The EMM and the spectral analysis of a non self-adjoint Hamiltonian on an infinite dimensional Hilbert space

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Abstract The Equation of Motion Method is used in the spectral analysis of a non self-adjoint bosonic Hamiltonian acting on an infinite dimensional Hilbert space. The presented operator has real eigenvalues and can be diagonalized when it is expressed in terms of pseudo-bosons, which do not behave as ordinary bosons under the adjoint transformation, but obey the Weil-Heisenberg commutation relations.

1 Introduction

In conventional formulations of non-relativistic quantum mechanics, the Hamiltonian operator is self-adjoint. However, certain relativistic extensions of quantum mechanics lead to the consideration of non self-adjoint Hamiltonian operators with a real spectrum. This motivated an intense research activity, both on the physical and mathematical level (see, e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9] and their references).

Throughout, we shall use synonymously the terms Hermitian and self-adjoint.

Let $\mathscr{D}$ be a certain domain, dense in a Hilbert space $\mathscr{H}$, endowed with an inner product $\langle , \rangle$. Let $a, b, a^*, b^* : \mathscr{D} \to \mathscr{D}$, be bosonic operators. We recall that, conventionally, a, b are said to be *annihilation operators*, while a^*, b^* are *creation operators*. It is worth noticing that these operators are unbounded. Moreover, a, b and of their adjoints satisfy the commutation rules (CR's),

$$[a, a^*] = [b, b^*] = \mathbf{1}, \tag{1}$$

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where **1** is the identity operator on $\mathscr{H}$. (This means that $aa^*f - a^*af = bb^*f - b^*bf = f$ for any $f \in \mathscr{D}$.) Furthermore,

$$[a,b^*] = [b,a^*] = [a^*,b^*] = [a,b] = 0. \tag{2}$$

As it is well-known, the canonical commutation relations (1) and (2) characterize an algebra of Weil-Heisenberg (W-H). Moreover, the existence of a vector $\Phi_0 \in \mathscr{D}$, a so-called *vacuum state*, satisfying,

$$a\Phi_0 = b\Phi_0 = 0,$$

is postulated. The set of vectors

$$\{\Phi_{m,n} = a^{*m}b^{*n}\Phi_0 : \ m,n \geq 0\}, \tag{3}$$

constitutes a basis of $\mathscr{H}$, that is, every vector in $\mathscr{H}$ can be uniquely expressed in terms of this vector system, which is *complete*, since 0 is the only vector orthogonal to all its elements.

The main goal of this note is to investigate spectral properties of a certain non self-adjoint operator which is expressed as a quadratic combination of bosonic operators.

2 Non self-adjoint operator and the EMM

We are concerned with bosonic operators $a_i^*, a_j,\ i,j = 1,\dots N$, which, as usual, act on an infinite dimensional Hilbert space $\mathscr{H}$. Ordinary bosons obey the Weil-Heisenberg commutation relations,

$$[a_i,a_j^*] = \delta_{ij}\mathbf{1}, \quad [a_i^*,a_j^*] = 0, \quad [a_i,a_j] = 0, \quad i,j = 1,\dots,N,$$

where δ_{ij} denotes the Kronecker symbol (δ_{ij} equals 1 for $i = j$ and 0 otherwise). Let us consider the non self-adjoint Hamiltonian

$$H = \sum_{i,j=1}^{N} \left(A_{ij}a_i^*a_j + \frac{1}{2}B_{ij}a_i^*a_j^* - \frac{1}{2}B_{ij}a_ia_j \right), \tag{4}$$

where $A = (A_{ij}),\ B = (B_{ij})$ are real symmetric matrices of size $N \times N$. In order to determine the eigenvalues of H, we use the equation of motion method (EMM). We investigate the condition

$$\left[H, \sum_{i=1}^{N} (X_i a_i^* - Y_i a_i) \right] = \lambda \sum_{i=1}^{N} (X_i a_i^* - Y_i a_i), \tag{5}$$

with λ a complex parameter and $[X,Y]=XY-YX$ denoting, as usually, the commutator of X and Y. From (5), we get the block matrix equation

