$ is inessential in T_i and it bounds a disk in ∂T_i that we denote by L. The disk L

intersects $\mathcal{D}^*$. In fact, suppose L is disjoint from $\mathcal{D}^*$, and consider a disk D of $L \cap C_i$ or $L \cap S$ with boundary intersecting ∂L at a single component. Then, if $D \subset C_i$ it is also a disk in C and we get a contradiction to the minimality of $|P \cap E|$, and if $D \subset S$ we obtain a contradiction with Lemma 2.4(e).
Consider the ball R in B_1 bounded by $\Gamma \cup L$. As L contains some component of $\mathcal{D}^*$ the ball R intersects $\mathcal{T}_1$; it contains at most two arcs, the string s' or a portion of the string s.
Suppose R contains the string s' only. As there are no local knots in the tangle, s' is parallel to L. By pushing L to S from ∂C_i we have that the string s' is unknotted in $(B_1, \mathcal{T}_1)$.
Suppose R contains also a portion of the string s. From Lemma 3.1(a), we have that the tangle $(R^c, R^c \cap K)$, where R^c is the complement of R in S^3, is essential. As $|\partial R \cap V| < |S \cap V|$, if the tangle $(R, R \cap K)$ is essential, we have a contradiction to the minimality of $|S \cap V|$. Therefore, $(R, R \cap K)$ is a trivial tangle. Then, the string s' is unknotted in R and parallel to the disk L. By an isotopy of L from T_i to S we have that the string s' is also unknotted in $(B_1, \mathcal{T}_1)$.
So, we can assume that $R \cap K$ is only a portion of the string s. Consider the solid torus $T_i' = T_i \cup R$, and the complement in B_1 of the ball obtained by cutting T_i' along D_i^* that we denote by Q. Then, Q is a ball in B_1 containing s' and a portion of s. The 2-sphere ∂Q is isotopic to S rel. $Q \cap S$ in B_1. Then, if s is unknotted in Q it is also unknotted in B_1. As $|\partial Q \cap V| < |S \cap V|$, following a similar reasoning as when R contains two arcs, we also have that some string of some tangle is unknotted.

Assume now that γ has only one end at D_k.
Suppose, without loss of generality, that the other end of γ is in D_i^*. We isotope S along a regular neighborhood of a disk in C intersecting K once, intersecting the disk D_k along a single arc, and separating D_i^* from D_j^* in ∂C. In this way, we split D_k in two disks D_k and $D_{k'}$, and C in two cylinders from D_k to D_i^*, C_{k,i^*}, and from $D_{k'}$ to D_j^*, C_{k',j^*}. The boundary of Γ lies in S, and in the boundaries of the balls C_{k,i^*} and C_{k',j^*}. The arcs of $\partial\Gamma \cap C_{k',j^*}$ have both ends attached to $D_{k'}$. Hence, we can isotope these arcs to S. Also, all but one arc of $\partial\Gamma \cap C_{k,i^*}$ has both ends in D_k. The other arc has one end in D_i^* and the other in D_k. We isotope all arcs of $\partial\Gamma \cap (C_{k,i^*} \cup C_{k',j^*})$ with both ends in D_k or both ends in $D_{k'}$ from $\partial C_{k,i^*}$ or $\partial C_{k',j^*}$ to S, respectively. We are left with the disk Γ with boundary defined by one arc in S and other arc in the boundary of C_{k,i^*} from D_i^* to D_k. Using this disk we can isotope C_{k,i^*} through S. We did an isotopy of V where we obtain a new 2-string tangle decomposition of K, that contains in each tangle a string from the original tangle decomposition defined by S. We also reduced $|S \cap V|$. So, the new tangle decomposition cannot be essential, which implies that some string of the original tangle decomposition defined by S is unknotted.

Assume at last that γ has one end in D_i^* and the other end in D_j^*.
Then each arc of $\Gamma \cap S$ co-bounds a disk in $S - intD_k$, with ∂D_k, containing $D_i^* \cup D_j^*$, and other disk containing none of these disks. Hence, we can isotope $\partial\Gamma$ to lie in ∂C with the exception of γ. So, after the isotopy $\partial\Gamma$ is defined by γ and an arc in C from D_i^* to D_j^*. The string s is trivial in C and therefore it is parallel to the arc $\partial\Gamma \cap C$. Therefore, the string s is unknotted in $(B_1, \mathcal{T}_1)$. □

4. Outermost disks over torus components of $V - V \cap S$

In this section, we prove several lemmas related with the existence of outermost disks over tori components of $V - V \cap S$ disjoint or intersecting K at a single arc. These lemmas are fundamental on the proof of Theorem 1.

Lemma 4.1. *There is no outermost disk over a solid torus component of $V - V \cap S$ containing a single disk of $V \cap S$ and disjoint from K.*

Proof. Denote the disk $T \cap S$ by D. Let δ be the outermost arc co-bounding an outermost disk Δ as in the statement. Consider, also, the corresponding arc β and a disk O cut by δ in $S - intD$. Let $L = O \cup \Delta$. The disk L is a meridian for the complement of T and intersects a meridian of T once. Consider a regular neighborhood of Δ in W, $N(\Delta)$. So, $N(\Delta) \cap S$ is a regular neighborhood of δ in S, $N(\delta)$, and $N(\Delta) \cap \partial T$ is a regular neighborhood of β in ∂T, $N(\beta)$. We isotope the annulus $N(\delta) \cup D$ through $N(\Delta)$ to the annulus $N(\beta) \cup D$. As β intersects a meridian of T once, the annulus $A = N(\beta) \cup D$ is such that $T = A \times I$. Therefore, we can isotope $A \subset S$ through T to $\partial T - A$ and out of V. Let S' be the 2-sphere obtained from S after this isotopy. The tangle decomposition of K obtained from S' is the same as the one given by S. However, $|S' \cap V| < |S \cap V|$ and we contradict the minimality of $|S \cap V|$. □

Lemma 4.2. *Assume $V - V \cap S$ has a solid torus component T intersecting K in a single string and with $T \cap \mathcal{D}^*$ being a single disk. If there is an outermost disk over T then some string of some tangle is unknotted.*

Proof. Suppose T is in the tangle $(B_1, \mathcal{T}_1)$. Let s be the string $T \cap K$, that is a component of $\mathcal{T}_1$, and D^* be the component of $T \cap S$ that intersects K. Then, both ends of s are in $D^* \subset \partial T$. Let Δ be an outermost disk over T and δ the respective outermost arc in E attached to the disk D of $T \cap S$. Consider also the disk O cut by δ in $S - intD$ and disjoint from D^*. Let $L = O \cap \Delta$. Isotope the disks (if any) of $T \cap O$ away from S in B_1, and denote the resulting solid torus by T'. By adding the 2-handle with core L to T' we define a ball Q that intersects S at disk components.

Suppose $D = D^*$. As δ is a sk-arc, and two ends of strings are in D, we have that O intersects $\mathcal{D}^*$ in a a single disk. In this case, the ball Q contains the string s, and also an unknotted portion of the other string of $\mathcal{T}_1$. From Lemma 3.1, some string of some tangle defined by S is unknotted or the tangle $(Q, Q \cap K)$ is trivial. So, we can assume that s is trivial in Q. As s has both ends at the same disk component of $Q \cap S$, we have that s is unknotted in $(B_1, \mathcal{T}_1)$.

Suppose D doesn't intersect K. If O intersects K at a single point, then following the argument used in the previous case we have that some string of some tangle is unknotted. So, assume that O intersects K at a collection of two points. In this case, the ball Q contains the string s, and also two portions of the other string, s' that are unknotted in Q. So, ∂Q defines a 3-string tangle decomposition of K. Let Q^c denote the complement of Q in S^3. From Ozawa's unicity theorem, either the tangle $(Q, Q \cap K)$ or the tangle $(Q^c, Q^c \cap K)$ isn't essential. As there are no local knots in the tangles defined by S and the tangles $(Q, Q \cap K)$ and $(Q^c, Q^c \cap K)$ are 3-string tangles, the tangle that isn't essential has a trivial string. If the tangle $(Q^c, Q^c \cap K)$ isn't essential then, following an argument as in the proof of Lemma 3.1, either some string of the tangle $(B_2, \mathcal{T}_2)$ is trivial, which is a contradiction, or the string $Q^c \cap s'$ is trivial in Q^c. In the latter case isotope Q from S in such a way that $Q \cap S$ is only $D \cup O$, and denote by Q' the ball after the isotopy. Then, $(Q', Q' \cap s')$ is the product tangle and $Q'^c \cap s'$ is isotopic to $\partial Q' - \partial Q' \cap S$. Therefore, after the isotopy of $Q'^c \cap s'$ to Q' we have that s' is unknotted in $(Q', Q' \cap K)$. As s' has both ends in the same disk component of $Q' \cap S$ we have that s' is unknotted in the tangle $(B_1, \mathcal{T}_1)$. Suppose now that the 3-string tangle $(Q, Q \cap K)$ isn't essential. Then one of the strings $Q \cap K$ is trivial in this tangle. If such a string is s then the string s is unknotted in $(B_1, \mathcal{T}_1)$. If such a string is one of the arcs obtained from $Q \cap s'$, then consider the compressing disk C for ∂Q in the interior of Q and the ball Q'', containing the string s, obtained after cutting Q along C. From Lemma 3.1, either some string of $(B_1, \mathcal{T}_1)$ is unknotted or the tangle $(Q'', Q'' \cap K)$ is trivial. Then s is trivial in Q'' and unknotted in Q (that is obtained from Q'' after gluing a ball along a disk). As s has both strings in the disk $T \cap \mathcal{D}^*$ we also have that s is unknotted in the tangle $(B_1, \mathcal{T}_1)$. □

Lemma 4.3. *Let T be a torus component of $V - V \cap S$ with more than one component from $V \cap S$ in its boundary. Then there is no ball Q, in the tangles defined by S, with the following properties,*

(1) *$T \subset Q$, $\partial Q \cap \partial T$ is an annulus A that contains at least two components of $T \cap S$ union with the disks of $T \cap S$ not in A;*
(2) *$(\partial Q \cap S) \cup A$ is an annulus A', $A' - A$ is a collection of disks attached to some disks of $T \cap S$ and contain the disks of $T \cap S$ not in A;*
(3) *the two strings of a tangle are in Q and the tangle in Q with these two strings is essential.*

Proof. Suppose there is a ball Q as in the statement. From Lemma 3.1(a) the complement of Q in S^3 contains an essential tangle. As $(Q, Q \cap K)$ is an essential tangle, from Ozawa's unicity theorem, we have that the tangle decomposition of K defined by S and ∂Q are isotopic. Note that as $A' - A$ is a collection of disks in S attached to $\partial T \cap S$, $A - A \cap S = A' - A' \cap S$. Consider an arc a in $A - A \cap S$ connecting the two components of ∂A. Then a is also an arc in $A' - A' \cap S$ connecting the two different components of $\partial A'$. Consider S' after an isotopy of ∂Q along a regular neighborhood of the arc a in the complement of Q. Then, all disks of $S \cap T$ that are in A are now in a single disk component of $S' \cap T$. The sphere S' defines the same tangle decomposition to K than S does. And also, as A contains at least two components of $T \cap S$, we have $|S \cap V| > |S' \cap V|$, which contradicts the minimality of $|S \cap V|$. □

For the next lemmas assume that $n_1 \geq 3$ and consider a solid torus component T of $V - V \cap S$ intersecting K at a single arc component. Suppose there is an outermost disk Δ over T, and let δ be the respective outermost arc attached to the disk D of $V \cap S$. Assume also without loss of generality that T is in $(B_1, \mathcal{T}_1)$. Denote by s_{11} and s_{12} the strings of $\mathcal{T}_1$, and by s_{21} and s_{22} the strings of $\mathcal{T}_2$. Suppose that s_{11} is the string of $\mathcal{T}_1$ that T contains.

Lemma 4.4. *If one of the disks separated by δ in $S - intD$ contains just one disk of $V \cap S$ and it intersects K once, some string of some tangle is unknotted.*

Proof. Suppose one of the disks cut by δ in $S - intD$, say O, contains a single disk of $V \cap S$. As δ is a st or sk-arc the disk of $V \cap S$ in O is a disk D^* of $\mathcal{D}^*$, which from the statement intersects K once.
Consider the disk $L = \Delta \cup O$. As we are in S^3, by attaching the 2-handle with core L to T we obtain a ball. Consequently, as D^* is a disk of $V \cap S$ (separating or non-separating in V), by attaching a regular neighborhood of the annulus $A = L - intD^*$ to V we have a handlebody V' also of genus three. Furthermore, as A is incompressible and non-separating in W, by cutting W along A we obtain a handlebody W' of genus three. Altogether, by cutting W along A and simultaneously adding a regular neighborhood of A to V, we obtain a Heegaard decomposition of S^3, $V' \cup W'$, of the same genus as the one defined by ∂V.

Let T' be a solid torus obtained from T by an ambient isotopy of $B_1 \cap V$ taking D^* away from S in B_1. We denote by Q the ball obtained by attaching a regular neighborhood of L to T'. As T intersects K at a single arc and as L intersects K at a single point, we have that Q intersects K at two arcs, with one being unknotted.
Let T'' be a solid torus obtained from T by an ambient isotopy of $(T \cap S) - D$ away from S in B_1. We denote by R the ball obtained by attaching a regular neighborhood of L to T''. As T intersects K at a single arc and as L intersects K at a single point, we have that R intersects K at two arcs, with one being unknotted.

Suppose $n_1 = 4$.

(1) Suppose D is in $\mathcal{D}$ and D^* is not in T. Then Q intersects each string of $\mathcal{T}_1$ at a single arc. Then by Lemma 3.1(b) some string of some tangle defined by S is unknotted or the tangle $(Q, Q \cap K)$ is trivial. So, we can assume the latter. Each disk of $Q \cap (V' - intQ)$ intersects K at most at a single point. Therefore, the arcs $Q \cap K$ can be isotoped to ∂Q intersecting $Q \cap (V' - intQ)$ only at the end points. From Lemma 2.3, all the other components of $(V' - V' \cap S) - Q$ intersecting K have the same property. Furthermore, if two consecutive arcs are in adjoint components of $V' - V' \cap S$ then, after the isotopy to the boundary of the arcs in the respective components, we can choose that the common ends are at the same point of the disk of intersection between the components. (In this case, this is a consequence from each component of $V' \cap S$ intersecting K at most once and the tangle in each component of $V' - V' \cap S$ being trivial.) So, with V' being the union of components with these properties, K is parallel to $\partial V'$. We also note that there is a meridian disk of V' intersecting K once. Altogether, we have that $(V' - N(K)) \cup W'$ is a genus three Heegaard decomposition of the knot K exterior. But $|S \cap V'| < |S \cap V|$, which is a contradiction to the minimality of $|S \cap V|$.

(2) Suppose D is in $\mathcal{D}$ and D^* is in T. Note that Q intersects $\mathcal{T}_1$ at s_{11} in two arcs, with one of the arcs being unknotted in Q. If the tangle $(Q, Q \cap K)$ is trivial then following a similar argument as in (1), we obtain contradiction with the minimality of $|S \cap V|$. So, we can assume that $(Q, Q \cap K)$ is essential. Consider the complement of Q in S^3, Q^c. If the tangle $(Q^c, Q^c \cap K)$ is also essential then the tangle decomposition defined by S is isotopic to the one defined by ∂Q; as $(Q, Q \cap K)$ contains an unknotted string, this means that some string of some tangle defined by S contains an unknotted string. So, we can assume that the tangle $(Q^c, Q^c \cap K)$ is trivial. Let s_1 be the intersection of Q^c with s_{11}, and s_2 the other string of $Q^c \cap K$. As $(Q^c, Q^c \cap K)$ is trivial and K is prime, s_1 or s_2 are trivial in Q^c. Suppose that s_2 is trivial in Q^c. By following a similar argument as in the proof of Lemma 3.1(a), we have that either s_{21} and s_{22} are trivial in $(Q_2, \mathcal{T}_2)$, or s_{12} is trivial in $(Q_1, \mathcal{T}_1)$, which is a contradiction to these tangles being essential. Suppose s_2 is knotted in Q^c. As $(Q^c, Q^c \cap K)$ is trivial, there is a proper disk in Q^c separating s_1 and s_2; let B be the ball separated by this disk containing s_2. Then s_2 is knotted in B. As K is prime, the string in the complement of B in S_3, $B^c \cap K$, is trivial. We have $B^c \cap K$ being s_1 and $Q \cap K$. Then, following a similar argument as in Lemma 3.1(a), we have that one of the strings of $Q \cap K$ is trivial in Q, which contradicts $(Q, Q \cap K)$ being essential, or the string s_1 is trivial in Q^c and the strings of $Q \cap K$ are parallel in Q. But one of the strings $Q \cap K$ is unknotted in Q, which is a contradiction to the assumption that $(Q, Q \cap K)$ is essential.

(3) Suppose D is in $\mathcal{D}^*$ and D^* is not in T. The ball R intersects each string of $\mathcal{T}_1$ at a single arc component, with one of them being unknotted in R. From Lemma 3.1(a), some string of some tangle defined by S is unknotted or the tangle $(R, R \cap K)$ is trivial. Let R_1 be the complement of R in B_1, and R_1^c the complement of R_1 in S^3. Suppose the tangle $(R_1, R_1 \cap K)$ is trivial then, as there are no local knots in $(B_1, \mathcal{T}_1)$, $R_1 \cap s_{11}$ is unknotted in R_1. As $R \cap s_{11}$ is also unknotted in R we have that s_{11} is unknotted in B_1. So, we can assume that $(R_1, R_1 \cap K)$ is essential. Again from Lemma 3.1(a), we have that the tangle $(R_1^c, R_1^c \cap K)$ is essential. Therefore, the tangle decompositions defined by S and ∂R_1 are isotopic. This means that the tangle $(R, R \cap K)$ is the following product tangle: it is ambient isotopic to the tangle in the ball $(D \cup O) \times I$, that is R, with strings being $((D \cup O) \cap K) \times I$. Let V' be obtained from V by replacing T'' by R, as in (1), and $W' = S^3 - intV'$. Then, the arcs $R \cap K$ can be isotoped to ∂R intersecting $R \cap (V' - intR)$ only at the end points. Also, if two arcs are in adjoint components of $V' - V' \cap S$ then, after the isotopy to the boundary of the respective components, we can assume that the common ends are at the same point of the disk of intersection between the components. (In this case, this a consequence from $(R, R \cap K)$ being the product tangle described, the tangle in each component of $V' - V' \cap S$ being trivial and each componnet of $V \cap S$ intersecting K at most

once.) So, as in (1), $(V' - intN(K)) \cup W'$ is a Heegaard decomposition of the knot exterior with $|S \cap V'| < |S \cap V|$, and we have a contradiction to the minimality of $|S \cap V|$.

(4) Suppose D is in $\mathcal{D}^*$ and D^* is in T. So, the ball R intersects s_{11} at two arcs, and R_1 intersects K at a portion of s_{11} and the string s_{12}. If the tangle $(R_1, R_1 \cap K)$ is trivial we have that the string s_{12} is trivial in R_1, and as it has ends in the same disk component of $R_1 \cap S$ it is unknotted in $(B_1, \mathcal{T}_1)$. So, we can assume that $(R_1, R_1 \cap K)$ is essential. From Lemma 3.1(a), the tangle $(R_1^c, R_1^c \cap K)$ is essential. Then the tangle decompositions defined by S and ∂R_1 are isotopic. This means that the tangle $(R, R \cap K)$ is the product tangle as in (3). Following a similar argument as in (3), we obtain a contradiction to the minimality of $|S \cap V|$.

Suppose $n_1 = 3$.

Assume that the ends of s_{11} are at the same disk of $T \cap S$. Then Q intersects each string of $\mathcal{T}_1$ at a single component. Therefore, from Lemma 3.1(b) some string of some tangle is unknotted or the tangle $(Q, Q \cap K)$ is trivial. In the latter case we have that s_{11} is trivial in Q and unknotted in B_1. In case the ends of s_{11} are in distinct components of $T \cap S$, we can follow a similar argument as in case (4). (Note that, as $n_1 = 3$ and the genus of V is three the solid torus T cannot contain two disks of $\mathcal{D}^*$ and components of $\mathcal{D}$; so, in this case we have necessarily D in $\mathcal{D}^*$ and D^* in T.) □

Lemma 4.5. *Suppose T intersects $\mathcal{D}^*$ at two disks, D and D', and is disjoint from $\mathcal{D}$. Then some string of some tangle is unknotted, or there is a ball Q, in B_1, where*

1. *$Q \cap S$ is a disk intersecting $\mathcal{D}^*$ in two components;*
2. *$Q \cap K$ is a collection of two arcs each with one end in $Q \cap S$;*
3. *$(Q, Q \cap K)$ is a product tangle with respect to the disk $Q \cap S$ and its intersection with K;*
4. *the complement of Q in B_1 intersects T either in a cylinder between D' and a disk parallel to it in V, or in a cylinder between two disks parallel to D' in V.*

Proof. As T contains a single component from the intersection with K, we have that D and D' intersects K once. As D intersects K at one point, one of the disks separated by δ in $S - intD$ intersects K once; denote by O this disk. Let T' be the solid torus obtained by an isotopy of T taking D' away from S in B_1. Consider the ball Q defined by adding the 2-handle with core $L = O \cup \Delta$ to T'. Denote by Q_1 the complement of Q in B_1.

First assume that $O \cap \mathcal{D}^*$ is a disk not in T. Then $Q_1 \cap T$ is a cylinder between D' and a disk parallel to it in V. The arc $Q \cap s_{12}$ is unknotted in Q. From Lemma 3.1, either some string in the tangle $(B_1, \mathcal{T}_1)$ is unknotted or the tangle $(Q, Q \cap K)$ is trivial. So, we can assume the latter. Also, from Lemma 3.1(a), the tangle in the complement of Q_1 in S^3 is essential. If the tangle defined in $(Q_1, Q_1 \cap K)$ is also essential then the tangle decompositions defined by S and ∂Q_1 are isotopic. Then the tangle in Q is the product tangle as in the statement. Otherwise, if the tangle $(Q_1, Q_1 \cap K)$ is not essential then, as the strings of $Q \cap K$ are trivial in Q, both strings of the tangle $(B_1, \mathcal{T}_1)$ are unknotted. So, we either have that one string of $(B_1, \mathcal{T}_1)$ is unknotted or that $(Q, Q \cap K)$ is the product tangle described with Q_1 intersecting T in a cylinder between D' and a disk parallel to it in V.

Assume now that $O \cap \mathcal{D}^*$ is a disk in T. In this case, $Q_1 \cap T$ is a cylinder having intersection with Q in two disks parallel to D' in V. From Lemma 3.1(a), the tangle in the complement of Q, or of Q_1, in S^3 is essential. If the tangle $(Q_1, Q_1 \cap K)$ is essential then the tangle decompositions defined by ∂Q_1 and S are isotopic. This implies that the tangle $(Q, Q \cap K)$ is the product tangle as in the statement. Otherwise, the tangle $(Q_1, Q_1 \cap K)$ is trivial. As the string s_{12} is in Q_1 with ends in the disk $Q_1 \cap S$, it is also unknotted in $(B_1, \mathcal{T}_1)$. Hence, we either have that one string of $(B_1, \mathcal{T}_1)$ is unknotted or that $(Q, Q \cap K)$ is the product tangle descrbed with Q_1 intersecting T in a cylinder between two disks parallel to D' in V. □

5. Outermost disks over components of $V - V \cap S$ when $n_1 = 3$

In this section we consider the several cases when $n_1 = 3$ with respect to the existence of a genus two or a genus one component of $V - V \cap S$.

So assume $n_1 = 3$ and let D_1^*, D_2^* and D_3^* be the disk components of $S \cap V$ that intersect K. As $|S \cap K| = 4$, without loss of generality, we assume that $|D_1^* \cap K| = 2$ and $|D_i^* \cap K| = 1$, for $i = 2, 3$. As no 2-sphere is non-separating in S^3, we have that D_1^* is not parallel to D_2^* or D_3^* in V. So, D_1^* isn't parallel in V to any other disk of $S \cap V$.

Lemma 5.1. *If $V - V \cap S$ has a genus two component then some string of some tangle is unknotted.*

Proof. Assume there is a component of $V - S \cap V$ with genus two that we denote by V_2. As the genus of V is three, $S \cap V_2$ is a collection of at most two disks.
If $S \cap V_2$ is a collection of two disks or a single disk disjoint from K, then, as the genus of V is three, some disk of $\mathcal{D}$ is parallel to a disk of $\mathcal{D}^*$, or D_1^* is parallel to D_2^* or D_3^* in V. This is impossible as observed before. Therefore, $S \cap V_2$ is a single disk intersecting K.

As $S \cap V_2$ is also separating, we can only have $S \cap V_2 = D_1^*$, as in Figure 6. So, the disks D_2^*

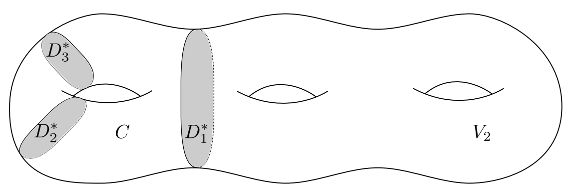


Figure 6

and D_3^* are necessarily parallel in the solid torus separated by D_1^* in V, and we have $n_2 = 0$. Also, as V_2 is the only non-ball component of $V - V \cap S$, from Remark 3, all outermost disks are over V_2 and attached to D_1^*.
Let C be the ball component of $V - V \cap S$ cut from V by $D_1^* \cup D_2^* \cup D_3^*$ and suppose it lies in the tangle $(B_1, \mathcal{T}_1)$. The ball C contains both strings of the tangle $(B_1, \mathcal{T}_1)$: the string s_{12} with one end in D_1^* and the other in D_2^*, and the string s_{13} with one end in D_1^* and the other end in D_3^*, and from Lemma 2.3 both strings are mutually trivial in C.
Between the arcs of $E \cap P$ with end in D_2^* or D_3^* we choose one that is outermost in E, say γ, as in Figure 7(a). We note that γ cannot have one end in D_2^* and the other in D_3^*, as, otherwise δ wouldn't be essential in P. (See Figure 7(b).) So, without loss of generality, assume that γ has one end in D_2^*. The disk Γ is in the complement of C in B_1 and its boundary intersects D_2^* only once. So, D_2^* is a primitive disk with respect to the complement of C in B_1, which is a handlebody. Then, by an isotopy of C along D_2^* away from S in B_1, we are left with with the ball $C_{1^*,3^*}$ that intersects S at D_1^* and D_3^*, whose complement in B_1 is a solid torus and with the string s_{13} as a core. Hence, the string s_{13} is unknotted in $(B_1, \mathcal{T}_1)$. □

Lemma 5.2. *If there is a solid torus component of $V - V \cap S$ then both strings of some tangle are μ-primitive.*

Proof. As the genus of V is three, and one component of $V - V \cap S$ is a solid torus, the components of $V - V \cap S$ are balls or solid tori. From Remark 3, all outermost disks are over solid torus

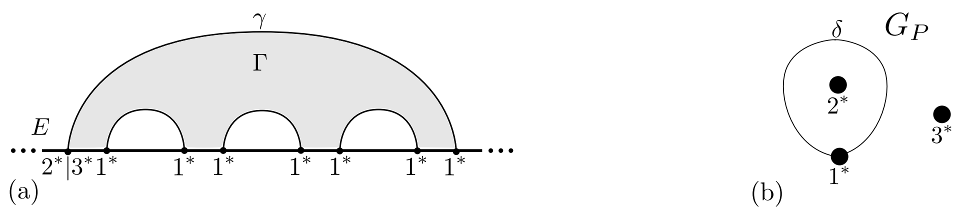


FIGURE 7: In (a) the arc γ represents an arc of $E \cap P$ outermost in E between the ones with at least one end in either D_2^* or D_3^*. The label $2^*|3^*$ at an end of the arc γ means that this end is either at the disk D_2^* or at the disk D_3^*.

components of $V - V \cap S$. Let T be a torus component of $V - S \cap V$ with a outermost disk over it, and suppose T is in B_1. The collection of disks $T \cap S$ cannot be bigger than four as the genus of V is three. If the number of disks in $T \cap S$ is four then D_1^* is parallel to some other disk of $V \cap S$, which is impossible as previously observed. So, $|T \cap S|$ is at most three.

Suppose $T \cap S$ is a single disk. If $T \cap S$ is disjoint from K we get a contradiction to Lemma 4.1. If $T \cap S$ intersects K, from Lemma 4.2 some string of some tangle is unknotted.

In case $T \cap S$ is a collection of two disks then we have several cases two consider. If these two disks don't intersect K then D_1^* is necessarily separating. Furthermore, one string from a tangle lies in a ball of $V - V \cap S$ cut by $T \cap S$ and D_1^* with the two ends in D_1^*. Then, from Lemma 2.3 this string is trivial in the respective tangle, which is a contradiction to the tangle being essential. If only one disk of $T \cap S$ intersects K then it is necessarily D_1^*, because K intersects $T \cap S$ an even number of times. In this situation, T intersects K at a single arc and from Lemma 4.2 some string of some tangle is unknotted.
If the two disks of $T \cap S$ intersect K then $T \cap S = D_2^* \cup D_3^*$. In this case, $D_2^* \cup D_3^*$ separate V in

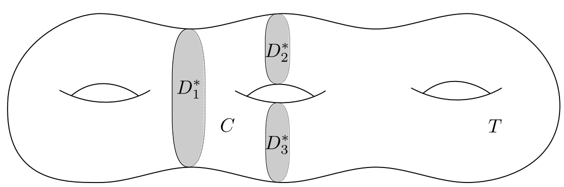
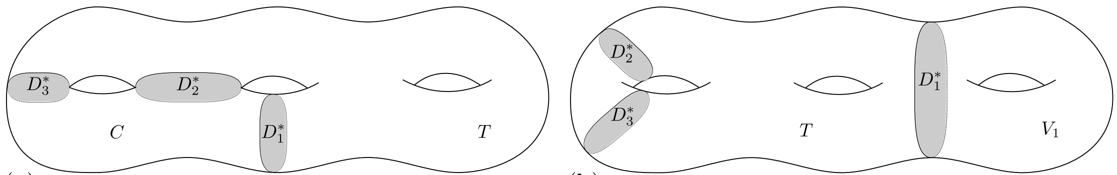


FIGURE 8

two solid tori components, T and V_1. The disk D_1^* is in V_1 and is necessarily separating. (See Figure 8.) We also have $n_2 = 0$. Then, for the respective outermost arc of an outermost disk over T we are always under the statement of Lemma 4.4, which means that some string of some tangle defined by S is unknotted.