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \lambda \begin{bmatrix} X \\ Y \end{bmatrix} \tag{6}$$

where $X=(X_i),\ Y=(Y_i)$ are column matrices with N entries. Since the block matrix

$$M = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

is real symmetric, its eigenvalues λ are real. From (6), it follows that

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} Y \\ -X \end{bmatrix} = -\lambda \begin{bmatrix} Y \\ -X \end{bmatrix},$$

so, if λ is an eigenvalue of (6), so is $-\lambda$.Thus, the eigenvalues appear in pairs of symmetric real numbers. Let us consider a set of orthogonal eigenvectors of (6). Let

$$\begin{bmatrix} X^{(r)} \\ Y^{(r)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Y^{(r)} \\ -X^{(r)} \end{bmatrix}$$

be the eigenvectors corresponding to the eigenvalues $\lambda_r > 0$ and $-\lambda_r$, respectively. Since they are associated to distinct eigenvalues, they are orthogonal. Orthogonality implies

$$\begin{aligned} X^{(r)T}X^{(s)} + Y^{(r)T}Y^{(s)} &= \delta_{rs}, \\ Y^{(r)T}X^{(s)} - X^{(r)T}Y^{(s)} &= 0. \end{aligned}$$

These orthogonality relations are matricially expressed as

$$\begin{bmatrix} X^{(s)T} & Y^{(s)T} \end{bmatrix} \begin{bmatrix} X^{(r)} \\ Y^{(r)} \end{bmatrix} = \delta_{rs}, \quad \begin{bmatrix} X^{(s)T} & Y^{(s)T} \end{bmatrix} \begin{bmatrix} Y^{(r)} \\ -X^{(r)} \end{bmatrix} = 0,$$

$$\begin{bmatrix} Y^{(s)T} & -X^{(s)T} \end{bmatrix} \begin{bmatrix} X^{(r)} \\ Y^{(r)} \end{bmatrix} = 0, \quad \begin{bmatrix} Y^{(s)T} & -X^{(s)T} \end{bmatrix} \begin{bmatrix} Y^{(r)} \\ -X^{(r)} \end{bmatrix} = \delta_{rs},$$

or, compactly, as

$$\begin{bmatrix} \mathscr{X} & -\mathscr{Y} \\ \mathscr{Y} & \mathscr{X} \end{bmatrix}^T \begin{bmatrix} \mathscr{X} & -\mathscr{Y} \\ \mathscr{Y} & \mathscr{X} \end{bmatrix} = I_{2N},$$

where I_{2N} is the $2N \times 2N$ identity matrix and

$$\mathscr{X} = \begin{bmatrix} X^{(1)} & \dots & X^{(N)} \end{bmatrix}, \quad \mathscr{Y} = \begin{bmatrix} Y^{(1)} & \dots & Y^{(N)} \end{bmatrix} \in M_N,$$

the algebra of $N \times N$ real matrices. The matrix

$$\begin{bmatrix} \mathscr{X} & \mathscr{Y} \\ -\mathscr{Y} & \mathscr{X} \end{bmatrix} = \exp \begin{bmatrix} S & T \\ -T & S \end{bmatrix}, \quad S = -S^T, \ T = T^T,$$

belongs to a certain subgroup of the real orthogonal group and the matrix

$$\begin{bmatrix} S & T \\ -T & S \end{bmatrix},$$

belongs to a certain sub-algebra of the algebra of the real skew-symmetric matrices. Consider the pseudo-bosons defined as

$$c_r^\ddagger = \sum_{i=1}^N (X_i^{(r)} a_i^* - Y_i^{(r)} a_i), \quad c_r = \sum_{i=1}^N (Y_i^{(r)} a_i^* + X_i^{(r)} a_i), \quad r,s = 1,\dots,N. \qquad (7)$$

Although $c_r^\ddagger \neq c_r^*$, pseudo-bosons obey the Weil-Heisenberg commutation relations,

$$[c_r, c_s^\ddagger] = \delta_{rs}\mathbf{1}, \quad [c_r^\ddagger, c_s^\ddagger] = 0, \quad [c_r, c_s] = 0, \quad r,s = 1,\dots,N,$$

where $\mathbf{1}$ is the identity on $\mathscr{H}$. The expressions (7) may be inverted, using the orthogonality relations, as

$$a_i^* = \sum_{r=1}^N (X_i^{(r)} c_r^\ddagger + Y_i^{(r)} c_r), \quad a_i = \sum_{r=1}^N (-Y_i^{(r)} c_r^\ddagger + X_i^{(r)} c_r).$$

From these expressions, we obtain

$$H = \sum_{i=1}^N \sum_{r=1}^N \lambda_r Y_i^{(r)2} \mathbf{1} + \sum_{r=1}^N \lambda_r c_r^\ddagger c_r.$$

The eigenvectors of H are of the form

$$\Psi_{n_1,\dots,n_N} = c^{\ddagger n_1} \cdots c^{\ddagger n_N} \Psi_0,$$

where Ψ_0 is such that

$$c_1 \Psi_0 = 0, \dots, c_N \Psi_0 = 0,$$

and the respective eigenvalues are of the form

$$E_{n_1,\dots,n_N} = \sum_{i=1}^N \sum_{r=1}^N \lambda_r Y_i^{(r)2} + \sum_{r=1}^N n_r \lambda_r,$$

so that

$$H \Psi_{n_1,\dots,n_N} = E_{n_1,\dots,n_N} \Psi_{n_1,\dots,n_N}.$$

Similarly, the eigenvectors of

$$H^* = \sum_{i,j=1}^{N} \left(A_{ij} a_i^* a_j - \frac{1}{2} B_{ij} a_i^* a_j^* + \frac{1}{2} B_{ij} a_i a_j \right).$$

are given by

$$\Psi'_{n_1,\dots,n_N} = c^{*n_1} \cdots c^{*n_N} \Psi'_0,$$

where Ψ'_0 is such that

$$c_1^{\ddagger *} \Psi'_0 = 0, \dots, c_N^{\ddagger *} \Psi'_0 = 0.$$

The eigenvalues of H and H^* coincide. The associated eigenvector systems are biorthogonal:

$$\langle \Psi'_{n_1,\dots,n_N}, \Psi_{m_1,\dots,m_N} \rangle = n_1! \cdots n_N! \delta n_1 m_1 \cdots \delta n_N m_N \langle \Psi'_0, \Psi_0 \rangle.$$

Next, the existence of the vacua vectors Ψ_0 and Ψ'_0 is discussed. The real skew-symmetric matrix

$$\begin{bmatrix} S & T \\ -T & S \end{bmatrix},$$

induces the operator

$$\mathscr{S} = -\frac{1}{2} \sum_{i,j=1}^{N} \left(s_{ij} (a_i^* a_j + a_j a_i^*) + t_{ij} a_i^* a_j^* + t_{ij} a_i a_j \right), \quad (s_{ij}) = S, \ (t_{ij}) = T$$

which satisfies

$$\begin{aligned} \mathrm{e}^{\mathscr{S}} a_r^* \mathrm{e}^{-\mathscr{S}} &= c_r^{\ddagger} = \sum_{i=1}^{N} (X_i^{(r)} a_i^* - Y_i^{(r)} a_i), \\ \mathrm{e}^{\mathscr{S}} a_r \mathrm{e}^{-\mathscr{S}} &= c_r = \sum_{i=1}^{N} (Y_i^{(r)} a_i^* + X_i^{(r)} a_i), \quad r, s = 1, \dots, N. \end{aligned}$$

By definition, we shall consider

$$\mathscr{D} = \left\{ \sum_{n_1,\dots,n_N} z_{n_1,\dots,N} \ a^{*\ n_1} \cdots a^{*\ n_N} \ \Phi_0 : z_{n_1,\dots,N} \in \mathbb{C}, \ n_i \geq 0 \right\},$$

where the sum is finite. Some considerations are in order. We observe that a_j^* and a_j map $\mathscr{D}$ into $\mathscr{D}$ and that $\mathscr{S}^n \Phi_0 \in \mathscr{D}, \ 0 \leq n \in \mathbf{Z}$, where $\Phi_0 \in \mathscr{D}$ is the vacuum of the operators a_i, i.e. a vector such that

$$a_1 \Phi_0 = 0, \dots, a_N \Phi_0 = 0,$$

whose existence is postulated. It follows that

$$\Psi_0 = \mathrm{e}^{\mathscr{S}} \Phi_0, \quad \Psi'_0 = \mathrm{e}^{-S} \Phi_0.$$

Consider the series expansion

$$\sum_{n=0}^{\infty} \frac{(\gamma \mathscr{S})^n}{n!} \Phi_0, \quad \gamma = \pm 1.$$

The following question naturally arises: does it converge? Can we ensure that Ψ_0 belongs to $\mathscr{H}$? This point must be investigated on a case by case basis. In the next section it is considered for a specific example in which these questions are affirmatively answered.