Assume now that $T \cap S$ is a collection of three disks. At least some disk of $T \cap S$ intersects K, as otherwise D_1^* would have to be parallel in V to some other disk of $V \cap S$, which is impossible as previously observed.
If only one disk of $T \cap S$ intersects K then this disk is D_1^*, and from Lemma 4.2 some string of some tangle is unknotted.
If two disks of $T \cap S$ intersect K then these disks have to be D_2^* and D_3^*. As the genus of V is three either $T \cap S$ or $D_2^* \cup D_3^*$ cuts a ball from V. In either case, D_1^* would have to be parallel to some other disk, which is impossible as previously observed.
The last case is when $T \cap S = D_1^* \cup D_2^* \cup D_3^*$. The disk D_1^* can be separating or non-separating. In the latter case $D_1^* \cup D_2^* \cup D_3^*$ separates a ball from V, and in the former case the disks D_2^*

and D_3^* are parallel and the disk D_1^* separates a solid torus V_1 in V. (See Figure 9.) So, from Lemma 2.3 and the fact that no disk of $\mathcal{D}$ is parallel to a disk of $\mathcal{D}^*$, we can assume that $n_2 = 0$. From $|S \cap V| = 3$ and Lemma 2.4(b), there is only one disk attached to outermost arcs.
Assume D_1^* is non-separating, then $D_1^* \cup D_2^* \cup D_3^*$ separates a ball C from V, which is in the tangle $(B_2, \mathcal{T}_2)$. If there is a string in C with both ends in D_1^* then, from Lemma 3.1(d), this



(a) (b)

FIGURE 9

string is unknotted in $(B_2, \mathcal{T}_2)$. So, we can assume that each string in C has only one end in D_1^*. Consider a second outermost disk Γ^*, and the respective outermost disk γ^*. Then Γ^* is in the complement of C in B_2. If γ^* has equal ends then, following the proof Lemma 3.4, we have that some string of some tangle is unknotted. If the ends of γ^* are distinct, then D_1^*, D_2^* or D_3^* is primitive in the complement of C in B_2. Suppose D_2^* (or D_3^*) is primitive with respect to the complement C in B_2. After an isotopy of C along D_2^* (resp., D_3^*) away from S, we have that the complement of a regular neighborhood of the string s_{13} (resp., s_{23}) is a solid torus, which implies that this string is unknotted in $(B_2, \mathcal{T}_2)$. Suppose D_1^* is primitive with respect to the complement of C in B_2. As the complement of C in B_2 is a handlebody, after an isotopy of C along D_1^* away from S, we obtain a cylinder from D_2^* to D_3^*, with core t, whose complement in B_2 is a solid torus. Then t is unknotted in B_2. As s_{12} and s_{13} are trivial in C, we have that C is the union of the regular neighborhoods of $t \cup s_{12}$, and also of $t \cup s_{13}$. Consequently, both s_{12} and s_{13} are μ-primitive.
Assume now that D_1^* is separating. Suppose D_2^* and D_3^* are the only disks attached to outermost arcs. As D_2^* is parallel to D_3^* by the finiteness of outermost arcs, if we consider a second-outermost arc we have that both disks have loops attached in G_P, which contradicts Lemma 2.4(b). So, D_1^* has outermost arcs attached and all second-outermost arcs are after outermost arcs attached to D_1^*. If there is an outermost disk over V_1, from Lemma 4.2 some string of some tangle is unknotted. So, we can assume that all outermost disks are over T. Let Γ^* be a second-outermost disk, then Γ^* is in the complement of V_1 in B_2. Suppose $\partial\Gamma^*$ is essential in $\partial V_1 \cup_{D_1^*} S$. Then the complement of V_1 in B_2 is also a solid torus (intersecting S at a single disk). From Lemma 2.3 the string s_{11} is trivial in V_1. Then, from Lemma 2.2, s_{11} is μ-primitive. We note also that $B_2 \cap V$ is V_1 together with the cylinder cut by $D_2^* \cup D_3^*$ in V, $C_{2^*3^*}$, where the string s_{23} is a core. As the complement of $B_2 \cap V$ in B_2 is a handlebody we have that s_{23} is trivial in the complement of V_1 in B_2. Therefore, from Lemma 2.2, s_{23} is also μ-primitive. Suppose now that $\partial\Gamma^*$ is inessential in $\partial V_1 \cup_{D_1^*} S$. Then $\partial\Gamma^*$ bounds a disk L in $\partial V_1 \cup_{D_1^*} S$. Let R be the ball in B_2 bounded by $\Gamma^* \cup L$. By similar arguments as in the proof of Lemma 3.4, we have that s_{23} is in R and is parallel to L. So, s_{23} is trivial in the complement of V_1 in B_2. As the complement of $B_2 \cap V$ in B_2 is a handlebody, this implies that the complement of V_1 in B_2 is a solid torus. Then, as when $\partial\Gamma^*$ is essential in $\partial V_1 \cup_{D_1^*} S$, we have that both s_{11} and s_{23} are μ-primitive. □

6. OUTERMOST DISKS OVER COMPONENTS OF $V - V \cap S$ WHEN $n_1 = 4$

Along this section we consider the several cases when $n_1 = 4$ with respect to the components of $V - V \cap S$ topology and their intersection with $S \cap V$.

So assume that $n_1 = 4$. As $|S \cap V| = 4$ we have $|D_i^* \cap K| = 1$, for $i = 1, 2, 3, 4$. Therefore, D_i^* is a non-separating disk in V.
Denote by γ_i^* the outermost arcs of $E \cap P$, in E, between the arcs with at least one end in D_i^*, for $i = 1, 2, 3, 4$. Also, let Γ_i^* denote the disk of $E - E \cap P$ co-bounded by γ_i^* in the outermost side of this arc in E, for $i = 1, 2, 3, 4$.

Lemma 6.1. *If $V - V \cap S$ contains a genus two component then some string in some tangle is unknotted.*

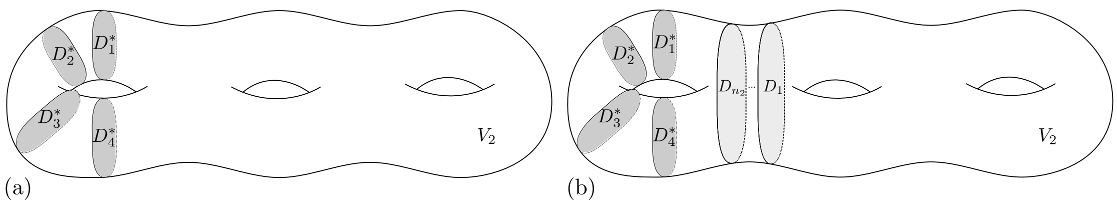


FIGURE 10

Proof. Assume that $V - V \cap S$ contains a genus two component, V_2. As the genus of V is three $S \cap V_2$ is a collection of at most two disks. If $S \cap V_2$ is a collection of two disks then $S \cap V$ are all parallel disks in V and, from Remark 3, all outermost disks are over V_2. Therefore, $n_2 = 0$ and all disks of $\mathcal{D}^*$ are parallel, as in Figure 10(a). Consequently, by the finiteness of outermost arcs, we have parallel type I d*-arcs in E, as in Figure 11(a1) or (a2), in contradiction to Corollary 2.2. Then, $S \cap V_2$ is a single disk. As each disk of $\mathcal{D}^*$ intersects K once, $S \cap V_2$ is a disk of $\mathcal{D}$. Then, all disks of $\mathcal{D}^*$ are parallel in the solid torus cut from V by $S \cap V_2$, and all disks of $\mathcal{D}$ are

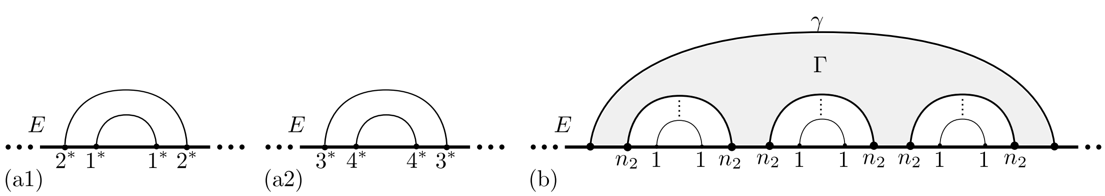


FIGURE 11

parallel to $S \cap V_2$ in V, as in Figure 10(b). Let $D_1, \ldots, D_{n_2}$ be the disks of $\mathcal{D}$, with $S \cap V_2$ being D_1. The outermost disks are all adjacent to D_1 and are over V_2. Consider a second-outermost arc γ, as in Figure 11(b). If the arc γ has at least one end in D_{n_2}, or has one end in D_1^* and the other in D_4^*, by Lemma 3.4, some string in the tangle decomposition defined by S is unknotted. Otherwise, if all second-outermost arcs have both ends in D_1^* or both ends in D_4^*, as when $S \cap V_2$ is two disks, by the finiteness of outermost arcs we have a contradiction to Corollary 2.2. □

Assume now that $V - V \cap S$ has a solid torus component T with some outermost disk over it. Hence, as the genus of V is three, the components of $V - V \cap S$ are solid tori or balls, and the solid torus T components intersect S at most in four disks. As each disk of $\mathcal{D}^*$ intersects K once, the solid torus T intersects $\mathcal{D}^*$ at an even number of disks.

Lemma 6.2. *Suppose $V - V \cap S$ contains a solid torus component intersecting $\mathcal{D}^*$ at the four disks. Then some string in some tangle is unknotted.*

Proof. Let T be the solid torus component of $V - V \cap S$ as in the statement, and suppose it lies in the tangle $(B_1, \mathcal{T}_1)$. As the genus of V is three and T intersects $\mathcal{D}^*$ at the four disks, we have that the disks of $\mathcal{D}^*$ are parallel two-by-two in V, say D_1^* parallel to D_2^* and D_3^* parallel to D_4^*. So, $n_2 = 0$, and $V \cap S$ is as in Figure 12. Also, from Remark 3, we can assume all outermost disks are over T. From Lemma 2.4(b), at most two disks are adjacent to outermost arcs.
If the outermost arcs are attached to a single disk or if they are attached to two non-parallel disks, by the finiteness of outermost arcs of $E \cap P$ in E we have a contradiction to Corollary 2.2. Then, the outermost arcs are attached to two parallel disks. Without loss of generality, assume

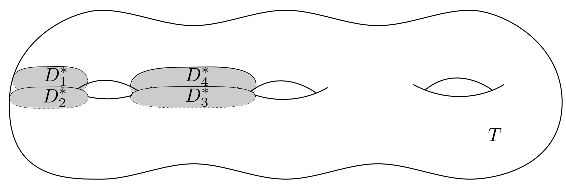


FIGURE 12

that the only disks adjacent to outermost arcs are D_1^* and D_2^*. Consider the second outermost

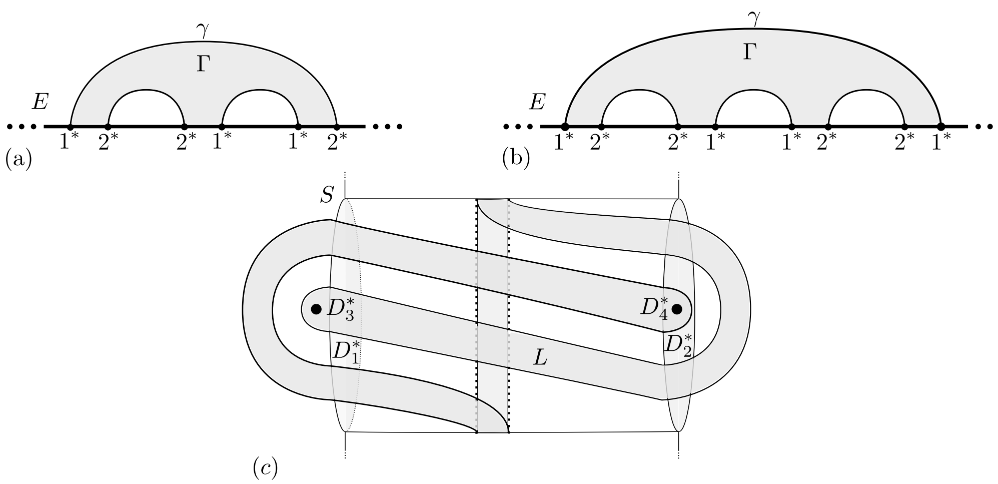


FIGURE 13

arc γ of $E \cap P$ in E, after the outermost arcs attached to D_1^* and D_2^*, and the disk component of $E - E \cap P$, Γ, co-bounded by γ on the outermost side of γ in E. Let C_{12} and C_{34} be the cylinders cut from V by $D_1^* \cup D_2^*$ and $D_3^* \cup D_4^*$, resp.. We have that Γ is in $B_2 - intC_{12}$. If Γ is essential in $S \cup \partial C_{12}$, as in Figure 13(a), then Γ is a meridian disk to $B_2 - intC_{12}$, which implies that the string s_{12}, a core of C_{12}, is unknotted in the tangle B_2. Otherwise, if Γ is inessential in $S \cup \partial C_{12}$, we have that $\partial\Gamma$ bounds a disk L in the torus $S \cup_{D_1^* \cup D_2^*} \partial C_{12}$. (See Figure 13(b), (c).) Let R be the ball in B_2 bounded by $\Gamma \cup L$. The string s_{34}, as a core of C_{34}, is in R and, as there are no trivial knots, it is trivial in R and parallel to L. Hence, as the complement of $C_{12} \cup C_{34}$ in B_2 is a handlebody, we have that the complement of C_{12} in B_2 is a solid torus. Therefore, in this case, the string s_{12} is also unknotted. □

Lemma 6.3. *Suppose $V - V \cap S$ contains a solid torus that intersects $\mathcal{D}^*$ in a collection of two disks and $\mathcal{D}$ in a single separating disk. Then both strings of some tangle are μ-primitive.*

Proof. Let T be the solid torus component of $V - V \cap S$ as in the statement, and suppose it lies in the tangle $(B_1, \mathcal{T}_1)$. Assume that $T \cap \mathcal{D}^* = D_1^* \cup D_4^*$ and that $\mathcal{D} \cap T = D_1$, and denote by V_1 the solid torus separated by D_1 in V. As $\mathcal{D} \cap T$ is separating, and K is connected, the four disks of $\mathcal{D}^*$ have to be parallel in V. If all outermost arcs are attached to D_1^* or to D_4^* then, by the finiteness of outermost arcs, we have a contradiction to Corollary 2.2. Hence, there is an outermost arc attached to D_1. The set $\mathcal{D}$ contains a collection of separating disks, $D_1, \ldots, D_k$ in V, and might also contain a collection of non-separating parallel disks $D_{k+1}, \ldots, D_{n_2}$ in V_1, as in Figure 14(a), (b). From Remark 3, the outermost disks are over T or V_1, and from Lemma 4.1 there are no outermost disks over V_1. So, all outermost disks are over T, attached to D_1^*, D_4^* or D_1, with no sequence of parallel arcs of $E \cap P$ in E after an outermost arc attached to D_1^* or D_4^*.