The following remark is in order. The spectral analysis of a non self-adjoint Hamiltonian quadratic in bosonic operators should be preceded by the spectral analysis of its self-adjoint part. Recall that a self-adjoint Hamiltonian quadratic in bosonic operators may not have a real spectrum, as is the case of the self-adjoint operator

$$x^2 + \frac{\mathrm{d}^2}{\mathrm{d}x^2} : \mathscr{D} \to \mathscr{D},$$

which does not have real eigenvalues. Indeed, for instance,

$$\left(x^2 + \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right) \mathrm{e}^{i\,x^2/2} = i\,\mathrm{e}^{i\,x^2/2}.$$

Notice that $\mathrm{e}^{i\,x^2/2} \notin \mathscr{H}$, because $\langle \mathrm{e}^{i\,x^2/2}, \mathrm{e}^{i\,x^2/2} \rangle = +\infty$. In general, a non self-adjoint Hamiltonian may have a real spectrum and a system of eigenvectors only if the spectrum of its self-adjoint part is real. Only then its system of eigenvectors will be biorthogonal to the system of eigenvectors of the adjoint Hamiltonian.

3 Example

As a simple illustrative example of application of the EMM developed in the previous section, we consider the Hamiltonian in (4) for case N even and with

$$A = \bigoplus_{i=1}^{N/2} A_i, \quad B = \bigoplus_{i=1}^{N/2} B_i,$$

where

$$A_i = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_i \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 & \beta_i \\ \beta_i & 0 \end{bmatrix}, \quad \alpha_i, \beta_i > 0.$$

The EMM condition $[H, Z] = \lambda Z$, for

$$Z = X_1 a_1^* + X_2 a_2^* - Y_1 a_1 - Y_2 a_2 + \ldots + X_{N-1} a_{N-1}^* + X_N a_N^* - Y_{N-1} a_{N-1} - Y_N a_N,$$

by this order, leads to the real symmetric matrix

$$M = \bigoplus_{i=1}^{N/2} \begin{bmatrix} \alpha_i & 0 & 0 & \beta_i \\ 0 & \alpha_i & \beta_i & 0 \\ 0 & \beta_i & -\alpha_i & 0 \\ \beta_i & 0 & 0 & -\alpha_i \end{bmatrix}, \tag{8}$$

whose positive eigenvalues are as follows

$$\lambda_1 = \lambda_2 = \sqrt{\alpha_1^2+\beta_1^2},\ \lambda_3 = \lambda_4 = \sqrt{\alpha_2^2+\beta_2^2},\ \ldots\ ,\ \lambda_{N-1} = \lambda_N = \sqrt{\alpha_{N/2}^2+\beta_{N/2}^2}.$$

Notice that α_i and $\pm\beta_i$ are the eigenvalues of the blocks A_i and B_i, respectively. In terms of pseudo-bosonic operators, which are determined by the eigenvectors of (8) associated to positive and negative eigenvalues, H is given by

$$H = \sum_{r=1}^{N/2} \left(\sqrt{\alpha_r^2+\beta_r^2} - \alpha_r + \sqrt{\alpha_r^2+\beta_r^2} \left(c_{2r-1}^{\ddagger} c_{2r-1} + c_{2r}^{\ddagger} c_{2r} \right) \right).$$

For

$$\mathscr{S} = \theta_1(a_1^* a_2^* + a_1 a_2) + \theta_2(a_3^* a_4^* + a_3 a_4) + \ldots + \theta_{N/2}(a_{N-1}^* a_N^* + a_{N-1} a_N),$$

where $-\pi/2 \le \theta_i \le \pi/2$, $i = 1, \ldots, N/2$, we obtain

$$\begin{aligned} &\mathrm{e}^{\mathscr{S}} \left(\sqrt{\alpha_1^2+\beta_1^2}(a_1^* a_1 + a_2 a_2^*) + \ldots + \sqrt{\alpha_{N/2}^2+\beta_{N/2}^2}(a_{N-1}^* a_{N-1} + a_N a_N^*) \right) \mathrm{e}^{-\mathscr{S}} \\ &= \sqrt{\alpha_1^2+\beta_1^2}\cos 2\theta_1 (a_1^* a_1 + a_2 a_2^*) + \ldots + \sqrt{\alpha_{N/2}^2+\beta_{N/2}^2}\cos 2\theta_{N/2}(a_{N-1}^* a_{N-1} + a_N a_N^*) \\ &+ \sqrt{\alpha_1^2+\beta_1^2}\sin 2\theta_1 (a_1 a_2 - a_1^* a_2^*) + \ldots + \sqrt{\alpha_{N/2}^2+\beta_{N/2}^2}\sin 2\theta_{N/2}(a_{N-1} a_N - a_{N-1}^* a_N^*) \end{aligned}$$