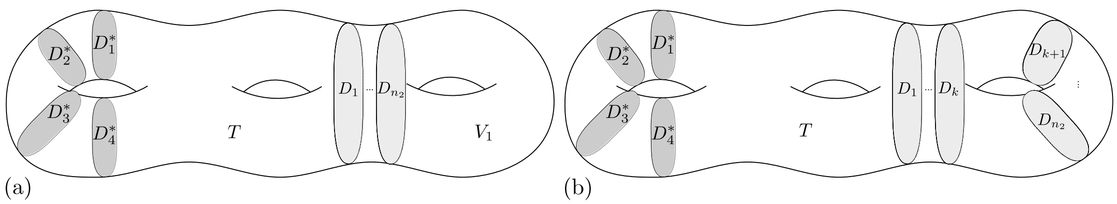


FIGURE 14

Case 1. *Assume that all disks of $\mathcal{D}$ are parallel and separating as in Figure 14(a).*

If $n_2 > 1$, by the finiteness of outermost arcs, there is a sequence of parallel arcs of $E \cap P$ in E, $\delta_1, \ldots, \delta_{n_2}$, as in the Figure 15(a), where δ_i has both ends in D_i and δ_1 is an outermost arc attached to D_1. Denote the outermost disk that δ_1 co-bounds by Δ, and the disk between δ_i and δ_{i+1} by Δ_i. Considering the disks Δ_i and the cylinder cut from V by $D_i \cup D_{i+1}$ we define a ball R_i as in Lemma 3.3. The balls R_i intersect S at disks O_i and O_{i+1}, co-bounded by δ_i and δ_{i+1} resp., each containing an end of the string in R_i. Then, in particular, O_1 contains a single disk of $\mathcal{D}^*$. If $n_2 = 2$, as R_1 contains a single string, we have that O_1 intersects $S \cap V$ at a single disk, that is of $\mathcal{D}^*$. Assume $n_2 \geq 3$. If D_2^*, or D_3^*, is in O_1 then R_2 contains T and consequently two strings of the tangle, which is a contradiction as R_2 contains a single string. Therefore, without loss of generality, we can assume that D_1^* is in O_1. Suppose that some disk of $\mathcal{D}$, say D_i, is in O_1. Then $O_i \subset O_1$ and $D_1^* \subset O_i$. Consequently, following the strings in the sequence of balls R_j, we have $T \subset R_{i-1}$ and D_1 in O_i, which is a contradiction as D_i is in O_1. Therefore, D_1^* is the only disk of $S \cap V$ in O_1. Then, by Lemma 4.4, if $n_2 > 1$ some string of some tangle is unknotted.

Suppose $n_2 = 1$. As before we denote by δ_1 an outermost arc attached to D_1. If a disk cut from $S - intD_1$ by δ_1 intersects $\mathcal{D}^*$ at a single disk from Lemma 4.4 some string of some tangle is unknotted. Therefore, we can assume that all outermost arcs δ_1 separate $S - intD_1$ in two disks each intersecting $\mathcal{D}^*$ at two disks. Consequently they are all parallel in P. Let Γ be a second-outermost disk. From Lemma 2.2, the disk Γ is in the complement of V_1 in B_2. If $\partial\Gamma$ is inessential in the solid torus $B_1 \cup_{D_1} V_1$ then Γ bounds a disk L in $S \cup_{D_1} \partial V_1$. Let R be the ball bounded by $\Gamma \cup L$ in B_2. By similar arguments as in the proof of Lemma 3.4, we have that the strings s_{12} and s_{34} are in R and are parallel to L. Hence, the complement of V_1 in B_2 is a solid torus intersecting S at a single disk. Altogether, from Lemma 2.2 we have that both strings s_{12} and s_{34} are μ-primitive. Suppose now that $\partial\Gamma$ is essential in the solid torus $B_1 \cup \partial_{D_1} V_1$. Then the complement of V_1 in B_2 is also a solid torus. Consider an outermost arc between the arcs with one end in D_2^* or D_3^*, and denote these arcs by γ^*. Suppose there are arcs γ^* with both ends in D_2^* and also in D_3^*. Then there are arcs γ_1^* and γ_4^* of Type II outermost between the d*-arcs, and the disks Γ_1^* and Γ_4^* are in B_1 and intersect D_1^* and D_4^*, resp., exactly once. Then, D_1^* and D_4^* are primitive with respect to the complement of $V \cap B_1$ in B_1. Let T' be the solid torus obtained by an isotopy of T along $D_1^* \cup D_4^*$ away from S. We also have that an outermost disk Δ intersects a meridian of T' once. Altogether, the complement of the cylinder from D_2^* to D_3^* in V is a solid torus; as the core of this cylinder is the string s_{23}, this string is unknotted in $(B_1, \mathcal{T}_1)$. Otherwise, without loss of generality, suppose there is an arc γ^* with only one end in D_2^*. This means γ^* is an γ_2^* arc, and we can consider the respective disk Γ_2^*. Let C_{12} (resp., C_{34}) be the cylinder from D_1^* to D_2^* (resp., D_3^* to D_4^*) in V. As D_2^* is primitive with respect to the complement of $V \cap B_2$ in B_2, a core of C_{34}, as the string s_{34}, is trivial in the complement of V_1 in B_2. If the other end of γ^* is in D_3^* then D_3^* is also primitive with respect to the complement of $V \cap B_2$ in B_2, and similarly a core of C_{12}, as the string s_{12}, is trivial in the complement of V_1 in B_2. Otherwise, if the other end of γ^* is not in D_3^*, using the disk Γ_2^*, we have that a core of C_{12}, as the string s_{12}, is trivial in the complement of V_1 in B_2. Then, from Lemma 2.2, both strings s_{12} and s_{34} are μ-primitive.

Case 2. *Assume now that $\mathcal{D}$ also has a collection of non-separating disks in V, as in Figure 14(b).*

Claim 6.3.1. *If the outermost arcs attached to D_1 are not parallel in P then some string of some tangle is unknotted.*

Proof of Claim 6.3.1. In fact, let δ_1 and δ_1' be outermost arcs attached to D_1, non-parallel in P. Consider the disjoint disks O_1 and O_1' co-bounded, respectively, by δ_1 and δ_1' in $S - D_1$, and also the respective outermost disk Δ_1, Δ_1'. Consider the disks $L_1 = O_1 \cup \Delta_1$ and $L_1' = O_1' \cup \Delta_1'$. Let Q be the ball obtained by attaching a regular neighborhood of L_1 and L_1' to T and adding a ball to the respective boundary component disjoint from S. If $\mathcal{D}^* \subset O_1 \cup O_1'$ then the arcs δ_1 and δ_1' are parallel. If $(O_1 \cup O_1') \cap \mathcal{D}^*$ is only D_1^* and D_4^*, then $\partial Q - \partial Q \cap S$ is a compressing disk for P. Otherwise, $D_2^* \cup D_3^*$ is in $O_1 \cup O_1'$ and the string s_{23} is in Q. From Lemma 4.3 the tangle $(Q, Q \cap K)$ is trivial. Therefore, the string s_{23} is trivial in Q. As the ends of s_{23} are in the same disk component of $Q \cap S$ we have that s_{23} is unknotted in $(B_1, \mathcal{T}_1)$. △

From the previous claim, we assume that the outermost arcs attached to D_1 are parallel in P.

If $k > 1$, by the finiteness of outermost arcs we have a sequence of arcs, δ_i for $i = 1, \dots, k$, after an outermost arc, δ_1, as in the Figure 15(b). Following the construction at the beginning of Case 1, from each sequence of parallel arcs after an outermost arc δ_1 we have a sequence of balls R_i, $i = 1, \dots, k$. Also, as there are no t-arcs, the outermost arc, δ_{k+1}, after these arcs is a st-arc. If some arc δ_{k+1} has both ends in D_k, following an argument as in Lemma 3.4, we have that some string of some tangle is unknotted. Then, the arcs δ_{k+1} have both ends in D_{k+1}, as in Figure 15(c), or in D_{n_2}.

For any k, suppose we have both situations, that there are arcs δ_{k+1} and δ_{k+1}' with both ends in D_{k+1} and D_{n_2}, resp.. Consider the component disks Δ_k and Δ_k' of $E - E \cap P$ co-bounded by δ_{k+1} and δ_{k+1}', resp., in the outermost side of these arcs in E. As the outermost arcs δ_1 are parallel in P, and the balls R_i, $i = 1, \dots, k$, contain only one string, the arcs of $\partial\Delta_k$ and $\partial\Delta_k'$ that have both ends in D_k are parallel in P. Let C be the ball cut from V by $D_k \cup D_{k+1} \cup D_{n_2}$, and $C_{k,k+1}$ (resp., C_{k,n_2}) be the ball obtained from C by an isotopy of D_{n_2} (resp., D_{k+1}) away from S. Let L_k and L_k' be the disks bounded by $\partial\Delta_k$ and $\partial\Delta_k'$, resp., in $\partial C_{k,k+1} \cup_{D_k \cup D_{k+1}} S$ and $\partial C_{k,n_2} \cup_{D_k \cup D_{n_2}} S$, resp.. Consider the balls R_k and R_k' bounded by $L_k \cup \Delta_k$ and $L_k' \cup \Delta_k'$, not containing S. Similarly, as observed in Case 1, the balls R_k and R_k' contain only one string.

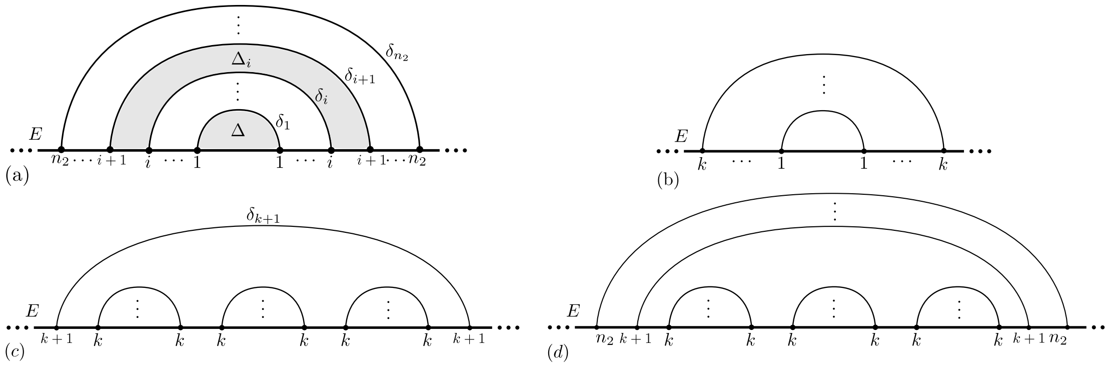


FIGURE 15

Suppose none of these balls contains the other, as in Figure 16(a). Hence, each of the disks L_k and L'_k intersection with S contains a disk component, O_k and O'_k resp., co-bounded, with ∂D_k, by a single arc of $\partial\Delta_k \cap S$, δ_k, and $\partial\Delta'_k \cap S$, δ'_k. Each of the arcs δ_k and δ'_k is in a sequence of arcs after an outermost arc, as in Figure 15(b). As observed before in this Claim, we are assuming that these arcs are parallel in P. Then one of the disks O_k or O'_k has to be contained in the other, which is a contradiction with the assumption that R_k and R'_k are disjoint. So, assume that, say, R'_k is contained in R_k, as in Figure 16(b). Then L_k contains D_{n_2} and L'_k. Therefore, from the minimality of $|E \cap P|$ and from the arcs of $\partial\Delta_k \cap S$ and $\partial\Delta'_k \cap S$ that have both ends in D_k being parallel in P, we have that $\partial\Delta_k$ intersects S in two arcs, one with two ends in D_k and the other with two ends in D_{k+1}; similarly, $\partial\Delta'_k$ intersects S in two arcs, one with two ends in D_k and the other with two ends in D_{k+1}. Let O_{k+1}, resp. O'_{k+1}, be the disks cut from $S - intD_{k+1}$, resp. $S - intD_{n_2}$, by δ_{k+1}, resp. δ'_{k+1}, disjoint from D_k. As there are no local knots, the string in $R'_k \subset R_k$ is trivial. Then, from the minimality of $|S \cap V|$, we have $|O'_{k+1} \cap V|$ the same as $|O'_k \cap V|$. Also, $O_k \cap (V \cap S)$ is the same as $O'_k \cap (V \cap S)$. Therefore, $|O_{k+1} \cap V|$ is bigger than $|O_k \cap (V \cap S)|$. So, we can isotope $D_{k+1} \cup O_{k+1}$ along R_k union the ball $C_{k,k+1}$ to reduce $|S \cap V|$, which is a contradiction.
So, assume without loss of generality that all arcs δ_{k+1} have both ends in D_{k+1}. By the finiteness of outermost arcs we have a sequence of parallel arcs, $\delta_{k+2}, \ldots, \delta_{n_2}$, as in Figure 15(d), and the respective sequence of balls $R_{k+2}, \ldots, R_{n_2-1}$. Then, we have a sequence of arcs parallel to an outermost arc, $\delta_1, \ldots, \delta_{n_2}$, and the respective balls $R_1, \ldots, R_{n_2-1}$. Following a similar argument as in Case 1, we have that δ_1 is as in Lemma 4.4, which means that some string of some tangle is unknotted. $\square$

Lemma 6.4. *Suppose $V - V \cap S$ contains a solid torus that intersects $\mathcal{D}^*$ at two disks and $\mathcal{D}$ at a single non-separating disk. Then both strings of some tangle are μ-primitive.*

Proof. Let T be the solid torus component of $V - V \cap S$ as in the statement, and suppose it lies in the tangle $(B_1, \mathcal{T}_1)$. Assume that $T \cap \mathcal{D}^* = D_1^* \cup D_4^*$ and that $T \cap \mathcal{D} = D_1$. The disks D_1^* and D_4^* are not parallel, otherwise D_1 would be separating. Then, $D_1 \cup D_1^* \cup D_4^*$ separate a ball from V, as in Figure 17, and all outermost disks are over T with corresponding outermost arcs attached to D_1^*, D_4^* or D_1. The disks $D_1, D_2, \ldots, D_{n_2}$ are all parallel and non-separating in V.

Claim 6.4.1. *If the disks of $\mathcal{D}^*$ are parallel two-by-two then some string of some tangle is unknotted.*

Proof of Claim 6.4.1. Assume that D_2^* is parallel to D_1^* and that D_3^* is parallel to D_4^* in V, as in Figure 17(a). If D_1^* or D_4^* are the only disks with outermost disks attached then by

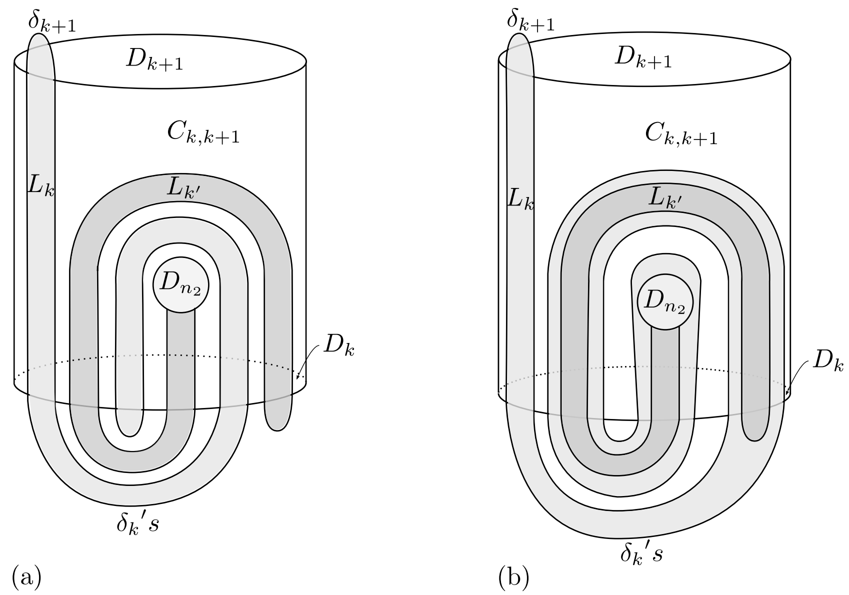


FIGURE 16: The disks L and L_k when R'_k is disjoint from R_k and when R'_k is contained in R_k, resp.: the arcs δ_k won't be parallel in P as previously observed.

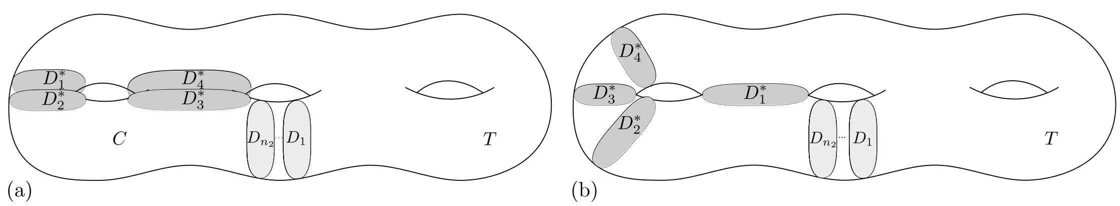


FIGURE 17

the finiteness of outermost arcs we have parallel sk-arcs in E, as in Figures 11(a1), (a2), which is a contradiction to Corollary 2.2. So, D_1 has an outermost arc attached. Furthermore, from Corollary 2.2, even if D_1^*, or D_4^*, has outermost arcs attached we cannot have a sequence of parallel sk-arcs in E after such outermost arcs. So, by the finiteness of outermost arcs only some outermost arc attached to D_1 is before a sequence of parallel arcs of $E \cap P$ in E, as in Figure 15(a). Consider a second-outermost arc γ, and the disk component of $E - E \cap P$, Γ, co-bounded by γ in the outermost side of this arc in E, as in Figure 11(b). The boundary of Γ intersects S in γ and arcs with both ends in D_{n_2}. If γ has at least one end in D_{n_2}, or one end in D_2^* and the other in D_3^*, then from Lemma 3.4 we have that some string in some tangle is unknotted. Otherwise, the ends of all second outermost arcs are both in D_2^* or both in D_3^*, and by the finiteness of outermost arcs we have a contradiction to Corollary 2.2. $\triangle$

From this claim, we can assume that the disks of $\mathcal{D}^*$ are not parallel two-by-two in V. Therefore, as no disk of $\mathcal{D}^*$ can be parallel in V to a disk of $\mathcal{D}$, without loss of generality, we assume that the disks D_2^* and D_3^* are parallel to D_4^*, as in Figure 17(b). Under this setting, we continue the lemma's proof in several steps with respect to which disks are attached to outermost arcs and to the value of n_2.

Claim 6.4.2. *The disks D_1 or D_1^* have outermost arcs attached; and the disks D_1^* and D_4^* cannot have simultaneously outermost arcs attached.*

Proof of Claim 6.4.2. If all outermost arcs are attached to D_4^* then there is a sequence of parallel sk-arcs, as in Figure 11(a2), which is a contradiction to Corollary 2.2. Then D_1 or D_1^* have outermost arcs attached.
Suppose D_1^* and D_4^* have simultaneously outermost arcs attached. Let δ_i^* be an outermost arc attached to D_i^*, and Δ_i^* the respective outermost disk, for $i = 1, 4$. Consider also the disjoint disks O_1^* and O_4^*, in $S - int\{D_1^* \cup D_4^*\}$, co-bounded by δ_1^* and δ_4^*, respectively. Let $L_i^* = \Delta_i^* \cup O_i^*$, for $i = 1, 4$. As the arcs δ_i^* are st-arcs, $D_2^* \cup D_3^*$ is in $O_1^* \cup O_4^*$. Taking a regular neighborhood of the disks L_i^* together with T, and by capping off the boundary component of $N(T) \cup_{i=1,4} N(L_i^*)$ disjoint from S with the ball it bounds, we get a ball Q in the tangle $(B_1, \mathcal{T}_1)$ containing both strings s_{14} and s_{23}. Each string of $\mathcal{T}_1$ in Q has ends in two distinct disk components of $\partial Q \cap S$, $D_1^* \cup O_1^*$ and $D_4^* \cup O_4^*$. Then with the tangle $(Q, \mathcal{T}_1)$ we have a contradiction between Lemma 4.3 and Lemma 3.1(c). △

Claim 6.4.3. *If D_1 or D_1^* is not attached to outermost arcs then $n_2 \leq 3$.*

Proof of Claim 6.4.3. If D_1 or D_1^* is not attached to outermost arcs then all outermost d-arcs have either both ends in D_{n_2} or in D_1. Then by the finiteness of outermost arcs there is a sequence of parallel arcs, δ_i, as in Figure 15(a). As in Case 1 of Lemma 6.3, using the disks Δ_i between the arcs δ_i and δ_{i+1} in E, attached to the disks D_i and D_{i+1}, resp., and the disk that $\partial\Delta_i$ bounds in the torus $C_{i,i+1} \cup_{D_i \cup D_{i+1}} S$, we define a ball R_i. Each of these balls contains a single string of the tangle decomposition and it is regular neighborhood of it. If $n_2 \geq 5$ then all components of $V - S \cap V$ are contained in some ball $R_i \cup C_{i,i+1}$. We note that these balls are either disjoint or intersect at a disk, wether the strings they contain are disjoint or intersect at an end. Then, taking the union of the largest balls $R_i \cup C_{i,i+1}$ for each each string, we have a solid torus with K as its core, V in its interior and boundary essential in W, which is a contradiction as W is a handlebody. So, given that n_2 is odd, $n_2 \leq 3$.
(If both D_1 and D_1^* have outermost arcs attached, in Claim 6.4.5 we also prove that $n_2 \leq 3$.) △

Claim 6.4.4. *If D_4^* and D_1 are the only disks with outermost arcs attached then some string of some tangle is unknotted.*

Proof of Claim 6.4.4. Suppose both disks D_1 and D_4^* are attached to outermost arcs, δ_1 and δ_4, resp., . If $n_2 = 1$ then either δ_1 or δ_4 are as in Lemma 4.4, which means that some string of some tangle is unknotted.
So, from Claim 6.4.3, we can assume that $n_2 = 3$. From Corollary 2.2 there are no parallel sk-arcs after an outermost arc attached to D_4^*, as in Figure 11(a2). Consequently, from the finiteness of outermost arcs, we have such a sequence of parallel arcs after an outermost arc attached to D_1, as in Figure 15(a), and consider the respective balls R_i, for $i = 1, 2$. Let O_i and O_{i+1} be the disk components of $R_i \cap S$ that are co-bounded by δ_i and δ_{i+1}, resp., for $i = 1, 2$. As R_1 contains a single string, O_1 intersects $\mathcal{D}^*$ at a single disk. Then, as D_4^* has a type I arc attached, this disk is not in O_1. If D_2^* or D_3^* are in O_1 then R_2 contains T and, consequently, two strings of the tangle, which we know is impossible. Then D_1^* is in O_1, the string s_{12} is in R_1 and the string s_{23} is in R_2. So, if D_2 or D_3 is in O_1, also O_2 or O_3 will be, and consequently the same for D_2^* or D_3^*, which is impossible as observed before. Then, $O_1 \cap (S \cap V)$ is only $\mathcal{D}_1^*$. So, δ_1 is an outermost arc as in Lemma 4.4. Then, some string of some tangle is unknotted. △

Claim 6.4.5. *If D_1^* and D_1 are both attached to outermost arcs then both strings of some tangle are μ-primitive.*

Proof of Claim 6.4.5. Suppose that both D_1 and D_1^* have outermost arcs attached, denoted by δ_1 and δ_1^* resp.. Let O_1 and O_1^* be the disjoint disks in $S - int(D_1 \cup D_1^*)$ co-bounded by δ_1 and δ_1^*, resp.. Consider also the disks $L_1^* = \Delta_1^* \cup O_1^*$ and $L_1 = \Delta_1 \cap O_1$. Let Q be the ball obtained by adding a regular neighborhood of L_1^* and L_1 to T, together with the ball that the boundary component of $N(T) \cup_{i=1,4} N(L_i^*)$, disjoint from S, bounds. As δ_1^* and δ_1 are sk-arcs, we have that O_1^* and O_1 intersect $\mathcal{D}^*$. As D_1^* is not in $O_1^* \cup O_1$, in this particular case $D_2^* \cup D_3^*$ is necessarily in $O_1^* \cup O_1$, and the string s_{23} is also in Q. The disk D_4^* may or not be in $O_1^* \cup O_1$. If D_4^* is in $O_1^* \cup O_1^*$ then Q intersects S in two components: $D_1^* \cup O_1^*$ and $D_1 \cup O_1$. From Lemma 4.3, the tangle $(Q, \mathcal{T}_1)$ is trivial, which is a contradiction to Lemma 3.1(c).
So, we can assume that D_4^* is not in $O_1^* \cup O_1$ and Q intersects S in three component disks: D_4^*, $D_1^* \cup O_1^*$ and $D_1 \cup O_1$. Also, $O_1 \cap \mathcal{D}^*$, and $O_1^* \cap \mathcal{D}^*$, is either D_2^* or D_3^*. Furthermore, from Lemma 4.3, both strings s_{14} and s_{23} are trivial in Q.
If $n_1 = 1$ then δ_1 is as in Lemma 4.4, which means that some string of some tangle is unknotted. So we can assume that $n_2 \geq 3$.
Suppose there is a sequence of parallel arcs in E after an outermost arc δ_1, $\delta_2, \dots, \delta_{n_2}$, as in Figure 15(a), and consider the balls R_i as in Case 1 of Lemma 6.3. Then, as $O_1 \cap \mathcal{D}^*$ is either D_2^* or D_3^* we have that the ball R_2 contains two strings, which is a contradiction to the balls R_i containing a single string. Consequently, there is no sequence of parallel arcs in E, $\delta_2, \dots, \delta_{n_2}$, after an outermost arc δ_1.
Consider an arc parallel to an outermost arc δ_1^* or otherwise a second-outermost arc, γ, and denote by Γ the disk of $E - E \cap S$, co-bounded by γ, in the outermost side of this arc in E. (See Figure 18(a).) As there is no sequence $\delta_2, \dots, \delta_{n_2}$ after the outermost arcs δ_1, as in Figure 18(a), we have that Γ intersects S in γ and outermost arcs δ_1^*. Note that γ cannot have only one end in D_{n_2}, otherwise γ would be a t-arc, which is a contradiction to Lemma 2.4(c). If γ has two ends in D_1^* or one end in D_1^* and the other end in D_2^*, following reasoning as in the proof of Lemma 3.4, we have that some string in some tangle is unknotted. If the ends of all arcs γ are both in D_2^*, by the finiteness of outermost arcs, we have a contradiction with Lemma 2.2. Hence, we can assume that some arc γ has both ends at D_{n_2}.

Let O_{n_2} be the disk in $S - intD_{n_2}$ cut by γ, disjoint from D_1^*. Denote by C the ball cut

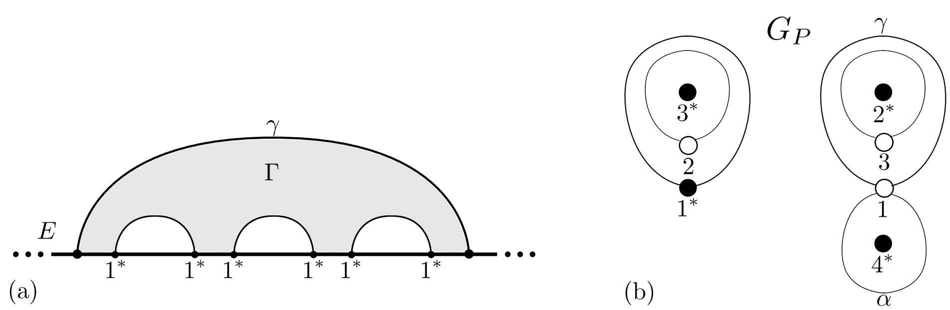


FIGURE 18

from V by $D_1^* \cup D_2^* \cup D_{n_2}$, and by C_{1^*,n_2} the cylinder obtained from C by an isotopy of C along D_2^* away from S. Note that C is in B_2. Consider the disk L bounded by $\partial\Gamma$ in the torus $\partial C_{1^*,n_2} \cup_{D_1^* \cup D_{n_2}} S$. Let R be the ball bounded by $\Gamma \cup L$ in B_2. If R intersects K in two components, then we can prove that γ is parallel to δ_1^* in E. By taking R together with C_{1^*,n_2} we

define a cylinder containing the two strings of $\mathcal{T}_2$ with ends in the disks $D_1^* \cup O_1^*$ and $D_{n_2} \cup O_{n_2}$. Then from Lemma 3.1(a), (c), and because $\partial C_{1^*,n_2} - L \cap \partial C_{1^*,n_2}$ is a single disk containing $D_1^* \cup D_{n_2}$, we obtain a contradiction to the minimality of $|S \cap V|$. So, we have that R intersects $\mathcal{T}_2$ at a single component. Naturally $O_{n_2} \subset L$, and also $O_{n_2} \cap \mathcal{D}^*$ is D_2^*. In fact, if D_3^* is in O_{n_2} then, s_{34} is in R. As R intersects $\mathcal{T}_2$ at a single component, and O_1^* intersects $\mathcal{D}^*$, we have D_4^* in O_1^*, which contradicts our assumption that $O_1^* \cap \mathcal{D}^*$ is only D_2^* or D_3^*. If D_4^* is in O_{n_2} then, following a similar reasoning, D_3^* is in O_1^* and D_2^* is in O_1. As before, with the existence of parallel arcs to γ or δ_1 in E we can define the balls R_{n_2-1} or R_1. But then, in this case, R_1 or R_{n_2} contain two strings, which is a contradiction. Then, D_2^* is in O_{n_2}. As O_{n_2} is disjoint from O_1^*, and $O_1^* \cap \mathcal{D}^*$ is either D_2^* or D_3^*, we have that D_3^* is in O_1^*. Then, D_2^* is in O_1 and if R_1 exists it has two strings, which is impossible. So, we can assume that there is a sequence of balls $R_{n_2-1}, \dots, R_2$ exists, related to a sequence of parallel arcs of $E \cap P$ to γ in E, $\delta_{n_2-1}, \dots, \delta_2$. As O_{n_2} contains D_2^*, if $n_2 \geq 5$ we have that the ball R_{n_2-3} contains T and consequently two strings, which is a contradiction. Therefore $n_2 = 3$, and the ball R_2 contains the string s_{23}. But R_2 cannot contain T, otherwise it would contain two strings. Hence, $O_2 \subset O_1^*$ and $O_3 \subset O_1$. (See Figure 18(b).)
Consider an arc α outermost after the outermost arcs δ_1 and parallel arcs to γ. Then α has ends in $D_1 \cup D_2$. If the arcs α have one end in D_1 and the other in D_2 then we get a contradiction to $D_2 \subset O_1^*$ and O_1 being disjoint from O_1^*. Then α has equal ends. If the ends of α are in D_2 then α is in O_1^* (because D_2 is in O_1^*). All loops attached to D_2, as α, have to be parallel in P to the arc parallel to γ in P attached to D_2. Otherwise, D_4^* is contained in O_1^*, which contradicts the assumption that it is not. Let A be the disk of $E - E \cap P$ co-bounded by α in the outermost side of the arc in E. Suppose α is attached to D_2 or is parallel to δ_1 in P. The boundary of A bounds a disk in $S \cup_{D_1 \cup D_2} \partial C_{1,2}$ that contains O_1, and the union of these two disks bounds a ball, R_1', in B_2. The ball R_1' has similar properties to the balls R_i; including containing a single string of $\mathcal{T}_2$, which is a consequence of Lemma 3.1 (a), (c), the arcs $\partial A \cap S - \gamma$ with both ends in D_1 and D_2 being parallel in P resp., and also from the minimality of $|S \cap V|$. But has R_1' contains O_1, it also contains two strings, which a contradiction to the previous observation. Then, α is attached to D_1 and is not parallel to δ_1. In this case, R_1' contains the string s_{34} as a core, that is parallel to the core of the cylinder $C_{1,2}$. Consider the outermost arcs γ' between the arcs of $E \cap P$ with distinct ends in $D_1^* \cup \mathcal{D}$. Given the configuration of G_P, as in Figure 18(b), the only possible ends for γ' are one end in D_1^* and the other in D_1, one end in D_1^* and the other in D_2 and one end in D_1 and the other in D_3. The only possible case, because the disks involved belong to the same component of $V - V \cap S$, is having γ' with one end in D_1^* and the other in D_1. Let Γ' be the disk, of $E - E \cap S$, co-bounded by γ', in the outermost side of γ' in E. Then Γ' is over Q and S, in B_1. All the arcs of $\partial \Gamma' \cap S$ that intersect D_1^* are either γ' or have both ends in D_1^* and are parallel to δ_1^* in P. By an isotopy of these arcs to Q we get that $D_1^* \cup O_1^*$ is primitive with respect to the complement of Q in B_1, that is a handlebody. Then the core of the cylinder from from D_4^* to D_1 is unknotted. As the string s_{14} is parallel to the core from D_1^* to D_4^* in Q and the string s_{23} is parallel to the core from D_1^* to D_1 in Q, we have that both strings are μ-primitive. △