Taking

$$\tan 2\theta_1 = -\frac{\beta_2}{\alpha_2},\ \ldots,\ \tan 2\theta_{N/2} = -\frac{\beta_{N/2}}{\alpha_{N/2}},$$

we find

$$\begin{aligned} &\mathrm{e}^{\mathscr{S}} \left(\sqrt{\alpha_1^2+\beta_1^2}(a_1^* a_1 + a_2 a_2^*) + \ldots + \sqrt{\alpha_{N/2}^2+\beta_{N/2}^2}(a_{N-1}^* a_{N-1} + a_N a_N^*) \right) \mathrm{e}^{-\mathscr{S}} \\ &= \alpha_1(a_1^* a_1 + a_2 a_2^*) + \beta_1(a_1^* a_2^* - a_1 a_2) + \ldots \\ &+ \alpha_{N/2}(a_{N-1}^* a_{N-1} + a_N a_N^*) + \beta_{N/2}(a_{N-1}^* a_N^* - a_{N-1} a_N). \end{aligned}$$

We have shown that the desired transformation is given by $e^{\mathscr{S}}$.

Recall that $\Phi_0 \in \mathscr{D}$ is the vacuum of the operators $a_i, i = 1\ldots, N$. Next we prove that $\langle \mathrm{e}^{\mathscr{S}}\Phi_0, \mathrm{e}^{\mathscr{S}}\Phi_0 \rangle < \infty$, so that the groundstate eigenvector of H is $\mathrm{e}^{\mathscr{S}}\Phi_0 \in$ span $\mathscr{D} = \mathscr{H}$. Indeed, it may be checked that the vector $\mathrm{e}^{\mathscr{S}}\Phi_0$ and the vector

$$\Xi_0 = \exp\left(\tan\theta_1\ a_1^* a_2^* + \ldots + \tan\theta_{N/2}\ a_{N-1}^* a_N^*\right)\Phi_0$$

$$= \sum_{n_1=1}^{\infty} \cdots \sum_{n_{N/2}=1}^{\infty} \frac{\tan^{n_1}\theta_1}{n_1!} \cdots \frac{\tan^{n_{N/2}}\theta_{N/2}}{n_{N/2}!} a_1^{*n_1} a_2^{*n_1} \cdots a_{N-1}^{*n_{N/2}} a_N^{*n_{N/2}} \ \Phi_0$$

may differ only by a numerical factor. Notice that

$$c_i \, \mathrm{e}^{\mathscr{S}} \Phi_0 = c_i \ \Xi_0 = 0, \ i = 1, \ldots, N.$$

Actually, $\mathrm{e}^{\mathscr{S}} \Phi_0$ reduces to Ξ_0 by a convenient rearrangement of the series. Now,

$$\tan\theta_i = \alpha_i/\beta_i - \sqrt{1 + (\alpha_i/\beta_i)^2},$$

so that $\tan^2\theta_i < 1$, and

$$\langle \Xi_0, \Xi_0 \rangle = \prod_{i=1}^{N/2} \sum_{n=0}^{\infty} \tan^{2n}\theta_i = \prod_{i=1}^{N/2} (1 - \tan^2\theta_i)^{-1} < \infty.$$

It follows that $\Xi_0 \in \mathscr{H}$. We observe that the geometric series $\sum_{n=0}^{\infty} \tan^{2n}\theta_i$ with ratio $\tan^2\theta_i < 1$, converges in the interior of the unitary disc.

4 Discussion

In Section 2, a non self-adjoint Hamiltonian, which is expressed as a quadratic combination of bosonic operators, is investigated. Its eigenvalues and eigenvectors have been determined with the help of a real symmetric matrix M of size $2N \times 2N$, where N is the number of bosonic states, that is determined by the EMM. The investigated Hamiltonian has a system of eigenvectors expressed in terms of the creation and annihilation operators of pseudo-bosons, which is biorthogonal to the system of eigenvectors of the adjoint Hamiltonian, constructed in terms of pseudo-bosonic operators acting on the associated vacuum state.

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