Claim 6.4.6. *If only D_1^* is attached to outermost arcs then some string of some tangle is unknotted.*

Proof of Claim 6.4.6. Denote by δ_1^* the outermost arcs attached to D_1^*. Consider a second outermost arc, γ, and let Γ be the disk of $E - E \cap V$ co-bounded by γ in the outermost side of this arc in E. (See Figure 18(a).) The curve $\partial \Gamma$ bounds a disk L in the torus $S \cup_{D_1^* \cup D_{n_2}} \partial C_{1^*,n_2}$. Following a similar argument as in Claim 6.4.5, we can assume γ has both ends in D_{n_2} and we define similarly the ball R in B_2 with boundary $\Gamma \cup L$. So, either the string s_{34} or a portion the string s_{12} with end in D_2^* is in R, and therefore, this string is parallel to the core of the cylinder C_{1^*,n_2}. Let O_1^* and O be the disjoint disks in $S^3 - int\{D_1^* \cup D_{n_2}\}$ co-bounded by δ_1^* and γ, resp.. Note that O is in $L \subset \partial R$. As R contains a single string, we have that O intersects $\mathcal{D}^*$ at a single disk. From Claim 6.4.3, we have $n_2 \leq 3$; also, when $n_2 = 3$ we consider the balls R_1, R_2 and the respective disks of intersection with S, O, O_2 and O_1, attached to D_3, D_2 and D_1, resp. .

Assume R contains the string s_{34}. In this case O_1^* is in R, and each O and O_1^* contain a single disk of $\mathcal{D}^*$, D_3^* or D_4^*. Then, if $n_2 = 3$ one of the balls R_1 or R_2 contains two strings of a tangle, which is impossible. Hence, $n_2 = 1$. As O_1^* is disjoint from D_1 we have that O_1^* intersects $S \cap V$ at a single disk of $\mathcal{D}^*$. Therefore, some arc δ_1^* is as in Lemma 4.4, which means that some string of some tangle is unknotted.

Assume now that R contains a portion of the string s_{12}.

Suppose $n_2 = 3$. We have $O \cap \mathcal{D}^* = D_2^*$ and consequently s_{23} is in R_2 and s_{34} is in R_1, which means that $O_2 \cap \mathcal{D}^* = D_3^*$ and $O_1 \cap \mathcal{D}^* = D_4^*$, as in Figure 19(a). Consider an outermost arc, α, between the arcs with ends in distinct disk components of $D_1^* \cup \mathcal{D}$, and A the disk of $E - E \cap P$ co-bounded by α in its outermost side in E. Note that α can only have ends in disks in the same component of $V - V \cap S$. So, α can only have ends in D_1^* and D_1, D_3 and D_2, D_2 and D_1, or also, D_1^* and D_3, as in Figure 19(b).
If the ends of α are in D_1^* and D_3, D_3 and D_2, or D_2 and D_1, then the strings s_{12}, s_{23} or s_{34} are unknotted, respectively.
So, assume that all arcs α have ends in D_1^* and D_1. Consider now the outermost arc α' between

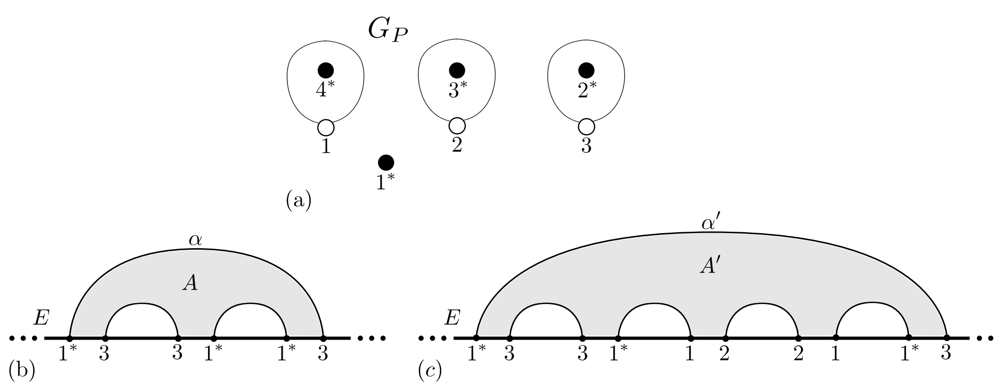


FIGURE 19

the ones with ends in distinct components of $D_1^* \cup \mathcal{D} - D_1$ or that have ends in distinct components of $\mathcal{D}$. Let A' be the disk of $E - E \cap S$ co-bounded by α' in the outermost side of the arc in E. (See Figure 19(c).) The arc α' can only connect components of $V - V \cap S$ with the disks D_1^* and D_1 in them . Hence, the disk A' is in the tangle with the strings s_{12}, s_{34}. Using the disk A' and depending on the ends of α' we can prove that s_{12} or s_{34} is unknotted.

Suppose $n_2 = 1$. Suppose that $O_1^* \cap \mathcal{D}^*$ is either D_3^* or D_4^*. As O_1^* and D_1 are disjoint, δ_1^* is as in Lemma 4.4, which means that some string of some tangle is unknotted.
Suppose, now, that O_1^* intersects $\mathcal{D}^*$ in $D_3^* \cup D_4^*$, as in Figure 20(c). Consider the arcs γ_3^* and γ_4^*, and the respective disks Γ_3^* and Γ_4^*. From Lemma 2.4(b), the two disks D_3^* or D_4^* cannot have simultaneously loops attached in G_P. Then, all arcs γ_3^* or all arcs γ_4^* have distinct ends.
Assume that all arcs γ_3^* have distinct ends. Suppose also that some Γ_3^* intersects D_4^* as in Figure

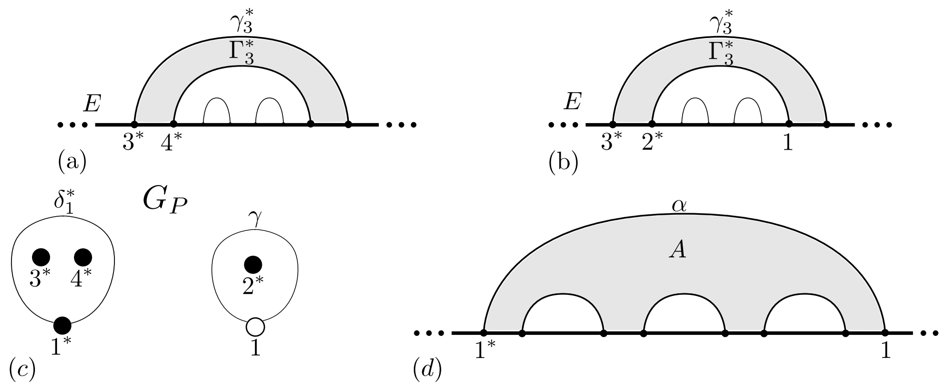


FIGURE 20

20(a). Then the disks D_3^* and D_4^* are primitive with respect to the complement of $V \cap B_2$ in B_2. Consequently, the complement of $C_{1^*,1}$ in B_2 is a solid torus. As s_{12} is parallel to the core of the ball $C_{1^*,1}$ we have that s_{12} is unknotted. Otherwise, suppose that all disks Γ_3^* intersect D_2^* as in Figure 20(b). Then, the disks D_2^* and D_3^* are primitive with respect to the complement of $V \cap B_1$ in B_1. Consider an outermost arc α between the arcs with one end in D_1^* and the other end in D_1. Let A be the disk, of $E - E \cap V$, co-bounded by α in the outermost side of α in E, as in Figure 20(d). Suppose that A is in B_2. The components of $\partial A \cap S$ that intersect D_1 are α and eventually arcs with both ends in D_1 parallel to γ. The disk D_2^* is primitive in the complement of $V \cap B_2$ in B_2. Then after adding the 2-handle with core D_2^* to the complement of $V \cap B_2$ in B_2 we are left with the complement of $C_{1^*,1} \cup C_{3^*,4^*}$. We isotope the arcs of $A \cap S$ parallel to γ, through O, to the boundary of the cylinder $C_{1^*,1}$. After this isotopy, A intersects

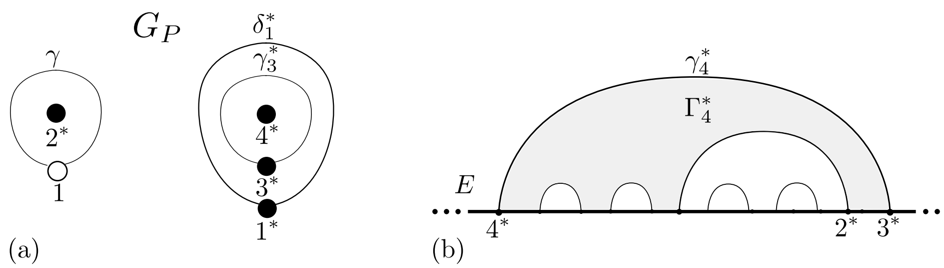


FIGURE 21

D_1 geometrically once. Then, the complement of $C_{3^*,4^*}$ in B_2 is a solid torus, which means that the string s_{34} is unknotted. Otherwise, assume that A is in B_1. The components of $\partial A \cap S$ that intersect D_1 are α and eventually arcs with both ends in D_1 parallel to γ, or arcs with one end in D_1 and the other in D_2^*. As Γ_3^* is in B_1, we have that A doesn't intersect any arc γ_3^*. Then we can proceed as follows. Take T union with a regular neighborhood of O. Isotope to $N(D_1 \cup O)$ the arcs of $A \cap S$ parallel to γ. Then, the disk A intersects $D_1 \cup O$ geometrically once, and Γ_3^* intersects D_2^* geometrically once. As A is disjoint from any γ_3^*, cut $T \cup N(O)$ along D_2^* and, afterwards, we isotope $T \cup N(O)$ along $D_1 \cup O$ away from S. Denote the solid torus after the isotopy as T'. Then, the complement of T' in B_1 is a handlebody. Let O_1^{*c} be the complement of O_1^* in $S - intD_1^*$. Denote by Q' the ball obtained by adding the two handle with core $O_1^{*c} \cup \Delta_1^*$ to T'. The ball Q' intersects S in $D_1^* \cup O_1^{*c}$ and D_4^*, and its complement in B_1 is

a solid torus. The ball Q' contains s_{14} and intersects the string s_{23} at an unknotted component. Then, by Lemma 3.1(b), either one string of the tangle decomposition given by S is unknotted or the tangle in $(Q', Q' \cap \mathcal{T}_1)$ is trivial. Hence, we can assume the latter and that the string s_{14} is trivial in Q'. As the string s_{14} has one end in each of the two components of $Q' \cap S$, it is a core of the cylinder Q'. Consequently, the string s_{14} is unknotted.
Suppose now that some γ_3^* has identical ends. Then, all arcs γ_4^* have distinct ends, and from Figure 21(a), the other end of γ_4^* is in D_3^*. As γ_4^* is the outermost d*-arc with one end in D_4^* and γ_4^* has one end in D_3^*, we have that Γ_4^* intersects D_2^* once. (See Figure 21(b).) This means that Γ_4^* is in B_1 and that D_2^* and D_3^* are primitive with respect to the complement of $V \cap B_2$ in B_2. Then, considering the arc α and disk A and proceeding as before, we have that some string of some tangle is unknotted. △

Claim 6.4.7. *If only D_1 is attached to outermost arcs then some string of some tangle is unknotted.*

Proof of Claim 6.4.7. Let δ_1 denote the outermost arcs attached to D_1. If $n_2 = 3$ by the finiteness of outermost arcs there is a sequence of parallel arcs to some δ_1, that is δ_2 and δ_3, as in Figure 15(a), and with this sequence we can consider the balls R_i as in Case 1 of Lemma 6.3. Let γ be a second-outermost arc of $E \cap P$ in E, as in Figure 11(b). From Lemma 3.4, if γ has one end in D_{n_2} or one end in D_1^* and the other in D_2^* then some string of some tangle defined by S is unknotted. If all arcs γ have both ends in D_2^*, then, by the finiteness of outermost arcs, we have a contradiction to Corollary 2.2. Then some arc γ has both ends in D_1^*. Consider this arc γ and let Γ be the disk component of $E - E \cap P$ co-bounded by γ in the outermost side of this arc in E. The disk Γ bounds a disk L in $\partial C_{1^*,n_2} \cup_{D_1^* \cup D_{n_2}} S$ that together with Γ bound a ball R in B_2. As in the previous claim, we have that either a portion of the string s_{12} with end in D_2^* is in R, or the string s_{34} is in R. Let O_1^* be the disk co-bounded by γ in $S - intD_1^*$, disjoint from D_{n_2}. Then $O_1^* \subset L$.

Assume R contains a portion of the string s_{12}. In this case, D_2^* is the only disk of $\mathcal{D}^*$ in L and $D_2^* \subset O_1^*$. Consider the ball C_{1^*} obtained by an isotopy of the ball C, cut from V by $D_1^* \cup D_2^* \cup D_{n_2}$, along $D_2^* \cup D_{n_2}$ away from S in B_2. From Lemma 2.3, the arc $C_{1^*} \cap s_{12}$ is trivial in C_{1^*}. We also have that the portion of s_{12} in the complement of C_{1^*} in B_2 is unknotted. In fact, we can assume that this arc is $R \cap s_{12}$. As there are no local knots, $R \cap s_{12}$ is trivial in R, and therefore, it is parallel to L. We can isotope the components of $L \cap S - O_1^*$ from S to $\partial C_{1^*,n_2}$. With the isotopy we verify that the arc $R \cap s_{12}$ is parallel to the boundary of C_{1^*}. Altogether, we have that s_{12} is unknotted in $(B_2, \mathcal{T}_2)$.

Assume now that R contains the string s_{34}. Following along an argument of the similar situation in Claim 6.4.6, we have some string of some tangle is unknotted. △

□

Lemma 6.5. *Suppose $V - V \cap S$ contains a solid torus that intersects $\mathcal{D}^*$ and $\mathcal{D}$ at two disks. Then some string of some tangle is unknotted.*

Proof. Let T be the solid torus component of $V - V \cap S$ as in the statement, and suppose it lies in the tangle $(B_1, \mathcal{T}_1)$. As the genus of V is three, all disks of $\mathcal{D}^*$ are parallel in V, and the same is true for the disks of $\mathcal{D}$. Assume that $\partial T \cap \mathcal{D}^* = D_1^* \cup D_4^*$ and $\partial T \cap \mathcal{D} = D_1 \cup D_{n_2}$, as in Figure 22). From Remark 3, we can assume that all outermost disks are over T with respective outermost arcs attached to D_1^*, D_4^*, D_1 or D_{n_2}. If all outermost arcs are attached to D_1^* or D_4^*, then by the finiteness of outermost arcs there are parallel sk-arcs in contradiction to Corollary 2.2. Then, some outermost arc is attached to D_1 or D_{n_2}. Furthermore, even if D_1^* or D_4^* is attached to outermost arcs the only sequences of arcs parallel to outermost arcs in E are with respect to outermost arcs attached to D_1 or D_{n_2}. Without loss of generality, we assume that D_1 is always attached to some outermost arc, and denote by δ_1 an outermost arc attached to D_1.

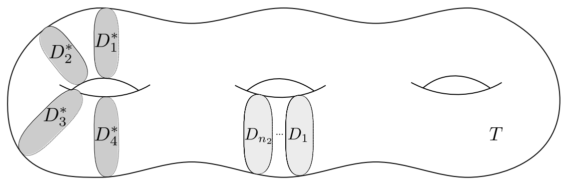


FIGURE 22

Claim 6.5.1. *If D_{n_2} is not attached to outermost arcs then some string of some tangle is unknotted.*

Proof of Claim 6.5.1. Assume that D_{n_2} is not attached to outermost arcs. By the finiteness of outermost arcs and Lemma 2.2, there is a sequence of arcs of $E \cap P$ in E parallel to an outermost arc δ_1, that is $\delta_2, \dots, \delta_{n_2}$, as in Figure 15(a). As in Case 1 of Lemma 6.3 we define the balls R_i; consider also the respective disks O_i and O_{i+1}. As R_1 contains a single string we have that O_1 intersects $\mathcal{D}^*$ at a single disk. If $n_2 = 2$ then as O_1 and O_2 are disjoint, we have that O_1 intersects $S \cap V$ at a single disk. Hence, δ_1 is as in Lemma 4.4, which means that some string of some tangle is unknotted. Suppose $n_2 \geq 4$. (Note that n_2 is necessarily even.) If D_2^* or D_3^* are in O_1 then R_2 contains two strings, which is impossible. Then, without loss of generality, we can assume that D_1^* is in O_1. Suppose some disk of $\mathcal{D}$ is in O_1, say D_i. Then D_1^* is also in O_i. This means that T is in R_{i-1}, and consequently, D_1 is in O_i, which is a contradiction as D_i is in O_1. Therefore, δ_1 is under the conditions of Lemma 4.4, which means that some string of some tangle is unknotted. $\triangle$

Claim 6.5.2. *If D_{n_2} is attached to outermost arcs then some string of some tangle is unknotted.*

Proof of Claim 6.5.2. Assume that both D_1 and D_{n_2} have outermost arcs attached, denoted by δ_1 and δ_{n_2} resp.. Let the outermost disk co-bounded by δ_1 (resp., δ_{n_2}) be denoted by Δ_1 (resp., Δ_{n_2}) and let O_1 (resp., O_{n_2}) be the disk in $S - int(D_1 \cup D_{n_2})$ separated by δ_1 (resp., δ_{n_2}). By adding a regular neighborhood of $O_{n_2} \cup \Delta_{n_2}$ and $O_1 \cup \Delta_1$ to T, and the ball bounded by the boundary component that is disjoint from S, we define a ball Q. If Q contains both strings of $\mathcal{T}_1$, from Lemma 4.3, we have that the tangle $(Q, Q \cap K)$ is trivial. If $\mathcal{D}^*$ is in $O_1 \cup O_{n_2}$ then we get a contradiction between Lemma 3.1(c) and Lemma 4.3. Then, O_1 or O_{n_2} intersects $\mathcal{D}^*$ at a single disk.
If $n_2 = 2$, O_1 and O_2 are disjoint, and δ_1 or δ_{n_2} are as in Lemma 4.4, which means that some string of some tangle is unknotted.
Assume $n_2 \geq 4$.
Suppose that $O_1 \cup O_{n_2}$ intersect $\mathcal{D}^*$ in three disks and, without loss of generality, that O_1 intersects $\mathcal{D}^*$ at a single disk. If there is any arc of $E \cap P$ parallel to δ_{n_2} in E, the respective ball R_{n_2-1} contains two strings, which is impossible as observed in Lemma 3.3. Then, there is a sequence of parallel arcs of $E \cap P$ in E, $\delta_1, \dots, \delta_{n_2-1}$ and we can consider the respective balls $R_1, \dots, R_{n_2-2}$. If $O_1 \cap \mathcal{D}^*$ is D_2^* or D_3^*, then R_2 contains two strings, which is impossible. Then, $O_{n_2} \cap \mathcal{D}^*$ is $D_2^* \cup D_3^*$. The string s_{23} is trivial in Q and has ends in the same disk component of $Q \cap S$, then s_{23} is unknotted in $(B_1, \mathcal{T}_1)$.
Suppose that $O_1 \cup O_{n_2}$ intersect $\mathcal{D}^*$ in two disks. Assume $D_2^* \cup D_3^*$ is in $O_1 \cup O_{n_2}$. If there are two consecutive balls R_i after O_1 or O_{n_2}, then some ball R_i contains two strings, which is impossible. Then, $n_2 = 4$ and both δ_1 and δ_4 have a parallel arc of $E \cap P$ in E, δ_2 and δ_3, resp.,

from where we define the balls R_1 and R_3. As R_1 and R_3 have a single string, we have that O_1 is disjoint from D_2, D_3 and D_4. Hence, δ_1 is as in Lemma 4.4 and some string of some tangle is unknotted. Assume now that $D_1^* \cup D_4^*$ is in $O_1 \cup O_{n_2}$. If there is a sequence of parallel arcs to δ_1 (or to δ_{n_2}) attached to all disks $D_1, \dots, D_{n_2}$, as in Figure 15(a), following the same argument as in Claim 6.5.1, we prove that some string of some tangle is unknotted. Otherwise, the sequences of parallel arcs from δ_1 go up to some arc δ_i and from δ_{n_2} go up to some arc δ_{i+1}. If $n_2 = 4$, then as before δ_1 or δ_{n_2} are as in Lemma 4.4, which means that some string of some tangle is unknotted. Suppose $n_2 \geq 6$. Then, again using arguments as in Claim 6.5.1, if O_1 intersects $\mathcal{D}$, it is in D_i or D_{i+1}. From the sequences of parallel arcs we can consider the respective balls $R_1, \dots, R_{i-1}$ and $R_{i+1}, \dots, R_{n_2-1}$. Denote by $C_{k,k+1}$ the cylinder in V between D_k and D_{k+1}. If D_i and D_{i+1} are not in O_1 then δ_1 resp., is as in Lemma 4.4, which means that some string of some tangle is unknotted. Otherwise, without loss of generality, suppose that D_i is in O_1. Then, then as R_{i-1} cannot be in Q, we have that $C_{i,i+1}$ is in Q. Each string of the tangles defined by S is in Q or is some ball R_k. Following as in the previous claim, consider Q union with $R_k \cup C_{k,k+1}$, for $k = 1, \dots, i-1, i+1, \dots, n_2$, we define a solid torus that is a neighborhood of K, containing V, and with boundary essential in W, which is a contradiction to W being a handlebody. △

□

Lemma 6.6. *Suppose $V - V \cap S$ contains a solid torus component disjoint from $\mathcal{D}$ and intersecting $\mathcal{D}^*$ at two disks. Then both strings of some tangle are μ-primitive.*

Proof. Let T be a solid torus component as in the statement and suppose $\mathcal{D}^* \cap T = D_1^* \cup D_4^*$. Assume that T is in the tangle $(B_1, \mathcal{T}_1)$. From Remark 3, all outermost disks are over solid torus components of $V - V \cap S$. Suppose some outermost disk is attached to some disk of $\mathcal{D}$. As the genus of V is three, this outermost disk is over a solid torus disjoint from K intersecting $S \cap V$ at a single disk, which is a contradiction to Lemma 4.1. Then all outermost disks are attached to disks of $\mathcal{D}^*$.

Claim 6.6.1. *If the disks of $\mathcal{D}^*$ are parallel two-by-two then some string in some tangle is unknotted.*

Proof of Claim 6.6.1. Suppose only one disk or two non-parallel disks are adjacent to outermost arcs. By the finiteness of outermost arcs we have parallel sk-arcs, as in Figure 11(a1), (a2), and we get a contradiction to Corollary 2.2.
Otherwise, we are left with the case when the outermost arcs are only adjacent to two parallel disks of $\mathcal{D}^*$. Following an argument of a similar situation in Lemma 6.2, we have that some string in some tangle is unknotted. △

Claim 6.6.2. *If the disks of $\mathcal{D}^*$ are not parallel two-by-two then both strings of some tangle are μ-primitive.*

Proof of Claim 6.6.2. Assume, without loss of generality, that no other disk of $S \cap V$ is parallel to D_1^*. If there are disks (of $\mathcal{D}^*$) parallel to D_4^*, and D_4^* or one disk parallel to it are the only disks with outermost arcs attached, then we get a contradiction to Corollary 2.2. So, without loss of generality, assume there is some outermost arc attached to D_1^*, and that it is over T. Under these conditions, we define a ball Q as in Lemma 4.5, using the outermost disk attached to D_1^* over T. From Lemma 4.5, the tangle $(Q, Q \cap K)$ is the product tangle. So, we can isotope S through Q, and we replace D_1^* with a disk parallel to D_4^*. So, if $n_2 = 0$ we reduce this case to either the case when there is a genus two component, as in Lemma 6.1 or to the case when $V - V \cap S$ contains a solid torus component intersecting $\mathcal{D}^*$ at the four disks as in Lemma 6.2. If

$n_2 > 0$ we also reduce to the cases when $V - V \cap S$ contains a solid torus component intersecting $\mathcal{D}^*$ in a collection of two disks and $\mathcal{D}$ in one or two disks, as in Lemmas 6.3, 6.4 and 6.5. From these lemmas we get that both strings in some tangle are μ-primitive. $\triangle$

$\square$

Lemma 6.7. *Suppose $V - V \cap S$ contains a solid torus component disjoint from K. Then both strings of some tangle are μ-primitive.*

Proof. As the genus of V is three and no disk of $\mathcal{D}^*$ is parallel to a disk of $\mathcal{D}$, $\mathcal{D} \cap T$ is a collection of at most three disks. If there is some solid torus component of $V - V \cap S$ intersecting $\mathcal{D}^*$ we follow as in the previous lemmas to get the conclusion that two strings of some tangle are μ-primitive. Otherwise, the solid torus components of $V - V \cap S$ are disjoint from K. From Remark 3, without loss of generality, we can assume that some outermost disk is over T.
If $\mathcal{D} \cap T$ is a single disk we get a contradiction to Lemma 4.1. Then, we have that T intersects $\mathcal{D}$ at more than one disk, in which case T is the only solid torus component of $V - V \cap S$ and all outermost disks are over T.
Assume that $\mathcal{D} \cap T$ is a collection of two disks, D_1 and D_1'. The outermost arcs are attached

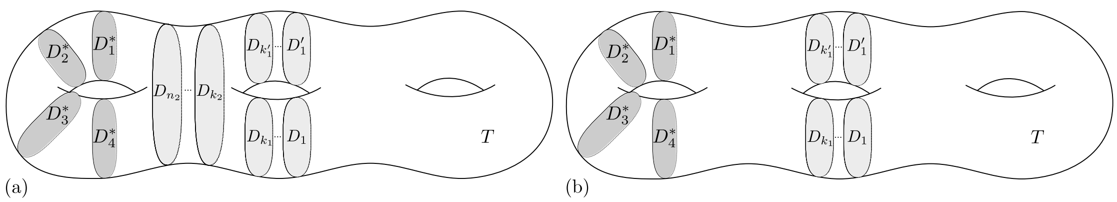


Figure 23

to D_1 or to D_1', with outermost disks over T. Let $D_2, D_3, \dots, D_k$ be the disks of $\mathcal{D}$ parallel to D_1 in V, in case there exists such a sequence. Without loss of generality, assume there is an outermost arc δ_1 attached to D_1, and that there is a sequence of arcs, δ_i, after an outermost arc, δ_1, as in Figure 15(a). Let Δ be the outermost disk bounded by δ_1, in E, and also, Δ_i be the disk of $E - E \cap S$ between δ_i and δ_{i+1}. As S^3 has no lens space summand, we have that $\partial\Delta$ intersects a meridian of T geometrically once. So, we can perform an isotopy of the annulus in S, $A = D_1 \cup (S \cap N(\Delta))$ through $N(\Delta)$ to the annulus $A' = D_1 \cup (\partial T \cap N(\Delta))$. As $\partial\Delta_1$ intersects a meridian of T geometrically once, we isotope A' through T to a disk in T parallel to D_1', that we also denote by D_1. Using the disk $\Delta_1 \cup \Delta$ we can perform a similar isotopy, and from the disk D_2 of $E \cap S$ we get a disk in T parallel to the new disk D_1. In this way we can perform a sequence of isotopies of S to get from the disks $D_1, D_2, \dots, D_k$ new disks in T parallel to D_1'. With this isotopy we reduce this case to other cases: If the disks of $\mathcal{D}^*$ are not parallel in V we can reduce this case to the case when $T \cap \mathcal{D}^*$ is a collection of two disks and $T \cap \mathcal{D}$ is one non-separating disk, as in Lemma 6.4. So, we are left with the situation when the disk components of $\mathcal{D}^*$ are parallel. The disk components of $\mathcal{D}$ in V can be parallel to D_1 or to D_1', or can be separating. Assume there is a disk of $\mathcal{D}$ that is separating in V, as in Figure 23(a). By the previous isotopy we reduce this case to the case, considered next, when $\mathcal{D}^* \cap T$ is empty and $\mathcal{D} \cap T$ is a collection of three disks. Otherwise, suppose that no disk of $\mathcal{D}$ is separating, as in Figure 23(b). Similarly, we reduce this case to the case when $\mathcal{D}^* \cap T$ and $\mathcal{D} \cap S$ is a collection of two disks, as in Lemma 6.5.
At last, suppose that $\mathcal{D} \cap T$ is a collection of three disks. We have a collection of parallel non-separating disks of $\mathcal{D}$, and a collection of separating disks of $\mathcal{D}$ in V, as in Figure 24. As in Lemma 3.4, let C be the ball component of $V - V \cap S$ cut from V by $D_1^* \cup D_4^* \cup D_{n_2}$. Every

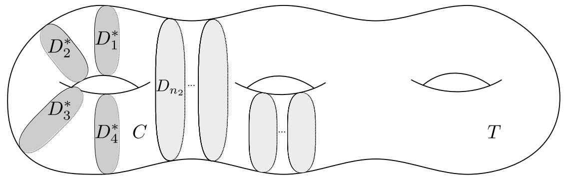


Figure 24

outermost arc of $E \cap S$ in E attached to D_{n_2} has both ends attached to it (otherwise, it would be a t-arc, which don't exist from Lemma 2.4(c)). By the finiteness of outermost arcs, we consider an outermost arc γ after outermost arcs with both ends in D_{n_2}. From Lemma 3.4, if γ has at least one end in D_{n_2} or one end in D_1^* and the other in D_4^*, some string of some tangle is unknotted. In case, all arcs γ have both ends in D_1^* or in D_4^*, by the finiteness of outermost arcs, we have a contradiction to Corollary 2.2. □

7. Proof of Theorem 1

For the proof of Theorem 1, we study all cases of $S \cap V$ with respect to the value n_1.

Proposition 1. *If $n_1 = 1$ then both strings of some tangle are μ-primitive.*

Proof. Suppose $n_1 = 1$. If $n_2 > 0$ we have a contradiction between Lemma 2.4(b) and (f). So, $n_2 = 0$, P is a disk and $|P \cap E| = 0$.
Let $D_1^* = S \cap V$. The 2-sphere $S = D_1^* \cup P$ is separating, then D_1^* is a separating disk in V.

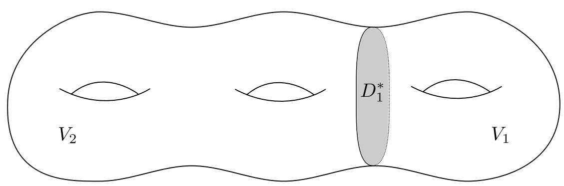


Figure 25

As the handlebody V has genus three, the disk D_1^* separates V in a solid torus V_1 and a genus two handlebody V_2, as in Figure 25. Let (B_1, T_1) denote the tangle containing V_1. The strings of this tangle lie in V_1, have end points in D_1^* and, by Lemma 2.3, are simultaneously parallel to ∂V_1. Also, the complement of V_1 in B_1 is a torus. Hence, from Lemma 2.2, both strings of the tangle (B_1, T_1) are μ-primitive. □

Proposition 2. $n_1 \neq 2$.

Proof. Suppose $n_1 = 2$. We denote by D_1^* and D_2^* the components of $\mathcal{D}^*$. From Lemma 2.4(b), (g) $n_2 > 0$ and every outermost arc is a st-arc.

Claim. *If $n_1 = 2$ there is no ball C of $V - V \cap S$ containing strings of a tangle.*
Proof of Claim. Suppose that there is a ball component of $V - V \cap S$, C, containing strings of a tangle.
Suppose, the ball C contains two strings. From Lemma 2.3, the strings are parallel to ∂C. Therefore, the tangle $(C, C \cap K)$ is trivial, which is a contradiction to Lemma 3.1(c).
Otherwise, suppose that C contains a single string. As $D_1^* \cup D_2^*$ intersects K in four points only one of these disks can be in ∂C, and both ends of the string in C are in this disk. Then, this

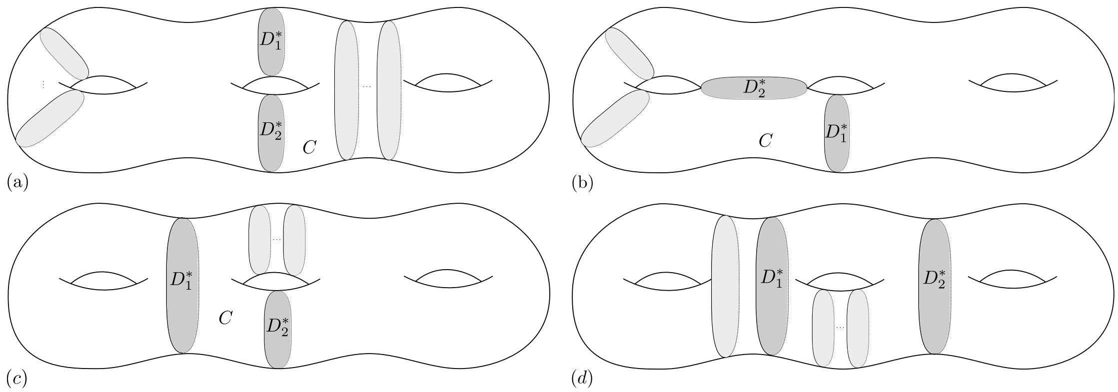


FIGURE 26

string is trivial in ∂C. Furthermore, as this is the only string in C it is also trivial in the respective tangle, which is a contradiction to the tangle decomposition defined by S being essential. $\triangle$

If D_1^* and D_2^* are parallel in V then the ball component of $V - S \cap V$ cut by $D_1^* \cup D_2^*$ is in contradiction to Claim.
Suppose now that D_1^* and D_2^* are not parallel, as in the examples of Figure 26. Then, the components of $V - D_1^* \cup D_2^*$ are solid tori. As $n_2 > 0$, the disks of $\mathcal{D}$ are in some of these solid tori. Then, some ball component of $V - V \cap S$ contains D_1^*, D_2^*, or both, which is a contradiction to Claim. □

Proposition 3. *If $n_1 = 3$ then both strings of some tangle are μ-primitive.*

Proof. Consider the components of $V - V \cap S$. From Remark 3 we can assume that some component of $V - S \cap V$ is not a ball.
If there is a genus two component of $V - V \cap S$ then, by Lemma 5.1 , some string of some tangle is unknotted. Otherwise, there is some solid torus component of $V - V \cap S$, and from Lemma 5.2 two strings of some tangle are μ-primitive. □

Proposition 4. *If $n_1 = 4$ then both strings of some tangle are μ-primitive.*

Proof. As in Proposition 3, we consider the components of $V - V \cap S$ and we assume that some component of $V - S \cap V$ is not a ball.
If $V - V \cap S$ has a genus two component then, by Lemma 6.1, some string of some tangle is unknotted.
Now, assume that $V - V \cap S$ has no genus two component. This means at least one of its components is a solid torus, T. The collection of disks $\mathcal{D}^* \cap T$ is always even, because ∂T is a separating torus in S^3. We consider several cases with respect to $\mathcal{D}^* \cap T$.
If $\mathcal{D}^* \subset T$, from Lemma 6.2, some string of some tangle is unknotted.
Suppose $\mathcal{D}^* \cap T$ is a collection of two disks. As the genus of V is three, there are at most two disks of $\mathcal{D}$ in ∂T. Then we are under Lemmas 6.3, 6.4, 6.5 and 6.6, and we have that both strings of some tangle are μ-primitive.
At last, suppose $\mathcal{D}^* \cap T = \emptyset$. From Lemma 6.7, we also have that both strings of some tangle are μ-primitive. □

We can now prove Theorem 1 and its Corollary 1.1.

Proof of Theorem 1. If K has an inessential 2-string free tangle decomposition then the tunnel number of K is one. This is a contradiction with the assumption that the tunnel number of K is two. Hence, the 2-string free tangle decomposition of K is essential.
We have that $0 \leq n_1 \leq 4$. If $n_1 = 0$ then, as $S \cap K \subset S \cap V$ we have $n_2 = 0$. Hence, $S \subset V$ which is a contradiction to Lemma 2.3(a). In case $n_1 \neq 0$, from Propositions 1, 2, 3 and 4, we have that two strings of some tangle are μ-primitive. □

Proof of Corollary 1.1. Let K be a knot with a 2-string free tangle decomposition where at least a string of each tangle is not μ-primitive.
From Corollary 2.4 in [16] by Morimoto, if a knot K has a n-string free tangle decomposition, then $t(K) \leq 2n - 1$. Hence, in this case $t(K) \leq 3$.
On the other hand, as no tangle in the decomposition of K has both strings being μ-primitive, from Theorem 1 we have $t(K) \geq 3$.
Altogether, from the two inequalities, $t(K) = 3$.

□

8. On the tunnel number degeneration under the connected sum of prime knots

In this section, we construct an infinite class of knots with a 2-string free tangle decomposition where no tangle has both strings being μ-primitive. With these collection of knots, Theorem 1 and the work of Morimoto [16] we prove Theorem 2.
A particular, simplified, version of Theorem 3.4 in [16] by Morimoto gives us the following proposition, which is relevant to the proof of Theorem 2.

Proposition 5 ([16], Morimoto)**.** *Let K_1 be a knot which has a 2-string free tangle decomposition and K_2 be a knot with a 3-bridge decomposition. Then $t(K_1 \# K_2) \leq 3$.*

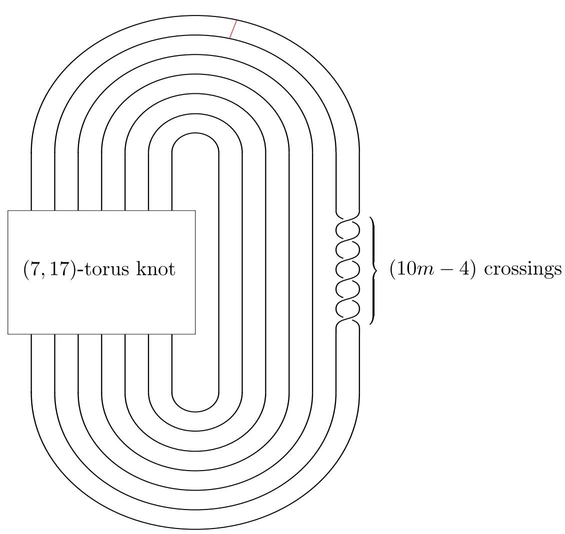


Figure 27: The knot $K(m)$ and one unknotting tunnel, with m a natural number.

For the construction of knots, K_1, as in Theorem 2 we consider 2-string free tangle decompositions. Suppose there are two 2-string free tangles $(B_1, \mathcal{T}_1)$ and $(B_2, \mathcal{T}_2)$ where one of the strings in each tangle is not μ-primitive. Identify $(\partial B_1, \partial \mathcal{T}_1)$ to $(\partial B_2, \partial \mathcal{T}_2)$, such that no string of $\mathcal{T}_1$ has its end identified to the ends of the same string of $\mathcal{T}_2$. Then $(B_1 \cup B_2, \mathcal{T}_1 \cup \mathcal{T}_2)$ is a knot

(S^3, K_1) under the conditions of Proposition 5. Furthermore, from Corollary 1.1, $t(K_1) = 3$. Hence, this procedure gives us a knot as in the statement of Theorem 2.
So, we need 2-string free tangles with one of the strings not μ-primitive. As observed in Remark 1, if a string s properly embedded in a ball B is μ-primitive, then by capping s along ∂B we get a μ-primitive knot. Then, for the construction of a 2-string free tangle where at least one of the strings is not μ-primitive we consider a tunnel number one knot K that is not μ-primitive, and one of its unknotting tunnels. For such a knot K, let s be a string in a ball B, that when capped off along ∂B we obtain K, together with one unknotting tunnel for K. If we slide the ends of the unknotting tunnel from s to ∂B we get an essential 2-string free tangle where one of the strings is not μ-primitive.
Then, we want tunnel number one knots that are not μ-primitive. Existence results of such knots are known by work Johnson and Thompson in [6] and also Moriah and Rubinstein in [12]. On the other hand, explicit or constructive examples of knots with tunnel number one that are not μ-primitive is given by work Eudave-Muñoz in [18] and [19], Ramírez-Losada and Valdez-Sánchez in [9], Minsky, Moriah and Schleimer in [10] and also Morimoto, Sakuma and Yokota in [17]. With any of these examples it is possible to construct knots as in the statement of Theorem 2. As an example of such construction we consider the class of knots $K(7, 17; 10m-4)$ from [17], where m is an integer, together with an unknotting tunnel. We denote these knots by $K(m)$, as in Figure 27.
As previously described, from the knots $K(m)$ and an unknotting tunnel we construct tangles $T(m)$ where at least one of the strings is not μ-primitive, as in Figure 28. From the construction

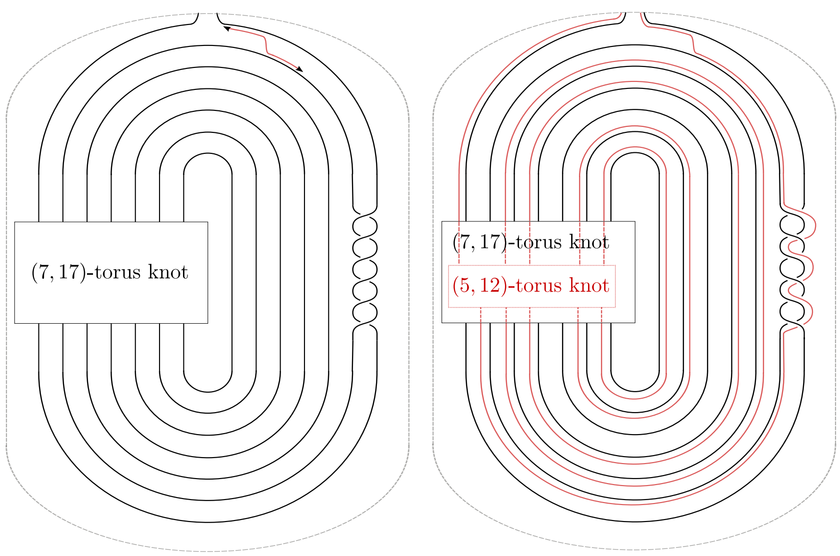


FIGURE 28: A possible construction of a tangle $T(m)$ from the knot $K(m)$ and one of its unknotting tunnels.

we have that the tangles $T(m))$ are free. With the tangles $T(m)$ and $T(m')$ we construct a knot $K_1(m, m')$, as explained before, that has a 2-string free tangle decomposition where no tangle has both strings being μ-primitive. With this construction we can now prove Theorem 2 and its Corollary 2.1.

Proof of Theorem 2. Consider the collection of knots $\{K_1(m, m') : m, m' \in \mathbb{N}, m \leq m'\}$. From Corollary 1.1, we have that $t(K_1(m, m')) = 3$. From Ozawa's unicity theorem, the knot

$K_1(m, m')$ is prime. And, from Proposition 5, for any 3-bridge knot K_2, $t(K_1(m, m')\# K_2) \leq 3$. □

Proof of Corollary 2.1. Consider the collection of knots $\{K_1(m, m') : m, m' \in \mathbb{N}, m \leq m'\}$. Let K_2 be any 3-bridge prime knot with tunnel number two. From Proposition 5, $t(K_1(m, m')\# K_2) \leq 3$. From tunnel number one knots being prime and the main theorem in [14], we also have that $t(K_1(m, m')\# K_2) \geq 3$. Then, $t(K_1(m, m')\# K_2) = 3 = t(K_1(m, m')) + t(K_2) - 2$. □

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